Hello and welcome to class!

Previously

We have discussed linear algebra.

This time

We start studying differential equations. We will begin with the study of first- and then second-order linear homogenous constant coefficient equations.

Differential equations

What is a differential equation?

An equation involving a function and its derivatives.

Often we are interested in solving for the (unknown) function.

Differential equations

How do they arise?

Say one has some real-world system. Its state at some given time is modeled by a function (perhaps vector-valued) $x(t)$. We can reason intuitively or scientifically about the rules governing the evolution of the system — i.e., we can describe how it is changing at a given time t, in terms of its current state at time t.

That's a differential equation! In fact, the vast majority of models in the sciences at least begin in this form.

Population growth

For instance, suppose you have a population of some kind, each member of which behaves identically. Then the rate of change of the population, whatever it is, must be proportional to the population.

$$y'(t) = ky(t)$$

The population might be of people or animals, in which case we are talking about population growth in the usual sense. It could also be a population of radioactive atoms, and we are talking about radioactive decay, or a population of invested dollars and we are talking about interest.

Newton's second law

Another sort of example is given by Newton's law of motion. This asserts that the path $y(t)$ of an object of mass m, subject to a force F, satisfies

$$F = my''(t)$$

Implicit in this formula is an assertion that the force F may depend on t, $y(t)$, and $y'(t)$, but depends on no higher derivatives of y.

I do not know any good intuitive explanation of this fact. Probably there isn't one: in all the history of the human race before Newton, no-one came up with this law and many very smart people came up with other, incorrect laws. It's just how the world works.


PlONK
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Gravity near earth

Any object is subject to the force of gravity exerted by other objects. According to Newton, this force has strength

$$|F| = \frac{Gm_1 m_2}{r^2}$$

Near the surface of the earth, $g := -\frac{Gm_{Earth}}{r^2} \sim 9.8\, m/s^2$, the negative sign meaning it goes in the downward direction.

Thus any other object subject to no other forces would move according to the differential equation $gm = my''(t)$, or just

$$y''(t) = g$$

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As it turns out, a spring stretched or compressed by some length y exerts a restorative force in the opposite direction, proportional to the stretching.

Thus an object of mass m attached to a spring obeys the law

$$my''(t) = -ky$$

for some constant k depending on the constitution of the spring.

The heat equation

The temperature in an object, varying over time, is recorded by a function in space and time variables $T(x, y, z, t)$.

The laws of thermodynamics assert that, all things being equal, the temperature wants to equalize.

I.e., the temperature at a given point will change according as to how different the temperature at nearby points is.

$$\frac{\partial T}{\partial t} \sim \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

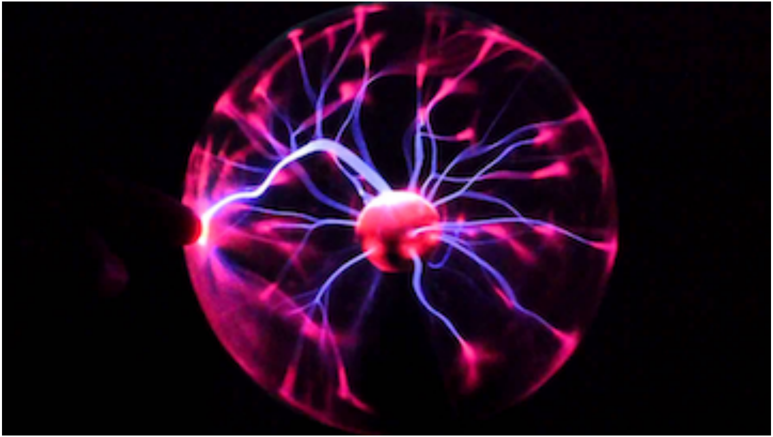
The wave equation

Imagine a substance which has some preferred density and everywhere locally responds to compression or expansion by exerting a restorative force in the opposite direction.

I.e., just about any substance.

