Regularity of linear waves at the Cauchy horizon of black hole spacetimes

Peter Hintz
joint with András Vasy

Luminy

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Cauchy horizon of charged black holes

(subextremal) Reissner-Nordström-de Sitter spacetime

- solution of Einstein-Maxwell system,
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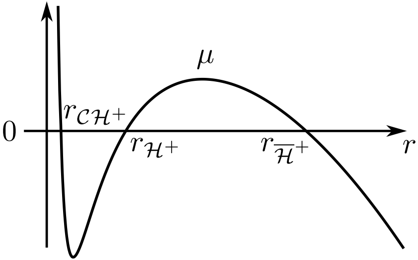
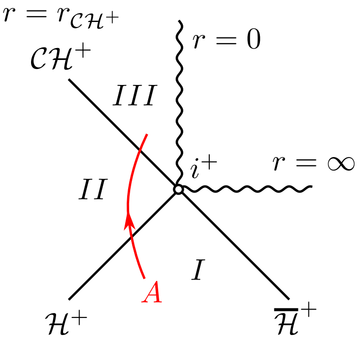
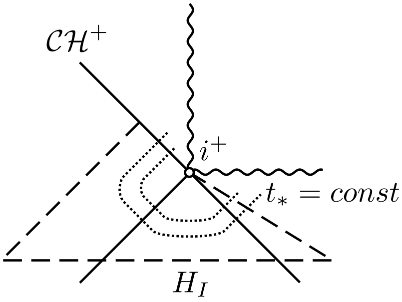
- solution of Einstein-Maxwell system,
- black hole mass $M > 0$, charge $Q > 0$,
- cosmological constant $\Lambda > 0$,
- topology: $\mathbb{R}_{t_*} \times (0, \infty)_r \times \mathbb{S}^2$,
- metric: $g = \mu(r)\, dt^2 - \mu(r)^{-1}\, dr^2 - r^2\, d\sigma^2$; $t_* = t - F(r)$.



Penrose diagram:





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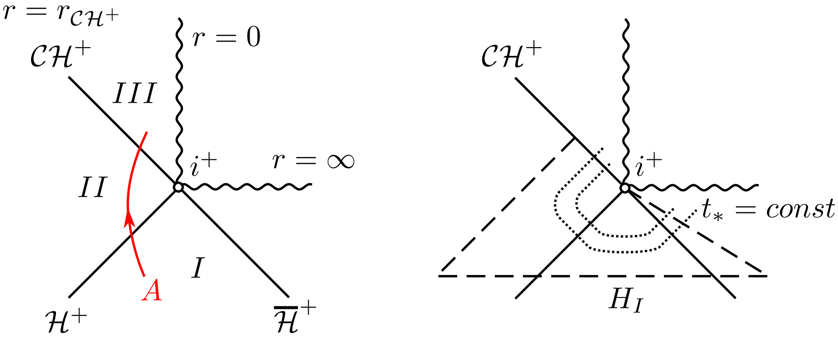
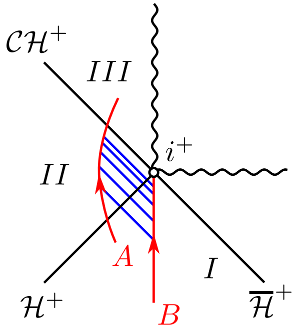


Cauchy horizon $\mathcal{CH}^+$: boundary of domain of uniqueness of solution u to wave equation $\Box_g u = 0$ with Cauchy data on H_I

Blue-shift effect and strong cosmic censorship

(Simpson–Penrose '73.)



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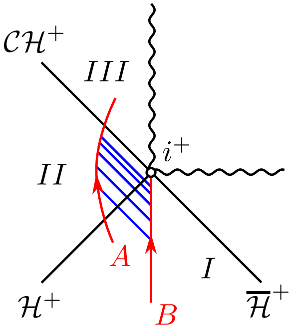

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Observer B lives forever.

