POSITIVE COMMUTATORS

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1. Introduction

In this short and very elementary note, we will discuss the technique of proving estimates for solutions to certain partial differential equations using positive commutators, which is a widely used and powerful technique in particular in microlocal analysis. In order to keep the necessary prerequisites at the bare minimum, we won't make any use of pseudodifferential operators and related microlocal machinery in the central parts of this note, but references will be given. Positive commutator estimates are particularly useful for proving energy and regularity estimates for hyperbolic equations like the wave equation, or, more generally, equations which involve some sort of "flow" (see remark 4.1 for references).

To be more specific, consider the wave equation on $\mathbb{R}\times M$, M a closed (compact, boundaryless) manifold, given by $\Box u \equiv (\partial_t^2 - \Delta)u = f$, and suppose $u|_{t<0} = 0 = f|_{t<0}$. Since the operator $\Box$ is not elliptic, we do not expect elliptic estimates to hold (which would tell us that u is 2 orders more regular than f, say in a Sobolev sense; here, 2 is the order of the operator $\Box$); instead, motivated by the intuition that the "energy" of our solution u (which is a travelling wave of some kind) in a time interval $[0,T]$ should be controlled by the the forcing f, we expect an estimate

$$\|u\|_{H^1([0,T]\times M)} \leq C\|f\|_{L^2([0,T]\times M)}, \tag{1.1}$$

where the choice of norms (H^1 for u, L^2 for f) takes into account that we are not in an elliptic setting, i.e. we lose one derivative. And indeed, this estimate holds; see proposition 3.1 for the precise statement. The idea of the proof is the following: We take a vector field $V = \chi\partial_t$, with a function χ to be chosen momentarily, such that the "commutator" $V^*\Box + \Box V$ is positive in an appropriate sense[1] (up to error terms we can control), that is,

$$\langle \Box u, Vu\rangle + \langle Vu, \Box u\rangle = \langle (V^*\Box + \Box V)u, u\rangle \geq C(\|\partial_t u\|^2 + \|\nabla u\|^2),$$

where C is a constant of order controlled by $|\chi'|$. (In this particular example, there is no error term on the right hand side; in proposition 3.6, we will have one though.) Now, using Cauchy-Schwartz on the left hand side and noting that $\|Vu\|_{L^2} \leq c\|u\|_{H^1}$ for a constant c of order controlled by $|\chi|$, we get an inequality of the form

$$C\|u\|_{H^1}^2 \leq c(\|\Box u\|_{L^2}^2 + \|u\|_{H^1}^2)$$

Now comes the kicker: Since we can dominate a function (in this case χ) by its derivative (in this case, it turns out, by $-\chi'$) if we choose χ wisely, we can dominate

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[1] As a justification why one could call this a commutator, note that $V^* = -V - \partial_t\chi$, i.e. $V^*\Box + \Box V = [\Box, V] - (\partial_t\chi)\Box$.

c by C, thus absorb the second term on the right into the left hand side and finally end up with the desired estimate (1.1).

Why does this work? By analogy with the Lie derivative of vector fields just being their commutator, our "commutator" $V^*\Box + \Box V$ can be thought of as the "derivative" of the commutant V "along" $\Box$, and by making V "decrease along" $\Box$, we get the desired positivity.[2]

In general, the commutant does not have to be a vector field; in fact, in section 2, it will just be a function. More generally, it is more or less going to be a ΨDO whose leading order term, when commuted with the operator in question, here $\Box$, gives rise to the desired positivy, modulo controllable errors. We have to admit that all estimates we will prove in this note can also be proven (perhaps more easily, depending on your taste) by arguments involving integration by parts directly (e.g. as in section 2.8 in [Tay11a]); but keeping in mind the robustness of the method explained here and the extendability to more general settings, it is definitely useful to know how positive commutators work, also since e.g. working with ΨDOs only adds technical difficulties (and only few of them) but no conceptual ones.

2. Two Poincaré inequalities

We begin our discussion of positive commutators with two simple examples which illustrate well how the main point of positive commutators arguments, namely dominating a function by its derivative, works.

So suppose $u \in C^\infty(\mathbb{R})$, and $u|_{t<0} = 0$. We want to obtain the inequality[3]

$$\|u\|_{L^2([0,T])} \leq C\|\partial_t u\|_{L^2([0,T])} \tag{2.1}$$

as a positive commutator estimate. To do so we need a function χ whose derivative "along" (here: with respect to) ∂_t has a definite sign and controls χ itself.

