Lecture 16:
Character isomorphisms via tempered cohomology

Peter J. Haine

January 25, 2022

Abstract

These are extended notes from a talk at the 202One Talbot Workshop on Ambidexterity. Their purpose is to explain how to use Lurie's work on elliptic cohomology to recover character isomorphisms in chromatic homotopy theory due to Hopkins–Kuhn–Ravenel and Stapleton. The main tool is a basechange result for tempered cohomology theories; the proof relies on an ambidexterity result for tempered local systems.

Plan/reminder

Here is the plan for the talk:

(1) Motivation from character theory (see §1).

(2) How the results in §1 relate to tempered cohomology (§2).

(3) *Orientations* of **P**-divisible groups (§3).

(4) Basechange for tempered cohomology and how to deduce character isomorphism from basechange (§4).

0.1 Example. Piotr introduced two key examples of tempered cohomology theories $(A, \mathbf{G})$.

(1) $A = \mathrm{KU}$ is complex K-theory and $\mathbf{G} = \mu_{\mathbf{P}^\infty}$ is the *multiplicative* **P***-divisible group*. The tempered cohomology theory $\mathrm{KU}_{\mu_{\mathbf{P}^\infty}}$ gave us a way to glue together G-equivariant K-theory for all finite abelian groups G. This is the motivating example guiding our definition of a tempered cohomology theory. In this talk, we'll examine addition features that $\mathrm{KU}_{\mu_{\mathbf{P}^\infty}}$ has and ask that our tempered cohomology theories satisfy these as well.

(2) $A = \mathrm{E}_n$ is height n Lubin–Tate theory and $\mathbf{G}$ is the Quillen p-divisible group.

It is important to keep these examples in mind throughout the talk.

0.2 Remark. The main reference for this talk is [Ell III]. However, we only cover a very small percentage of the text and do so somewhat nonlinearly. For ease of navigation, reference to specific results in [Ell III] are hyperlinked to the exact spot in the PDF where the result is stated.

1 Motivation: Character theory

In the last talk, Piotr defined the notion of a *tempered cohomology theory* to capture the features of equivariant K-theory for finite groups. We've captured many features of equivariant K-theory, but there is one result we haven't seen the analogue of: the relationship between equivariant K-theory and representation theory via *characters*.

1.1 Character theory for finite groups

We begin by recalling the fundamental result about the character theory of a finite group.

1.1 Theorem. *Let G be a finite group, and write* $\mathrm{Rep}(G)$ *for the representation ring of G. The assignment taking a G-representation to its character defines an isomorphism*

$$\mathbf{C} \otimes_{\mathbf{Z}} \mathrm{Rep}(G) \xrightarrow{\sim} \{\text{class functions } G \to \mathbf{C}\} .$$

There are two important ways that we can inject homotopy theory into this theorem:

(1) Notice that the representation ring $\mathrm{Rep}(G)$ is the G-equivariant K-theory $\mathrm{KU}^0_G(*)$ of the point.

(2) Notice that class functions $G \to \mathbf{C}$ are just functions $G/^{\mathrm{ad}}G \to \mathbf{C}$ from the quotient of G acting on itself by the adjoint action. In other words, there is an isomorphism

$$\{\text{class functions } G \to \mathbf{C}\} \cong \mathrm{H}^0(G_{\mathrm{h}G}; \mathbf{C})$$

between class functions and the 0-th cohomology of the homotopy orbit space $G_{\mathrm{h}G}$.

That is, Theorem 1.1 provides an isomorphism

$$\mathbf{C} \otimes_{\mathbf{Z}} \mathrm{KU}^0_G(*) \xrightarrow{\sim} \mathrm{H}^0(G_{\mathrm{h}G}; \mathbf{C}) . \tag{1.2}$$

Thus it is natural to ask:

1.3 Question. In the isomorphism (1.2), can we replace the point by a more general G-space?

To answer this, we need to give a reinterpretation the term $G_{\mathrm{h}G}$ when the point is replaced by a general G-space X. To do this, note that G with the conjugation action can be written as the disjoint union $\coprod_{g \in G} *$ where G-acts by permuting the factors of the coproduct. This interpretation makes sense for more general G-spaces:

1.4 Notation. Let G be a finite group and $X \colon \mathrm{Orb}(G)^{\mathrm{op}} \to \mathbf{Spc}$ a G-space. Recall that for each orbit $H \backslash G$, we write X^H for the value of X on $H \backslash G$. For each $g \in G$, we simply write X^g for $X^{\langle g \rangle}$. Write

$$\coprod_{g \in G} X^g$$

for the space with G-action given by conjugation on the factors and residual G-actions on X^g.

1.5. When $X = *$ note that $\coprod_{g \in G} X^g$ is G with the conjugation action.

The *equivariant Chern character* provides an affirmative answer to Question 1.3.

1.6 Theorem (equivariant Chern character)**.** *Let G be a finite group and X a finite G-space. The equivariant Chern character defines an isomorphism*

$$\mathrm{ch}_G \colon \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{KU}^0_G(X) \xrightarrow{\sim} \mathrm{H}^{\mathrm{even}}\left(\left(\coprod_{g \in G} X^g\right)_{\mathrm{h}G}; \mathbf{C}\right).$$

Since we're interested in *chromatic* homotopy theory in this workshop, it is natural to ask:

1.7 Question. What is the 'p-complete' analogue of Theorem 1.6?

1.8. To answer this, note that since G is finite there are isomorphisms

$$G \cong \mathrm{Hom}(\mathbf{Z}, G) \cong \mathrm{Hom}(\widehat{\mathbf{Z}}, G).$$

Thus to replace the coproduct over G in Theorem 1.6 by a 'p-complete' variant, we might try replacing $\mathrm{Hom}(\widehat{\mathbf{Z}}, G)$ by $\mathrm{Hom}(\mathbf{Z}_p, G)$.

Since G is finite, every homomorphism $\mathbf{Z}_p \to G$ factors through a finite quotient $\mathbf{Z}/p^k$. Hence there is a natural isomorphism

$$\mathrm{Hom}(\mathbf{Z}_p, G) \cong \left\{ g \in G \,\middle|\, g^{p^k} = 1 \text{ for } k \gg 0 \right\}.$$

An element of G whose order is a power of p is called a *p-singular* element.

