VISCOSITY SOLUTIONS

PETER HINTZ

We follow Han and Lin, *Elliptic Partial Differential Equations*, §5.

1. Motivation

Throughout, we will assume that $\Omega \subset \mathbb{R}^n$ is a bounded and connected domain and that $a_{ij} \in C(\Omega)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \ x \in \Omega, \xi \in \mathbb{R}^n,$$

for some $\Lambda, \lambda > 0$. We consider the operator L defined by

$$Lu = a_{ij}(x)D_{ij}u$$

for $u \in C^2(\Omega)$.

Suppose $u \in C^2(\Omega)$ is a supersolution in Ω, i.e. $Lu \leq 0$. Then if $\phi \in C^2(\Omega)$ satisfies $L\phi > 0$, we get $L(u - \phi) < 0$ in Ω, hence by the maximum principle, $u - \phi$ does not have interior local minima in Ω. Put differently, if $\phi \in C^2(\Omega)$ is such that $u - \phi$ has a local minimum at $x_0 \in \Omega$, then necessarily $L\phi(x_0) \leq 0$. Geometrically, $u - \phi$ having a local minimum at x_0 means that the graph of ϕ touches the graph of u from below at x_0, if we shift ϕ by a constant.

Definition 1.1. Let $f \in C(\Omega)$. We say that $u \in C(\Omega)$ is a *viscosity supersolution* (resp. *subsolution*) of $Lu = f$ in Ω if for all $x_0 \in \Omega$ and all functions $\phi \in C^2(\Omega)$ such that $u - \phi$ has a local minimum (resp. maximum) at x_0, the inequality

$$L\phi(x_0) \leq f(x_0) \quad (\text{resp. } \geq f(x_0))$$

holds. We say that u is a *viscosity solution* if it is both a viscosity supersolution and a viscosity subsolution.

Since L is a second order operator, we in fact obtain an equivalent definition if we restrict ϕ to be a quadratic polynomial.

Lemma 1.2. *If $u \in C^2(\Omega)$, then u is a classical supersolution (resp. subsolution) if and only if is a viscosity supersolution (resp. subsolution).*

Proof. The 'only if' part follows from the discussion preceding Definition 1.1. For the 'if' part, we can simply choose $\phi = u \in C^2(\Omega)$ in the definition of viscosity solutions; note that in this case $u - \phi \equiv 0$ has a local minimum (resp. maximum) at every point $x_0 \in \Omega$. □

The point of viscosity solutions is that the regularity requirements on u are very weak, and spaces of viscosity (super-/sub-)solutions have good closure/compactness properties (as do the space $\mathscr{S}^{\pm}$ defined below), which is particularly important in the study of nonlinear (degenerate elliptic) PDEs.

Date: April 25, 2014.

Connecting to the discussion preceding Definition 1.1, for any function ϕ which is C^2 at x_0, the condition $L\phi(x_0) \le 0$ is equivalent to $\sum_{ij} a_{ij}(x_0) D_{ij}\phi(x_0) \le 0$, and hence by Lemma 1.3 below, we obtain

$$\sum_k \alpha_k e_k \le 0,$$

where $\alpha_1 \ge \cdots \ge \alpha_n > 0$ are the eigenvalues of $a_{ij}(x_0)$, and $e_1 \le \cdots \le e_n$ are the eigenvalues of $D_{ij}\phi(x_0)$ (notice the reversed order). Splitting the sum, this gives

$$\sum_{e_k>0} \alpha_k e_k \le \sum_{e_k<0} \alpha_k(-e_k),$$

and thus, using $\lambda \le \alpha_k \le \Lambda$ for all k,

$$\lambda \sum_{e_k>0} e_k \le \Lambda \sum_{e_k<0} (-e_k), \tag{1.1}$$

which is to say that the positive eigenvalues of $D^2\phi(x_0)$ are controlled by its negative eigenvalues.

Lemma 1.3. *Let A be a positive definite $n \times n$ matrix with eigenvalues $\alpha_1 \ge \cdots \ge \alpha_n \ge 0$, and suppose B is a symmetric $n \times n$ matrix with eigenvalues $\beta_1 \ge \cdots \ge \beta_n$. Then*

$$\sum_{k=1}^n \alpha_k \beta_{n-k+1} \le \mathrm{Tr}(A^T B) \le \sum_{k=1}^n \alpha_k \beta_k.$$

