# Hello and welcome to class!

## Last time

We learned what linear equations were, and discussed how to solve them by row reduction.

## Today

We'll start by discussing finer points of this technique, and then discuss other ways to conceptualize systems of linear equations.

# Echelon

This term originally meant a diagonal military formation

This formation has been used since antiquity for the large range of vision it offers to each member. It also offers reduced drag for airplanes, birds, and cyclists.

# Echelon form for matrices

A matrix is in echelon form when every nonzero entry of every row (other than the first) is strictly to the right of a nonzero entry in the row above.

$$\begin{bmatrix} 0 & 0 & \blacksquare & * & * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\blacksquare$ means a nonzero entry, and $*$ means any entry (zero or not).

# Reduced echelon form

A echelon matrix is in reduced echelon form when the first nonzero entry of every row is 1, and every other entry in its column is 0.

$$\begin{bmatrix} 0 & 0 & 1 & * & * & 0 & * & * & 0 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Echelon versus reduced echelon

$$\begin{bmatrix} 0 & 0 & \blacksquare & * & * & * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & * & * & 0 & * & * & 0 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row reduction algorithm

- Choose a nonzero entry in the leftmost nonzero column.
- Bring it to the top row by exchanging rows if necessary. This entry is now said to be in a pivot position.
- Divide the row by the value of the pivot.
- Add multiples of this row to the other rows (including "covered rows") to zero out entries in the pivot column.
- Cover up this row and all rows above it, and repeat the procedure on the remaining matrix.

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 1 & 3 & 5 & 4 & 11 \\ 0 & 1 & 4 & 7 & 5 & 14 \\ 0 & 2 & 8 & 17 & 10 & 31 \end{array}\right]$$

Choose a nonzero entry in the leftmost nonzero column.

$$\left[\begin{array}{ccccc|c} 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & \boxed{1} & 3 & 5 & 4 & 11 \\ 0 & 1 & 4 & 7 & 5 & 14 \\ 0 & 2 & 8 & 17 & 10 & 31 \end{array}\right]$$

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & \boxed{1} & 3 & 5 & 4 & 11 \\ 0 & 1 & 4 & 7 & 5 & 14 \\ 0 & 2 & 8 & 17 & 10 & 31 \end{array}\right]$$

Bring it to the top row by exchanging rows if necessary.

$$\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 3 & 5 & 4 & 11 \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 1 & 4 & 7 & 5 & 14 \\ 0 & 2 & 8 & 17 & 10 & 31 \end{array}\right]$$

This entry is now said to be in a pivot position.

# Let's do an example

$$\left[\begin{array}{ccccc|c} \color{red}{0} & \boxed{\color{red}{1}} & \color{red}{3} & \color{red}{5} & \color{red}{4} & \color{red}{11} \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 1 & 4 & 7 & 5 & 14 \\ 0 & 2 & 8 & 17 & 10 & 31 \end{array}\right]$$

Add multiples of this row to the other rows (including "covered rows") to zero out entries in the pivot column.

$$\left[\begin{array}{ccccc|c} \color{red}{0} & \boxed{\color{red}{1}} & \color{red}{3} & \color{red}{5} & \color{red}{4} & \color{red}{11} \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 7 & 2 & 9 \end{array}\right]$$

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 3 & 5 & 4 & 11 \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 7 & 2 & 9 \end{array}\right]$$

Cover up this row and all rows above it, and repeat the procedure on the remaining matrix.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 3 & 5 & 4 & 11 \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 7 & 2 & 9 \end{array}\right]$$

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 1 & 3 & 5 & 4 & 11 \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 2 & 7 & 2 & 9 \end{array}\right]$$

Choose a nonzero entry in the leftmost nonzero column.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 3 & 5 & 4 & 11 \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 0 & \boxed{1} & 2 & 1 & 3 \\ 0 & 0 & 2 & 7 & 2 & 9 \end{array}\right]$$

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 1 & 3 & 5 & 4 & 11 \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 0 & \boxed{1} & 2 & 1 & 3 \\ 0 & 0 & 2 & 7 & 2 & 9 \end{array}\right]$$

Bring it to the top row by exchanging rows if necessary.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 3 & 5 & 4 & 11 \\ 0 & 0 & \boxed{1} & 2 & 1 & 3 \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 0 & 2 & 7 & 2 & 9 \end{array}\right]$$

This entry is now said to be in a pivot position.