Theorem 1.6 implies the following 'p-complete' variant.

1.9 Theorem. *Let G be a finite group and X a finite G-space. Fix a prime p and an embedding $\mathbf{Z}_p \hookrightarrow \mathbf{C}$. There is a canonical isomorphism*

$$\mathbf{C} \otimes_{\mathbf{Z}_p} (\mathrm{KU}^\wedge_p)^0(X_{\mathrm{h}G}) \xrightarrow{\sim} \mathrm{H}^{\mathrm{even}}\left(\left(\coprod_{\alpha \colon \mathbf{Z}_p \to G} X^{\mathrm{im}(\alpha)}\right)_{\mathrm{h}G}; \mathbf{C}\right).$$

Note that Theorem 1.9 explains a nontrivial relationship between a height 1 cohomology theory ($\mathrm{KU}^\wedge_p$) and a height 0 cohomology theory (singular cohomology with coefficients in $\mathbf{C}$). In particular, it tells us that after tensoring with $\mathbf{C}$, the information captured by a height 1 cohomology theory can be computed as singular cohomology of a *different* space.

1.10 Question. What about relating height n phenomena and height 0 phenomena?

The answer to Question 1.10 is the content of *Hopkins–Kuhn–Ravenel character theory*; we discuss this next. Given integers $m \leq n$, one might also wonder if it is possible to relate height n phenomena to height m phenomena. Stapleton has provided a wonderful generalization of Hopkins–Kuhn–Ravenel character theory that does exactly this; see [13; 14]. In order to simplify the exposition, we focus on relating heights n and 0.

1.2 Character theory in chromatic homotopy theory

In order to state the main result of Hopkins–Kuhn–Ravenel character theory, let us first recall some notation.

1.11 Notation. We fix some notation for the remainder of this section:

(1) Let k be a perfect field of characteristic $p > 0$. Let $\widehat{\mathbf{G}}_0$ be a height n formal group. Let $\widehat{\mathbf{G}}$ be the universal deformation of $\widehat{\mathbf{G}}_0$, so that $\widehat{\mathbf{G}}$ is the identity component of a p-divisible group $\mathbf{G}$.

(2) Write R for the Lubin–Tate ring, so that R is noncanonically isomorphic to $\mathbf{W}(k)[[v_1, \ldots, v_{n-1}]]$.

(3) Let C_0 denote the R-algebra classigying isomorphisms of p-divisible groups

$$\underline{\mathbf{Q}_p/\mathbf{Z}_p} \cong \mathbf{G} .$$

(4) Let E be the Lubin–Tate spectrum of $\widehat{\mathbf{G}}_0$. Recall that there are isomorphisms

$$\pi_0(\mathrm{E}) \cong \mathrm{R} \qquad \text{and} \qquad \mathrm{Spf}(\mathrm{E}^0(\mathbf{C}\mathrm{P}^\infty)) \cong \widehat{\mathbf{G}}_0 .$$

(5) Write $\mathrm{R}_{\mathbf{Q}} := \mathrm{R}[1/p]$ and $\mathrm{E}_{\mathbf{Q}} := \mathrm{E}[1/p]$ for the rationalizations of R and E.

1.12 Theorem (Hopkins–Kuhn–Ravenel [4; 5])**.** *Let G be a finite group and X a finite G-space. There is a canonical isomorphism of graded* C_0*-algebras*

$$\mathrm{C}_0 \otimes_{\mathrm{R}} \mathrm{E}^*(X_{\mathrm{h}G}) \xrightarrow{\sim} \mathrm{C}_0 \otimes_{\mathrm{R}_{\mathbf{Q}}} \mathrm{E}^*_{\mathbf{Q}}\left(\left(\coprod_{\alpha\colon \mathbf{Z}_p \to G} X^{\mathrm{im}(\alpha)}\right)_{\mathrm{h}G}\right) .$$

Like Theorem 1.9, Theorem 1.12 tells us that after tensoring with C_0, the information captured by a height n cohomology theory can be computed as height 0 cohomology of a *different* space.

1.13 Remark. For a nice exposition of Hopkins–Kuhn–Ravenel character theory, see [12, §6].

1.14 Goal (for the rest of the talk)**.** Explain how to deduce/reprove all of the results stated so far via tempered cohomology. The key steps are:

(1) Understand $A_{\mathbf{G}_0 \oplus (\underline{\mathbf{Q}_p/\mathbf{Z}_p})^n}$. See §2.

(2) Understand how tempered cohomology interacts with basechange: $B \otimes_A A^{\mathsf{X}}_{\mathbf{G}}$ vs. $B^{\mathsf{X}}_{\mathbf{G}}$. See §§3 and 4.

2 Formal loop spaces & character isomorphisms

The first goal of the section is to answer the following question.

2.1 Question. What really 'are' the spaces

$$\left(\coprod_{\alpha\colon \mathbf{Z}_p \to G} X^{\mathrm{im}(\alpha)}\right)_{\mathrm{h}G}$$

appearing in Hopkins–Kuhn–Ravenel character theory?

To do this, we will give an orbispace refinement of these spaces (see Example 2.6). After that, we begin to reinterpret the character isomorphisms of Theorem 1.12 in terms of tempered cohomology (§§2.3 and 2.4).

2.1 Reminder on orbispaces

To get started, let's recall the basics of orbispaces.

2.2 Recollection (orbispaces [Ell III, §3.1])**.** We write $\mathcal{T} \subset \mathbf{Spc}$ for the full subcategory spanned by those spaces equivalent to the classifying space of a finite abelian group. We write

$$\mathbf{OSpc} \coloneqq \operatorname{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Spc})$$

for the ∞-category of presheaves of spaces on $\mathcal{T}$. We refer to objects of $\mathbf{OSpc}$ as *orbispaces*.

There is a chain of four adjoints relating spaces and orbispaces

$$\mathbf{OSpc} \quad \begin{array}{c} \xrightarrow{\ \operatorname{colim}_{\mathcal{T}^{\mathrm{op}}}\ } \\ \xleftarrow{\ \underline{(-)}\ } \\ \xrightarrow{\ |-|\ } \\ \xleftarrow[\ X^{(-)} \leftarrow\!\shortmid X\ ]{} \end{array} \quad \mathbf{Spc}\ .$$

Here functors lie above their right adjoints.