Proof. This is closely related to *von Neumann's trace inequality.* We give a proof in the spirit of Leon Mirsky, *A trace inequality of John von Neumann.* Let $T, S \in O(n)$ be such that $T^T A T = \tilde{A}$ and $S^T B S = \tilde{B}$ are diagonal, which diagonal entries $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, respectively, in this order. Then

$$\mathrm{Tr}(A^T B) = \mathrm{Tr}(T\tilde{A}T^T S^T \tilde{B} S) = \mathrm{Tr}(ST\tilde{A}(ST)^T \tilde{B}) = \mathrm{Tr}((\tilde{A}Q)^T Q\tilde{B})$$

with $Q = (ST)^T \in O(n)$. The claim reduces to the inequalities

$$\sum_{k=1}^n \alpha_k \beta_{n-k+1} \le \sum_{i,j=1}^n d_{ij}\alpha_i\beta_j \le \sum_{k=1}^n \alpha_k\beta_k \tag{1.2}$$

for $d_{ij} = q_{ij}^2$. Now notice that $\sum_i q_{ij}^2 = 1 = \sum_j q_{ij}^2$ because of $Q \in O(n)$; hence the matrix $D = (d_{ij})$ is doubly stochastic, thus, by the Birkhoff-von Neumann theorem, a convex combination of permutation matrices. But the set of matrices D for which (1.2) holds is easily seen to contain all permutation matrices (this uses $\alpha_k \ge 0$), and moreover is convex, and the proof is complete. □

The inequality (1.1) is independent of the particular matrix $a_{ij}(x_0)$, and we use it to define a general class of functions which is designed to capture 'all solutions to all elliptic equations' (which satisfy an appropriate ellipticity condition):

Definition 1.4. Let $f \in C(\Omega)$, $\Lambda, \lambda > 0$. We define the space $\mathscr{S}^+(\lambda, \Lambda, f)$ (resp. $\mathscr{S}^-(\lambda, \Lambda, f)$) to consist of all $u \in C(\Omega)$ such that the following holds: If $\phi \in C^2(\Omega)$ is such that $u - \phi$ has a local minimum (resp. maximum) at x_0, then

$$\mathscr{M}^-(\lambda, \Lambda, D^2\phi) := \lambda \sum_{e_k>0} e_k(x_0) + \Lambda \sum_{e_k<0} e_k(x_0) \le f(x_0),$$

resp.

$$\mathscr{M}^+(\lambda, \Lambda, D^2\phi) := \Lambda \sum_{e_k>0} e_k(x_0) + \lambda \sum_{e_k<0} e_k(x_0) \geq f(x_0),$$

where[1] $e_1(x_0), \ldots, e_n(x_0)$ are the eigenvalues of the Hessian $D^2\phi(x_0)$. We denote

$$\mathscr{S}(\lambda, \Lambda, f) = \mathscr{S}^+(\lambda, \Lambda, f) \cap \mathscr{S}^-(\lambda, \Lambda, f).$$

By the above discussion, any viscosity supersolution (resp. subsolution) of $Lu = f$ with $L = a_{ij}D_{ij}$ as before belongs to the class $\mathscr{S}^+(\lambda, \Lambda, f)$ (resp. $\mathscr{S}^-(\lambda, \Lambda, f)$).

Remark 1.5. By Lemma 1.3, we can define $\mathscr{M}^\pm$ (still assuming $\Lambda \geq \lambda > 0$) alternatively by

$$\mathscr{M}^-(\lambda, \Lambda, M) = \inf_{\lambda \leq A \leq \Lambda} \sum_{ij} A_{ij}M_{ij},$$
$$\mathscr{M}^+(\lambda, \Lambda, M) = \sup_{\lambda \leq A \leq \Lambda} \sum_{ij} A_{ij}M_{ij},$$

with M a symmetric $n\times n$ matrix, where the inf and sup are taken over all symmetric $n\times n$ matrices A for which $\lambda|\xi|^2 \leq \sum_{ij} A_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$. In particular, this gives

$$\mathscr{M}^-(\lambda, \Lambda, M + N) \leq \mathscr{M}^-(\lambda, \Lambda, M) + \mathscr{M}^-(\lambda, \Lambda, N). \tag{1.3}$$

2. Alexandroff maximum principle

We now prove the Alexandroff maximum principle for viscosity solutions. For a continuous function v on an open convex set Ω, recall the *convex envelope* of v, defined by

$$\Gamma(v)(x) = \sup_L \{L(x) \mid L \leq v \text{ in } \Omega, L \text{ affine function}\}, \ x \in \Omega.$$

Clearly, $\Gamma(v)$ is a convex function in Ω. The set $\{v = \Gamma(v)\}$ is called the *(lower) contact set* of v. The points in the contact set are called *contact points*.