Conjecture (Penrose's Strong Cosmic Censorship)

The maximal globally hyperbolic development of generic initial data for Einstein's field equations is inextendible as a suitably regular Lorentzian manifold.

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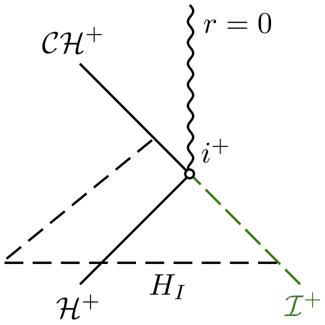
Related work:

- Christodoulou (. . . ,'99, '08),
- Dafermos ('03, '05, '13),
- ongoing work by Dafermos–Luk, Luk–Oh.

Toy model: linear wave equation

Cauchy problem for $\Box_g u = 0$ on Reissner–Nordström.



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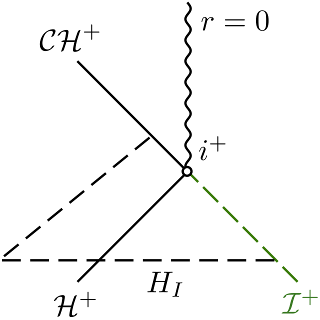

Theorem (Franzen, '14)

For $\mathcal{C}^\infty$ initial data, u remains uniformly bounded near $\mathcal{CH}^+$.

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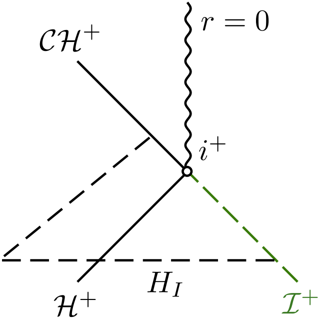

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Theorem (Luk–Oh, '15)

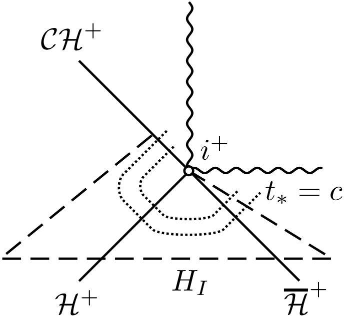
For generic $\mathcal{C}^\infty$ initial data, u is not in H^1_{loc} near any point of $\mathcal{CH}^+$.

Theorem (H.–Vasy, '15)

For $\mathcal{C}^\infty$ initial data on Reissner–Nordström–de Sitter, u solving $\Box_g u = 0$ has a partial asymptotic expansion near i^+,

$$u = u_0 + u', \quad u_0 \in \mathbb{C},\ |u'(t_*)| \lesssim e^{-\alpha t_*},$$

where $\alpha > 0$ depends only on the spacetime.



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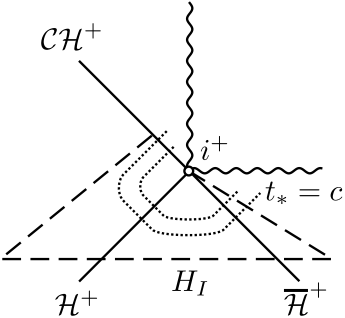

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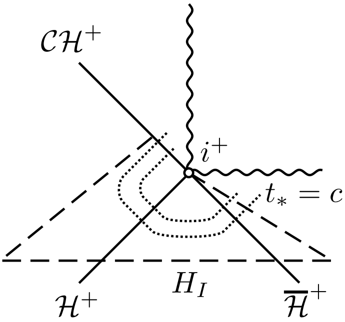

$\kappa > 0$: surface gravity of the Cauchy horizon

Previous work

Microlocal analysis/scattering theory approach:

- Melrose ('93)
- Sá Barreto–Zworski ('97), Bony–Häfner ('08)
- Melrose–Sá Barreto–Vasy ('14)
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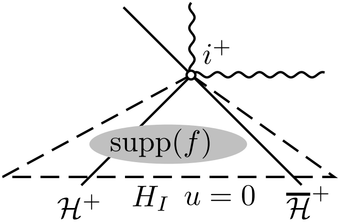
Energy estimates:

- Dafermos–Shlapentokh-Rothman ('15)
- Luk–Sbierski ('15)

Analysis near the exterior region



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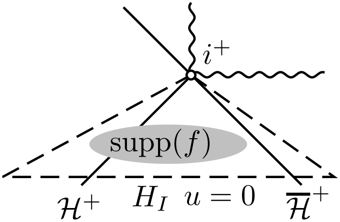

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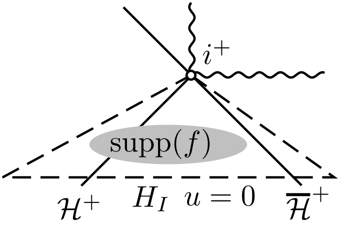

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u is $\mathcal{C}^\infty$. Quantitative bounds as $t_* \to \infty$?

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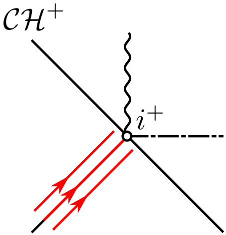

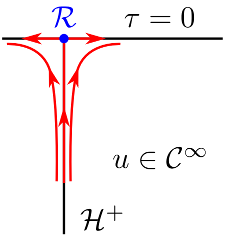

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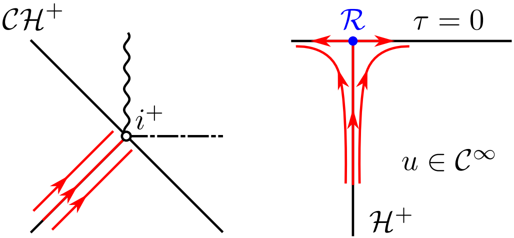

Propagation of singularities on ‘uniform’ version ${}^{\mathrm{b}}T^*M$ of the cotangent bundle down to $\tau = 0$. $\mathcal{R}$: saddle point for the null-geodesic flow lifted to ${}^{\mathrm{b}}T^*M$.

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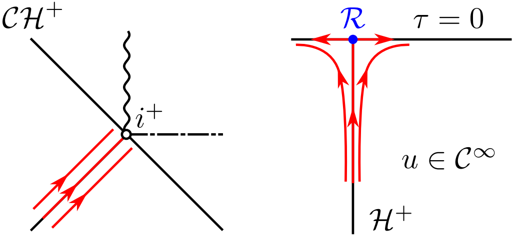

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Resonance expansions

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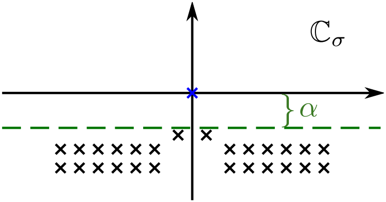

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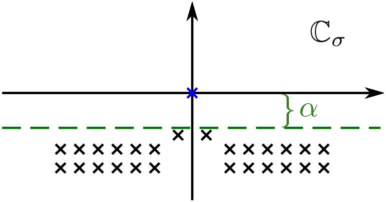
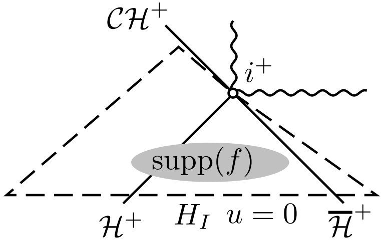


Obtain:

$$u = u_0 + u', \quad u_0 \in \mathbb{C}, \; u' \in e^{-\alpha t_*} H^\infty.$$

This gives asymptotics and decay in $r \geq r_{\mathcal{CH}^+} + \epsilon$, $\epsilon > 0$.