(1) The extreme left adjoint sends an orbispace X to the colimit $\operatorname{colim}_{\mathcal{T}^{\mathrm{op}}} \mathsf{X}$ of X over $\mathcal{T}^{\mathrm{op}}$.

(2) Its right adjoint $\underline{(-)}\colon \mathbf{Spc} \to \mathbf{OSpc}$ carries a space X to the constant functor $\mathcal{T}^{\mathrm{op}} \to \mathbf{Spc}$ with value X. The functor $\underline{(-)}$ is fully faithful.

(3) The right adjoint of $\underline{(-)}$ is denoted by $|-|\colon \mathbf{OSpc} \to \mathbf{Spc}$ and given by evaluation on the point:

$$|\mathsf{X}| \coloneqq \mathsf{X}(*)\ .$$

We call $|\mathsf{X}|$ the *underlying space* of the orbispace X.

(4) The right adjoint of $|-|$ is the functor $\mathbf{Spc} \to \mathbf{OSpc}$ given by sending a space X to the restricted Yoneda functor

$$\begin{aligned} X^{(-)}\colon \mathcal{T}^{\mathrm{op}} &\to \mathbf{Spc} \\ T &\mapsto X^{T} \coloneqq \operatorname{Map}_{\mathbf{Spc}}(T, X)\ . \end{aligned}$$

Since the functor $\underline{(-)}$ is fully faithful, this extreme right adjoint is also fully faithful.

2.3 Recollection (orbispace quotient [Ell III, §3.2])**.** Let G be a finite group and X a G-space. The *orbispace quotient* $X /\!\!/ G$ is the orbispace given by the colimit

$$X /\!\!/ G \coloneqq \operatorname*{colim}_{\substack{H\backslash G \to X \\ H \text{ abelian}}} \mathrm{B}H\ .$$

Explicitly, the orbispace quotient can be described by the formula

$$(X /\!\!/ G)^{\mathrm{B}K} = \left(\coprod_{\alpha\colon K \to G} X^{\operatorname{im}(\alpha)} \right)_{\mathrm{h}G}\ ;$$

here K is a finite abelian group. In particular, the underlying space $|X /\!\!/ G|$ of the orbispace quotient recovers the homotopy orbits $X_{\mathrm{h}G}$. Also note that given a subgroup $H \subset G$, there is an equivalence

$$(H\backslash G) /\!\!/ G \simeq \mathrm{B}H^{(-)}\ .$$

Thus the values of the orbispace quotient on classifying spaces of *finite* abelian groups recovers spaces given by 'the same formulas' as the spaces appearing on the right-hand sides of Theorems 1.6, 1.9 and 1.12. The difference is that in Theorems 1.6, 1.9 and 1.12, the finite abelian group K is replaced with an infinite group ($\mathbf{Z}$ or $\mathbf{Z}_p$).

2.2 Formal loop spaces

In this subsection, we answer Question 2.1.

2.4 Notation. Given a torsion abelian group Λ, we write $\Lambda^\vee$ for the Pontryagin dual of Λ.

2.5 Definition. Let X be an orbispace and Λ a torsion abelian group. The *formal loop space* $\mathcal{L}^\Lambda(\mathsf{X})$ is the orbispace defined by

$$\begin{aligned} \mathcal{T}^{\mathrm{op}} &\to \mathbf{Spc} \\ T &\mapsto \operatorname*{colim}_{\substack{\Lambda_0 \subset \Lambda \\ \text{finite}}} \mathsf{X}^{T \times \mathrm{B}\Lambda_0^\vee} . \end{aligned}$$

The following key example is an exercise in the definitions.

2.6 Example [Ell III, Example 3.4.5]**.** Let G be a finite group and X a G-space. There is a natural equivalence

$$\mathcal{L}^\Lambda(X /\!\!/ G) \simeq \Bigg(\coprod_{\alpha \colon \Lambda^\vee \to G} X^{\mathrm{im}(\alpha)} \Bigg) /\!\!/ G .$$

Recall that

(1) If $\Lambda = \mathbf{Q}_p/\mathbf{Z}_p$, then $\Lambda^\vee = \mathbf{Z}_p$. In this case,

$$\mathcal{L}^{\mathbf{Q}_p/\mathbf{Z}_p}(X /\!\!/ G) \simeq \Bigg(\coprod_{\alpha \colon \mathbf{Z}_p \to G} X^{\mathrm{im}(\alpha)} \Bigg) /\!\!/ G .$$

Hence the underlying space $|\mathcal{L}^{\mathbf{Q}_p/\mathbf{Z}_p}(X /\!\!/ G)|$ is the space appearing on the right-hand side of Theorem 1.12.

(2) If $\Lambda = \mathbf{Q}/\mathbf{Z}$, then

$$\Lambda^\vee = \hat{\mathbf{Z}} \cong \prod_{p \in \mathbf{P}} \mathbf{Z}_p .$$

Moreover,

$$\mathrm{Hom}(\Lambda^\vee, G) \cong \mathrm{Hom}(\mathbf{Z}, G) \cong G .$$

In this case,

$$\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(X /\!\!/ G) \simeq \Bigg(\coprod_{g \in G} X^g \Bigg) /\!\!/ G .$$

Hence the underlying space $|\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(X /\!\!/ G)|$ is the space appearing on the right-hand side of Theorem 1.6.

We are most interested in the formal loop spaces $\mathcal{L}^\Lambda(\mathsf{X})$ when Λ is $\mathbf{Q}_p/\mathbf{Z}_p$, $\mathbf{Q}/\mathbf{Z}$, or a direct sum of groups of this form. The following notion captures this class of group:

2.7 Definition. An abelian group Λ is a *colattice* if:

(1) Λ is torsion,

(2) and for all $n > 0$, the multiplication by n map $\Lambda \to \Lambda$ is a surjection with finite kernel.

2.8. Equivalently, Λ is a colattice if for each prime p, the p-localization $\Lambda_{(p)}$ is isomorphic to $(\mathbf{Q}_p/\mathbf{Z}_p)^{n_p}$ for some integer n_p depending on p. (The integer n_p should be thought of as the *height* at p.)

2.3 Character isomorphisms

Our next goal is to compare formal loop spaces and free loop spaces. We start by constructing a map comparing these two types of objects.

