What is linear algebra?

In its most concrete form

Linear algebra is the study of systems of equations like this one:

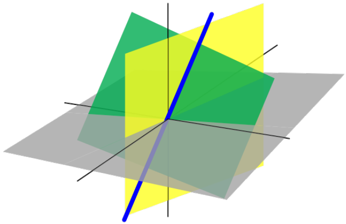
$$\begin{aligned} x + 2y + 3z &= 6 \\ x - y + z &= 1 \\ 2x + 3y + 4z &= 9 \end{aligned}$$

We will spend the first few days on the concrete manipulation of such equations

But let me give you a hint of what is to come:

What is linear algebra?

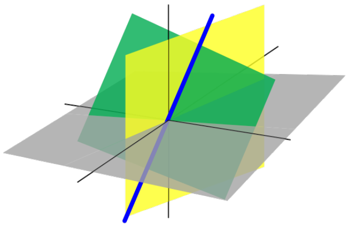
Systems of equations have a geometric meaning:



The region where each equation is satisfied is a plane, so the simultaneous solution to all the equations is where the planes intersect.

What is linear algebra?

Systems of equations have a geometric meaning:



Linear algebra is the basic tool for understanding such geometric configurations, say in computer graphics or computer vision.

What is linear algebra?

More abstractly, linear algebra is the study of transformations of spaces which carry lines to lines.

What is a space? What is a line? What is a transformation?

We will not try to give the general answers yet.

First we will study many examples of the above phenomenon.

What is linear algebra?

The more abstract perspective is worth the effort.

Many phenomena are linear in this abstract sense, and they all can be studied concretely using systems of linear equations.

Schrödinger's equation for a quantum mechanical particle:

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t) = \left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(\mathbf{x},t)\right]\Psi(\mathbf{x},t)$$

Here the space is a space of functions which might be our desired $\Psi(x,t)$, and the linear transformations are the partial differential operators $i\hbar\frac{\partial}{\partial t}$ and $-\frac{\hbar^2}{2\mu}\nabla^2 + V(\vec{x},t)$.

What is linear algebra?

But this equation is still linear!

And by the end of the class, Schrödinger's equation

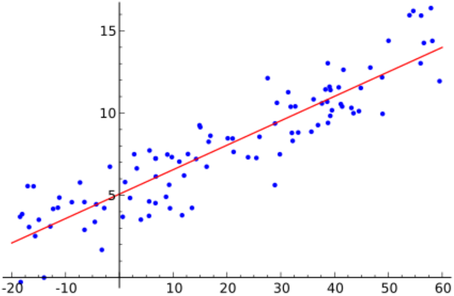
$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t) = \left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(\mathbf{x},t)\right]\Psi(\mathbf{x},t)$$

will look no worse to you than this one:

$$\begin{aligned} x + 2y + 3z &= 6 \\ x - y + z &= 1 \\ 2x + 3y + 4z &= 9 \end{aligned}$$

What is linear algebra?

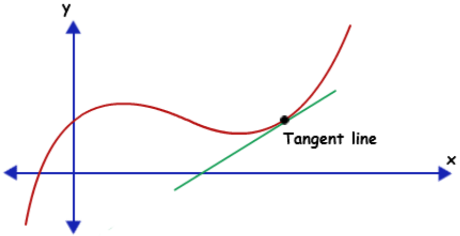
Many phenomena are approximately linear:

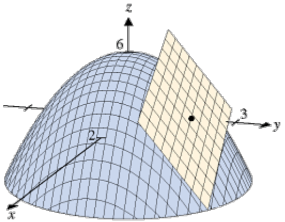


Discovering such statistical linearities plays a major role in the experimental sciences of all kinds, and in search algorithms and economic forecasting.

What is linear algebra?

Many phenomena are approximately linear:





Linear algebra is the language for discussing the differential and integral calculus, especially in higher dimensions.

What will you learn in this class?

Concrete procedures for manipulating linear equations

Abstract notions capturing and organizing the idea of linearity

Many real world examples of linear phenomena, especially in the form of differential equations

It will be a lot of work — both in terms of the raw amount of new concepts to process, and correspondingly, in terms of the number of exercises assigned to help you master them (20-30 per week) — but you will leave this class equipped with a powerful conceptual framework on which the vast majority of mathematics, science, engineering, etc., depend.

Let's get to work

Linear equations

Example.

The equation $x + 2y + 3z = 6$ is linear in x, y, z.

Definition. An equation in variables $x_1, x_2, \ldots, x_n$ is linear if it can be put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b do not depend on any of the x_i.

Usually, the a_i and b will just be explicit real or complex numbers.

Linear equations

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Nonexample.

The equation $x^3 = 6$ is not linear in x. You might try writing it as $(x^2)x = 6$ and pretend x^2 is a coefficient, but this is no good because x^2 depends on x.

Linear equations

Definition. An equation in variables $x_1, x_2, \ldots, x_n$ is linear if it can be put in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b do not depend on any of the x_i.

