Hello and welcome to class!

Last time

We started discussing differential equations. We found a complete set of solutions to the second order linear homogenous constant coefficient ordinary differential equation.

This time

We'll say more about the initial value problem and discuss some methods to approach the inhomogenous case.

The initial value problem

For a differential equation, in our case $ay'' + by' + cy = 0$,

Given some starting time t_0 and constants y_0 and y_0',

We want to find a function $y(t)$ such that the differential equation is satisfied, $y(t_0) = y_0$, and $y'(t_0) = y_0'$.

The initial value problem

Theorem

For an ordinary, linear, constant coefficient, homogenous, second-order differential equation, the initial value problem has a unique solution.

The initial value problem

Another way to say this: the following map is an isomorphism:

$$\begin{array}{rcl} \{\text{Solutions to } ay'' + by' + cy = 0\} & \to & \mathbb{R}^2 \\ y & \mapsto & (y(t_0), y'(t_0)) \end{array}$$

More precisely, existence asserts that this map is surjective, i.e., that there is always a solution with specified value and first derivative, whereas uniqueness asserts that it is injective, i.e., there is at most one such solution.

The initial value problem

We won't prove uniqueness in this class although I gave some ideas last time about why it might be true. Let us just accept it.

Last time, we saw there was always a two dimensional space of solutions to a differential equation of the form $ay'' + by' + cy = 0$.

Thus we have an injective linear map between 2-dimensional vector spaces which is therefore an isomorphism. This settles existence.

Solving the initial value problem

Solving the initial value problem in practice uses the same ideas as the above argument. For example, consider the equation

$$y'' + y = 0$$

Last time, you learned (or perhaps could have guessed) that $\cos(t), \sin(t)$ give a basis for the space of solutions.

Let us now "solve the initial value problem" of finding a solution which has $y(0) = 3$ and $y'(0) = 4$.

Solving the initial value problem

Begin with the general solution $y(t) = A\cos(t) + B\sin(t)$. We want to determine the values of A and B for which $y(0) = 3$ and $y'(0) = 4$. So we compute:

$$3 = y(0) = A\cos(0) + B\sin(0) = A$$

$$4 = y'(0) = A\cos'(0) + B\sin'(0) = -A\sin(0) + B\cos(0) = B$$

So the solution to this "initial value problem" is

$$y(t) = 3\cos(t) + 4\sin(t)$$

Solving the initial value problem

In general, the last step may involve more complicated linear algebra. Suppose instead we wanted a solution with $y(1) = 1$ and $y'(1) = 2$. Then we would have

$$1 = y(0) = A\cos(1) + B\sin(1)$$
$$2 = y'(0) = -A\sin(1) + B\cos(1)$$

Or in other words,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

Solving the initial value problem

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos(1) & \sin(1) \\ -\sin(1) & \cos(1) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

Fortunately, you know how to solve this. Inverting the matrix,

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos(1) - 2\sin(1) \\ \sin(1) + 2\cos(1) \end{bmatrix}$$

and the solution to the initial value problem is:

$$y(t) = (\cos(1) - 2\sin(1))\cos(t) + (\sin(1) + 2\cos(1))\sin(t)$$

Inhomogenous equations

We will now try and solve equations of the form

$$ay''(t) + by'(t) + cy(t) = f(t)$$

for some pre-given function f.

Let us write this as

$$\left(a\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right) y(t) = f(t)$$

This is a linear equation just like our matrix equations $A\mathbf{x} = \mathbf{b}$.

Inhomogenous equations

Recall that $A\mathbf{x} = \mathbf{b}$ could be solved if and only if $\mathbf{b}$ was in the range of A. Moreover, solving this equation was the same as taking the given spanning set for the range, namely the columns of A, and finding a way to write $\mathbf{b}$ as a linear combination of these columns.

In the finite dimensional case, we had sitting in front of us a spanning set for the range. In this present, infinite dimensional case, we not have such a basis, but any such would be infinite.