2.9 Observation. Let X be a space and Λ a torsion abelian group. Write $\mathrm{B}\Lambda^\vee$ for the classifying space of the *discrete* group obtained from $\Lambda^\vee$ by ignoring the profintie topology. For each finite subgroup $\Lambda_0 \subset \Lambda$, the natural map $\mathrm{B}\Lambda^\vee \to \mathrm{B}\Lambda_0^\vee$ induces a map

$$X^{\mathrm{B}\Lambda_0^\vee} \to X^{\mathrm{B}\Lambda^\vee}$$

These maps are compatible with inclusions of subgroups and assemble into a natural map

(2.10) $$\left|\mathcal{L}^\Lambda(X^{(-)})\right| \simeq \operatorname*{colim}_{\substack{\Lambda_0 \subset \Lambda \\ \text{finite}}} \mathrm{X}^{T \times \mathrm{B}\Lambda_0^\vee} \to X^{\mathrm{B}\Lambda^\vee} .$$

By adjunction, the map (2.10) corresponds to a map $\mathcal{L}^\Lambda(X^{(-)}) \longrightarrow (X^{\mathrm{B}\Lambda^\vee})^{(-)}$.

The following is the main nontrivial result that we need. For this comparison, it is important that Λ is not an arbitrary torsion abelian group, but is a *colattice*.

2.11 Proposition [Ell III, Proposition 3.4.7]**.** *Let X be a π-finite space and Λ a colattice. Then the natural map*

$$\mathcal{L}^\Lambda(X^{(-)}) \longrightarrow (X^{\mathrm{B}\Lambda^\vee})^{(-)}$$

is an equivalence of orbispaces.

For π-finite spaces, Proposition 2.11 allows us to recognize formal loop spaces as free loop spaces.

2.12 Remark (comparison of formal and free loop spaces)**.** Let X be a space. Write

$$\mathcal{L}(X) := X^{\mathrm{B}\mathbf{Z}}$$

for the *free loop space* of X. The inclusion $\mathbf{Z} \hookrightarrow \hat{\mathbf{Z}}$ induces a map of classifying spaces $u \colon \mathrm{B}\mathbf{Z} \to \mathrm{B}\hat{\mathbf{Z}}$. The map u induces a comparison map

$$X^{\mathrm{B}\hat{\mathbf{Z}}} \to \mathcal{L}(X) .$$

Thus we have natural maps of orbispaces

$$\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(X^{(-)}) \to (X^{\mathrm{B}\hat{\mathbf{Z}}})^{(-)} \to \mathcal{L}(X)^{(-)} .$$

If X is π-finite, then Proposition 2.11 shows that the left-hand map is an equivalence. In this case, the fact that $u\colon \mathrm{B}\mathbf{Z} \to \mathrm{B}\hat{\mathbf{Z}}$ becomes an equivalence after profinite completions implies that the right-hand map is also an equivalence. That is, for every π-finite space X, there is a natural equivalence

$$\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(X^{(-)}) \xrightarrow{\sim} \mathcal{L}(X)^{(-)} .$$

Similarly, if X is a p-finite space, then there is a natural equivalence

$$\mathcal{L}^{\mathbf{Q}_p/\mathbf{Z}_p}(X^{(-)}) \xrightarrow{\sim} \mathcal{L}(X)^{(-)} .$$

See [Ell III, Variant 3.4.8] for more details.

Given Proposition 2.11, the following is formal:

2.13 Corollary (character isomorphism [Ell III, Theorem 4.3.2])**.** *Let* $\mathbf{G}_0$ *be a preoriented* $\mathbf{P}$*-divisible group over* A*,* Λ *a colattice, and write* $\mathbf{G} \coloneqq \mathbf{G}_0 \oplus \underline{\Lambda}$*. Then for any orbispace* X*, there is a natural identification*

$$\chi\colon A_{\mathbf{G}}^{\mathsf{X}} \xrightarrow{\sim} A_{\mathbf{G}_0}^{\mathcal{L}^{\Lambda}(\mathsf{X})} .$$

2.14 Corollary [Ell III, Corollary 4.3.4]**.** *Keep the notation of Corollary 2.13, and let* G *be a finite group and* X *a* G*-space. Then* χ *induces an isomorphism of graded rings*

$$\chi\colon A_{\mathbf{G}}^*(X /\!\!/ G) \xrightarrow{\sim} A_{\mathbf{G}_0}^*\left(\left(\coprod_{\alpha\colon \mathbf{Z}_p \to G} X^{\mathrm{im}(\alpha)}\right) /\!\!/ G\right) .$$

2.4 The Chern character via tempered cohomology

The goal of the remainder of this section is to explain how to use Corollary 2.13 to recover the equivariant Chern character of Theorem 1.6.

2.15 Notation. Write $\mathrm{KU}_{\mathbf{C}} \coloneqq \mathbf{C} \otimes \mathrm{KU}$.

2.16. Over $\mathbf{C}$ there is an isomorphism of $\mathbf{P}$-divisible groups

$$\begin{aligned} \exp\colon \underline{\mathbf{Q}/\mathbf{Z}} &\xrightarrow{\sim} \mu_{\mathbf{P}^\infty} \\ \lambda &\mapsto \exp(2\pi i \lambda) . \end{aligned}$$

Since $\underline{\mathbf{Q}/\mathbf{Z}}$ is constant, combining the exponential with the character isomorphism of Corollary 2.13 provides an equivalence

$$\theta\colon (\mathrm{KU}_{\mathbf{C}})_{\mu_{\mathbf{P}^\infty}}^{\mathsf{X}} \xrightarrow[\exp]{\sim} (\mathrm{KU}_{\mathbf{C}})_{\underline{\mathbf{Q}/\mathbf{Z}}}^{\mathsf{X}} \xrightarrow[\chi]{\sim} \mathrm{KU}_{\mathbf{C}}^{|\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(\mathsf{X})|} .$$

2.17 Definition. Let X be an orbispace. The *orbispace Chern character* is the composite

$$\mathrm{ch}\colon \mathrm{KU}_{\mu_{\mathbf{P}^\infty}}^{\mathsf{X}} \longrightarrow (\mathrm{KU}_{\mathbf{C}})_{\mu_{\mathbf{P}^\infty}}^{\mathsf{X}} \xrightarrow[\theta]{\sim} \mathrm{KU}_{\mathbf{C}}^{|\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(\mathsf{X})|} .$$

2.18. On homotopy, the orbispace Chern character induces a map

$$\mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(\mathsf{X}) \longrightarrow \mathrm{H}^*(|\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(\mathsf{X})|;\mathbf{C})(\!(\beta^{-1})\!)\ .$$

If X is a G-space, taking $\mathsf{X} = X /\!\!/ G$, we get a map of graded rings

$$\mathrm{KU}^*_G(X) \cong \mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(X /\!\!/ G) \longrightarrow \mathrm{H}^*\left(\left(\coprod_{\alpha\colon \mathbf{Z}\to G} X^{\mathrm{im}(\alpha)}\right)_{\mathrm{h}G};\mathbf{C}\right)(\!(\beta^{-1})\!)\ .$$

In degree 0, this recovers the equivariant Chern character of Theorem 1.6!