Take any $\chi \in C^\infty(\mathbb{R})$ (to be chosen later, when we know what exactly we need) vanishing for $t > T_e > T > 0$. We then have

$$\langle \partial_t u, \chi u\rangle + \langle \chi u, \partial_t u\rangle = \langle [\chi, \partial_t]u, u\rangle = -\langle \chi' u, u\rangle,$$

where we integrate by parts for the first equality. Choosing χ non-negative and such that $-\chi' \geq 0$ for $0 \leq t \leq T_e$, this together with Cauchy-Schwartz and AM-GM gives

$$\begin{aligned}\|\sqrt{-\chi'}u\|^2_{L^2([0,T_e])} &\leq 2\|\sqrt{\chi}\partial_t u\|_{L^2([0,T_e])}\|\sqrt{\chi}u\|_{L^2([0,T_e])} \\ &\leq \|\sqrt{\chi}\partial_t u\|^2_{L^2([0,T_e])} + \|\sqrt{\chi}u\|^2_{L^2([0,T_e])},\end{aligned} \tag{2.2}$$

where we split $\chi = \sqrt{\chi}\cdot\sqrt{\chi}$ in order to be able to take L^2 norms on the right hand side without worries. Rewriting this as

$$\int_0^{T_e} (-\chi' - \chi)|u|^2\,dt \leq \|\sqrt{\chi}\partial_t u\|^2_{L^2([0,T_e])},$$

we now see what we need, namely, say, $-\chi' - \chi \geq -\chi'/2$ on $[0, T_e]$.

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[2]In the language of microlocal analysis, the principal symbol r of $\frac{1}{i}[\Box, V]$ is given by the derivative of the symbol of V along the Hamilton vector field H_p of the principal symbol $p = \sigma_2(\Box)$ of $\Box$, and if we can make r positive, it can be written as q^2 for some symbol $q > 0$, thus $\frac{1}{i}[\Box, V]u = Q^*Qu+$ lower order terms. A priori control on the lower order terms (paired with a solution u of $\Box u = f$) then gives, with some additional technical details, control on Qu, thus we get regularity information on u where q is positive.

[3]A related inequality is the Poincaré inequality $\|u\|_{L^2([0,T])} \leq C\|\partial_t u\|_{L^2([0,T])}$ for $u \in H^1_0([0,T])$.

The common way to find such a χ is to consider the $C^\infty(\mathbb{R})$ function

$$\phi(t) = \begin{cases} e^{-1/t}, & t > 0 \\ 0, & t \leq 0, \end{cases} \tag{2.3}$$

satisfying

$$\phi'(t) = t^{-2}\phi(t).$$

Throughout the rest of this note, ϕ is always going to denote this function. We then choose

$$\chi(t) = \phi(\digamma^{-1}(T_e - t))$$

and compute

$$\chi'(t) = -\digamma^{-1} \cdot (\digamma^{-1}(T_e - t))^{-2}\chi(t) = -\digamma(T_e - t)^{-2}\chi(t).$$

Since for $0 \leq t \leq T_e$, $\digamma(T_e - t)^{-2} \geq \digamma T_e^{-2}$, we can choose $\digamma$ sufficiently large to get $-\chi' - \chi \geq -\chi'/2$ on $[0, T_e]$, as required. Plugging this into (2.2), we can absorb the $\|\sqrt{\chi}u\|_{L^2([0,T_e])}$ term into the left hand side and thus obtain

$$\|\sqrt{-\chi'}u\|^2_{L^2([0,T_e])} \leq C\|\sqrt{\chi}\partial_t u\|^2_{L^2([0,T_e])} \tag{2.4}$$

for some $C > 0$. Now we see that we do not quite get (2.1), since $-\chi'$ is not bounded from below on all of $[0, T_e]$; instead, using that $-\chi'$ *does* have a positive lower bound on $[0, T]$ $(T < T_e)$, we only get the weaker estimate

$$\|u\|_{L^2([0,T])} \leq C\|\partial_t u\|_{L^2([0,T_e])}.$$

To fix this insufficiency, note that if we cut off χ at time T, i.e. replaced $\chi(t)$ by $\chi(t)H(T - t)$, H being the Heaviside function, and still got the estimate (2.4), interpreted in a distributional sense (the left hand side being the distributional pairing $-\langle\chi' u, u\rangle$), we would get an additional term involving the δ-distribution at $t = T$, i.e.