We recall the classical version of the Alexandroff maximum principle:

Proposition 2.1. *Suppose $u \in C^{1,1}(B_1) \cap C^0(\overline{B_1})$, $u|_{\partial B_1} \geq 0$. Then*

$$\sup_{B_1} u^- \leq c(n) \left(\int_{B_1 \cap \{u = \Gamma_u\}} \det D^2 u \right)^{1/n},$$

where $u^- = \max(-u, 0)$ is the negative part of u (which is a positive function!), and $\Gamma_u = \Gamma(-u^-)$ is the convex envelope of $-u^-$.

The viscosity version is:

Theorem 2.2. *Suppose $u \in \mathscr{S}^+(\lambda, \Lambda, f)$ in B_1 with $u \geq 0$ on ∂B_1, where $f \in C(\overline{B_1})$. Then*

$$\sup_{B_1} u^- \leq c(n, \lambda, \Lambda) \left(\int_{B_1 \cap \{u = \Gamma_u\}} (f^+)^n \right)^{1/n},$$

where $u^- = \max(-u, 0)$ and $\Gamma_u = \Gamma(-u^-)$ as before, and $f^+ = \max(f, 0)$.

[1] $\mathscr{M}^\pm$ are called the *Pucci extremal operators*.

Proof. We will prove that at a contact point x_0, the inequalities

$$f(x_0) \geq 0 \tag{2.1}$$

and

$$L(x) \leq \Gamma_u(x) \leq L(x) + C(f(x_0) + \epsilon(x))|x - x_0|^2 \tag{2.2}$$

hold for some affine function L and any x close to x_0, where $\epsilon(x) = o(1)$ as $x \to x_0$, and $C = C(n, \lambda, \Lambda)$ is a positive constant. By Alexandroff's theorem on convex functions (yielding that Γ_u has a second derivative almost everywhere), the estimate (2.2) together with (2.1) implies

$$\det D^2\Gamma_u(x) \leq C(n, \lambda, \Lambda) f^+(x)^n$$

for a.e. $x \in \{u = \Gamma_u\}$, and hence applying Proposition 2.1 to Γ_u gives the result.

To prove (2.1) and (2.2), suppose x_0 is a contact point, i.e. $u(x_0) = \Gamma_u(x_0)$. By translating the domain, we may assume $x_0 = 0$, and by subtracting a supporting plane from u, i.e. an affine linear function L with $L(x_0) = u(x_0)$ and $L(x) \leq -u^-(x)$ for all $x \in B_1$, we may moreover assume $u \geq 0$ in B_1 and $u(0) = 0$. Notice that changing the function ϕ in the definition of the space $\mathscr{S}^+(\lambda, \Lambda, f)$ by an affine linear function does not affect the Hessian of ϕ, hence the new function u, after translating it and subtracting a supporting plane from it, still belongs to $\mathscr{S}^+(\lambda, \Lambda, f)$. Let w be the corresponding translated and shifted version of Γ_u.

The estimate (2.1) is now obvious since u has a local minimum at 0, which by definition of $\mathscr{S}^+(\lambda, \Lambda, f)$ gives $0 \leq f(0)$, as desired. We will prove the estimate (2.2) in the form

$$0 \leq w(x) \leq C(n, \lambda, \Lambda)(f(0) + \epsilon(x))|x|^2, \ x \in B_1,$$

where $\epsilon(x) = o(1)$ as $x \to 0$. We need to estimate

$$C_r = \frac{1}{r^2} \max_{B_r} w$$

for $r > 0$ small. Since in the case $C_r = 0$, there is nothing to do, we may assume $C_r > 0$. By convexity, w attains its maximum in $\overline{B_r}$ at some point on the boundary $\partial\overline{B_r}$, say at $(0, \ldots, 0, r)$. We claim that this implies

$$w(x', r) \geq w(0, \ldots, 0, r) = C_r r^2$$

for all $x = (x', r) \in B_1$. Indeed, if $w(x', r) < w(0, r)$, then by continuity of w, we would also have $w(x', r+\epsilon) < w(0, r)$ for some small $\epsilon > 0$, but then by convexity of w, the value of w at the second intersection point of the line through $(x', r+\epsilon)$ and $(0, r)$ with $\partial\overline{B_r}$ would have to be larger than the value of w at $(0, r)$, contradicting the definition of the point $(0, r)$. We use this information to construct a quadratic polynomial that touches u from below in

$$R_r = \{(x', x_n) \mid |x'| \leq Nr, |x_n| \leq r\},$$

with $N = N(n, \lambda, \Lambda)$ to be chosen below, which curves upwards quickly in the x_n-direction, but curves downwards very slowly in the x'-directions. Concretely, define

$$\tilde{\phi}(x) = (x_n + r)^2 - \frac{4}{N^2}|x'|^2.$$

Then we have

(1) for $x_n = -r$, $\tilde{\phi} \leq 0$,
(2) for $|x'| = Nr$, $\tilde{\phi} \leq 4r^2 - 4r^2 = 0$,

(3) for $x_n = r$, $\tilde{\phi} \leq 4r^2$.