3 Orientations

Let G be a finite group and X a finite G-space. Then there is a comparison map

$$\mathrm{KU}^*_G(X) \to \mathrm{KU}^*(X_{\mathrm{h}G})$$

from the G-equivariant K-theory of X to the K-theory of the homotopy orbits of X. The *Atiyah–Segal Completion Theorem* states that $\mathrm{KU}^*(X_{\mathrm{h}G})$ is the completion of $\mathrm{KU}^*_G(X)$ with respect to the *augmentation ideal* [1; 2]. This is the remaining feature of complex K-theory that we have not yet asked our tempered cohomology theories to satisfy. The purpose of this section is to identify an analogue of the conclusion of the Atiyah–Segal Completion Theorem that makes sense in the context of tempered cohomology theories. The tempered cohomology theories that satisfy the conclusion of this Atiyah–Segal Completion Theorem are called *oriented*.

3.1 The Atiyah–Segal comparison map

To start, we need to situate the comparison map $\mathrm{KU}^*_G(X) \to \mathrm{KU}^*(X_{\mathrm{h}G})$ in the context of tempered cohomology theories. Note that $\mathrm{KU}^*_G(X)$ is the $\mathrm{KU}_{\mu_{\mathbf{P}^\infty}}$-tempered cohomology of the orbispace quotient $X /\!\!/ G$, whereas $\mathrm{KU}^*(X_{\mathrm{h}G})$ is the KU-cohomology of the realization $|X /\!\!/ G|$. We'll more generally constrict a comparison map where we replace $X /\!\!/ G$ by an arbitrary orbispace:

3.1 Goal. Given a preoriented $\mathbf{P}$-divisible group $\mathbf{G}$ over A and an orbispace X, construct a comparison map

$$\zeta\colon A^{\mathsf{X}}_{\mathbf{G}} \to A^{|\mathsf{X}|}\ .$$

To start, we reinterpret $A^{|\mathsf{X}|}$ in terms of the tempered cohomology theory $A_{\mathbf{G}}$.

3.2 Observation. For an $\mathbf{E}_\infty$-ring A, the functor

$$\begin{aligned} \mathbf{Spc}^{\mathrm{op}} &\to \mathrm{CAlg}_A \\ X &\mapsto A^X \end{aligned}$$

is uniquely determined by the requirements that it preserves limits and sends the point to A. Since the constant functor $\underline{(-)}\colon \mathbf{Spc} \hookrightarrow \mathbf{OSpc}$ is a left adjoint, the composite

$$\mathbf{Spc}^{\mathrm{op}} \xhookrightarrow{\underline{(-)}} \mathbf{OSpc}^{\mathrm{op}} \xrightarrow{A_{\mathbf{G}}} \mathrm{CAlg}_A$$

also has this property! That is, there is a natural identification

$$A_{\mathbf{G}}^{X} \simeq A^{X} .$$

In particular, for any orbispace X, we have a natural identification $A_{\mathbf{G}}^{\underline{|\mathsf{X}|}} \simeq A^{|\mathsf{X}|}$.

3.3 Observation. Since we desire to construct a map $A_{\mathbf{G}}^{\mathsf{X}} \to A_{\mathbf{G}}^{\underline{|\mathsf{X}|}}$, it suffices to construct a map $\underline{|\mathsf{X}|} \to \mathsf{X}$. We take this to be the counit of the adjunction

$$\underline{(-)} : \mathbf{Spc} \rightleftarrows \mathbf{OSpc} : |-| .$$

3.4 Definition (Atiyah–Segal comparison map)**.** Let A be an $\mathbf{E}_\infty$-ring and $\mathbf{G}$ a preoriented $\mathbf{P}$-divisible group over A. The *Atiyah–Segal comparison map* associated to an orbispace X is the composite

$$\zeta : A_{\mathbf{G}}^{\mathsf{X}} \xrightarrow{A_{\mathbf{G}}^{\mathrm{counit}}} A_{\mathbf{G}}^{\underline{|\mathsf{X}|}} \simeq A^{|\mathsf{X}|} .$$

3.5 Example. Let X be a finite space. The Sullivan Conjecture (proven by Carlsson [3], Lannes [6], and Miller [8; 9; 10; 11]) implies that the natural map

$$\underline{X} \to X^{(-)}$$

is an equivalence of orbispaces. In particular, taking $\mathsf{X} = X^{(-)}$ we see that the Atiyah–Segal comparison map

$$\zeta : A_{\mathbf{G}}^{X} \to A^{X}$$

is an equivalence.

3.6 Example. Let G be a finite group and X a G-space. Since $|X /\!\!/ G| \simeq X_{\mathrm{h}G}$, there are natural maps of orbispaces

$$\underline{X_{\mathrm{h}G}} \to X /\!\!/ G \to X_{\mathrm{h}G}^{(-)} .$$

These induce comparison maps

$$A_{\mathbf{G}}^{X_{\mathrm{h}G}} \longrightarrow A_{\mathbf{G}}^{X /\!\!/ G} \xrightarrow{\zeta} A^{X_{\mathrm{h}G}} .$$

In the the examples from chromatic homotopy theory that we're interested in, it turns out that the Atiyah–Segal comparison map is *always* an equivalence. The proofs at height 0 and height $n > 0$ are slightly different; we start with the positive height case.