Inhomogenous equations

In a systematic treatment, we would try understand the range of the linear transformation

$$\left(a\frac{d^2}{dt^2} + b\frac{d}{dt} + c \right)$$

But, this would involve infinite dimensional linear algebra in a serious way, and is beyond the scope of this class.

Inhomogenous equations

Instead, we will just feed some (perhaps somewhat arbitrary) functions into the linear transformation

$$\left(a\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right)$$

and thereby learn some elements of the range.

Then, whenever we want to try and solve

$$\left(a\frac{d^2}{dt^2} + b\frac{d}{dt} + c\right) y(t) = f(t)$$

we will ask whether the $f(t)$ in question is in the span of those elements of the range we have found.

First order inhomogenous equations

Let's do the first-order case.

I.e., we want to study the range of $\frac{d}{dt} - r$.

We'll do so by just computing $(\frac{d}{dt} - r)f(x)$ for various functions.

First order inhomogenous equations: polynomials

Let's begin with polynomials.

$$\left(\frac{d}{dt} - r\right) 1 = -r$$

$$\left(\frac{d}{dt} - r\right) t = 1 - rt$$

$$\left(\frac{d}{dt} - r\right) t^2 = 2t - rt^2$$

$$\left(\frac{d}{dt} - r\right) t^3 = 3t^2 - rt^3$$

First order inhomogenous equations: polynomials

We see that all polynomials are in the range. More precisely,

$$\left(\frac{d}{dt} - r\right)\mathbb{P}_n = \begin{cases} \mathbb{P}_n, & r \neq 0 \\ \mathbb{P}_{n-1}, & r = 0 \end{cases}$$

Example

Solve the differential equation $y' + y = t^2$.

We know that t^2 is contained in $\left(\frac{d}{dt} + 1\right)\mathbb{P}_2$.

To find the elements which map to it is now a linear algebra problem.

We could solve it by e.g. choosing a basis in $\mathbb{P}_2$ and writing $\left(\frac{d}{dt} + 1\right)$ as a matrix.

Example

Or in other words, we write

$$t^2 = \left(\frac{d}{dt} + 1\right)(At^2 + Bt + C) = A(2t + t^2) + B(1 + t) + C$$

and then solve for A, B, C.

This is called "the method of undetermined coefficients".

It could also just be called linear algebra. In this case, by inspection $A = 1, B = -2, C = 2$, and a solution is given by $y(t) = t^2 - 2t + 2$.

First order inhomogenous equations: exponentials

Now let's try an exponential function.

$$\left(\frac{d}{dt} - r\right) e^{st} = se^{st} - re^{st} = (s - r)e^{st}$$

So, e^{st} is in the range, at least if $s \neq r$.

First order inhomogenous equations: poly times exp

Now let's try a polynomial times an exponential function.

$$\left(\frac{d}{dt} - r\right) e^{st} = se^{st} - re^{st} = (s-r)e^{st}$$

$$\left(\frac{d}{dt} - r\right) te^{st} = e^{st} + ste^{st} - rte^{st} = e^{st} + (s-r)te^{st}$$

$$\left(\frac{d}{dt} - r\right) t^2e^{st} = 2te^{st} + st^2e^{st} - rt^2e^{st} = 2te^{st} + (s-r)t^2e^{st}$$

$$\left(\frac{d}{dt} - r\right) t^3e^{st} = 3t^2e^{st} + st^3e^{st} - rt^3e^{st} = 3t^2e^{st} + (s-r)t^3e^{st}$$

First order inhomogenous equations: poly times exp

Thus any polynomial times e^{st} is in the range.