Hence, if we set

$$\phi(x) = \frac{C_r}{4}\tilde{\phi}(x),$$

we obtain $\phi \leq w \leq u$ on ∂R_r and $\phi(0) = C_r r^2/4 > 0 = w(0) = u(0)$. Therefore, $u - \phi$ has a local minimum somewhere inside R_r. The eigenvalues of $D^2\phi$ at any point are given by $C_r/2, -2C_r/N^2, \ldots, -2C_r/N^2$, thus by definition of $\mathscr{S}^+(\lambda, \Lambda, f)$, we have

$$\lambda\frac{C_r}{2} - 2\Lambda(n-1)\frac{C_r}{N^2} \leq \max_{R_r} f.$$

By choosing $N = N(n, \lambda, \Lambda)$ large ($N = \sqrt{8\Lambda(n-1)/\lambda}$ works), this gives $\lambda C_r/4 \leq \max_{R_r} f$, or put differently,

$$\max_{B_r} w \leq \frac{4}{\lambda} r^2 \max_{R_r} f.$$

Since $\max_{R_r} f \to f(0)$ as $r \to 0$, we obtain (2.2). □

3. Harnack inequality

We will prove:

Theorem 3.1. *Suppose u belongs to $\mathscr{S}(\lambda, \Lambda, f)$ in B_1 with $u \geq 0$ in B_1 for some $f \in C(B_1)$. Then*

$$\sup_{B_{1/2}} u \leq C\Big(\inf_{B_{1/2}} u + \|f\|_{L^n(B_1)}\Big), \tag{3.1}$$

where $C = C(n, \lambda, \Lambda)$ is a positive constant.

This immediately gives interior Hölder continuity:

Corollary 3.2. *Suppose u belongs to $\mathscr{S}(\lambda, \Lambda, f)$ in B_1 for some $f \in C(B_1)$. Then $u \in C^\alpha(B_1)$ for some $\alpha = \alpha(n, \lambda, \Lambda) \in (0, 1)$. Moreover, there is a positive constant $C = C(n, \lambda, \Lambda)$ such that for all $x, y \in B_{1/2}$, the estimate*

$$|u(x) - u(y)| \leq C|x - y|^\alpha \Big(\sup_{B_1} |u| + \|f\|_{L^n(B_1)}\Big)$$

holds.

Proof. The estimate (3.1) scales as

$$\sup_{B_{R/2}} u \leq C\Big(\inf_{B_{R/2}} u + R\|f\|_{L^n(B_R)}\Big). \tag{3.2}$$

Let $M(r) = \max_{B_r} u$ and $m(r) = \min_{B_r} u$ for $r \in (0, 1)$, and put $\omega(r) = M(r) - m(r)$. It suffices to show that

$$\omega(r) \leq Cr^\alpha \Big(\sup_{B_{1/2}} |u| + \|f\|_{L^n(B_1)}\Big)$$

for $r < 1/2$. Applying (3.2) to $M(r) - u \geq 0$ in B_r, we obtain

$$\sup_{B_{r/2}} (M(r) - u) \leq C\Big(\inf_{B_{r/2}} (M(r) - u) + r\|f\|_{L^n(B_r)}\Big),$$

that is,

$$M(r) - m(r/2) \leq C\big(M(r) - M(r/2) + r\|f\|_{L^n(B_r)}\big).$$

Similarly, apply (3.2) to $u - m(r) \geq 0$ in B_r, which gives

$$M(r/2) - m(r) \leq C\big(m(r/2) - m(r) + r\|f\|_{L^n(B_r)}\big).$$

Adding these two inequalities gives

$$\omega(r) + \omega(r/2) \leq C\big(\omega(r) - \omega(r/2) + r\|f\|_{L^n(B_r)}\big),$$

thus, with $\gamma = (C-1)/(C+1) < 1$,

$$\omega(r/2) \leq \gamma\omega(r) + Cr\|f\|_{L^n(B_r)}.$$

Iterating this inequality easily gives

$$\omega(\rho) \leq C\rho^\alpha\big(\omega(1/2) + \|f\|_{L^n(B_1)}\big)$$

for $\rho \in (0, 1/2]$; but $\omega(1/2) \leq 2\sup_{B_{1/2}} |u|$, finishing the proof. □