3.7 Recollection [Ell III, Example 3.5.7]**.** Let p be a prime number and $n \geq 1$ an integer. Let A be a $\mathrm{K}(n)$-local $\mathbf{E}_\infty$-ring. Then the functor

$$\begin{aligned} \mathcal{T}^{\mathrm{op}} &\to \mathrm{CAlg}_A \\ T &\mapsto A^T \end{aligned}$$

is $\mathbf{P}$-divisible. Moreover, the $\mathbf{P}$-divisible group associated to this functor is the *Quillen p-divisible group* $\mathbf{G}_A^{\mathrm{Q}}$. The Quillen p-divisible group comes equipped with a canonical preorientation.

3.8 Theorem [Ell III, Theorem 4.2.5]**.** *Let p be a prime number and $n \geq 1$ an integer. Let A be a $\mathrm{K}(n)$-local $\mathbf{E}_\infty$-ring, and let $\mathbf{G} = \mathbf{G}_A^{\mathrm{Q}}$ be the Quillen p-divisible group of A. Then for every orbispace X, the Atiyah–Segal comparison map*

$$\zeta : A_{\mathbf{G}}^{\mathsf{X}} \to A^{|\mathsf{X}|}$$

is an equivalence of $\mathbf{E}_\infty$-A-algebras.

Proof. Since **OSpc** is generated under colimits by $\mathcal{T} \subset \mathbf{OSpc}$ and both of the functors

$$\mathsf{X} \mapsto A_{\mathbf{G}}^{\mathsf{X}} \qquad \text{and} \qquad \mathsf{X} \mapsto A^{|\mathsf{X}|}$$

carry colimits of orbispaces to limits in CAlg_A, it suffices to prove the claim in the special case where $\mathsf{X} = T^{(-)}$ is representable by an object $T \in \mathcal{T}$. In this case, the result is immediate from the definition of the Quillen p-divisible group. □

3.9 Theorem [Ell III, Variant 4.2.6]**.** *Let A be an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra, and let $\mathbf{G} = 0$ denote the trivial $\mathbf{P}$-divisible group over A. Then for every orbispace X, the Atiyah–Segal comparison map*

$$\zeta : A_{\mathbf{G}}^{\mathsf{X}} \to A^{|\mathsf{X}|}$$

is an equivalence of $\mathbf{E}_\infty$-A-algebras.

Proof. As in the proof of Theorem 3.8, it suffices to prove the claim in when $\mathsf{X} = T^{(-)}$ is representable by an object $T \in \mathcal{T}$. In this case, the result follows from the assumption that A is an $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra and the fact that the classifying space T is rationally acyclic. □

3.2 Encoding the Atiyah–Segal Completion Theorem

Now we encode the conclusion of the Atiyah–Segal Completion Theorem. To do so, let us first identify the *augmentation ideal* in the setting of tempered cohomology theories.

3.10 Definition. Let G be a finite group, A an $\mathbf{E}_\infty$-ring, and $\mathbf{G}$ a preoriented $\mathbf{P}$-divisible group over A. The point $* \to \mathrm{B}G$ induces a surjective homomorphism

$$A_{\mathbf{G}}^0(\mathrm{B}G) \twoheadrightarrow A_{\mathbf{G}}^0(*) \cong \pi_0(A)\,.$$

The *augmentation ideal* is the kernel

$$\mathrm{I}_G \coloneqq \ker\left(A_{\mathbf{G}}^0(\mathrm{B}G) \twoheadrightarrow \pi_0(A)\right)\,.$$

3.11 Remark. If G is abelian, then $A_{\mathbf{G}}^0(\mathrm{B}G)$ is finite projective over $\pi_0(A)$. Hence I_G is also finite projective over $\pi_0(A)$. In particular, I_G is finitely generated.

3.12 Recollection. Let R be an $\mathbf{E}_2$-ring and $I \subset \pi_0(R)$ a finitely generated ideal.

(1) An R-module M is *I-complete* if for each $x \in I$, the limit of the tower

$$\cdots \longrightarrow M \xrightarrow{x\cdot} M \xrightarrow{x\cdot} M$$

is zero. The inclusion of I-complete R-modules into all R-modules admits a left adjoint called *I-completion*.

(2) An R-module N is *I-local* if for each I-complete R-module M, the space $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ is contractible.

See [SAG, Chapter 7] for a nice treatment of complete and local modules.

3.13 Definition (oriented)**.** Let A be an $\mathbf{E}_\infty$-ring and $\mathbf{G}$ a preoriented $\mathbf{P}$-divisible group over A. We say that $\mathbf{G}$ is *oriented* if for each prime p the following hold:

(1) The Atiyah–Segal comparison map

$$\zeta : A_{\mathbf{G}}^{\mathrm{BC}_p} \to A^{\mathrm{BC}_p}$$

exhibits A^{BC_p} as the $\mathrm{I}_{\mathrm{C}_p}$-completion of $A_{\mathbf{G}}^{\mathrm{BC}_p}$.

(2) The Tate construction A^{tC_p} is $\mathrm{I}_{\mathrm{C}_p}$-local as an $A_{\mathbf{G}}^{\mathrm{BC}_p}$-module.

3.14 Remark. There are many ways to reformulate the condition that a preoriented $\mathbf{P}$-divisible group be oriented; for ease of exposition, we have chosen the one that is the most simple to state. See [Ell III, Definition 2.6.12, Proposition 4.2.8, Proposition 4.2.15, & Theorem 4.6.2] for other formulations. At least when A is Noetherian, the condition that A be oriented is exactly that $A_{\mathbf{G}}$ satisfy the conclusion of the Atiyah–Segal Completion Theorem; see [Ell III, Theorem 4.9.2].

3.15 Example. The $\mathbf{P}$-divisible group $\mu_{\mathbf{P}^\infty}$ over KU is oriented.

3.16 Example. Let p be a prime number, $n \geq 1$ an integer, and A a $\mathrm{K}(n)$-local $\mathbf{E}_\infty$-ring. If A is *complex periodic*, then the Quillen p-divisible group $\mathbf{G}_A^{\mathrm{Q}}$ is oriented.

4 Basechange

In this section we finally state the fundamental basechange property for *oriented* $\mathbf{P}$-divisible groups (Theorem 4.3). We then sketch how to deduce the character isomorphisms of §1 from this basechange result. We begin with some motivation.