Usually, the a_i and b will just be explicit real or complex numbers.

Example-nonexample.

The equation $xy = 1$ is "linear in the variable x", and it is "linear in the variable y", but it is not "linear in the variables x and y".

Try it yourself!

Which of these equations are linear in $x_1, x_2, \ldots, x_n$?

- $x_1 = 5$ linear
- $x_1 + x_2 + \cdots + x_n = 1$ linear
- $4x_1 + 17x_2 = -x_3$ linear
- $4x_1 + 17x_2 = -x_3 \cos(s)$ linear
- $x_1 + x_2 + \cdots + x_n = x_1 x_2 \cdots x_n$ nonlinear
- $x_1/x_2 = 4$ I will try not to ask this question on an exam

The equation $x_1/x_2 = 4$ is equivalent to the equation $x_1 = 4x_2$, subject to the condition that $x_2 \neq 0$. Depending on the context, one might or might not want to call this linear.

Systems of linear equations

Definition. A system of linear equations in $x_1, x_2, \ldots, x_n$ is a finite collection of linear equations in $x_1, x_2 \ldots, x_n$.

It is helpful to "line up the x's" and write such systems in the form

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

We say this is a system of m linear equations in n unknowns.

Systems of linear equations: examples

We now have a description for our old friend

$$\begin{aligned} x + 2y + 3z &= 6 \\ x - y + z &= 1 \\ 2x + 3y + 4z &= 9 \end{aligned}$$

It is a system of 3 linear equations in 3 unknowns (namely x, y, z).

Systems of linear equations: examples

An example of 2 equations in 3 unknowns

$$\begin{aligned} x + y + z &= 0 \\ x + y &= 1 \end{aligned}$$

An example of 2 equations in 1 unknown

$$\begin{aligned} x &= 1 \\ x &= 2 \end{aligned}$$

Solutions of systems of linear equations

Definition. The set of solutions to a system of linear equations in $x_1, \dots, x_n$ is the set of all tuples of numbers $(s_1, \dots, s_n)$ such that substituting s_i for x_i gives an identity.

Examples.

- The equation $x_1 = 5$ has solution set $\{5\}$.
- The system $x_1 = 2$ and $x_1 + x_2 = 7$ has solution set $\{(2, 5)\}$.
- The system $x_1 = 2$ and $x_1 = 7$ has the empty solution set.
- The system $x_1 + x_2 = 0$ has the solution set $\{(s, -s)\}$ where s takes any value.

Consistency and inconsistency.

Definition. A system of linear equations is *consistent* if it has solutions, and inconsistent otherwise.

We saw examples of both consistent and inconsistent systems already. In all the examples so far, there were either $0, 1$, or ∞ solutions. We will learn soon that this is always the case.

Try it yourself!

Find all solutions to the following. Is it a consistent system? How many solutions are there?

$$\begin{aligned} x &= 1 \\ x &= 2 \end{aligned}$$

The solution set is the empty set. The system is inconsistent, with zero solutions.

Try it yourself!

Find all solutions to the following. Is it a consistent system? How many solutions are there?

$$\begin{aligned} x + y + z &= 0 \\ x + y &= 1 \end{aligned}$$

The solution set of possible (x, y, z) is

$$\{(s, 1 - s, -1) \mid \text{ any number } s\}$$

The system is consistent, with infinitely many solutions.

I'll do one

Find the solution set. Is it a consistent system? How many solutions are there?

$$\begin{aligned} x + 2y + 3z &= 6 \\ x - y + z &= 1 \\ 2x + 3y + 4z &= 9 \end{aligned}$$

Solving a system

Here is one way to arrive at the solution. Of the equations,

$$\begin{aligned} x + 2y + 3z &= 6 \\ x - y + z &= 1 \\ 2x + 3y + 4z &= 9 \end{aligned}$$

we can pick one, say $x - y + z = 1$, and use it to express x in terms of the other variables: $x = 1 + y - z$. Now, we substitute this back into the other two, giving

$$(1 + y - z) + 2y + 3z = 6 \qquad \text{and} \qquad 2(1 + y - z) + 3y + 4z = 9$$

Solving a system

These two equations

$$(1 + y - z) + 2y + 3z = 6 \qquad \text{and} \qquad 2(1 + y - z) + 3y + 4z = 9$$

simplify into

$$3y + 2z = 5 \qquad \text{and} \qquad 5y + 2z = 7$$

We can use the first to write $z = (5 - 3y)/2$, and then substitute this into the second to find $5y + 2(5 - 3y)/2 = 7$, which we simplify to $2y = 2$, then $y = 1$. We substitute back into either one of the two equations above to find $z = 1$, which we substitute into any of the original equations to get $x = 1$.

That worked

But it was a bit of a mess.

Let's clean it up.