$$\|\sqrt{-\chi'}u\|^2_{L^2([0,T])} + \chi(T)|u(T)|^2 \leq C\|\sqrt{\chi}\partial_t u\|^2_{L^2([0,T])},$$

and the boundary term $\chi(T)|u(T)|^2$ would be positive (put differently, it has the same sign as $-\chi'$), thus could just be dropped, and we would get (2.1)! In fact, this idea works. To see this formally, let $\check{\chi}(t) = \chi(t)H(T-t)$, then we can compute just as above (now with $\langle\ ,\ \rangle$ denoting the distributional pairing, one side being a distribution, the other side a test function, made antilinear in the second slot)

$$\langle\partial_t u, \check{\chi}u\rangle + \langle\check{\chi}u, \partial_t u\rangle = \langle[\check{\chi}, \partial_t]u, u\rangle = -\langle\check{\chi}' u, u\rangle = \chi(T)|u(T)|^2 - \langle\chi' u, u\rangle_{L^2([0,T])},$$

where, for the first equality, we just use the definition of the distributional derivative (which of course does the same as integration by parts), and in the last equality we use $-\check{\chi}' = -\chi' H(T - \cdot) + \chi(T)\delta_T$. We now proceed as before, dropping the boundary term $\chi(T)|u(T)|^2$, and finally obtain[4] (2.1).

Remark 2.1. We note that (2.1) and (2.4) hold more generally for functions $u \in C^\infty(\mathbb{R}_t, L^2(M))$ vanishing for $t < 0$, where M is any measure space, e.g. a Riemannian manifold.

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[4]If one is suspicious about the use of distributions here, one can also write everything out in terms of integrals from 0 to T, and the boundary term $\chi(T)|u(T)|^2$ is precisely the boundary term one gets from integrating by parts. However, this procedure makes it seem like there was something special about the boundary term, although there is not, really.

Next, we look at a propagation estimate.

Proposition 2.2. *Let $-\infty < t_0 < t_1 < t_2 < \infty$. Then there is a constant $C > 0$ such that for all $u \in C^\infty(\mathbb{R})$,*

$$\|u\|_{L^2([t_1,t_2])} \leq C(\|u\|_{L^2([t_0,t_1])} + \|\partial_t u\|_{L^2([t_0,t_2])}). \tag{2.5}$$

Proof. We need to choose our commutant χ a little more carefully: Namely, while we still need to be able to dominate χ by $-\chi'$ on the interval $[t_1, t_2]$, we also have to require $\chi|_{t<t_0} = 0$. This is easily arranged, as we will see shortly. Conceptually, this means that we cannot absorb a term $\|\sqrt{\chi}u\|_{L^2([t_0,t_1])}$ into $\|\sqrt{|\chi'|}u\|_{L^2([t_0,t_1])}$, which is fine, since it appears on the right hand side of (2.5) anyway. Put differently, we only require the commutant χ to have the favorable properties as used in the previous proof away from a region (here $[t_0, t_1]$) where we have a priori control on u.

Guided by these considerations, we let $\check{\chi}(t) = \chi(t)H(t_2-t)$ and $\chi(t) = \chi_0(t)\chi_1(t)$; here, $\chi_0(t) = \phi(\digamma^{-1}(t_3-t))$, where $t_3 > t_2$ is fixed, and $\chi_1(t) = \phi_1((t-t_0)/(t-t_1))$ with ϕ_1 non-negative, identically 0 on $(-\infty, 0]$ and identically 1 on $[1, \infty)$. We then compute

$$\begin{aligned}\langle\check{\chi}u, \partial_t u\rangle + \langle\partial_t u, \check{\chi}u\rangle &= -\langle\check{\chi}'u, u\rangle = -\langle\chi'u, u\rangle_{L^2([t_0,t_2])} + \chi(t_2)|u(t_2)|^2 \\ &= \|\sqrt{-\chi'}u\|^2_{L^2([t_1,t_2])} - \langle\chi'u, u\rangle_{L^2([t_0,t_1])} + \chi(t_2)|u(t_2)|^2,\end{aligned}$$

hence, again using Cauchy-Schwartz and AM-GM,

$$\begin{aligned}\|\sqrt{-\chi'}u\|^2_{L^2([t_1,t_2])} &\leq \|\sqrt{\chi}u\|^2_{L^2([t_0,t_2])} + \|\sqrt{\chi}\partial_t u\|^2_{L^2([t_0,t_2])} + \|\sqrt{|\chi'|}u\|^2_{L^2([t_0,t_1])} \\ &= \|\sqrt{\chi}u\|^2_{L^2([t_1,t_2])} + \|\sqrt{\chi + |\chi'|}u\|^2_{L^2([t_0,t_1])} + \|\sqrt{\chi}\partial_t u\|^2_{L^2([t_0,t_2])}.\end{aligned} \tag{2.6}$$