4.1 Recollection. Given a space X and map of $\mathbf{E}_\infty$-rings $f : A \to B$, the map $f^X : A^X \to B^X$ extends to a B-linear comparison map

$$c_X : B \otimes_A A^X \to B^X .$$

The comparison map is *rarely* an equivalence. For example, if X is a π-finite space (such as $\mathrm{K}(\mathrm{C}_p, n)$), then c_X is not usually an equivalence. Note, however, that there are two useful situations in which c_X is an equivalence:

(1) *The $\mathbf{E}_\infty$-ring B is perfect as an A-module.* To see this, note that $B \otimes_A (-)$ preserves colimits, so both sides preserve colimits in X and have the same value on the point.

(2) *The space X is finite.* To see this, note that both sides preserve finite colimits in X and agree on the point.

4.2 Idea. Tempered cohomology is designed to correct the failure of c_X to be an equivalence by replacing A^X by a spectrum $A_{\mathbf{G}}^X$ for which it is more likely that the comparison map

$$B \otimes_A A_{\mathbf{G}}^X \to B_{\mathbf{G}}^X$$

is an equivalence. The point here is that that we want to replace A^X be a 'decompletion'.

4.3 Theorem (basechange for tempered cohomology [Ell III, Theorem 4.7.1]). *Let $f\colon A \to B$ be a map of $\mathbf{E}_\infty$-rings and $\mathbf{G}$ an oriented $\mathbf{P}$-divisible group over A. Then for every π-finite space X, the comparison map*

$$B \otimes_A A_{\mathbf{G}}^X \to B_{\mathbf{G}}^X$$

is an equivalence.

Next lecture, Arpon will sketch a proof of Theorem 4.3 as a consequence of an ambidexterity result for *tempered local systems* [Ell III, Theorem 7.2.10].

4.1 Some consequences of basechange

Now we explain how to deduce the character isomorphisms of §1 from Theorem 4.3. We begin by noting that tempered basechange holds for orbispace quotients:

4.4 Corollary [Ell III, Corollary 4.7.5]. *Let $f\colon A \to B$ be a map of $\mathbf{E}_\infty$-rings and $\mathbf{G}$ an oriented $\mathbf{P}$-divisible group over A. Let G be a finite group and X a finite G-space. Then the natural map*

$$B \otimes_A A_{\mathbf{G}}^{X/\!\!/G} \to B_{\mathbf{G}}^{X/\!\!/G}$$

is an equivalence.

Proof. Both sides carry finite colimits in X to finite limits in CAlg_B. Thus we are reduced to the case where $X = H\backslash G$ for a subgroup $H \subset G$. In this case,

$$(H\backslash G)/\!\!/G \simeq \mathrm{B}H^{(-)} .$$

The desired claim is now a special case of Theorem 4.3. □

Now let us explain consequences of tempered basechange on the level of homotopy groups. To do this, recall that we need one of the modules appearing in the tensor product to be *flat*.

4.5 Recollection [HA, Proposition 7.2.2.13]. Let R be an $\mathbf{E}_1$-ring, M a right R-module, and N a left R-module. If M is flat over R, then for each integer $n \in \mathbf{Z}$, the natural map

$$\pi_0(M) \underset{\pi_0(R)}{\otimes} \pi_n(N) \to \pi_n(M \otimes_R N)$$

is an isomorphism of $\pi_0(R)$-modules.

4.6 Corollary [Ell III, Corollary 4.7.3]. *Under the hypotheses of Theorem 4.3, if either $A_{\mathbf{G}}^X$ or B is flat over A, then the natural morphism of graded rings*

$$\pi_0(B) \underset{\pi_0(A)}{\otimes} A_{\mathbf{G}}^*(X) \to B_{\mathbf{G}}^*(X)$$

is an isomorphism.

Proof. Combine Theorem 4.3 and Recollection 4.5. □

4.7 Corollary [Ell III, Corollary 4.7.6]. *In the setting of Corollary 4.4, if either $A_{\mathbf{G}}^{X/\!\!/G}$ or B is flat over A, the natural natural morphism of graded rings*

$$\pi_0(B) \underset{\pi_0(A)}{\otimes} A_{\mathbf{G}}^*(X/\!\!/G) \to B_{\mathbf{G}}^*(X/\!\!/G)$$

is an isomorphism.

Proof. Combine Corollary 4.4 and Recollection 4.5. ☐

Consider the case $A = \mathrm{KU}$, $B = \mathrm{KU}_{\mathbf{C}}$, $\mathbf{G} = \mu_{\mathbf{P}^\infty}$, and $\mathsf{X} = X /\!\!/ G$. Recall that in this setting

$$\mathrm{KU}_G^*(X) \cong \mathrm{KU}^*_{\mu_{\mathbf{P}^\infty}}(X /\!\!/ G) \qquad \text{and} \qquad \left(\coprod_{g\in G} X^g\right)_{\mathrm{h}G} \simeq |\mathcal{L}^{\mathbf{Q}/\mathbf{Z}}(X /\!\!/ G)| .$$

Applying (consequences of) Theorem 4.3 reproves Theorem 1.6:

4.8 Corollary [Ell III, Corollary 4.7.7]**.** *Let G be a finite group and X a finite G-space. The orbispace Chern character induces an isomorphism of graded rings*

$$\mathrm{ch}_G \colon \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{KU}_G^*(X) \xrightarrow{\sim} \mathrm{H}^*\left(\left(\coprod_{g\in G} X^g\right)_{\mathrm{h}G}; \mathbf{C}\right)(\!(\beta^{-1})\!) .$$

Finally, let us sketch how to deduce Theorem 1.12 from Theorem 4.3.

