Hello and welcome to class!

Last time

We finished talking about second order constant coefficient linear ordinary differential equations.

This time

We discuss the generalization to the higher order case. This is relatively straightforward at least if you completely understood the second order case.

Warm-up

Solve the differential equation

$$y''(x) + 6y'(x) + 5y(x) = e^{-x}$$

subject to the initial values $y(0) = 1$ and $y'(0) = 2$.

The auxiliary polynomial factors as $z^2 + 6z + 5 = (z + 1)(z + 5)$.

So does the differential operator: the equation can be written as:

Warm-up

$$\left(\frac{d}{dx}+5\right)\left(\frac{d}{dx}+1\right)y = e^{-x}$$

Thus a basis for the solution space of the homogenous equation is given by e^{-x}, e^{-5x}.

To find one solution of the inhomogenous equation, we know we should try xe^{x}.

$$\left(\frac{d}{dx}+5\right)\left(\frac{d}{dx}+1\right)xe^{-x} = \left(\frac{d}{dx}+5\right)e^{-x} = 4e^{-x}$$

So one solution to the inhomogenous equation is given by $\frac{1}{4}xe^{-x}$.

Warm-up

Thus the general solution to the inhomogenous equation is

$$y(x) = Ae^{-x} + Be^{-5x} + \frac{1}{4}xe^{-x}$$

We now want to find A and B such that

$$1 = y(0) = Ae^{-x} + Be^{-5x} + \frac{1}{4}xe^{-x}\Big|_{x=0} = A + B$$

$$2 = y'(0) = -Ae^{-x} - 5Be^{-5x} + \frac{1}{4}e^{-x} - \frac{1}{4}xe^{-x}\Big|_{x=0} = -A - 5B + \frac{1}{4}$$

Warm-up

It remains to solve the linear equation

$$\begin{pmatrix} 1 \\ 7/4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -5 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Inverting the matrix,

$$\begin{pmatrix} A \\ B \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} -5 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 7/4 \end{pmatrix} = \begin{pmatrix} 27/16 \\ -11/16 \end{pmatrix}$$

Thus the final answer is

$$y(x) = \frac{27}{16} e^{-x} - \frac{11}{16} e^{-5x} + \frac{1}{4} x e^{-x}$$

Some linear operations on functions

Multiplying by a function is linear, by the distributive property.

$$\begin{aligned} a(x)\cdot : \text{functions} &\rightarrow \text{functions} \\ f(x) &\mapsto a(x)f(x) \end{aligned}$$

Taking a derivative is also linear:

$$\begin{aligned} \frac{d}{dx} : \text{differentiable functions} &\rightarrow \text{functions} \\ f(x) &\mapsto f'(x) \end{aligned}$$

Linear differential operators

Definition

An n'th order linear differential operator is a linear map of the form

$$f(x) \mapsto a_n(x)f^{(n)}(x)+a_{n-1}(x)f^{(n-1)}(x)+\cdots+a_1(x)f'(x)+a_0(x)f(x)$$

We denote this linear map by

$$a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

Linear differential operators

The operator $f \mapsto \frac{d}{dx}(a(x)f(x))$ is also a linear map.

Indeed, it is the composition of two linear maps, namely multiplication and differentiation.

However, they are placed in the opposite order as we have allowed in the definition.

Why isn't it allowed?

Linear differential operators

The answer is that we did allow this operator.

Observe:

$$\frac{d}{dx}\left(a(x)f(x)\right) = a'(x)f(x) + a(x)f'(x) = \left(a'(x) + a(x)\frac{d}{dx}\right) f(x)$$

In fact, any product of linear differential operators is again a linear differential operator, though some work must be done to write it in the form specified in the definition.

Linear differential equations

Definition

An n'th order homogenous linear ordinary differential equation is

$$\mathbf{L}y(x) = 0$$

where $\mathbf{L}$ is an n'th order linear differential operator.