Choosing $\digamma$ big enough, we can absorb the first term on the right into the left hand side; thus, since $-\chi'$ is bounded from below on $[t_1, t_2]$ and χ, χ' are bounded in absolute value on all of $[t_0, t_2]$, we obtain (2.5). $\square$

Sure enough, using positive commutators to prove these simple inequalities is a little overkill (although the given proofs are conceptually clearer than hands-on proofs), but we will see in the next section that without doing anything new, we can obtain similar energy estimates for the wave equation.

3. Energy estimates for the wave equation on $\mathbb{R} \times M$

To keep things simple and computations easily manageable, we focus on the wave equation on a product manifold, that is, we consider the operator $\Box = \partial_t^2 - \Delta_M$, where Δ_M is the Laplace-Beltrami operator on the closed Riemannian manifold M.

Proposition 3.1. *There exists $C > 0$ such that for all $u \in C^\infty(\mathbb{R} \times M)$, $u|_{t<0} = 0$, we have the estimate*

$$\|u\|_{H^1([0,T]\times M)} \leq C\|\Box u\|_{L^2([0,T]\times M)}. \tag{3.1}$$

Proof. We consider the commutant $V = \check{\chi}\partial_t$, where $\check{\chi}(t) = \chi(t) \cdot H(T-t)$, where $\chi(t) = \phi(\digamma^{-1}(T_e - t))$, $T_e > T$ fixed. Then for $u \in C^\infty(\mathbb{R} \times M)$, $u|_{t<0} = 0$, we compute, using the fact that $[\Delta_M, \partial_t] = 0$ and $[\nabla, \chi] = 0$ (∇ being the gradient operator on M)

$$\begin{aligned}\langle Vu, \Box u\rangle + \langle \Box u, Vu\rangle &= \langle(-\partial_t \check{\chi}\Box + \Box\check{\chi}\partial_t)u, u\rangle \\ &= \langle(-\partial_t\check{\chi}\partial_t\partial_t + \partial_t\partial_t\check{\chi}\partial_t)u, u\rangle + \langle(\check{\chi}\partial_t - \partial_t\check{\chi})\Delta_M u, u\rangle \\ &= -\langle \partial_t\check{\chi}\cdot\partial_t u, \partial_t u\rangle - \langle\partial_t\check{\chi}\cdot\Delta_M u, u\rangle \\ &= -\langle\partial_t\chi\cdot\partial_t u, \partial_t u\rangle_{L^2([0,T]\times M)} + \chi(T)\|\partial_t u\|^2_{L^2(\{T\}\times M)} \\ &\qquad - \langle\partial_t\chi\cdot\nabla u, \nabla u\rangle_{L^2([0,T]\times M)} + \chi(T)\|\nabla u\|^2_{L^2(\{T\}\times M)}.\end{aligned}$$

Thus, dropping the second and fourth term on the right hand side, both being positive, we arrive at the estimate

$$\|\sqrt{-\partial_t\chi}\partial_t u\|^2_{L^2} + \|\sqrt{-\partial_t\chi}\nabla u\|^2_{L^2} \leq \|\sqrt{\chi}\Box u\|^2_{L^2} + \|\sqrt{\chi}\partial_t u\|^2_{L^2}. \tag{3.2}$$

We again make $-\partial_t\chi - \chi \geq -\partial_t\chi/2$ on $[0,T]$ by choosing $\digamma$ large to absorb the second term on the right into the first term on the left hand side. Using the already established inequality $\|\sqrt{-\partial_t\chi}u\|_{L^2} \leq C\|\sqrt{\chi}\partial_t u\|_{L^2}$ (with C independent of χ), see remark 2.1, and, if necessary, choosing $\digamma$ even bigger to absorb this additional term $C\|\sqrt{\chi}\partial_t u\|_{L^2}$ into the left hand side of (3.2), we get our energy estimate (3.1), using that $-\chi'$ is bounded from below on $[0,T]$. $\Box$