Proof sketch of Theorem 1.12. We use the conventions of Notation 1.11. We regard the Quillen p-divisible group $\mathbf{G}$ as a p-divisible group over the $\mathbf{E}_\infty$-ring E rather than the Lubin–Tate ring $\mathrm{R} \cong \pi_0(\mathrm{E})$. Let $\mathbf{G}_{\mathrm{E}_{\mathbf{Q}}}$ denote the p-divisible group over the rationalization of E obtained from $\mathbf{G}$ by extending scalars. There is a connected-étale sequence

$$0 \longrightarrow \mathbf{G}' \longrightarrow \mathbf{G}_{\mathrm{E}_{\mathbf{Q}}} \xrightarrow{q} \mathbf{G}'' \longrightarrow 0 ,$$

where $\mathbf{G}'$ is an oriented p-divisible group of height 0 and $\mathbf{G}''$ is an étale p-divisible group of height n. Let B be universal among those $\mathbf{E}_\infty$-$\mathrm{E}_{\mathbf{Q}}$-algebras equipped with a map $u \colon \underline{(\mathbf{Q}_p/\mathbf{Z}_p)}^n \to \mathbf{G}_B$ for which the composition

$$\underline{(\mathbf{Q}_p/\mathbf{Z}_p)}^n \xrightarrow{u} \mathbf{G}_B \xrightarrow{q} \mathbf{G}''_B$$

is an equivalence. Then B is flat over both E and $\mathrm{E}_{\mathbf{Q}}$. Moreover, $\pi_0(B)$ can be identified with the commutative ring C_0 appearing in Theorem 1.12. By construction, $\mathbf{G}_B$ splits as

$$\mathbf{G}_B \simeq \mathbf{G}'_B \oplus \underline{(\mathbf{Q}_p/\mathbf{Z}_p)}^n .$$

Hence $\mathbf{G}'_B$ is the trivial $\mathbf{P}$-divisible group. Thus, given a finite group G and finite G-space X, we have isomorphisms

$$\begin{aligned}
\mathrm{C}_0 \underset{\mathrm{R}}{\otimes} \mathrm{E}^*(X_{\mathrm{h}G}) &\cong \pi_0(B) \underset{\pi_0(\mathrm{E})}{\otimes} \mathrm{E}^*_{\mathbf{G}}(X /\!\!/ G) \\
&\cong B^*_{\mathbf{G}}(X /\!\!/ G) && \text{(Corollary 4.7)} \\
&\cong B^*_{\mathbf{G}'}\left(\left(\coprod_{\alpha \colon \mathbf{Z}_p \to G} X^{\mathrm{im}(\alpha)}\right) /\!\!/ G\right) && \text{(Corollary 2.14)} \\
&\cong \pi_0(B) \underset{\pi_0(\mathrm{E}_{\mathbf{Q}})}{\otimes} (E_{\mathbf{Q}})^*_{\mathbf{G}'}\left(\left(\coprod_{\alpha \colon \mathbf{Z}_p \to G} X^{\mathrm{im}(\alpha)}\right) /\!\!/ G\right) && \text{(Corollary 4.7)} \\
&\cong \mathrm{C}_0 \otimes_{\mathrm{R}_{\mathbf{Q}}} \mathrm{E}^*_{\mathbf{Q}}\left(\left(\coprod_{\alpha \colon \mathbf{Z}_p \to G} X^{\mathrm{im}(\alpha)}\right)_{\mathrm{h}G}\right) && \text{(Theorem 3.9)} .
\end{aligned}$$

The composite isomorphism can be identified with the transchromatic character map of Theorem 1.12. □

4.9 Remark. Using similar arguments one can deduce Stapleton's character isomorphisms [13; 14] from Theorem 4.3. See [Ell III, Remark 1.1.20].

References

Ell III J. Lurie, *Elliptic cohomology III: Tempered cohomology*, Preprint available at math.ias.edu/~lurie/papers/Elliptic-III-Tempered.pdf, Jul. 2019.

HA ______, *Higher algebra*, Preprint available at math.ias.edu/~lurie/papers/HA.pdf, Sep. 2017.

SAG ______, *Spectral algebraic geometry*, Preprint available at math.ias.edu/~lurie/papers/SAG-rootfile.pdf, Feb. 2018.

1. M. F. Atiyah, *Characters and cohomology of finite groups*, Inst. Hautes Études Sci. Publ. Math., no. 9, pp. 23–64, 1961.
2. M. F. Atiyah and G. B. Segal, *Equivariant* K*-theory and completion*, J. Differential Geometry, vol. 3, pp. 1–18, 1969.
3. G. Carlsson, *Equivariant stable homotopy and Sullivan's conjecture*, Invent. Math., vol. 103, no. 3, pp. 497–525, 1991. DOI: 10.1007/BF01239524.
4. M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel, *Morava* K*-theories of classifying spaces and generalized characters for finite groups*, in *Algebraic topology (San Feliu de Guíxols, 1990)*, Lecture Notes in Math. Vol. 1509, Springer, Berlin, 1992, pp. 186–209. DOI: 10.1007/BFb0087510.
5. ______, *Generalized group characters and complex oriented cohomology theories*, J. Amer. Math. Soc., vol. 13, no. 3, pp. 553–594, 2000. DOI: 10.1090/S0894-0347-00-00332-5.
6. J. Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p-groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math., no. 75, pp. 135–244, 1992, With an appendix by Michel Zisman.
7. J. Lurie, *Character theory and tempered cohomology*, Talk at MSRI, www.msri.org/workshops/873/schedules/26330, Mar. 2019.
8. H. Miller, *The Sullivan conjecture*, Bull. Amer. Math. Soc. (N.S.), vol. 9, no. 1, pp. 75–78, 1983. DOI: 10.1090/S0273-0979-1983-15159-5.
9. ______, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2), vol. 120, no. 1, pp. 39–87, 1984. DOI: 10.2307/2007071.
10. ______, *Correction to: "The Sullivan conjecture on maps from classifying spaces" [Ann. of Math. (2)* **120** *(1984), no. 1, 39–87; MR0750716 (85i:55012)]*, Ann. of Math. (2), vol. 121, no. 3, pp. 605–609, 1985. DOI: 10.2307/1971212.
11. ______, *The Sullivan conjecture and homotopical representation theory*, in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, Amer. Math. Soc., Providence, RI, 1987, pp. 580–589.
12. N. Stapleton, *Lubin–Tate theory, character theory, and power operations*, in *Handbook of homotopy theory*, CRC Press/Chapman Hall Handb. Math. Ser. CRC Press, Boca Raton, FL, [2020] ©2020, pp. 891–929.
13. ______, *Transchromatic generalized character maps*, Algebr. Geom. Topol., vol. 13, no. 1, pp. 171–203, 2013. DOI: 10.2140/agt.2013.13.171.
14. ______, *Transchromatic twisted character maps*, J. Homotopy Relat. Struct., vol. 10, no. 1, pp. 29–61, 2015. DOI: 10.1007/s40062-013-0040-9.