We now turn to the proof of Theorem 3.1; we will work with cubes instead of balls for simplicity (we will be using a Calderon-Zygmund decomposition later) and deduce the theorem from the following lemma:

Lemma 3.3. *Suppose u belongs to $\mathscr{S}(\lambda, \Lambda, f)$ in*[2] *$B_{2\sqrt{n}}$ with $u \geq 0$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist two positive constants ϵ_0 and C, depending only on λ, Λ and n, such that if $\inf_{Q_{1/4}} u \leq 1$ and $\|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$, then $\sup_{Q_{1/4}} u \leq C$.*

Proof of Theorem 3.1 given Lemma 3.3. For $u \in \mathscr{S}(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ with $u \geq 0$ in $B_{2\sqrt{n}}$, consider

$$u_\delta = \frac{u}{\inf_{Q_{1/4}} u + \frac{1}{\epsilon_0}\|f\|_{L^n(B_{2\sqrt{n}})} + \delta}$$

for $\delta > 0$. Then Lemma 3.3 applies to u_δ (with f replaced by f_δ, defined analogously to u_δ), which in the limit $\delta \to 0$ gives

$$\sup_{Q_{1/4}} u \leq C\Big(\inf_{Q_{1/4}} u + \|f\|_{L^n(B_{2\sqrt{n}})}\Big).$$

A covering argument then gives (3.1). □

Roughly speaking, our strategy consists of the following three steps:

1. Show that if a viscosity solution u is small somewhere in the large cube Q_3, then it can be bounded on a large subset of Q_1 (Lemma 3.4);
2. use this to show a quantitative rate of decay for the size of the set where u is larger than $t > 0$ as $t \to \infty$ (Lemma 3.5);
3. considering $\delta(C - u)$ ($\delta > 0$ small, $C > 0$ large) instead of u, this gives that if u is very large at some point in a big cube, it has a large lower bound on a large subset of a smaller cube, contradicting the established quantitative decay rate and proving Lemma 3.3.

We begin with

Lemma 3.4. *Suppose u belongs to $\mathscr{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist constants $\epsilon_0 > 0$, $\mu \in (0, 1)$ and $M > 1$, depending only on λ, Λ, n, such that if*

$$u \geq 0 \text{ in } B_{2\sqrt{n}}, \quad \inf_{Q_3} u \leq 1, \quad \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0,$$

[2] The factors of $\sqrt{n}$ appear to make the cubes fit nicely into balls with 'nice rational' diameters.

then

$$|\{u \le M\} \cap Q_1| > \mu.$$

Proof. The idea is to construct a C^2 function g that only depends on λ, Λ and n, which satisfies

$$g \ge 2 \text{ on } Q_3, \quad g|_{\partial B_{2\sqrt{n}}} = 0, \tag{3.3}$$

so that $w = u - g$ is ≤ -1 at some point in Q_3 but $w \ge 0$ on $\partial B_{2\sqrt{n}}$. Then the Alexandroff maximum principle, Theorem 2.2, will give a lower bound on the measure of the lower contact set of $-w^-$, which proves the lemma.

If $\phi \in C^2(B_{2\sqrt{n}})$ is such that $w - \phi$ has a local minimum at $x_0 \in B_{2\sqrt{n}}$, then $u - (\phi + g)$ has a local minimum at x_0. By definition of $\mathscr{S}^+(\lambda, \Lambda, f)$, this implies

$$\mathscr{M}^-(\lambda, \Lambda, D^2\phi(x_0) + D^2 g(x_0)) \le f(x_0),$$

hence by (1.3)

$$\mathscr{M}^-(\lambda, \Lambda, D^2\phi(x_0)) \le f(x_0) - \mathscr{M}^-(\lambda, \Lambda, D^2 g(x_0)).$$

We thus want to choose g such that

$$\mathscr{M}^-(\lambda, \Lambda, D^2 g(x_0)) \ge 0 \text{ for all } x_0 \in B_{2\sqrt{n}} \setminus B_{1/2}. \tag{3.4}$$

With such g, which we will construct below, we obtain

$$w \in \mathscr{S}^+(\lambda, \Lambda, f + \eta) \text{ in } B_{2\sqrt{n}}$$

for some $\eta \in \mathcal{C}^\infty_{\mathrm{c}}(Q_1^\circ)$, $0 \le \eta \le C(n, \lambda, \Lambda)$. (Indeed, just take η to be a function in $\mathcal{C}^\infty_{\mathrm{c}}(Q_1^\circ)$ with $\eta + \mathscr{M}^-(\lambda, \Lambda, D^2 g) \ge 0$ in $B_{1/2}$.) Now, since $\inf_{Q_3} w \le -1$ and $w|_{\partial B_{2\sqrt{n}}} \ge 0$, we may apply Theorem 2.2 to w and obtain

$$1 \le C\left(\int_{B_{2\sqrt{n}} \cap \{w = \Gamma_w\}} (|f| + \eta)^n\right)^{1/n} \le C\|f\|_{L^n(B_{2\sqrt{n}})} + C|\{w = \Gamma_w\} \cap Q_1|^{1/n}.$$

