RECONSTRUCTION OF SCHEMES FROM THEIR ÉTALE TOPOI

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ABSTRACT. Let k be a field that is finitely generated over its prime field. In Grothendieck's anabelian letter to Faltings, he conjectured that sending a k-scheme to its étale topos defines a fully faithful functor from the localization of the category of finite type k-schemes at the universal homeomorphisms to a category of topoi. We prove Grothendieck's conjecture for infinite fields of arbitrary characteristic. In characteristic 0, this shows that seminormal finite type k-schemes can be reconstructed from their étale topoi, generalizing work of Voevodsky. In positive characteristic, this shows that perfections of finite type k-schemes can be reconstructed from their étale topoi.

CONTENTS

0. INTRODUCTION

Let k be a finitely generated field of characteristic 0. In Grothendieck's influential *anabelian letter* to Faltings [10], he made three classes of conjectures regarding the reconstruction of k-schemes. The first class is about describing isomorphisms between 'anabelian' k-schemes in terms of outer isomorphisms between their étale fundamental groups; the second is the *section conjecture* about describing the set of rational points of an 'anabelian' k-scheme X in terms of sections of the natural homomorphism $\pi_1^{\text{ét}}(X) \to \mathrm{Gal}(\bar{k}/k)$. This paper concerns the third conjecture, which is about reconstructing k-schemes from their étale topoi [10, p. 7]. The main result of this paper is to prove an extension of Grothendieck's conjecture.

Write $\mathbf{Sch}_k^{\mathrm{ft}}$ for the category of finite type k-schemes and $\mathbf{RTop}_k$ for the category of topoi over the étale topos of $\mathrm{Spec}(k)$. Grothendieck's conjecture asks to what extent the functor

$$(-)_{\text{ét}} \colon \mathbf{Sch}_k^{\mathrm{ft}} \to \mathbf{RTop}_k .$$

which takes a scheme X to its étale topos $X_{\text{ét}}$ is fully faithful. In order to make this plausible, two technical modifications are needed. For the first, note that every morphism between finite type k-schemes sends closed points to closed points, but this is not true of an arbitrary geometric morphism between étale topoi (see Example 2.30). Hence we must add this restriction to the morphisms of topoi considered. We say that a geometric morphism of topoi $\mathcal{X} \to \mathcal{Y}$ is *pinned* if the induced map on underlying spaces sends closed points to closed points. Write $\mathbf{RTop}_k^{\mathrm{pin}}$ for the category of topoi over $\mathrm{Spec}(k)_{\text{ét}}$ and pinned geometric morphisms.

The second is that by the *topological invariance* of the étale topos [STK, Tag 03SI], any universal homeomorphism of schemes induces an equivalence of étale topoi. Hence the optimal result would be that the

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functor $X \mapsto X_{\text{ét}}$ becomes fully faithful after localizing $\mathbf{Sch}_k^{\text{ft}}$ by inverting the universal homeomorphisms. Grothendieck's precise conjecture is that this is true when k is finitely generated of characteristic 0. Our main result is a proof of this conjecture, together with its natural extension to infinite fields of positive characteristic.

Denote the collection of universal homeomorphisms by UH.

0.1 Theorem. *Let k be a field that is finitely generated over its prime field. If $\mathrm{char}(k) > 0$, assume that k has transcendence degree at least 1. Then the functor*

$$(-)_{\text{ét}} \colon \mathbf{Sch}_k^{\text{ft}}[\mathrm{UH}^{-1}] \to \mathbf{RTop}_k^{\text{pin}}$$

sending a finite type k-scheme X to its étale topos $X_{\text{ét}}$ is fully faithful.

The first ingredient in our proof of Theorem 0.1 is an explicit description of the source category. For this, recall that a scheme X is *absolutely weakly normal* if every universal homeomorphism $X' \to X$ is an isomorphism. Over fields, this reduces to more well-known notions. A characteristic 0 scheme is absolutely weakly normal if and only if it is *seminormal*, i.e., does not have cuspidal cubic singularities. A positive characteristic scheme is absolutely weakly normal if and only if it is *perfect*. The inclusion of the full subcategory of absolutely weakly normal schemes into all schemes admits a right adjoint

$$(-)^{\text{awn}} \colon \mathbf{Sch} \to \mathbf{Sch}^{\text{awn}}$$

called *absolute weak normalization*. Moreover, absolute weak normalization induces an equivalence of categories $\mathbf{Sch}[\mathrm{UH}^{-1}] \xrightarrow{\sim} \mathbf{Sch}^{\text{awn}}$. As a result, in order to prove Theorem 0.1 we equivalently need to show that for all finite type k-schemes X and Y, the natural map

$$(0.2) \qquad\qquad \mathrm{Hom}_k(X^{\text{awn}}, Y) \to \mathrm{Hom}_k^{\text{pin}}(X_{\text{ét}}^{\text{awn}}, Y_{\text{ét}})$$

is bijective; here the right-hand side is the set of isomorphism classes of pinned geometric morphisms $X_{\text{ét}}^{\text{awn}} \to Y_{\text{ét}}$ over $\mathrm{Spec}(k)_{\text{ét}}$. By the Mordell–Weil Theorem, Theorem 0.1 follows from the following more refined result:

0.3 Theorem (see Proposition 2.28 and Theorem 6.24). *Let k be a field of exponential characteristic p. Assume that:*

(0.3.1) *For any torus T over k, the only element of $T(k)$ divisible by all integers coprime to p is the identity.*

(0.3.2) *For any regular finite type k-scheme X, the only torsion element of $\mathrm{Pic}(X)$ that is divisible by all integers coprime to p is the identity.*

(0.3.3) *The field k admits a nontrivial discrete valuation.*

Then for all finite type k-schemes X and Y the natural map

$$\mathrm{Hom}_k(X^{\text{awn}}, Y) \to \mathrm{Hom}_k^{\text{pin}}(X_{\text{ét}}^{\text{awn}}, Y_{\text{ét}})$$

is bijective.

Voevodsky's work. We conclude the introduction by noting that Theorem 0.3 improves upon a result of Voevodsky [28]. Voevodsky proved that if k is a finitely generated field of characteristic 0 and X and Y are finite type k-schemes with X normal, then the natural map

$$\mathrm{Hom}_k(X, Y) \to \mathrm{Hom}_k^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$$

is bijective. Theorem 0.3 generalizes this result in two directions. First, Theorem 0.3 also holds in positive characteristic. Second, even in charactistic 0, Theorem 0.3 applies when X is only seminormal, and seminormal schemes are quite a lot more general than normal schemes. For example, the nodal cubic is not normal, but *is* seminormal.

In Voevodsky's proof of Theorem 0.1 for normal schemes over a finitely generated field of characteristic 0, the fact that the group $\mathrm{Pic}(X)$ is finitely generated is crucial. However, for seminormal schemes, $\mathrm{Pic}(X)$ is in general not finitely generated and it is not clear how to adapt Voevodsky's argument in this more general

setting.[1] Our proof bypasses these issues by proceeding as follows. We show that both the source and target of (0.2) are h-local in X. Therefore, using de Jong's alterations, we may reduce to the case that X is regular. In particular, in characteristic 0, we can reduce to Voevodsky's argument. More generally, the assumption that k is amenable and admits a nontrivial discrete valuation allows us to adapt Voevodsky's methods to work in arbitrary characteristic.

Remark. Let k be an infinite finitely generated field of positive characteristic. Since this paper was written, we learned that in forthcoming work Zach Berens independently shows that the proof in Voevodsky's paper can be adapted to show that perfections of finite type k-schemes can be reconstructed from their étale topoi.

Linear overview. Section 1 provides background on absolute weak normalization, seminormalization, and perfection. Part of the difficulty in proving the positive characteristic cases of Theorems 0.1 and 0.3 is that absolute weak normalizations of finite type schemes are typically no longer of finite type or even noetherian. To effectively deal with this issue we introduce morphisms *topologically of finite type*, i.e., morphisms that factor as a universal homeomorphism followed by a morphism of finite type, and prove some basic results about them.

In §2, we introduce pinned geometric morphisms as well as what it means for a field k to satisfy the conclusion of Theorem 0.1. We call the latter the *étale reconstruction property*. We give a number of equivalent formulations of étale reconstruction (see Proposition 2.28) and discuss some immediate consequences (see Proposition 2.36). In §3, we prove that the injectivity part of the reconstruction theorem holds over any field, in fact any qcqs base scheme S (see Theorem 3.9 and Corollary 3.10). In §4, we prove that for any S-scheme Y, the functor $X \mapsto \mathrm{Hom}_S(X^{\mathrm{awn}}, Y)$ satisfies h-descent. We furthermore show that the étale topos satisfies h-descent. Using alterations, this lets us reduce proving Theorem 0.3 to the case where X is an affine, regular, connected, finite type k-scheme that admits a rational point and $Y = \mathbf{G}_{\mathrm{m}}$ (Theorem 4.18). Section 5 introduces the class of fields satisfying condition (0.3.1) and a slight generalization of (0.3.2); we call these *amenable* fields, see Definition 5.15. We also prove some basic results about amenable fields. In §6, we prove Theorem 0.3. The proof given in §6 generalizes the main argument in Voevodsky's paper [28, §3] to work in positive characteristic. It also streamlines a number of points.

Throughout the paper we use the language of topoi in our results and proofs, in contrast to Voevodsky who chose to work with sites. In Appendix A, we prove that the two perspectives are equivalent for our purposes: any geometric morphism between étale topoi of topologically noetherian schemes is induced by a morphism of (quasicompact separated) étale sites.