More precisely, writing

$$\mathbb{P}_n e^{st} := \{(\text{degree} \leq n \text{ polynomial}) e^{st}\}$$

we have

$$\left(\frac{d}{dt} - r\right) \mathbb{P}_n e^{st} = \begin{cases} \mathbb{P}_n e^{st}, & r \neq s \\ \mathbb{P}_{n-1} e^{st}, & r = s \end{cases}$$

Helpful fact

$$\left(\frac{d}{dt} - r\right)\left(f(t)e^{rt}\right) = f'(t)e^{rt} + f(t)re^{rt} - f(t)re^{rt} = f'(t)e^{rt}$$

Example

Solve the equation $y' - 3y = te^{3t}$.

We know that te^{3t} is in the image of $\mathbb{P}_2 e^{3t}$ so we should try a general element of this space.

$$te^{3t} = \left(\frac{d}{dt} - 3\right)(At^2e^{3t} + Bte^{3t} + Ce^{3t}) = A \cdot 2te^{3t} + B \cdot e^{3t}$$

so $A = 1/2$ and $B = 0$ and a solution is

$$y(t) = \frac{1}{2}t^2e^{3t}$$

Second order equations

We return now to the second order case

$$ay''(t) + by'(t) + cy(t) = f(t)$$

We should now study the range of the operator

$$a\left(\frac{d}{dt}\right)^2 + b\left(\frac{d}{dt}\right) + c$$

Factoring

Let $r_{\pm}$ be the roots of the equation $ax^2 + bx + c$. Then

$$a\left(\frac{d}{dt}\right)^2 + b\left(\frac{d}{dt}\right) + c = a\left(\frac{d}{dt} - r_+\right)\left(\frac{d}{dt} - r_-\right)$$

What does that mean?

Each of the above items are linear transformations on the space of (sufficiently) differentiable functions to itself. We are asserting that the composition of the two on the right is the one on the left. (Note that we could have written them in the other order.)

Factoring

We already saw

$$\left(\frac{d}{dt} - r\right)\mathbb{P}_n e^{st} = \begin{cases} \mathbb{P}_n e^{st}, & r \neq s \\ \mathbb{P}_{n-1} e^{st}, & r = s \end{cases}$$

Doing it twice,

$$\left(\frac{d}{dt} - r_+\right)\left(\frac{d}{dt} - r_-\right)\mathbb{P}_n e^{st} = \begin{cases} \mathbb{P}_n e^{st}, & s \notin \{r_+, r_-\} \\ \mathbb{P}_{n-1} e^{st}, & s = r_+ \,\mathrm{or}\, r_-, \text{ not both} \\ \mathbb{P}_{n-2} e^{st}, & s = r_+ = r_- \end{cases}$$

Factoring

Let's just rewrite that:

$$\left(a\left(\frac{d}{dt}\right)^2 + b\left(\frac{d}{dt}\right) + c\right)\mathbb{P}_n e^{st} = \begin{cases} \mathbb{P}_n e^{st}, & s \notin \{r_+, r_-\} \\ \mathbb{P}_{n-1} e^{st}, & s = r_+ \,\mathrm{or}\, r_-, \text{ not both} \\ \mathbb{P}_{n-2} e^{st}, & s = r_+ = r_- \end{cases}$$

where $r_\pm$ are the roots of $ax^2 + bx + c$.

Example

Solve the equation $y'' - 4y' + 4 = e^{2t}$.

Let's rewrite that as

$$\left(\frac{d}{dt} - 2\right)^2 y(t) = e^{2t}$$

We know that e^{2t} is in the image of $\mathbb{P}_2 e^{2t}$. So we should try the general element of this space.

Example

$$e^{2t} = \left(\frac{d}{dt} - 2\right)^2 \left((At^2 + Bt + C)e^{2t}\right) = 2Ae^{2t}$$

So a solution is given by

$$y(t) = \frac{1}{2}t^2e^{2t}$$

Example

Solve the equation $y'' - 4y' + 4 = e^{2t} + e^{3t}$.

Observe that if we solve, separately, the equations $y'' - 4y' + 4 = e^{2t}$ and $y'' - 4y' + 4 = e^{3t}$ then we can add the solutions to get a solution to the equation above, by linearity.

In the context of differential equations, linearity is sometimes called "the superposition principle".