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 1 & 3 & 5 & 4 & 11 \\ 0 & 0 & \boxed{1} & 2 & 1 & 3 \\ 0 & 0 & 4 & 10 & 4 & 14 \\ 0 & 0 & 2 & 7 & 2 & 9 \end{array}\right]$$

Add multiples of this row to the other rows (including "covered rows") to zero out entries in the pivot column.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

Cover up this row and all rows above it, and repeat the procedure on the remaining matrix.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

Choose a nonzero entry in the leftmost nonzero column.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & \boxed{2} & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & \boxed{2} & 0 & 2 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

Divide the row by the value of the pivot

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

# Let's do an example

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 3 \end{array}\right]$$

Add multiples of this row to the other rows (including "covered rows") to zero out entries in the pivot column.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

# Uniqueness of the result

We made choices in the algorithm, but the resulting reduced echelon matrix will always be the same.

If we had not bothered dividing through or zeroing out entries above pivots, the resulting echelon matrix is not uniquely determined, but its shape is.

This shape can be encoded by the location of the pivot entries.

# Try it yourself!

Consider this matrix.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Is it in echelon form? yes

Is it in reduced echelon form? no

Where are the pivots?

# Try it yourself!

Consider this matrix.

$$\begin{bmatrix} 4 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Is it in echelon form? no

Is it in reduced echelon form? no

Where are the pivots? we have to row reduce it to find out

$$\begin{bmatrix} 4 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 1 \\ 0 & -5 & -2 \end{bmatrix}$$

# Row reduction for solving systems

## Last time

We saw that row reduction of the augmented matrix of a system of linear equations encodes a series of transformations of the system into simpler, but equivalent, systems.

But what do you do at the end?

# Reading off the answer

Suppose you have a linear system whose augmented matrix is already in reduced echelon form.

Like this one:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(Perhaps you arrived at this system by row reduction.)

# Reading off the answer

First, look for any rows like this:

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The system is inconsistent if and only if you can find such a row.

In the case at hand, there are no such rows

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So this system is consistent.

# Reading off the answer

Assuming the system was consistent

now introduce free parameters for every variable whose corresponding column has no pivot

In this case

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

that's the first and fifth column. We assign the free parameters $s$ and $t$ to to $x_1$ and $x_5$.

# Reading off the answer

## Express the pivot variables

In terms of the free parameters and constants.

## In this case

$$\left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we have taken $x_1 = s$ and $x_5 = t$; the remaining equations say $x_2 = 3 - t$, $x_3 = 2 - t$, and $x_4 = 1$. Thus the solution set is

$$\{(s, 3 - t, 2 - t, 1, t) \,|\, \text{any } s, t\}$$

# Number of solutions: all about the pivots

For a system with reduced echelon augmented matrix

There are no solutions
If and only if there is a pivot in the last column.

In other words, there is a row like this

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

# Number of solutions: all about the pivots

For a system with reduced echelon augmented matrix

There is exactly one solution

If and only if there are no pivots in the last column
and a pivot in every other column.

In other words, the matrix looks like this

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

# Number of solutions: all about the pivots

For a system with reduced echelon augmented matrix

There are infinitely many solutions

If and only if there are no pivots in the last column,
and no pivots in at least one other column.

In other words, the matrix looks like this

$$\left[\begin{array}{ccccc|c} 1 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

# Vectors

A column vector is a matrix with just one column.

Some column vectors:

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

We write $\mathbb{R}^n$ for the set of column vectors with $n$ rows. The above vectors are in $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, and $\mathbb{R}^4$, respectively.