Notational conventions. Let $f : X \to Y$ be a morphism of schemes. We write:

(1) $|X|$ for the underlying topological space of X.

(2) $\mathbf{Sch}$ for the category of schemes and $\mathbf{Sch}_X$ for the category of X-schemes.

(3) $\mathbf{\acute{E}t}_X \subset \mathbf{Sch}_X$ for the full subcategory spanned by the étale X-schemes. We regard $\mathbf{\acute{E}t}_X$ as a site equipped with the étale topology.

(4) $X_{\mathrm{\acute{e}t}} := \mathrm{Sh}_{\mathrm{\acute{e}t}}(\mathbf{\acute{E}t}_X)$ for the étale topos of X.

(5) $f_* : X_{\mathrm{\acute{e}t}} \to Y_{\mathrm{\acute{e}t}}$ for the geometric morphism induced by the morphism of sites $X \times_Y (-) : \mathbf{\acute{E}t}_Y \to \mathbf{\acute{E}t}_X$.

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[1] More precisely, it suffices to know that $\mathrm{Pic}(X)$ has no infinitely divisible torsion elements. If X is seminormal and S_2, [11, Theorem 6.5] implies that $\mathrm{Pic}(X)$ has no infinitely divisible torsion elements. Without the S_2 assumption, we are neither aware of a proof nor a counterexample. This is also observed in forthcoming work by Zach Berens.

1. Background on absolute weak normalization

As mentioned in the introduction, if $f : X \to Y$ is a universal homeomorphism of schemes, then by the topological invariance of the étale topology [STK, Tag 03SI], the induced geometric morphism of étale topoi $f_* : X_{\text{ét}} \to Y_{\text{ét}}$ is an equivalence of categories. Thus, in order to have a reconstruction theorem, we must invert universal homeomorphisms of schemes. Said differently, we must work with schemes with the property that universal homeomorphisms between them are automatically isomorphisms.

In this section, we explain how to do this. The solution is to work with *absolutely weakly normal schemes*. This notion was originally introduced by Rydh [22, Appendix B] in order to study effective descent for étale morphisms, and further studied by Barwick [1] for the purpose of inverting universal homeomorphisms. For schemes over fields, the theory reduces to previously studied notions. A scheme X in positive characteristic is absolutely weakly normal if and only if X is *perfect*, i.e., the absolute Frobenius $X \to X$ is an isomorphism. In characteristic 0, absolute weak normality reduces to *seminormality*. Seminormality was originally introduced by Traverso [27] in order to characterize the rings for which the Picard group is homotopy-invariant, and later simplified by Swan [25].

In §1.1, we recall the definitions of absolute weak normality, seminormality, and perfection. In §1.2, we explain why absolute weak normalization inverts universal homeomorphisms. In §1.3, we discuss the interaction between seminormalization, perfection, and finite type hypotheses on morphisms of schemes. In particular, the perfection of a finite type $\mathbf{F}_p$-scheme is almost never of finite type (see Proposition 1.22). As a result, we introduce the class of schemes *topologically of finite type* (Definition 1.25). This notion behaves well with respect to perfection and is needed for the positive characteristic case of Theorem 0.1.

1.1 Seminormalization, perfection, and absolute weak normalization.

An elementary example of a universal homeomorphism is the normalization of the cuspidal cubic. Seminormal schemes are the schemes that do not have cuspidal cubic singularities:

1.1 Definition (seminormality). Let A be a ring. We say that A is *seminormal* if for all $x, y \in A$ such that $x^3 = y^2$, there exists an element $a \in A$ such that $x = a^2$ and $y = a^3$.

1.2 Proposition. *Let A be a ring.*

(1.2.1) *Assume that A is seminormal and let $x, y \in A$ be such that $x^3 = y^2$. Then there exists a* unique *element $a \in A$ such that $x = a^2$ and $y = a^3$.*

(1.2.2) *If A is seminormal, then A is reduced.*

(1.2.3) *The ring A is seminormal if and only if for each maximal ideal $\mathfrak{m} \subset A$, the local ring $A_{\mathfrak{m}}$ is seminormal.*

(1.2.4) *If A is a normal ring, then A is seminormal.*

Proof. For (1.2.1), see [STK, Tag 0EUK, Footnote 1] or [8]. Item (1.2.2) is [STK, Tag 0EUQ]. Item (1.2.3) is [25, Proposition 3.7].