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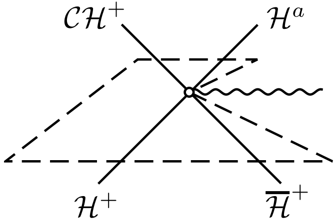

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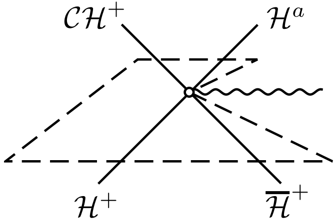
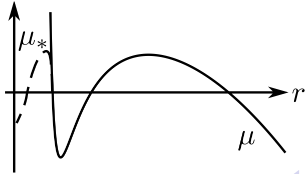
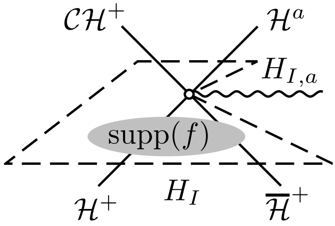


$$\widetilde{g} = \mu_* \, dt^2 - \mu_*^{-1} \, dr^2 - r^2 \, d\sigma^2.$$



Setup for the extended problem

Study forcing problem $\Box_{\widetilde{g}} u = f$, with $u = 0$ near $H_I \cup H_{I,a}$.



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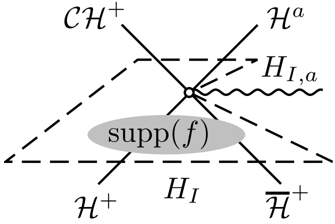

(Add complex absorbing potential $\mathcal{Q} \in \Psi_{\mathrm{b}}^2$ beyond $\mathcal{CH}^+$ to hide additional trapping and $\mathcal{H}^a$. Study $\Box_{\widetilde{g}} - i\mathcal{Q}$.)

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$\Box_{\widetilde{g}} u = f \in \mathcal{C}^\infty_{\mathrm{c}}$. Work near $\mathcal{CH}^+$. Recall $\tau = e^{-t_*}$.

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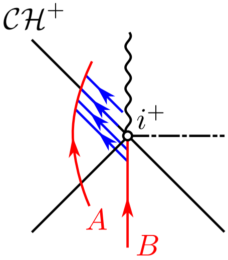

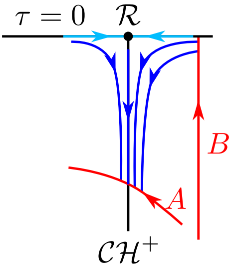

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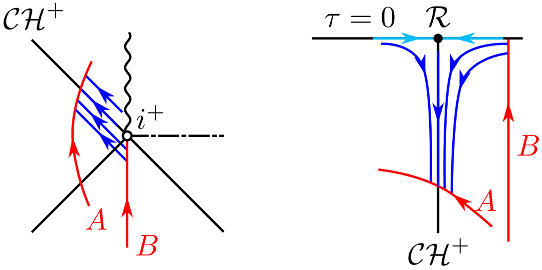
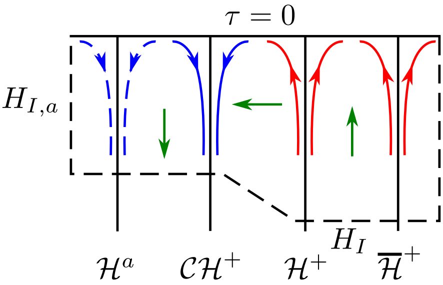


If $u \in e^{-\ell t_*} H^{-\infty}$ near $\mathcal{R}$, and $u \in e^{-\ell t_*} H^s$ in a punctured neighborhood of $\mathcal{R}$ within $\{\tau = 0\}$, then $u \in e^{-\ell t_*} H^s$ near $\mathcal{R}$, provided $s < 1/2 + \ell/\kappa$.

Microlocal analysis of the extended problem



Green arrows: Future directed timelike vectors.