Remark 3.2. Note that (3.1) is only an a priori estimate: It only tells us something about $u \in C^\infty(\mathbb{R}\times M)$ with $u|_{t<0} = 0$. It does *not* say that any $u \in L^2(\mathbb{R}\times M)$ with $u|_{t<0} = 0$ and $\Box u \in L^2([0,T]\times M)$ is in fact in $H^1([0,T])$. This means that in order to prove higher regularity (i.e. in particular that a weak solution $u \in L^2(\mathbb{R}\times M)$ to $\Box u = f, u|_{t<t_0} = 0$, with $f \in L^2([0,T]\times M)$, lies in $H^1([0,T]\times M)$), our estimate alone does not suffice. However, since $\Box = \partial_t^2 - \Delta_M$ commutes with convolutions in t, we can use a simple regularization argument[5] to prove higher regularity. We will do this for the forward solution of the wave equation on a product manifold $\mathbb{R}_t \times M$ in section 4.

Remark 3.3. The proof goes through without change and further thought if we just require $u \in C^\infty(\mathbb{R}, H^2(M))$ with $u|_{t<0} = 0$, since u only needs to have enough regularity in the spatial variables to ensure that we can integrate by parts in these, and that $\Box u \in L^2$. This will be used in section 4 to prove higher regularity for solutions of the wave equation.

Remark 3.4. Proposition 3.1 holds in much greater generality (cf. chapter 2.8 in [Tay11a], which however is in a slightly different setting and does not use commutator arguments). We provide some hints how to prove such more general estimates using positive commutator arguments:

- If we replace Δ_M by a t-dependent operator $L(t)$ satisfying $-\langle L(t)u, u\rangle \geq c\|\nabla u\|^2$ for $0 \leq t \leq T$ with $c > 0$, we can perform a similar commutator computation and get additional terms where the coefficients of $L(t)$ get differentiated with respect to t, but their prefactor χ is not hit by a derivative for these terms, thus they can be absorbed into the term $\|\sqrt{-\partial_t\chi}\nabla u\|$ on the left hand side in (3.2).

[5]In more general settings, a regularization argument still works, only one needs a little more machinery.

- In a similar vein, adding lower order (that is, 1st and 0th order) terms to $\Box$, i.e. considering $\Box+E$, we can again perform the above computation, where however we just leave the additional term $\langle Eu, Vu\rangle + \langle Vu, Eu\rangle$ unchanged, so that again this term only involves χ but not χ', hence can be absorbed into the left hand side.
- For the d'Alembert operator on a general Lorentzian manifold (X, g) on which we are given a proper function $t\colon X \to \mathbb{R}$ such that $t^{-1}(t_0)$ is a compact spacelike hypersurface for each $t_0 \in [0, T]$, we can take $V = \chi W$, where $g(W, -) = dt$, as our commutant. To deal with all the terms we get, it is useful to introduce a stress-energy tensor, which has a positive definiteness property that allows us to obtain the desired energy estimate. See section 3.3 of [Va12] for details in a slightly more elaborate setting.

Remark 3.5. As a final remark on proposition 3.1, note that we can use its straightforward generalization to vector-valued $\bar{u} \in C^\infty$ to obtain a priori estimates with higher order Sobolev spaces: Namely, we can put $\bar{u} = (u, \partial_t u, \ldots, \partial_t^{k-1} u)$ and obtain, using $[\Box, \partial_t] = 0$:

$$\|(\partial_t^j u)_{0\le j\le k-1}\|^2_{H^1([0,T]\times M)} \le C\|(\partial_t^j \Box u)_{0\le j\le k-1}\|^2_{L^2([0,T]\times M)},$$

which gives

$$\|u\|_{H^k([0,T]\times M)} \le C\|\Box u\|_{H^{k-1}([0,T]\times M)}.$$

Here, as before, u is assumed to vanish for $t < 0$.

We now prove an analogue of proposition 2.2 for the wave equation.