We transformed

$$\begin{aligned} x + 2y + 3z &= 6 \\ x - y + z &= 1 \\ 2x + 3y + 4z &= 9 \end{aligned}$$

into

$$\begin{aligned} 3y + 2z &= 5 \\ x - y + z &= 1 \\ 5y + 2z &= 7 \end{aligned}$$

by solving the original second equation for x, plugging into the others, and then simplifying. Instead: subtract the second equation from the first, and twice the second equation from the third.

Let's clean it up

We did some more substitutions to find $y = 1$. Instead, we can transform

$$\begin{aligned} 3y + 2z &= 5 \\ x - y + z &= 1 \\ 5y + 2z &= 7 \end{aligned}$$

into

$$\begin{aligned} 3y + 2z &= 5 \\ x - y + z &= 1 \\ 2y &= 2 \end{aligned}$$

by subtracting the first equation from the third.

Let's clean it up

After dividing the last equation by 2, we have

$$\begin{aligned} 3y + 2z &= 5 \\ x - y + z &= 1 \\ y &= 1 \end{aligned}$$

Now we can add multiples of the bottom equation to the top two,

$$\begin{aligned} 2z &= 2 \\ x + z &= 2 \\ y &= 1 \end{aligned}$$

Let's clean it up

Divide the top equation by 2,

$$\begin{aligned} z &= 1 \\ x + z &= 2 \\ y &= 1 \end{aligned}$$

and finally, subtract the top equation from the middle one:

$$\begin{aligned} z &= 1 \\ x &= 1 \\ y &= 1 \end{aligned}$$

Why did that work?

Solving one linear equation for a given variable, and then plugging that in to another linear equation

is the same as

adding a multiple of the first equation to the second. This is only true of linear equations!

What did we do?

We solved a system of linear equations,

by transforming them into simpler but equivalent — i.e., having the same solution set — systems of equations.

We only used the following "elementary" equivalences

adding a multiple of one equation to another and multiplying an equation by a nonzero number. We also allow ourselves to re-order the equations.

Augmented matrix

We can abbreviate

$$\begin{aligned} x + 2y + 3z &= 6 \\ x - y + z &= 1 \\ 2x + 3y + 4z &= 9 \end{aligned}$$

as

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & -1 & 1 & 1 \\ 2 & 3 & 4 & 9 \end{array}\right]$$

This is called the augmented matrix of the system. There are as many rows as equations, and one more column than the number of unknowns.

Let's clean it up

We can just write our solution as a series of augmented matrices:

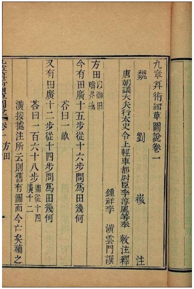
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & -1 & 1 & 1 \\ 2 & 3 & 4 & 9 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 3 & 2 & 5 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 5 \\ 1 & -1 & 1 & 1 \\ 0 & 5 & 2 & 7 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 0 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 3 & 2 & 5 \\ 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

Row reduction

At each stage, we reduced the number of non-zero entries in the matrix, by adding multiples of one row to another. This procedure is called row reduction, or Gaussian elimination.



九章筭術細草圖說卷一
魏 劉徽 注
唐朝議大夫行太史令上輕車都尉臣李淳風等奉 敕注釋
鍾祥李 潢雲門譔
方田以御田疇界域
今有田廣十五步從十六步問爲田幾何
荅曰一畝
又有田廣十二步從十四步問爲田幾何
荅曰一百六十八步圖從十四廣十二
潢按據注所云則舊有圖而今亡矣補之

It was in 9 chapters on the mathematical art, 2000 years before.

Try it yourself!

For what numerical values of s are the systems

$$\begin{aligned} x + sy &= 0 \\ -x + y &= s + 1 \end{aligned}$$

and

$$x - y = 0$$

equivalent?

Solution

We write and reduce the augmented matrix for the first system:

$$\left[\begin{array}{cc|c} 1 & s & 0 \\ -1 & 1 & s+1 \end{array}\right] \quad \to \quad \left[\begin{array}{cc|c} 1 & s & 0 \\ 0 & 1+s & s+1 \end{array}\right]$$

If $s = -1$, then the second equation says $0 = 0$ and the first says $x - y = 0$; i.e., in this case, it is equivalent to the second system.

Otherwise, we can divide the second equation by $1 + s$ to find

$$\left[\begin{array}{cc|c} 1 & s & 0 \\ 0 & 1 & 1 \end{array}\right] \quad \to \quad \left[\begin{array}{cc|c} 1 & 0 & -s \\ 0 & 1 & 1 \end{array}\right]$$

i.e., $x = -s$ and $y = 1$. This is not equivalent to the second system, which has solutions $(s, -s)$ for any s.

The systems are equivalent if and only if $s = -1$.

Next time:

We will continue to refine our solution method — transforming a given linear system into equivalent, but increasingly easy to solve, linear systems. We will also consider various reformulations and interpretations of linear equations.

See you next time!