Thus, for $\epsilon_0 = 1/2C$ and f with $\|f\|_{L^n(B_{2\sqrt{n}})} \le \epsilon_0$, we obtain

$$1/2 \le C|\{w = \Gamma_w\} \cap Q_1|^{1/n}.$$

Recalling that $\Gamma_w = \Gamma(-w^-)$ is non-positive on $B_{2\sqrt{n}}$, the equality $w(x) = \Gamma_w(x)$ implies $w(x) \le 0$, hence $u(x) \le g(x) \le M := \sup g$, thus $|\{u(x) \le M\} \cap Q_1| \ge (2C)^{-n}$, as desired.

It remains to construct the function g satisfying (3.3) and (3.4). We put

$$g(x) = M\left(1 - \frac{|x|^2}{4n}\right)^\beta$$

with $\beta > 0$ chosen below, and $M = M(\beta)$ chosen such that $g \le -2$ in Q_3. Notice that automatically $g|_{\partial B_{2\sqrt{n}}} = 0$. In order to ensure (3.4), we need the eigenvalues of $D^2 g$. We compute

$$D_{ij} g(x) = -\frac{M}{2n}\beta\left(1 - \frac{|x|^2}{4n}\right)^{\beta-1}\delta_{ij} + \frac{M}{(2n)^2}\beta(\beta-1)\left(1 - \frac{|x|^2}{4n}\right)^{\beta-2} x_i x_j.$$

By spherical symmetry, it suffices to compute the eigenvalues of $D^2 g$ at a point $x = (|x|, 0, \dots, 0)$, in which case they are

$$e^+(x) = \frac{M}{2n}\beta\left(1 - \frac{|x|^2}{4n}\right)^{\beta-2}\left(\frac{2\beta - 1}{4n}|x|^2 - 1\right)$$

with multiplicity 1 and

$$e^-(x) = -\frac{M}{2n}\beta\left(1-\frac{|x|^2}{4n}\right)^{\beta-1}$$

with multiplicity $n-1$; thus $e^+(x) > 0$ for $|x| \geq 1/2$ if β is large enough, and $e^-(x) < 0$. Thus

$$\begin{aligned}\mathscr{M}^-(\lambda, \Lambda, D^2 g(x)) &= \lambda e^+(x) + (n-1)\Lambda e^-(x) \\ &= \frac{M}{2n}\beta\left(1-\frac{|x|^2}{4n}\right)^{\beta-2}\left[\lambda\left(\frac{2\beta-1}{4n}|x|^2-1\right)-(n-1)\Lambda\left(1-\frac{|x|^2}{4n}\right)\right] \\ &\geq 0\end{aligned}$$

for $\beta = \beta(n, \lambda, \Lambda)$ large enough. Thus, (3.4) holds, and the proof is finished. □

Following our strategy, we deduce a quantitative decay of the distribution functions:

Lemma 3.5. *Suppose u belongs to $\mathscr{S}^+(\lambda, \Lambda, f)$ in $B_{2\sqrt{n}}$ for some $f \in C(B_{2\sqrt{n}})$. Then there exist positive constants ϵ_0, ϵ and C, depending only on λ, Λ, n, such that if*

$$u \geq 0 \text{ in } B_{2\sqrt{n}}, \quad \inf_{Q_3} u \leq 1, \quad \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0,$$

then

$$|\{u \geq t\} \cap Q_1| \leq Ct^{-\epsilon} \text{ for } t > 0. \tag{3.5}$$

Proof. With M and μ as in Lemma 3.4, we will prove

$$|\{u > M^k\} \cap Q_1| \leq (1-\mu)^k \tag{3.6}$$

for $k = 1, 2, \ldots$, which implies (3.5): Indeed, given $t > M$ (with $0 < t \leq M$ handled trivially), choose $k \in \mathbb{N}$ such that $M^k \leq t \leq M^{k+1}$, then

$$\begin{aligned}|\{u \geq t\} \cap Q_1| &\leq |\{u \geq M^k\} \cap Q_1| \leq (1-\mu)^k = (M^{2\epsilon}(1-\mu))^k M^{-2\epsilon k} \\ &\leq t^{-2\epsilon k/(k+1)} \leq t^{-\epsilon}\end{aligned}$$

if $\epsilon > 0$ is chosen such that $M^{2\epsilon}(1-\mu) \leq 1$.