An n'th order inhomogenous linear ordinary differential equation is similarly

$$\mathbf{L}y(x) = f(x)$$

Existence and uniqueness

Consider a linear differential operator

$$\mathbf{L} = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1(x)\frac{d}{dx} + a_0(x)$$

Assume that the coefficients $a_n(x)$ are continuous on an interval (x_{min}, x_{max}).

For any $x_0 \in (x_{min}, x_{max})$, and specified values $y_0, y_0', \ldots, y_0^{(n-1)}$,

There exists a unique function y defined on (x_{min}, x_{max}) with $\mathbf{L}y = 0$ and $y^{(i)}(x_0) = y_0^{(i)}$.

Existence and uniqueness

Equivalently, for every $x \in (x_{min}, x_{max})$, the map

$$\begin{aligned} ev_x : \mathrm{Kernel}(\mathbf{L}) &\to \mathbb{R}^n \\ y &\mapsto (y(x), y'(x), \ldots, y^{(n-1)}(x)) \end{aligned}$$

is an isomorphism.

Existence and uniqueness

Note this is saying something rather surprising. In general, for a collection of functions, there is no reason to expect that recording the derivatives at a point should determine an isomorphism, let alone that this should be true at every point. That is, the existence and uniqueness theorem is asserting that the kernel of a linear differential operator has rather special properties.

In fact, it is possible to show that this property characterizes such kernels. That is, given a collection of n sufficiently differentiable functions such that recording their derivatives at every point in (x_{min}, x_{max}) determines an isomorphism from their span to $\mathbb{R}^n$, there is an n'th order linear differential operator such that these functions span the kernel.

The Wronskian

Another way to express this notion: the Wronskian determinant.
Given n functions $y_1, \ldots, y_n$, and a point x, one forms:

$$\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

If $y_1, \ldots, y_n$ solve an n'th order linear differential equation, then this determinant is zero if and only if $y_1(x), \ldots, y_n(x)$ are linearly dependent as functions.

Wrong Wronskians

Consider the functions x, x^2. Their Wronskian at 0 is

$$\begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

But x and x^2 are linearly independent as functions!

What happened?

The Wronskian only works to detect linear dependence of n functions when they are solutions to some n'th order linear differential equation.

Constant coefficient equations

We now restrict ourselves to the study of operators of the form

$$\mathbf{L} = a_n \frac{d^n}{dx^n} + a_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_1 \frac{d}{dx} + a_0$$

We will construct the complete set of solutions to the homogenous equation $\mathbf{L}y = 0$.

Constant coefficient equations

We begin by factoring $\mathbf{L}$ into linear operators. To do this, first form the auxiliary polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Let r_i be the roots, and let m_i be the multiplicity of the root r_i. I.e., suppose

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = \prod_i (x - r_i)^{m_i}$$

The fundamental theorem of algebra assures us such a factorization exists, at least if we allow the r_i to be complex numbers.

Constant coefficient equations

The operator **L** factors in the same way as its auxiliary polynomial:

$$\mathbf{L} = \prod_i \left(\frac{d}{dx} - r_i \right)^{m_i}$$

This factorization holds because composing derivatives behaves, formally, just like multiplying polynomials, and in particular commutes with scalar multiplication.

The factors commute and so any function in the kernel of some collection of factors will also be in the kernel of **L**.

Constant coefficient equations

We know that the kernel of $\left(\frac{d}{dx} - r\right)$ is spanned by e^{rx}.

In our study of order 2 equations, we saw that the kernel of $\left(\frac{d}{dx} - r\right)^2$ is spanned by e^{rx} and xe^{rx}

More generally, $\left(\frac{d}{dx} - r\right)^n (f(x)e^{rx}) = f^{(n)}(x)e^{rx}$, so all of $\mathbb{P}_{n-1}e^{rx}$ is annihilated by this operator.

