Hello and welcome to class!

Last time

We talked about higher order linear ODE.

This time

We will discuss systems of linear ODE.

Systems of linear ODE

Recall that a linear ODE was something like this:

$$y''(t) + \sin(t)y(t) = \cos(t)$$

A system of linear ODE is something like this:

$$y''(t) + \sin(t)x(t) = e^t$$

$$x'(t) + t^2 y(t) = 10t$$

Systems of linear ODE

More formally, a linear ODE was something like this:

$$\left(a_n(t)\frac{d^n}{dt^n} + \cdots + a_0(t)\right) y(t) = f(t)$$

A system of linear ODE is something like this:

$$\left(A_n(t)\frac{d^n}{dt^n} + \cdots + A_0(t)\right) \mathbf{y}(t) = \mathbf{f}(t)$$

I.e., exactly the same sort of thing, except now the functions are vector-valued (i.e., they are maps $\mathbb{R} \to \mathbb{R}^n$ rather than $\mathbb{R} \to \mathbb{R}$) and the coefficients are matrix-valued functions.

Systems of linear ODE

For example, the linear system

$$y''(t) + \sin(t)x(t) = e^t$$

$$x'(t) + t^2 y(t) = 10t$$

could be also written as

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{d^2}{dt^2} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{d}{dt} + \begin{pmatrix} 0 & t^2 \\ \sin(t) & 0 \end{pmatrix} \right) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 10t \\ e^t \end{pmatrix}$$

Vector valued functions

Implicit in this discussion has been the understanding that the set of functions $f : \mathbb{R} \to \mathbb{R}^n$ is a vector space, with componentwise addition and scalar multiplication.

E.g., if $\mathbf{f}(t), \mathbf{g}(t)$ are maps $\mathbb{R} \to \mathbb{R}^3$, we might write a linear combination of them $a\mathbf{f}(t) + b\mathbf{g}(t)$ as:

$$a \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} + b \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = \begin{pmatrix} af_1(x) + bg_1(x) \\ af_2(x) + bg_2(x) \\ af_3(x) + bg_3(x) \end{pmatrix}$$

Matrix valued functions

Some, but by no means all, linear transformations on the space of vector valued functions are given by matrix multiplication by matrix valued functions.

For example,

$$\begin{pmatrix} \sin(t) & t^2 \\ e^t & 4 \end{pmatrix} \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} = \begin{pmatrix} \sin(t)f(t) + t^2 g(t) \\ e^t f(t) + 4g(t) \end{pmatrix}$$

Why?

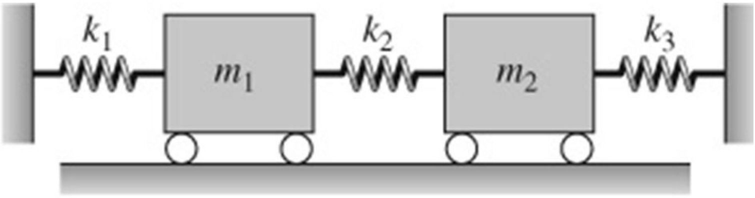
Systems of ODE arise when several quantities are varying simultaneously and depend on each other.

For example, consider a planet of mass m orbiting a star of mass M. We will take our coordinates so that the star is fixed, and disregard the pull of the planet on the star. Then the planet has coordinates $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$, and Newton's law of gravitation asserts

$$m\mathbf{x}''(t) = -\frac{GMm}{||\mathbf{x}||^3}\mathbf{x}$$

That's a system of 3 nonlinear ODE.

Why?



This system is governed by the equations

$$\begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) + k_2(x_2(t) - x_1(t)) \\ m_2 x_2''(t) &= -k_2(x_2(t) - x_1(t)) - k_3 x_2(t) \end{aligned}$$

Why?

Systems of linear ODE arise in, e.g., questions involving:

- springs attached to other springs
- more generally, complicated mechanical systems
- electrical circuits with resistors, inductors, and capacitors
- chemical processes not at equilibrium
- predator-prey models
- ... and in many more situations!