Proposition 3.6. *Let $t_0 < t_1 < t_2$. Then there exists $C > 0$ such that for all $u \in C^\infty(\mathbb{R}\times M)$, we have the estimate*

$$\|u\|_{H^1([t_1,t_2]\times M)} \le C(\|u\|_{H^1([t_0,t_1]\times M)} + \|\Box u\|_{L^2([t_0,t_2]\times M)}). \tag{3.3}$$

Proof. With the same $\check{\chi}$ as in the proof of proposition 2.2, we put $V = \check{\chi}\partial_t$ and compute

$$\begin{aligned}\langle \Box u, Vu\rangle + \langle Vu, \Box u\rangle &= -\langle \partial_t\check{\chi}\cdot\partial_t u, \partial_t u\rangle - \langle \partial_t\check{\chi}\cdot\nabla u, \nabla u\rangle \\ &\ge \|\sqrt{-\partial_t\chi}\partial_t u\|^2_{L^2([t_1,t_2])} - \|\sqrt{|\partial_t\chi|}\partial_t u\|^2_{L^2([t_0,t_1])} \\ &\quad + \|\sqrt{-\partial_t\chi}\nabla u\|^2_{L^2([t_1,t_2])} - \|\sqrt{|\partial_t\chi|}\nabla u\|^2_{L^2([t_0,t_1])},\end{aligned}$$

where we use that the boundary terms (arising from the δ-distribution which comes from the differentiation of $\check{\chi}$) are positive. Rearranging, we get

$$\begin{aligned}\|\sqrt{-\partial_t\chi}(\partial_t,\nabla)u\|^2_{L^2([t_1,t_2])} &\le \|\sqrt{|\partial_t\chi|}(\partial_t,\nabla)u\|^2_{L^2([t_0,t_1])} \\ &\quad + \|\sqrt{\chi}\Box u\|^2_{L^2([t_0,t_2])} + \|\sqrt{\chi}\partial_t u\|^2_{L^2([t_0,t_1])} + \|\sqrt{\chi}\partial_t u\|^2_{L^2([t_1,t_2])}.\end{aligned} \tag{3.4}$$

Again, we can dominate χ by $-\chi'$ for $t_1 \le t \le t_2$ by choosing F large, thus absorb the last term on the right into the left hand side. Moreover, adding to (3.4) the estimate

$$\begin{aligned}\|\sqrt{-\partial_t\chi}u\|^2_{L^2([t_1,t_2])} \le C(&\|\sqrt{\chi+|\chi'|}u\|^2_{L^2([t_0,t_1])} + \|\sqrt{\chi}\partial_t u\|^2_{L^2([t_0,t_1])} \\ &+ \|\sqrt{\chi}\partial_t u\|^2_{L^2([t_1,t_2])})\end{aligned}$$

from the proof of proposition 2.2 (for F big enough) and absorbing the last term on the right of this estimate into the left hand side of (3.4), we obtain the energy estimate (3.3). $\Box$

Remark 3.7. A remark similar to remark 3.3 applies here as well: It is enough to require $u \in C^\infty(\mathbb{R}, H^2(M))$.

4. Solving the wave equation on $\mathbb{R} \times M$; higher regularity

Now that we have some energy estimates, we want to show how to use them to prove existence, uniqueness and regularity to solutions of the wave equation

$$\begin{cases} \Box u &= (\partial_t^2 - \Delta_M)u = f \text{ in } (-\infty, T] \times M, \text{ where } f \in L^2(\mathbb{R}_t \times M), f|_{t<0} = 0, \\ u|_{t<0} &= 0. \end{cases} \tag{4.1}$$

The basic idea is that we use analogous energy estimates, which are in particular injectivity (or almost-injectivity) statements, for the (formal) adjoint $\Box^*(=\Box)$, which, combined with abstract functional analysis, will produce a solution to the original problem. The guiding principle here is that the annihilator of the range of $\Box$ is the same as the kernel of $\Box^*$; since we have an estimate for $\Box^*$, the range of $\Box$ (on appropriate spaces) is in fact closed, as we will see below.

Reversing the time direction, we already have an a priori estimate

$$\|v\|_{H^1([0,T]\times M)} \leq C\|\Box^* v\|_{L^2([0,T]\times M)}$$

for $v \in \mathcal{V}_T(\mathbb{R}, H^2(M)) := \{w \in C^\infty(\mathbb{R}, H^2(M)) : w|_{t>T} = 0\}$, since $\Box^* = \Box$ (we write $\Box^*$ here to be pedantic, though). Thus, given $f \in L^2([0,T] \times M)$, we can define a linear map[6]

$$\ell \colon \Box^* v \mapsto \langle v, f\rangle_{L^2([0,T]\times M)}, \quad v \in \mathcal{V}_T(\mathbb{R}, H^2(M)),$$

satisfying

$$|\langle v, f\rangle| \leq \|f\|_{(H^1([0,T]\times M))^*} \cdot C\|\Box^* v\|_{L^2([0,T]\times M)}.$$