E.g. by pretending this is "an infinite collection of springs", it's possible to show that the density $u(x, y, z, t)$ obeys

$$\frac{\partial^2 u}{\partial t^2} \sim \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

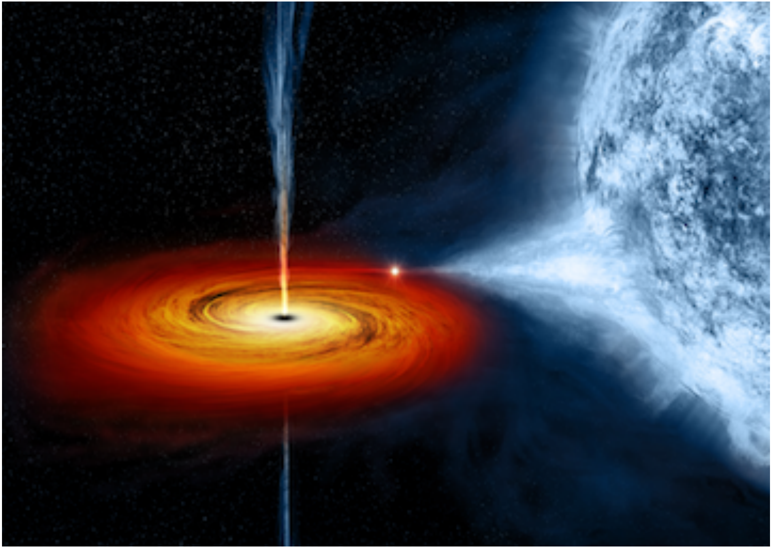


Maxwell's equations

Everywhere space and time are permeated by electric and magnetic vector fields **E** and **B** which moreover interact with each other and electric and magnetic charges...

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial B}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial E}{\partial t} \right)
\end{aligned}$$

We won't really discuss these in this class but they are in fact linear equations, and can be treated by the methods we develop.



Einstein's equations

Everywhere in space and time the very notion of distance is variable, and is recorded at each point by a matrix $g(x, y, z, t)$ (the metric). The fact that the notion of distance varies from point to point means that space is curved and these curvatures are recorded as derivatives of the metric. Two of these are called the scalar curvature $R(g)$ and the Ricci curvature $Ric(g)$.

The notion of distance itself is subject to Einstein's law

$$Ric(g) - \frac{1}{2}R(g)g + \Lambda g = \frac{8\pi G}{c^4}T$$

where Λ is the cosmological constant, G is the gravitational constant, c is the speed of light, and T is a thing which records the mass, energy, etc. at each point. This is a nonlinear partial differential equation, and we certainly won't talk about it here.

First order equations

The general form of a first order linear homogenous linear differential equation is:

$$a(t)\frac{dy}{dt} + b(t)y = 0$$

To solve this, we rearrange it into the form

$$\frac{dy}{y} = -\frac{b(t)}{a(t)}dt$$

and integrate to get

$$\log y = -\int \frac{b(t)}{a(t)}dt \qquad y = e^{-\int \frac{b(t)}{a(t)}dt}$$

First order equations

In particular, in the constant coefficient case

$$ay' + by = 0$$

Integrating gives

$$y(t) = Ce^{-\frac{b}{a}t}$$

for some undetermined constant C coming from the constant of integration. Of course, plugging in $t = 0$ reveals $C = y(0)$.

General solution versus initial value problem

Given a differential equation e.g., $y' + y = 0$, there are various things one can ask.

One possibility: ask for all functions which satisfy the differential equation. In this case, we say we want the general solution, which for this example is $y(t) = Ce^{-t}$, for an arbitrary constant C.

Another possibility: we have in mind some particular initial data; e.g. we know that $y(0) = 5$. This is called solving the initial value problem for the equation, and the solution in this case is $y(t) = 5e^{-t}$.

Existence and uniqueness

Given a differential equation, there are two basic questions. First, do solutions exist, and second, how many are there?

Existence. If you are modeling some system by a differential equation then, since the system behaves after all in some way over time it had better be the case that your equations have solutions. Moreover, it may happen that the solutions are defined only over some time interval $[0, T)$. This necessarily means that your model no longer makes sense after time T, and you should be somewhat worried about times near T.

Existence and uniqueness

Uniqueness.

Maybe you are modeling some system. You have observed that it obeys some differential equation.

Now you want to know how much data about it you must measure now in order to predict its future behavior.

Existence and uniqueness

In other words, in the space of all possible solutions to your equation, you have to pick out one.

In the setting of linear differential equations the solution space will always be a vector space and generally a finite dimensional one.

Thus to specify which solution you have now you have to name a vector in this space which you can do by choosing an isomorphism to $\mathbb{R}^n$.