All linear ODE are first order ODE

You can also turn a single n'th order linear ODE into a system of first order linear ODE.

E.g., the second order linear ODE $y''(t) - ty(t) = 0$ is equivalent to the first order system

$$y'(t) = z(t)$$

$$z'(t) = ty(t)$$

which we could also write as

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$$

All linear ODE are first order ODE

For that matter, any system of linear ode can be written as a first order system, by introducing variables which take the place of the higher derivatives. E.g.,

$$x_1''(t) = x_1(t) + x_1'(t) + x_2(t)$$
$$x_2''(t) = 2x_1(t) + x_2(t) + x_2'(t)$$

can also be viewed as a first-order system by introducing functions y_1, y_2 which play the roles of the x_1', x_2':

$$\begin{aligned}
x_1'(t) &= y_1(t) \\
x_2'(t) &= y_2(t) \\
y_1'(t) &= x_1(t) + y_1(t) + x_2(t) \\
y_2'(t) &= 2x_1(t) + x_2(t) + y_2(t)
\end{aligned}$$

Try it yourself!

Write $y'''(t) + y''(t) + y'(t) + y(t) = 0$ as a first order system.

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \\ y''(t) \end{pmatrix}$$

I didn't rename the derivatives of y, which is common practice.

Normal form

Thus any system of linear ODE can be written in the form

$$\frac{d}{dt}\mathbf{v}(t) = A(t)\mathbf{v}(t) + \mathbf{f}(t)$$

where $\mathbf{v}(t)$ is a vector valued indeterminate function (i.e., that we are interested in solving for), $A(t)$ is a matrix valued function, and $\mathbf{f}(t)$ is a given function.

This is called an equation in normal form, and it is homogenous when $\mathbf{f}(t) = 0$.

Existence and uniqueness

For any continuous $A(t)$ and $\mathbf{f}(t)$, any time t_0, and any given vector $\mathbf{v}_0 \in \mathbb{R}^n$, the equation $\mathbf{v}'(t) = A(t)\mathbf{v}(t) + \mathbf{f}(t)$ has a unique solution with $\mathbf{v}(t_0) = \mathbf{v}_0$.

Equivalently, for any fixed number s, the following linear morphism is an isomorphism.

$$\begin{array}{rcl} ev_s : solutions & \to & \mathbb{R}^n \\ \mathbf{v} & \mapsto & \mathbf{v}(s) \end{array}$$

The Wronskian

So if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a collection of n solutions to a system of n linear ODE, then the morphism

$$\begin{aligned} ev_s : Span(\mathbf{v}_1, \ldots, \mathbf{v}_n) &\rightarrow Span(\mathbf{v}_1(s), \ldots, \mathbf{v}_n(s)) \\ \mathbf{v} &\mapsto \mathbf{v}(s) \end{aligned}$$

is an isomorphism for every s.

In particular, the $\mathbf{v}_i$ span the solution space if and only if the $\mathbf{v}_i(s)$ span $\mathbb{R}^n$, which happens if and only if the determinant of the matrix whose columns are the $\mathbf{v}_i(s)$ is nonzero. This is called the Wronskian determinant.

Fundamental matrix

Given a homogenous system $\mathbf{v}'(t) = A(t)\mathbf{v}(t)$, we can collect a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ for the solution space into a matrix $V(t)$ whose columns are the $\mathbf{v}_i$.

Note that such a matrix satisfies the matrix equation

$$V'(t) = A(t)V(t)$$

because each of its columns does.

Conversely, any matrix satisfying the above equation has columns which satisfy the vector equation.

Fundamental matrix

Consider a matrix $V(t)$ satisfying $V'(t) = A(t)V(t)$.

By the existence and uniqueness theorem, the following are equivalent:

- The columns of $V(t)$ are linearly independent as vector valued functions
- The columns of $V(t)$ are linearly independent as vectors for some t
- The determinant of $V(t)$ never vanishes
- The determinant of $V(t)$ is nonzero for some t.

A matrix $V(t)$ satisfying the above is called a fundamental matrix for the system.