For (1.2.4), note that since A is normal, for each maximal ideal $\mathfrak{m} \subset A$, the localization $A_{\mathfrak{m}}$ is a normal domain. Hence by (1.2.3) it suffices to prove the claim in the case where A is a normal domain. In this case, since A is a domain, it suffices to show that if $x, y \in A \smallsetminus \{0\}$ and $x^3 = y^2$, there exists an element $a \in A$ such that $x = a^2$ and $y = a^3$. In the fraction field of A, we have

$$\left(\frac{y}{x} \right)^2 = \frac{x^3}{x^2} = x \,.$$

Hence the polynomial $t^2 - x \in A[t]$ is an integral equation for y/x. Since A is integrally closed, we see that $y/x \in A$. Hence $a = y/x$ is an element such that $x = a^2$ and $y = a^3$. $\square$

In positive characteristic, another example of a universal homeomorphism is the Frobenius:

1.3 Recollection (perfect $\mathbf{F}_p$-schemes; see [5, §3]). Let p be a prime number, and let X be an $\mathbf{F}_p$-scheme. Write $\mathrm{Frob}_X : X \to X$ for the *absolute Frobenius*: on underlying spaces, Frob_X is the identity, and on structure sheaves Frob_X is given by $a \mapsto a^p$. Also recall that Frob_X is a universal homeomorphism [STK, Tag 0CC8]. We say that X is *perfect* if Frob_X is an isomorphism.

The *perfection* of X is the cofiltered limit

$$X^{\mathrm{perf}} \cong \lim \left(\cdots \xrightarrow{\mathrm{Frob}_X} X \xrightarrow{\mathrm{Frob}_X} X \right).$$

(Note that since the absolute Frobenius is an affine morphism, this limit exists in the category of schemes.) The $\mathbf{F}_p$-scheme X^{perf} is always perfect. Write

$$\mathbf{Sch}_{\mathbf{F}_p}^{\mathrm{perf}} \subset \mathbf{Sch}_{\mathbf{F}_p}$$

for the full subcategory spanned by the perfect $\mathbf{F}_p$-schemes. Then the assignment $X \mapsto X^{\mathrm{perf}}$ is right adjoint to the inclusion $\mathbf{Sch}_{\mathbf{F}_p}^{\mathrm{perf}} \subset \mathbf{Sch}_{\mathbf{F}_p}$.

To invert universal homeomorphisms for schemes over $\mathrm{Spec}(\mathbf{Z})$, it turns out that we just need to make sure that our schemes do not have a simple class of plane singularity types involving prime powers.

1.4 Definition (absolute weak normality)**.** A ring A is *absolutely weakly normal* if the following conditions are satisfied:

(1.4.1) The ring A is seminormal.

(1.4.2) For each prime number ℓ and all elements $x, y \in A$ such that $\ell^\ell x = y^\ell$, there is a unique element $a \in A$ such that $x = a^\ell$ and $y = \ell a$.

1.5 Observation. Let A be a ring and ℓ a prime number. If ℓ is invertible in A, then given elements $x, y \in A$ such that $\ell^\ell x = y^\ell$, there is a unique element $a \in A$ such that $x = a^\ell$ and $y = \ell a$; namely $a = y/\ell$.

1.6 Lemma [STK, Tag 0EUM]**.** *Seminormality and absolute weak normality are Zariski local properties of rings.*

1.7 Definition. A scheme X is *seminormal* (resp., *absolutely weakly normal*) if for every affine open $U \subset X$, the ring $\Gamma(U; \mathcal{O}_X)$ is seminormal (resp., absolutely weakly normal).

1.8 Lemma [STK, Tag 0EUP]**.** *Seminormality and absolutely weakly normality are affine local properties of schemes. In particular, if X is a seminormal (resp., absolutely weakly normal) scheme, then every open subscheme of X is seminormal (resp., absolutely weakly normal).*

Over fields, absolute weak normality reduces to seminormality or perfectness:

1.9 Theorem (absolutely weakly normal schemes over fields)**.**

(1.9.1) *A $\mathbf{Q}$-scheme X is absolutely weakly normal if and only if X is seminormal.*

(1.9.2) *An $\mathbf{F}_p$-scheme X is absolutely weakly normal if and only if X is perfect.*

Proof. For (1.9.1), since absolute weak normality and seminormality are affine-local properties, it suffices to check the claim when $X = \mathrm{Spec}(A)$ is the spectrum of a $\mathbf{Q}$-algebra. Since absolutely weakly normal rings are seminormal, we just need to show that if A is seminormal, then A is absolutely weakly normal. Since A is a $\mathbf{Q}$-algebra, this follows from Observation 1.5.

Item (1.9.2) is [STK, Tag 0EVV]. □

1.10 Corollary. *A field k is perfect if and only if k is absolutely weakly normal.*

1.2 Inverting universal homeomorphisms. Let $\mathcal{C}$ be a category and $W \subset \mathrm{Mor}(\mathcal{C})$ a collection of morphisms. Recall that the *localization* of $\mathcal{C}$ at W is a category $\mathcal{C}[W^{-1}]$ equipped with a functor $L \colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ that sends morphisms in W to isomorphisms satisfying the following universal property: for any category $\mathcal{D}$, precomposition with L defines a fully faithful functor

$$- \circ L \colon \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

with essential image those functors $F \colon \mathcal{C} \to \mathcal{D}$ that send morphisms in W to isomorphisms in $\mathcal{D}$.