# Adding vectors

You add vectors term by term.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a+x \\ b+y \\ c+z \end{bmatrix}$$

This only makes sense if they have the same number of entries.

# Try it yourself!

Add these vectors.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Those vectors are not the same size, you can't add them.

# Multiplying vectors by scalars

To multiply a vector by a constant, multiply each entry.

$$c \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}$$

The constants are also called scalars. They scale the vectors.

# Try it yourself!

Multiply the vector by the scalar

$$3 * \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$11 * \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 33 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} * \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

Neither one of those is a scalar.

# Vectors

We will use boldface type to denote vectors:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

# Vector arithmetic

Vector addition and scalar multiplication satisfy the usual commutative, associative, and distributive properties, just because the same is true entry by entry.

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

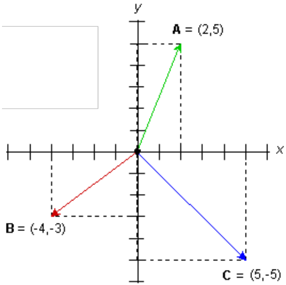
$$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$$

$$c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$$

$$(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$$

# Geometrically

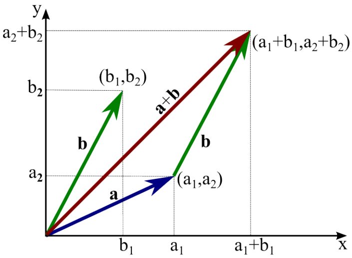
You can visualize a vector as an arrow




This works in $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$. You could do the same for $\mathbb{R}^n$ if you could visualize higher dimensional spaces.

# Addition, geometrically

Vector addition looks like this:




You move one vector to the end of the other to add them.
If this confuses you, think about the case of $\mathbb{R}^1$.

# Scalar multiplication, geometrically

Scalar multiplication means keeping the direction of the arrow, but changing its length.

For instance, $2\mathbf{v} = \mathbf{v} + \mathbf{v}$ is a vector in the same direction as $\mathbf{v}$, but twice as long.

# Linear combinations and linear span

### Definition

Given a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$, and scalars $a_1, a_2, \ldots, a_n \in \mathbb{R}$, an expression of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$$

is said to be a linear combination of the vectors. The set of all such expressions is called the linear span of the $\mathbf{v}_i$.

# Linear span

## Example

The linear span of the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is the set of all vectors of the form

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

# Try it yourself

What is the linear span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$? All vectors $\begin{bmatrix} t \\ 2t \end{bmatrix}$.

What is the linear span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$?

Probably some of you wrote "all vectors of the form $\begin{bmatrix} t + 3s \\ 2t + 4s \end{bmatrix}$"

This is in some sense correct, but a better answer is: all of $\mathbb{R}^2$.

# Linear span and linear equations

Is $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ in the linear span of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$?

I.e., do there exist $x, y$ with $x \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$?

I.e., do there exist $x, y$ with $\begin{bmatrix} x + y \\ x + 3y \\ 2x + 7y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$?

# Linear span and linear equations

I.e., does the system of linear equations with the following augmented matrix have any solutions:

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 2 & 7 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 5 & 1 \end{array}\right]$$

We do a little row reduction, and then see that the bottom two lines say respectively $2y = 1$ and $5y = 1$; there is no solution.

So, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ in not in the linear span of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$.

# Linear combinations and linear equations

The equations

$$\begin{aligned} 5x + 4y + 3z &= 2 \\ x + y + z &= 6 \\ x - y + z &= -3 \end{aligned}$$

are equivalent to the vector equation

$$x \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

# Linear combinations and linear equations

Finding all solutions to

$$\begin{aligned} 5x + 4y + 3z &= 2 \\ x + y + z &= 6 \\ x - y + z &= -3 \end{aligned}$$

is the same as finding all ways to express

$$\begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix} \text{ as a linear combination of } \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

the necessary coefficients are the $x, y, z$.