Thus, extending ℓ to all of $L^2([0,T] \times M)$ by requiring it to be 0 on the orthocomplement of $\Box^*(\mathcal{V}_T(\mathbb{R}, H^2(M)))$, we in fact get $\ell \in (L^2)^* = L^2$, i.e. we obtain $u \in L^2([0,T] \times M)$ such that

$$\langle v, f\rangle_{L^2([0,T]\times M)} = \langle \Box^* v, u\rangle_{L^2([0,T]\times M)}, \quad v \in \mathcal{V}_T(\mathbb{R}, H^2(M)). \tag{4.2}$$

Note that since $C_c^\infty((0,T) \times M) \subset \mathcal{V}_T(\mathbb{R}, H^2(M))$, this in particular implies that $\Box u = f$ in $\mathscr{D}'((0,T) \times M)$.

A closer look at equation (4.2) gives immediately that if we extend u to an L^2 function (which we still call u) on $(-\infty, T] \times M$ by setting $u|_{t<0} = 0$ and analogously extend f by 0, we get $\Box u = f$ in $\mathscr{D}'((-\infty, T) \times M)$; thus, we have found a weak solution to (4.1). For obvious reasons, we call u a *forward-solution.* Note that the uniqueness of u still needs to be settled.

We proceed by showing that any solution u to (4.1) must in fact be in $H^1([0, T - 2\delta] \times M)$, where $\delta > 0$ is arbitrary but fixed. As indicated in remark 3.2, we use a mollifier to achieve this: Choose $\omega \in C_c^\infty(\mathbb{R})$ with support in $[-1, 1]$ and put $\omega_\epsilon(t) = \epsilon^{-1}\omega(\epsilon^{-1}t)$; further, set $u_\epsilon := \omega_\epsilon * u$ and $f_\epsilon := \omega_\epsilon * f$ on $I_\delta := (-\infty, T - \delta]$ for $\epsilon < \delta$. Since $[\Box, \omega_\epsilon *] = 0$ (the coefficients of $\Box$ being independent of t), we have $\Box u_\epsilon = f_\epsilon$. Since $u_\epsilon, f_\epsilon \in C^\infty(I_\delta, L^2(M))$, this implies $\Delta_M u_\epsilon(t) = \partial_t^2 u_\epsilon(t) - f_\epsilon(t) \in L^2(M)$; elliptic regularity (cf. chapter 5 in [Tay11a]) then gives $u_\epsilon(t) \in H^2(M)$.

[6] Of course, we could take $f \in (H^1([0,T] \times M))^*$, the dual taken with respect to the L^2 inner product, but we are going to be interested in $f \in L^2$ later on anyway.

We contend that in fact $u_\epsilon \in C^\infty(I_{2\delta}, H^2(M))$. Indeed, defining the difference quotient operator $(D_h f)(t) := (f(t+h) - f(t))/h$ and noting that $[\Delta_M, D_h] = 0 = [\partial_t, D_h]$, we obtain at a fixed time t:

$$(\partial_t^2 - \Delta_M)(D_h u_\epsilon) = D_h f_\epsilon,$$

i.e.

$$\Delta_M(D_h u_\epsilon) = \partial_t^2 D_h u_\epsilon - D_h f_\epsilon.$$

As $h \to 0$, the right hand side is Cauchy in $L^2(M)$, since $u_\epsilon, f_\epsilon \in C^\infty(I_\delta, L^2(M))$. Therefore, elliptic regularity tells us that $D_h u_\epsilon$ is Cauchy in $H^2(M)$ as $h \to 0$, whence $\partial_t u_\epsilon = \lim_{h\to 0} D_h u_\epsilon$ with convergence in $H^2(M)$, i.e. $u_\epsilon \in C^1(I_{2\delta}, H^2(M))$, and $\partial_t u_\epsilon$ solves $(\partial_t^2 - \Delta_M)(\partial_t u_\epsilon) = \partial_t f_\epsilon$. Thus, we can iterate the above argument to prove our contention.