1.11 Example. Let $F \colon \mathcal{C} \to \mathcal{C}_0$ be a functor. If F admits a fully faithful left or right adjoint, then F exhibits $\mathcal{C}_0$ at the localization at those morphisms w in $\mathcal{C}$ such that $F(w)$ is an isomorphism in $\mathcal{C}_0$. See [6, Proposition 5.3.1] or [Ker, Tags 04JM & 04JR].

We now state the main theorem characterizing the category of absolutely weakly normal schemes as the localization of **Sch** at the universal homeomorphisms.

1.12 Notation. Write $\mathbf{Sch}^{\mathrm{sn}} \subset \mathbf{Sch}$ for the full subcategory spanned by the seminormal schemes and $\mathbf{Sch}^{\mathrm{awn}} \subset \mathbf{Sch}^{\mathrm{sn}}$ for the full subcategory spanned by the absolutely weakly normal schemes. We write $\mathrm{UH} \subset \mathrm{Mor}(\mathbf{Sch})$ for the collection of universal homeomorphisms of schemes and write $\mathrm{UH}_{\mathrm{res}} \subset \mathrm{UH}$ for those universal homeomorphisms that induce isomorphisms on residue fields.

1.13 Theorem (existence of absolute weak normalization & seminormalization)**.**

(1.13.1) *The inclusions* $\mathbf{Sch}^{\mathrm{awn}} \hookrightarrow \mathbf{Sch}$ *and* $\mathbf{Sch}^{\mathrm{sn}} \hookrightarrow \mathbf{Sch}$ *admit right adjoints*

$$(-)^{\mathrm{awn}} : \mathbf{Sch} \to \mathbf{Sch}^{\mathrm{awn}} \qquad and \qquad (-)^{\mathrm{sn}} : \mathbf{Sch} \to \mathbf{Sch}^{\mathrm{sn}}$$

Moreover, the counit $X^{\mathrm{awn}} \to X$ *is a universal homeomorphism and the counit* $X^{\mathrm{sn}} \to X$ *is a universal homeomorphism that induces isomorphisms on residue fields.*

(1.13.2) *If X is a reduced scheme, Y is an absolutely weakly normal (resp., seminormal) scheme, and $f : X \to Y$ is a universal homeomorphism (resp. universal homeomorphism that induces isomorphisms on residue fields), then f is an isomorphism.*

(1.13.3) *A morphism of schemes $f : X \to Y$ is a universal homeomorphism (resp. universal homeomorphism that induces isomorphisms on residue fields) if and only if the morphism f^{awn} (resp., f^{sn}) is an isomorphism.*

(1.13.4) *The functors*

$$(-)^{\mathrm{awn}} : \mathbf{Sch} \to \mathbf{Sch}^{\mathrm{awn}} \qquad and \qquad (-)^{\mathrm{sn}} : \mathbf{Sch} \to \mathbf{Sch}^{\mathrm{sn}}$$

identify $\mathbf{Sch}^{\mathrm{awn}}$ *and* $\mathbf{Sch}^{\mathrm{sn}}$ *with the localizations* $\mathbf{Sch}[\mathrm{UH}^{-1}]$ *and* $\mathbf{Sch}[\mathrm{UH}_{\mathrm{res}}^{-1}]$, *respectively.*

Proof. Items (1.13.1)–(1.13.2) are [STK, Tags 0EUS & 0H3G] for absolute weak normalization and [STK, Tags 0EUS & 0H3G] for seminormalization. Thre rest of the proof is exactly the same in both cases, so we only treat the case of absolute weak normalization. For (1.13.3), consider the commutative square

$$\begin{array}{ccc} X^{\mathrm{awn}} & \xrightarrow{\ f^{\mathrm{awn}}\ } & Y^{\mathrm{awn}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ f\ } & Y \,. \end{array}$$

By (1.13.1), the vertical morphisms are universal homeomorphisms, so by (1.13.2) we see that f is a universal homeomorphism if and only if the morphism f^{awn} is an isomorphism. To conclude, note that (1.13.4) follows from items (1.13.1) and (1.13.3) combined with Example 1.11. $\qquad\square$

1.14 Definition. Given a scheme X, we call X^{awn} the *absolute weak normalization* of X and call X^{sn} the *seminormalization* of X.