Existence and uniqueness

Often (i.e. in the context of the initial value problem) this isomorphism takes the following form:

$$\begin{array}{rcl} \textit{Solutions} & \mapsto & \mathbb{R}^n \\ f & \mapsto & (f(0), f'(0), \ldots, f^{(n-1)}(0)) \end{array}$$

E.g. for the first-order equations, we took $f \mapsto f(0)$.

Solving linear, constant coefficient, second order equations

Consider the equation $ay'' + by' + cy = 0$.

What can we say about the space of solutions before finding any?

Linearity

The solutions form a vector space: since the derivative is linear, if $y(t)$ and $z(t)$ are solutions, then

$$a(my(t) + nz(t))'' + b(my(t) + nz(t))' + c(my(t) + nz(t)) =$$

$$m(ay'' + by' + cy) + n(az'' + bz' + cz) = 0$$

So $ay(t) + bz(t)$ is also a solution.

How many solutions do we expect?

One way to read the equation $ay'' + by' + cy = 0$ is

$$y''(t) = \frac{1}{a}(by'(t) + cy(t))$$

In other words: if y is a solution to this equation,

> *knowing the value $y(t)$ and the first derivative of $y'(t)$ tells you the second derivative $y''(t)$!*

How many solutions do we expect?

Taking another derivative, $y'''(t) = \frac{1}{a}(by''(t) + cy'(t))$, we see that the third derivative can be written in terms of the first and second derivatives, which in turn can be written in terms of the first derivative and the value of the function itself.

$$y'''(t) = \frac{1}{a}(by''(t) + cy'(t)) = \left(\frac{b^2 + ac}{a^2}\right) y'(t) + \left(\frac{bc}{a^2}\right) y(t)$$

Continuing this process...

We can recover all derivatives at t from just $y(t)$ and $y'(t)$!

How many solutions do we expect?

In other words, for a fixed number t_0,

$$\begin{array}{rcl} \text{Solns of } ay''(t) + by'(t) + cy(t) = 0 & \to & \mathbb{R}^{n+1} \\ f & \mapsto & (f(t_0), f'(t_0), \ldots, f^{(n)}(t_0)) \end{array}$$

always has image of dimension at most two.

Or in other words, whatever the dimension of the space of solutions, the dimension of the space of their Taylor expansions around a given point is at most two.

How many solutions are there actually?

The above is not actually an argument that the space of solutions is two dimensional. This is because there exist infinitely differentiable functions Taylor series expansion at a point vanishes, but are nonetheless nonzero, e.g.

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

However, in fact there are at most two solutions to a linear second order ODE. This is a special case of a general uniqueness theorem which we will later state but not prove in this class.

Finding the solutions

Consider the equation $ay'' + by' + cy = 0$.

Exponential functions worked for solving the first-order equation, maybe let's try them again.

Some people would say we are making an exponential ansatz.

Solving second order equations

So suppose $y = e^{rt}$. When is $ay'' + by' + cy = 0$?

$0 = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = ar^2e^{rt} + bre^{rt} + ce^{rt} = e^{rt}(ar^2 + br + c)$

Since $e^{rt} \neq 0$, this is true if and only if

$$ar^2 + br + c = 0$$

This is called the auxilliary equation.

Solving second order equations

Taking $r_\pm$ to be the roots of $ar^2 + br + c = 0$, i.e.,

$$r_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We have found solutions $e^{r_+ t}$ and $e^{r_- t}$. Since the equations are linear, any linear combination $Ae^{r_+ t} + Be^{r_- t}$ is again a solution. So we have a two dimensional solution space which therefore must be all the solutions.

... unless $r_+ = r_-$ or the $r_\pm$ not real numbers!

Linear independence of exponentials

Recall that the functions e^{at} for differing a are linearly independent e.g., because they are eigenvectors of $\frac{d}{dt}$ with different eigenvalues.

Solving second order equations

There are three possibilities.

- ($b^2 - 4ac > 0$) The numbers r_+ and r_- are real and distinct. The solution space is spanned by $e^{r_+ t}$ and $e^{r_- t}$.
- ($b^2 - 4ac = 0$) We have $r_+ = r_-$, and a one dimensional space of solutions, spanned by $e^{r_+ t}$. Maybe there are other solutions.
- ($b^2 - 4ac < 0$) The numbers $r_\pm$ are complex, non-real numbers. The functions $e^{r_\pm t}$ are solutions in the sense that they formally satisfy the differential equation but we have yet to find a real solution.