Now note that $u_\epsilon|_{t<-\epsilon} = 0$; therefore, taking remark 3.3 into account, we can use the a priori estimate from proposition 3.1 to get

$$\|u_\epsilon\|_{H^1(I_{2\delta}\times M)} \leq C\|\Box u_\epsilon\|_{L^2(I_{2\delta}\times M)} = C\|f_\epsilon\|_{L^2(I_{2\delta}\times M)} \xrightarrow{\epsilon\to 0} C\|f\|_{L^2(I_{2\delta}\times M)}.$$

The weak compactness of the unit ball in H^1 thus shows that there is a subsequence of u_ϵ which converges weakly to some $\bar{u} \in H^1(I_{2\delta}\times M)$, hence in norm in L^2. Since on the other hand $u_\epsilon \to u$ in $L^2(I_{2\delta}\times M)$, we conclude that $u = \bar{u} \in H^1(I_{2\delta}\times M)$. Moreover, since $\|\cdot\|$ is weakly lower semicontinuous, we also get the estimate

$$\|u\|_{H^1(I_{2\delta}\times M)} \leq C\|f\|_{L^2(I_{2\delta}\times M)} \tag{4.3}$$

for a solution $u \in L^2((-\infty, T]\times M)$ of $\Box u = f$. Note that this is much stronger than an a priori estimate: It is to be understood as a regularity statement, namely any L^2-solution u to $\Box u = f \in L^2$ in fact lies in H^1, with a bound on the H^1-norm.

We can now easily prove that a solution u to the PDE (4.1) is unique: Indeed, given two solutions u_1, u_2, their difference $u_0 := u_1 - u_2$ satisfies $\Box u_0 = 0$, $u_0|_{t<0} = 0$, $u_0 \in H^1(I_{2\delta}\times M)$ for all $\delta > 0$. Thus, using (4.3),

$$\|u_0\|_{H^1(I_{2\delta}\times M)} \leq C\|\Box u_0\|_{L^2(I_{2\delta}\times M)} = 0$$

for all $\delta > 0$, whence $u_0 = 0$ on $(-\infty, T]$, proving uniqueness.

The uniqueness gives us an easy way to prove that a solution u to (4.1) lies in $H^1([0,T]\times M)$: Namely, just extend f beyond $t = T$ to an L^2-function $\bar{f}$ on $[0, T+1]\times M$ and produce an L^2 solution $\bar{u}$ vanishing for negative times. By uniqueness, $\bar{u}|_{[0,T]\times M} = u|_{[0,T]\times M}$. But we already know $\bar{u} \in H^1([0, T+1-2\delta]\times M)$ for every $\delta > 0$. In particular, for $\delta \leq 1/2$, this implies $u \in H^1([0,T]\times M)$.

Of course, we can apply the above argument iteratively to prove that the forward solution u to $\Box u = f$ with $f \in H^k((-\infty, T]\times M)$ vanishing in $t < 0$ in fact lies in H^{k+1}.

Remark 4.1. The above hands-on construction is an example that shows how to combine regularizers with a priori estimates to obtain higher regularity (and thus uniqueness) of solutions. There are other ways to achieve this:

- In a real setting, one could approximate all the data of the PDE by real-analytic functions and use Cauchy-Kovalevskaya to obtain local real-analytic solutions, whose Sobolev norms are well-behaved in view of the a priori estimates; a limiting argument similar to the above then gives higher regularity statements for solutions to the original PDE (see e.g. chapter 6 in [Tay11a]).

- Another tool that in many cases is easier to deal with is the "fundamental theorem of microlocal analysis", Hörmander's theorem on the propagation of singularities, which could also be called "propagation of regularity", since it in particular tells us that a solution u to $\Box u = f \in L^2$ which is H^1-regular for negative times (e.g. if it vanishes there) will be H^1-regular at all positive times (a little more precisely, at every point that can be connected to a point with negative time coordinate by a flow[7] associated to the operator $\Box$). A proof (modulo some details) of Hörmander's result using positive commutators can be found in section 4.2 [Wu08]. There exist many excellent sources which provide the required background on microlocal analysis, for example [Ho83] (in particular chapter XVIII), [Me09] and [Tay11b].
- An argument involving pseudodifferential regularizers and the calculus of ΨDOs is presented in chapter 7 of [Tay11b].

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[7] This actually is a flow in the cotangent bundle of $\mathbb{R} \times M$, and one needs a notion of microlocal Sobolev regularity for the precise statement of Hörmander's theorem. In the present situation, it is enough to think of the "flow" as being the collection of all null-geodesics, i.e. light rays, within $\mathbb{R} \times M$, endowed with the metric $dt^2 - g$, g being the Riemannian metric on M.