As an immediate consequence of Theorems 1.9 and 1.13, we deduce:

1.15 Corollary.

(1.15.1) *For each $\mathbf{Q}$-scheme X, we have $X^{\mathrm{awn}} = X^{\mathrm{sn}}$. Moreover, the functor $(-)^{\mathrm{sn}} : \mathbf{Sch}_{\mathbf{Q}} \to \mathbf{Sch}_{\mathbf{Q}}^{\mathrm{sn}}$ identifies $\mathbf{Sch}_{\mathbf{Q}}^{\mathrm{sn}}$ with the localization $\mathbf{Sch}_{\mathbf{Q}}[\mathrm{UH}^{-1}]$.*

(1.15.2) *For each $\mathbf{F}_p$-scheme X, we have $X^{\mathrm{awn}} = X^{\mathrm{perf}}$. Moreover, the functor $(-)^{\mathrm{perf}} : \mathbf{Sch}_{\mathbf{F}_p} \to \mathbf{Sch}_{\mathbf{F}_p}^{\mathrm{perf}}$ identifies $\mathbf{Sch}_{\mathbf{F}_p}^{\mathrm{perf}}$ with the localization $\mathbf{Sch}_{\mathbf{F}_p}[\mathrm{UH}^{-1}]$.*

1.16 Remark. For normal varieties, (1.15.2) was proven by Stix [24, Lemma 4.1.2].

1.17 Notation. Let $X = \mathrm{Spec}(A)$ be an affine scheme. Since universal homeomorphisms are affine [STK, Tag 04DE], both X^{awn} and X^{sn} are affine. We write

$$A^{\mathrm{awn}} := \Gamma(X^{\mathrm{awn}}; \mathcal{O}_{X^{\mathrm{awn}}}) \qquad \text{and} \qquad A^{\mathrm{sn}} := \Gamma(X^{\mathrm{sn}}; \mathcal{O}_{X^{\mathrm{sn}}}) \,.$$

If A is an $\mathbf{F}_p$-algebra, we write $A^{\mathrm{perf}} := A^{\mathrm{awn}}$.

For future use, we record the fact that absolute weak normalization preserves Zariski covers.

1.18 Lemma. *Let $j : U \hookrightarrow X$ be an open immersion of schemes. Then the natural square*

$$
\begin{CD}
U^{\mathrm{awn}} @>{j^{\mathrm{awn}}}>> X^{\mathrm{awn}} \\
@VVV @VVV \\
U @>{j}>> X
\end{CD}
$$

is cartesian. Hence, j^{awn} is an open immersion and the functor $(-)^{\mathrm{awn}} : \mathbf{Sch} \to \mathbf{Sch}$ preserves Zariski covers.

Proof. Write $c : X^{\mathrm{awn}} \to X$ for the counit. Since j is an open immersion,

$$
U \times_X X^{\mathrm{awn}} \cong c^{-1}(U)\,,
$$

where $c^{-1}(U)$ is given the unique subscheme structure. Since open subschemes of absolutely weakly normal schemes are again absolutely weakly normal (Lemma 1.8), $U \times_X X^{\mathrm{awn}}$ is absolutely weakly normal. Moreover, since $c : X^{\mathrm{awn}} \to X$ is a universal homeomorphism, the natural morphism $U \times_X X^{\mathrm{awn}} \to U$ is also a universal homeomorphism. Since the natural morphism $U^{\mathrm{awn}} \to U$ is a universal homeomorphism and universal homeomorphisms satisfy the 2-of-3 property, the natural morphism

$$
U^{\mathrm{awn}} \to U \times_X X^{\mathrm{awn}}
$$

is a universal homeomorphism. Since both U^{awn} and $U \times_X X^{\mathrm{awn}}$ are absolutely weakly normal, Theorem 1.13 shows that the natural morphism $U^{\mathrm{awn}} \to U \times_X X^{\mathrm{awn}}$ is an isomorphism. $\qquad\square$

1.3 Finiteness properties. We now analyze the interaction between seminormalization, perfection, and finite type hypotheses. The first result is that, like normalization, seminormalization preserves the property of a scheme being (locally) of finite type over a field.

1.19 Recollection (normalization). Let X be a scheme such that every quasicompact open of X has finitely many irreducible components (e.g., X is locally topologically noetherian). We write $\nu : X^{\mathrm{norm}} \to X$ for the normalization of X. Recall that the natural morphism $\nu : X^{\mathrm{norm}} \to X$ is an integral surjection. In particular, ν is affine; if $X = \mathrm{Spec}(A)$ is affine, we write

$$
A^{\mathrm{norm}} := \Gamma(X^{\mathrm{norm}}; \mathcal{O}_{X^{\mathrm{norm}}})\,.
$$

Also recall that the natural closed immersion $X_{\mathrm{red}} \hookrightarrow X$ induces an isomorphism $(X_{\mathrm{red}})^{\mathrm{norm}} \xrightarrow{\sim} X^{\mathrm{norm}}$.

1.20. Notice that since normal schemes are seminormal (Proposition 1.2), the normalization morphism $\nu : X^{\mathrm{norm}} \to X$ factors through the seminormalization of X. Since the seminormalization $c : X^{\mathrm{sn}} \to X$ is a universal homeomorphism, c is affine [STK, Tag 04DE]. Since ν is also affine, by cancellation we deduce that the induced morphism $X^{\mathrm{norm}} \to X^{\mathrm{sn}}$ is affine.