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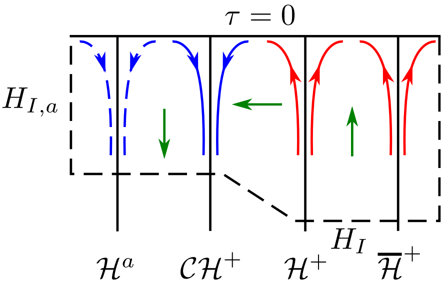

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Control u solving $\Box_{\widetilde{g}} u = f$ using standard hyperbolic theory near dashed surfaces, and microlocal elliptic regularity and propagation of singularities.

Obtain

$$\|u\|_{e^{-\ell t_*}H^s} \lesssim \|\Box_{\widetilde{g}} u\|_{e^{-\ell t_*}H^{s-1}} + \|u\|_{e^{-\ell t_*}H^{s_0}},$$

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Get solvability of extended problem $\Box_{\widetilde{g}} u = f$, and control of u near $\mathcal{CH}^+$.

Resonance expansion

Solution u of $\Box_{\widetilde{g}} u = f$ has partial expansion

$$u|_{r>r_{\mathcal{CH}^+}} = u_0 + u', \quad u_0 \in \mathbb{C},\ u' \in e^{-\alpha t_*} H^{1/2+\alpha/\kappa-0} \text{ near } \mathcal{CH}^+.$$

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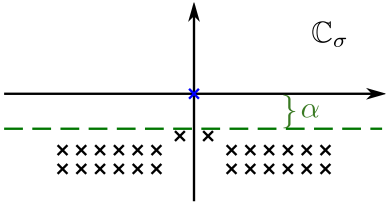
$\alpha > 0$: spectral gap of $\Box_g$.



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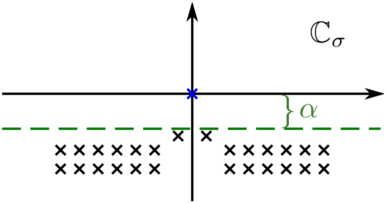

$u'(t_*, r, \omega)$ is smooth in t_*, ω. (Haber-Vasy '13.)

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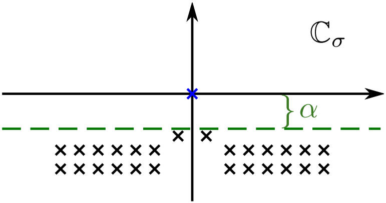
$$u|_{r>r_{\mathcal{CH}^+}} = u_0 + u', \quad u_0 \in \mathbb{C}, \; u' \in e^{-\alpha t_*} H^{1/2+\alpha/\kappa-0} \text{ near } \mathcal{CH}^+.$$

$\alpha > 0$: spectral gap of $\Box_g$.



$u'(t_*, r, \omega)$ is smooth in t_*, ω. (Haber-Vasy '13.)
$H^{1/2+0}(\mathbb{R}_r) \hookrightarrow L^\infty(\mathbb{R}_r)$ yields $|u'(t_*)| \lesssim e^{-\alpha t_*}$.

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Only potential issue for large a: resonances in $\operatorname{Im} \sigma \geq 0$ ('mode stability,' see Whiting '89, Shlapentokh-Rothman '14 for Kerr)

Outlook

Shallow resonances. Mode solution $\Box_g(e^{-i\sigma t_*} v(x)) = 0$ has $v \in H^{1/2-\operatorname{Im}\sigma/\kappa-0}$ at $\mathcal{CH}^+$; could be $\mathcal{C}^\infty$ in principle. Study location and regularity properties of shallow resonances.

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$\Lambda = 0$. Localize analysis near black hole region; in the exterior, polynomial decay according to Price's law (Tataru '13, Metcalfe–Tataru–Tohaneanu '12). Get polynomial decay near $\mathcal{CH}^+$. (Luk–Sbierski '15, H. '15)

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Nonlinear problems. Einstein's field equations. Work in progress by Luk–Rodnianski, Dafermos–Luk, Luk–Oh.