Example

For example, consider the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, where

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

Let us check that matrix

$$X(t) = \begin{pmatrix} e^{3t} & -e^{3t} & -e^{-3t} \\ 0 & e^{3t} & -e^{-3t} \\ e^{3t} & 0 & e^{-3t} \end{pmatrix}$$

is a fundamental matrix.

Example

There are two things to check. First, that the columns of $X(t)$ are solutions, or in other words, that $X'(t) = AX(t)$. Second, that the columns are linearly independent. Let us check the first thing:

$$X(t) = \begin{pmatrix} e^{3t} & -e^{3t} & -e^{-3t} \\ 0 & e^{3t} & -e^{-3t} \\ e^{3t} & 0 & e^{-3t} \end{pmatrix} \quad X'(t) = \begin{pmatrix} 3e^{3t} & -3e^{3t} & 3e^{-3t} \\ 0 & 3e^{3t} & 3e^{-3t} \\ 3e^{3t} & 0 & -3e^{-3t} \end{pmatrix}$$

$$\begin{aligned} AX(t) &= \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} e^{3t} & -e^{3t} & -e^{-3t} \\ 0 & e^{3t} & -e^{-3t} \\ e^{3t} & 0 & e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} 3e^{3t} & -3e^{3t} & 3e^{-3t} \\ 0 & 3e^{3t} & 3e^{-3t} \\ 3e^{3t} & 0 & -3e^{-3t} \end{pmatrix} \end{aligned}$$

Example

Now that we know $X'(t) = AX(t)$, we know that the columns of $X(t)$ are linearly independent if and only if this is true at some given value of t.

$t = 0$ is a particularly good choice:

$$X(t) = \begin{pmatrix} e^{3t} & -e^{3t} & -e^{-3t} \\ 0 & e^{3t} & -e^{-3t} \\ e^{3t} & 0 & e^{-3t} \end{pmatrix} \qquad X(0) = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

It is easy to see that $X(0)$ is invertible e.g. by computing its determinant, or by row reducing. Thus we have checked that $X(t)$ is a fundamental matrix.

Try it yourself

Show that the equation

$$\mathbf{x}'(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}(t)$$

has a fundamental matrix

$$X(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

Using a fundamental matrix

Suppose you know a fundamental matrix for an equation $V'(t) = A(t)V(t)$ and now want to solve the initial value problem $\mathbf{v}'(t) = A(t)\mathbf{v}(t)$ subject to some initial values $\mathbf{v}(t_0) = \mathbf{v}_0$.

Because $V(t)$ is a fundamental matrix, any solution is a linear combination of its columns, i.e. takes the form $\mathbf{v}(t) = V(t)\mathbf{c}$ for some coefficient vector $\mathbf{c}$.

We want to find $\mathbf{v}(t)$ such that $\mathbf{v}(t_0) = V(t_0)\mathbf{c} = \mathbf{v}_0$. Thus $\mathbf{c} = V(t_0)^{-1}\mathbf{v}_0$ and so $\mathbf{v}(t) = V(t)V(t_0)^{-1}\mathbf{v}_0$.

Using a fundamental matrix

For example, we saw that the equation $\mathbf{x}'(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x}(t)$ has a fundamental matrix $X(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$.

Let us solve the initial value problem for a $\mathbf{x}(t)$ with $\mathbf{x}(0) = (1, 2)$.

$$\mathbf{x}(t) = X(t)X(0)^{-1}\mathbf{x}(0) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} e^t + e^{-t} \\ e^t + 3e^{-t} \end{pmatrix}$$

Try it yourself

We saw that the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$, where

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

has a fundamental matrix

$$X(t) = \begin{pmatrix} e^{3t} & -e^{3t} & -e^{-3t} \\ 0 & e^{3t} & -e^{-3t} \\ e^{3t} & 0 & e^{-3t} \end{pmatrix}$$

Find some $\mathbf{x}(t)$ satisfying $\mathbf{x}'(t) = A\mathbf{x}(t)$ such that $\mathbf{x}(0) = (1, 2, 3)$.