1.21 Proposition. *Let X be a locally noetherian scheme.*

(1.21.1) *If $\nu : X^{\mathrm{norm}} \to X$ is finite, then $c : X^{\mathrm{sn}} \to X$ is finite.*

(1.21.2) *If X is Nagata, then both $\nu : X^{\mathrm{norm}} \to X$ and $c : X^{\mathrm{sn}} \to X$ are finite.*

(1.21.3) *If X is (locally) of finite type over a field k, then both $\nu : X^{\mathrm{norm}} \to X$ and $c : X^{\mathrm{sn}} \to X$ are finite. Hence the seminormalization X^{sn} is also (locally) of finite type over k.*

Proof. For (1.21.1), first note that since X is locally noetherian, the natural closed immersion $X_{\mathrm{red}} \hookrightarrow X$ is finite. Since the natural morphisms

$$
(X_{\mathrm{red}})^{\mathrm{norm}} \to X^{\mathrm{norm}} \qquad \text{and} \qquad (X_{\mathrm{red}})^{\mathrm{sn}} \to X^{\mathrm{sn}}
$$

are isomorphisms, it suffices to prove the claim under the additional assumption that X is reduced. In this case, factor the normalization morphism as a composite

$$
\nu : X^{\mathrm{norm}} \longrightarrow X^{\mathrm{sn}} \xrightarrow{\ c\ } X \,.
$$

Both of morphisms in the composite are affine, hence it suffices to prove the claim under the additional assumption that $X = \mathrm{Spec}(A)$ is affine, noetherian, and reduced.

Since A is reduced, the normalization homomorphism $A \to A^{\mathrm{norm}}$ is injective. Thus [25, Theorem 4.1] shows that the induced morphism $A^{\mathrm{sn}} \to A^{\mathrm{norm}}$ is also injective. Since A^{norm} is a finite A-module and A is noetherian, we deduce that A^{sn} is also a finite A-module. Thus the seminormalization morphism $A \hookrightarrow A^{\mathrm{sn}}$ is finite, as desired.

For (1.21.2), the finiteness of the normalization morphism is the content of [STK, Tag 035S]. Since Nagata schemes are locally noetherian, the finiteness of the seminormalization morphism follows from (1.21.1). For (1.21.3), note that since X is (locally) of finite over a field, X is Nagata [STK, Tag 035B]. Thus (1.21.2) shows that the counit $X^{\mathrm{sn}} \to X$ is a finite morphism. Since finite morphisms are of finite type, the composite $X^{\mathrm{sn}} \to X \to \mathrm{Spec}(k)$ is of (locally) of finite type. $\qquad\square$

Unfortunately, perfection (hence, also absolute weak normalization) does not preserve the property of being finite type. In fact, a perfect $\mathbf{F}_p$-scheme is basically never noetherian. The following precise formulation of this fact is well-known:

1.22 Proposition. *Let X be a perfect $\mathbf{F}_p$-scheme. Then X is noetherian if and only if X is isomorphic to the spectrum of a finite product of perfect fields.*

Proof. Clearly a finite product of perfect fields is noetherian. So it remains to show that if X is noetherian, then X is isomorphic to the spectrum of a finite product of perfect fields. See [21] for a proof of the claim under the additional assumption that X is affine. If X is not assumed to be affine, note that

$$\dim(X) = \sup_{\substack{U \subset X \\ \text{affine open}}} \dim(U)\,.$$

Since the spectrum of a product of fields is 0-dimensional, by the affine case we see that $\dim(X) = 0$. Since every 0-dimensional qcqs scheme is affine, X is also affine. $\qquad\square$

As a result, for the positive characteristic version of our reconstruction results, we need to work with schemes that are only *universally homeomorphic* to finite type schemes.

1.23 Definition. We say that a scheme X is *(locally) topologically noetherian* if the topological space $|X|$ is (locally) noetherian.

1.24 Example. Let X be a (locally) noetherian scheme. Then the absolute weak normalization X^{awn} is (locally) topologically noetherian.

1.25 Definition. We say that a morphism of schemes $f : X \to S$ is *(locally) topologically of finite type* if there exists a universal homeomorphism of S-schemes $X \to Y$ such that Y is (locally) of finite type over S.

1.26 Observation.

(1.26.1) Since universal homeomorphisms and morphisms (locally) of finite type are both stable under pullback, the class of morphisms (locally) topologically of finite type is stable under pullback.

(1.26.2) Since universal homeomorphisms are closed under composition, if $f : X \to S$ is (locally) topologically of finite type, then for any universal homeomorphism $h : X' \to X$, the composite $fh : X' \to S$ is (locally) topologically of finite type.