We prove (3.6) inductively: For $k = 1$, this is just Lemma 3.4. Assuming (3.6) holds for $k-1$, set

$$A = \{u > M^k\} \cap Q_1, \quad B = \{u > M^{k-1}\} \cap Q_1,$$

then $|B| \leq (1-\mu)^{k-1}$ by the inductive hypothesis, and we aim to prove

$$|A| \leq (1-\mu)|B|. \tag{3.7}$$

Now $A \subset B \subset Q_1$ and $|A| \leq 1-\mu$. We claim that if $Q = Q_r(x_0)$ is a cube in Q_1 such that

$$|A \cap Q| > (1-\mu)|Q|, \tag{3.8}$$

then $\tilde{Q} \cap Q_1 \subset B$ for $\tilde{Q} = Q_{3r}(x_0)$, i.e. if A has large measure in some cube Q, then B contains a 'parent cube' $\tilde{Q}$; by Lemma 3.6 below, this implies (3.7). We argue by contradiction, assuming there is $\tilde{x} \in \tilde{Q}$ such that $u(\tilde{x}) \leq M^{k-1}$. The idea is to consider a rescaled version $\tilde{u}$ of u in $Q_{3r}(x_0)$ and apply Lemma 3.4 to see that $|A^c \cap Q| > \mu|Q|$, contradicting the assumption (3.8).

More precisely, let $x = x_0 + ry$, $y \in Q_1$, $x \in Q = Q_r(x_0)$, and put

$$\tilde{u}(y) = \frac{1}{M^{k-1}} u(x).$$

Then $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$ and $\inf_{Q_3} \tilde{u} \leq 1$. Moreover, $\tilde{u} \in \mathscr{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$ in view of

$$\tilde{f}(y) = \frac{r^2}{M^{k-1}} f(x), \ y \in B_{2\sqrt{n}},$$

thus

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{M^{k-1}} \|f\|_{L^n(B_{2\sqrt{n}})} \leq \|f\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0.$$

By Lemma 3.4,

$$\mu < |\{\tilde{u}(y) \leq M\} \cap Q_1| = r^{-n} |\{u(x) \leq M^k\} \cap Q|,$$

thus $|A^c \cap Q| > \mu |Q|$, providing the desired contradiction to (3.8). □

To finish the proof, we need the following lemma. We first introduce some terminology: Cut the unit cube Q_1 equally into 2^n cubes, which we call the first generation; cut these cubes equally into 2^n cubes, obtaining the second generation, and so on. The cubes from all generations are called *dyadic cubes*. Every $(k+1)$-th generation cube Q is a 2^n-th part of a k-th generation cube $\tilde{Q}$, called its *predecessor*.

Lemma 3.6. *Suppose $A \subset B \subset Q_1$ are measurable sets such that*

(1) $|A| < \delta \in (0, 1)$;

(2) for any dyadic cube Q, $|A \cap Q| \geq \delta |Q|$ implies $\tilde{Q} \subset B$ for the predecessor $\tilde{Q}$ of Q.

Then $|A| \leq \delta |B|$.

Proof. Apply the Calderon-Zygmund decomposition to the characteristic function of A. We obtain a sequence of dyadic cubes $\{Q^j\}$ such that

$$\delta \leq \frac{|A \cap Q^j|}{|Q^j|} < 2^n \delta, \quad \frac{|A \cap \tilde{Q}^j|}{|\tilde{Q}^j|} < \delta$$

for the predecessor $\tilde{Q}^j$ of Q^j; moreover, $A \subset \bigcup_j Q^j$ except for a set of measure zero (this is because $\delta < 1$, and by the Lebesgue differentiation theorem, the set of points in A for which the lim sup of the averages over cubes containing the point is $\leq \delta < 1$ has measure 0). By assumption, we have $\tilde{Q}^j \subset B$ for each j. Hence

$$A \subset \bigcup_j \tilde{Q}^j \subset B.$$

Relabelling the $\tilde{Q}^j$ so that they are mutually disjoint, we get

$$|A| = \sum_i |A \cap \tilde{Q}^i| \leq \delta \sum_i |\tilde{Q}^i| \leq \delta |B|.$$ □

We are now in a position to prove Lemma 3.3, and thus Theorem 3.1.