Example

We already found a solution to $y'' - 4y' + 4 = e^{2t}$, namely $y(t) = \frac{1}{2}t^2e^{2t}$.

Let us now solve $y'' - 4y' + 4 = e^{3t}$. This time, 3 is not a root of the auxilliary equation, so we know $e^{3t} \in \left(\frac{d}{dt} - 2\right)^2 \mathbb{P}_0 e^{3t}$. So we try the general element of this space. Since

$$\left(\frac{d}{dt} - 2\right)^2 Ae^{3t} = Ae^{3t}$$

a solution is given by $y(t) = e^{3t}$.

Example

Finally, adding the formulas

$$\left(\frac{d}{dt} - 2\right)^2 \left(\frac{1}{2}t^2 e^{2t}\right) = e^{2t}$$

$$\left(\frac{d}{dt} - 2\right)^2 e^{3t} = e^{3t}$$

we find

$$\left(\frac{d}{dt} - 2\right)^2 \left(\frac{1}{2}t^2 e^{2t} + e^{3t}\right) = e^{2t} + e^{3t}$$

or in other words, $y(t) = \frac{1}{2}t^2 e^{2t} + e^{3t}$ solves the equation $y'' - 4y' + 4y = e^{2t} + e^{3t}$.

The general solution to an inhomogenous equation

For the equation $A\mathbf{x} = \mathbf{b}$, we observed on the one hand that

$$A\mathbf{x}_0 = \mathbf{b} \qquad \& \qquad A\mathbf{y} = 0 \qquad \Longrightarrow \qquad A(\mathbf{x}_0 + \mathbf{y}) = \mathbf{b}$$

and on the other hand that

$$A\mathbf{x}_0 = \mathbf{b} \qquad \& \qquad A\mathbf{x}_1 = \mathbf{b} \qquad \Longrightarrow \qquad A(\mathbf{x}_1 - \mathbf{x}_0) = 0$$

The general solution to an inhomogenous equation

In other words: given one solution to the inhomogenous equation, all other solutions can be found by adding to it a solution of the homogenous equation.

The same is true in the context of linear differential equations, and for exactly the same reason.

Existence and uniqueness: inhomogenous case

Theorem

Assume $ay'' + by' + cy = f(t)$ has a solution $\tilde{y}(t)$. Then for any t_0 and specified values y_0, y_0', there exists a unique solution to the initial value problem, i.e., a unique $y(t)$ satisfying the differential equation such that $y(t_0) = y_0$ and $y'(t_0) = y_0'$.

Proof.

The desired y is the sum of $\tilde{y}$ and the unique y_h satisfying the homogenous equation $ay'' + by' + cy = 0$ subject to the initial value condition $y_h(t_0) = y_0 - \tilde{y}(t_0)$ and $y_h'(t_0) = y_0' - \tilde{y}'(t_0)$. □

Example

Find a solution to $y'' - 4y' + 4y = e^{2t} + e^{3t}$ satisfying the conditions $y(0) = 1$ and $y'(0) = 2$.

We already found one solution to the inhomogenous equation, namely $\frac{1}{2}t^2e^{2t} + e^{3t}$.

From last time, we know how to find solutions to the homogenous equation, a basis is given by e^{2t}, te^{2t}

Example

So our desired solution takes the form

$$\frac{1}{2}t^2e^{2t} + e^{3t} + Ae^{2t} + Bte^{2t}$$

This has derivative

$$te^{2t} + t^2e^{2t} + 3e^{3t} + 2Ae^{2t} + Be^{2t} + 2tBe^{2t}$$

Plugging in $t = 0$ and comparing to our desired values:

$$1 = y(0) = 1 + A$$

$$2 = y'(0) = 3 + 2A + B$$

hence $A = 0$ and $B = -1$ and the solution to the initial value problem is given by

$$y(t) = \frac{1}{2}t^2e^{2t} + e^{3t} - te^{2t}$$