Repeated roots: $b^2 - 4ac = 0$, i.e. $r_+ = r_-$

Consider what happens to $Span(e^{r_+ t}, e^{r_- t})$ as $r_+ - r_- \to 0$.

Or equivalently, to $Span(e^{rt}, e^{(r+\epsilon)t})$ as $\epsilon \to 0$.

In particular, consider the function $\frac{1}{\epsilon}(e^{(r+\epsilon)t} - e^{rt})$.

$$\lim_{\epsilon\to 0} \frac{e^{(r+\epsilon)t} - e^{rt}}{\epsilon} = \frac{d}{dr} e^{rt} = te^{rt}$$

Repeated roots: $b^2 - 4ac = 0$, i.e. $r_+ = r_-$

More generally, in a sense we will not make precise here,

$$Span(e^{rt}, e^{(r+\epsilon)t}) \to Span(e^{rt}, te^{rt})$$

While the above argument could be made into rigorous mathematics it is much easier to just check!

Repeated roots: $b^2 - 4ac = 0$, i.e. $r_+ = r_-$

Thus let us consider also the function te^{rt}, where $r = r_+ = r_-$.

Note $(te^{rt})' = e^{rt} + rte^{rt}$ and $(te^{rt})'' = 2re^{rt} + r^2te^{rt}$

Plugging it back into the original equation

$$a(te^{rt})'' + b(te^{rt})' + cte^{rt} = a(2re^{rt} + r^2te^{rt}) + b(e^{rt} + rte^{rt}) + cte^{rt}$$

$$= (ar^2 + br + c)te^{rt} + (2ar + b)e^{rt}$$

Repeated roots: $b^2 - 4ac = 0$, i.e. $r_+ = r_-$

Thus for te^{rt} to be a solution, we should have

$$(ar^2 + br + c)te^{rt} + (2ar + b)e^{rt} = 0$$

But by assumption $ar^2 + br + c = 0$ and, since $b^2 = 4ac$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a}$$

we have $2ar + b = 0$.

So it's a solution.

Solving second order equations

There are three possibilities.

- ($b^2 - 4ac > 0$) The numbers r_+ and r_- are real and distinct. The solution space is spanned by $e^{r_+ t}$ and $e^{r_- t}$.
- ($b^2 - 4ac = 0$) We have $r_+ = r_- = -\frac{b}{2a}$. The solution space is spanned by $e^{-\frac{b}{2a}t}$ and $te^{-\frac{b}{2a}t}$.
- ($b^2 - 4ac < 0$) The numbers $r_\pm$ are complex, non-real numbers. The functions $e^{r_\pm t}$ are solutions in the sense that they formally satisfy the differential equation but we have yet to find a real solution.

Complex roots: $b^2 - 4ac < 0$

In some sense, $e^{r_\pm t}$ are solutions to $ay'' + by' + cy = 0$. However, if we want a real valued function, these do not qualify.

However, we can just take the real and imaginary parts. That is because, in order for $ay'' + by' + cy = 0$, both the real and imaginary parts must vanish.

Complex roots: $b^2 - 4ac < 0$

Observe $e^{r_\pm t} = e^{\left(\frac{-b}{2a} \pm i \frac{\sqrt{4ac-b^2}}{2a}\right) t} =$

$$e^{\left(\frac{-b}{2a}\right) t} \left(\cos\left(\left(\frac{\sqrt{4ac - b^2}}{2a} \right) t \right) \pm i \sin\left(\left(\frac{\sqrt{4ac - b^2}}{2a} \right) t \right) \right)$$

Taking real and imaginary parts gives

$$e^{\left(\frac{-b}{2a}\right) t} \cos\left(\left(\frac{\sqrt{4ac - b^2}}{2a} \right) t \right), \qquad e^{\left(\frac{-b}{2a}\right) t} \sin\left(\left(\frac{\sqrt{4ac - b^2}}{2a} \right) t \right)$$

Complex roots: $b^2 - 4ac < 0$

As a sanity check, let's look at the case $b = 0$ and $a = c = 1$:

$$y'' + y = 0$$

Then our formula

$$e^{\left(\frac{-b}{2a}\right)t}\cos\left(\left(\frac{\sqrt{4ac - b^2}}{2a}\right)t\right), \qquad e^{\left(\frac{-b}{2a}\right)t}\sin\left(\left(\frac{\sqrt{4ac - b^2}}{2a}\right)t\right)$$

just gives

$$\cos(t) \qquad \sin(t)$$