Constant coefficient equations

Let us return to the operator

$$\mathbf{L} = \prod_i \left(\frac{d}{dx} - r_i \right)^{m_i}$$

We have just seen that

$$\{e^{r_1 x}, xe^{r_1 x}, \ldots, x^{m_1 - 1} e^{r_1 x}, e^{r_2 t}, xe^{r_2 x}, \ldots, x^{m_2 - 1} e^{r_2 x}, \ldots\}$$

are in the kernel.

There are $\sum m_i = \mathrm{order}(\mathbf{L})$ of these, so to see that we have a basis for the space of solutions, it remains only to see that these are linearly independent.

Constant coefficient equations

Indeed, suppose there were some linear dependence

$$\sum_{i,j} c_{i,j} x^i e^{r_j x} = 0$$

Some coefficient must be nonzero; without loss of generality it may as well be the coefficient $c_{k,1}$ of $x^k e^{r_1 x}$ for some k.

In fact we choose k such that it is the largest number such that $c_{k,1}$ does not vanish.

Constant coefficient equations

Now we study

$$\left(\prod_{i>1}\left(\frac{d}{dx}-r_i\right)^{m_i}\right)\left(\frac{d}{dx}-r_1\right)^k \sum_{i,j} c_{i,j}x^i e^{r_j x}$$

The differential operator annihilates all terms $x^i e^{r_j x}$ for $j > 1$.

It also annihilates terms $x^i e^{r_1 x}$ for $i < k$.

And, the coefficients of terms $x^i e^{r_1 x}$ for $i > k$ are zero, because of how we chose k.

The only remaining term comes from $c_{k,1}x^k e^{r_1 x}$.

Constant coefficient equations

$$\left(\prod_{i>1}\left(\frac{d}{dx}-r_i\right)^{m_i}\right)\left(\frac{d}{dx}-r_1\right)^k c_{k,1}x^k e^{r_1 x}$$

$$=\left(\prod_{i>1}\left(\frac{d}{dx}-r_i\right)^{m_i}\right)k!c_{k,1}e^{r_1 x}$$

$$=\left(\prod_{i>1}(r_1-r_i)^{m_i}\right)k!c_{k,1}e^{r_1 x}$$

On the one hand, this has to be zero, since we got it by applying a linear operator to an expression which witnessed a linear dependence — by being zero. On the other hand, this can only happen if $c_{k,1}$ is zero which is a contradiction. Thus there is no linear dependence among the functions.

Systems of linear ODE

Recall that a linear ODE was something like this:

$$y''(t) + \sin(t)y(t) = \cos(t)$$

A system of linear ODE is something like this:

$$y''(t) + \sin(t)x(t) = e^t$$

$$x'(t) + t^2 y(t) = 10t$$

Systems of linear ODE

More formally, a linear ODE was something like this:

$$\left(a_n(t)\frac{d^n}{dt^n} + \cdots + a_0(t)\right) y(t) = f(t)$$

A system of linear ODE is something like this:

$$\left(A_n(t)\frac{d^n}{dt^n} + \cdots + A_0(t)\right) \mathbf{y}(t) = \mathbf{f}(t)$$

I.e., exactly the same sort of thing, except now the functions are vector-valued (i.e., they are maps $\mathbb{R} \to \mathbb{R}^n$ rather than $\mathbb{R} \to \mathbb{R}$) and the coefficients are matrix-valued functions.

Systems of linear ODE

For example, the linear system

$$y''(t) + \sin(t)x(t) = e^t$$

$$x'(t) + t^2 y(t) = 10t$$

could be also written as

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{d^2}{dt^2} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{d}{dt} + \begin{pmatrix} 0 & t^2 \\ \sin(t) & 0 \end{pmatrix} \right) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 10t \\ e^t \end{pmatrix}$$

All linear ODE are first order ODE

You can also turn a single n'th order linear ODE into a system of first order linear ODE.

E.g., the second order linear ODE $y''(t) - ty(t) = 0$ is equivalent to the first order system

$$y'(t) = z(t)$$

$$z'(t) = ty(t)$$

which we could also write as

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$$