Proof of Lemma 3.3. We prove that there exist $\theta \gtrapprox 1$ and $M_0 \gg 1$, depending only on λ, Λ and n, such that if $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$, there exists a sequence $\{x_k\} \in B_{1/2}$ such that $u(x_k) \geq \theta^k P$ for $k \in \mathbb{N}_0$, contradicting the boundedness of u; we thus conclude $\sup_{B_{1/4}} u \leq M_0$. The idea is simple: If x_1 did not exist, then $\theta P - u$ (appropriately rescaled) would be positive with small infimum over a certain small cube, thus it would be quantitatively bounded from above on a large subset of a smaller cube, hence u would be bounded from below on a large subset of a smaller cube, which contradicts the decay established in Lemma 3.5.

Concretely, assuming $u(x_0) = P > M_0$ for some $x_0 \in B_{1/4}$, we choose r such that $\{u > P/2\}$ covers less than half of $Q_r(x_0)$; to accomplish this, note that $\inf_{Q_3} u \leq \inf_{Q_{1/4}} u \leq 1$, hence Lemma 3.5 gives

$$|\{u > P/2\} \cap Q_1| \leq C(P/2)^{-\epsilon} \leq r^n/2 \tag{3.9}$$

for $r^n/2 \geq C(P/2)^{-\epsilon}$. Choosing $P > M_0$ large, we may assume $r \leq 1/4$. For such r, we have $Q_r(x_0) \subset Q_1$ and

$$\frac{1}{|Q_r(x_0)|}|\{u > P/2\} \cap Q_r(x_0)| \leq 1/2$$

We now show that for $\theta \gtrapprox 1$, there exists a point in $B_{2\sqrt{n}r}(x_0)$ at which $u \geq \theta P$. Assuming that $u < \theta P$ in $B_{2\sqrt{n}r}(x_0)$, we rescale as usual, putting $x = x_0 + ry$ for $y \in B_{2\sqrt{n}}$, $x \in B_{2\sqrt{n}r}(x_0)$, and

$$\tilde{u}(y) = \frac{\theta P - u(x)}{(\theta - 1)P}.$$

By assumption, $\tilde{u} \geq 0$ in $B_{2\sqrt{n}}$, and $\inf_{Q_3} \tilde{u} \leq \tilde{u}(0) = 1$; moreover $\tilde{u} \in \mathscr{S}^+(\lambda, \Lambda, \tilde{f})$ in $B_{2\sqrt{n}}$ with $\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \epsilon_0$, as is seen from

$$\tilde{f}(y) = -\frac{r^2}{(\theta - 1)P} f(x),\ y \in B_{2\sqrt{n}},$$

which gives

$$\|\tilde{f}\|_{L^n(B_{2\sqrt{n}})} \leq \frac{r}{(\theta - 1)P}\|f\|_{L^n(B_{2\sqrt{n}r}(x_0))} \leq \epsilon_0$$

provided $r \leq (\theta - 1)P$. By Lemma 3.5,

$$\frac{1}{|Q_r(x_0)|}|\{u \leq P/2\} \cap Q_r(x_0)| = \left|\left\{\tilde{u} \geq \frac{\theta - 1/2}{\theta - 1}\right\} \cap Q_1\right| \leq C\left(\frac{\theta - 1/2}{\theta - 1}\right)^{-\epsilon} < 1/2$$

if θ is chosen close to 1. This contradicts (3.9). To summarize what this gives: There exists $\theta = \theta(n, \lambda, \Lambda) > 1$ such that if $u(x_0) = P$ for some $x_0 \in B_{1/4}$, then $u(x_1) \geq \theta P$ for some $x_1 \in B_{2\sqrt{n}r}(x_0)$, provided

$$C(n, \lambda, \Lambda)P^{-\epsilon/n} \leq r \leq (\theta - 1)P.$$

Thus, we choose P such that $P \geq (C/(\theta - 1))^{n/(n+\epsilon)}$, and then take $r = CP^{-\epsilon/n}$.

We may iterate this to get a sequence $\{x_k\}$ such that for any $k \in \mathbb{N}$, there exists $x_k \in B_{2\sqrt{n}r_k}(x_{k-1})$ such that $u(x_k) \geq \theta^k P$, where $r_k = C(\theta^{k-1}P)^{-\epsilon/n} = C\theta^{-(k-1)\epsilon/n}P^{-\epsilon/n}$. In order to ensure that $x_k \in B_{1/2}$ for all $k \in \mathbb{N}$, we need $\sum 2\sqrt{n}r_k < 1/4$, which is satisfied if we choose M_0 such that

$$M_0^{\epsilon/n} \geq 8\sqrt{n}C\sum_{k=1}^{\infty}\theta^{-(k-1)\epsilon/n}$$

and $M_0 \geq (C/(\theta - 1))^{n/(n+\epsilon)}$ (which is the above condition on P) and then take $P > M_0$. The proof is complete. □