The main example we care about is the following:

1.27 Lemma. *Let $f : X \to S$ be a morphism of schemes. If f is (locally) topologically of finite type, then the induced morphisms $X^{\mathrm{awn}} \to S$ and $X^{\mathrm{awn}} \to S^{\mathrm{awn}}$ are (locally) topologically of finite type.*

Proof. The claim for the morphism $X^{\mathrm{awn}} \to S$ follows from (1.26.2) and the fact that the counit $X^{\mathrm{awn}} \to X$ is a universal homeomorphism. For the second claim, note that f^{awn} factors as

$$X^{\mathrm{awn}} \to X \times_S S^{\mathrm{awn}} \to S^{\mathrm{awn}}\,.$$

By the 2-of-3 property for universal homeomorphisms, the left-hand morphism is a universal homeomorphism. Since $X \to S$ is (locally) topologically of finite type, (1.26.1) shows that the right-hand morphism

is (locally) topologically of finite type. Applying (1.26.2) again shows that the composite $X^{\mathrm{awn}} \to S^{\mathrm{awn}}$ is (locally) topologically of finite type. □

1.28 Example. Let k be a field of characteristic $p > 0$ and X a scheme (locally) topologically of finite type over k. Then X^{perf} is (locally) topologically of finite type over both k and k^{perf}.

For seminormal schemes over a field of characteristic 0, the properties '(locally) of finite type' and '(locally) topologically of finite type' coincide:

1.29 Lemma. *Let k be a field of characteristic* 0 *and let X be a k-scheme.*

(1.29.1) *If X is (locally) topologically of finite type over k, then X^{sn} is (locally) of finite type over k.*

(1.29.2) *If X is seminormal, then X is (locally) topologically of finite type over k if and only if X is (locally) of finite type over k.*

Proof. For (1.29.1), factor the structure morphism $X \to \mathrm{Spec}(k)$ as a composite of a universal homeomorphism $X \to Y$ followed by a morphism $Y \to \mathrm{Spec}(k)$ (locally) of finite type. Since $\mathrm{char}(k) = 0$, the universal homeomorphism $X \to Y$ also induces an isomorphism on residue fields. By (1.13.3), the induced morphism $X^{\mathrm{sn}} \to Y^{\mathrm{sn}}$ is an isomorphism. To conclude, note that since Y is (locally) of finite type over k, by (1.21.3), the seminormalization $Y^{\mathrm{sn}} \cong X^{\mathrm{sn}}$ is also locally of finite type over k.

For the nontrivial implication of (1.29.2), assume that X is (locally) topologically of finite type over k. Then since X is seminormal, by (1.29.1) we conclude that $X = X^{\mathrm{sn}}$ is (locally) of finite type over k. □

We conclude this section by showing that the localization of the category of finite type k-schemes at the universal homeomorphisms is the category of k-schemes that are absolutely weakly normal and topologically of finite type over k.

1.30 Notation. Let S be a scheme. We write $\mathbf{Sch}^{\mathrm{ft}}_S \subset \mathbf{Sch}_S$ for the full subcategory spanned by the finite type S-schemes. We write $\mathbf{Sch}^{\mathrm{awn,tft}}_S \subset \mathbf{Sch}_S$ for the full subcategory spanned by the S-schemes that are absolutely weakly normal and topologically of finite type over S.

1.31 Observation. The full subcategory $\mathbf{Sch}^{\mathrm{awn,tft}}_S \subset \mathbf{Sch}_S$ is the smallest full subcategory closed under isomorphisms and containing the absolute weak normalizations of finite type S-schemes.

1.32 Lemma. *Let k be a field. Then the functor $(-)^{\mathrm{awn}} : \mathbf{Sch}^{\mathrm{ft}}_k \to \mathbf{Sch}^{\mathrm{awn}}_k$ induces an equivalence of categories*

$$\mathbf{Sch}^{\mathrm{ft}}_k[\mathrm{UH}^{-1}] \xrightarrow{\sim} \mathbf{Sch}^{\mathrm{awn,tft}}_k .$$

Proof. If $\mathrm{char}(k) = 0$, then the claim follows from Theorem 1.13 and Proposition 1.21.

If $\mathrm{char}(k) > 0$, note that if $f : X \to Y$ is a universal homeomorphism of finite type k-schemes, it follows from Corollary 1.15 that $f^{\mathrm{perf}} : X^{\mathrm{perf}} \to Y^{\mathrm{perf}}$ is an isomorphism of k-schemes. Thus there is an inverse map $g^{\mathrm{perf}} : Y^{\mathrm{perf}} \to X^{\mathrm{perf}}$ over k. For $n \geq 0$, write Y_n for the k-scheme defined by

$$Y \xrightarrow{\mathrm{Frob}^n_Y} Y \longrightarrow \mathrm{Spec}(k) .$$

Since X and Y are of finite type over k, there is some $n \gg 0$ such that composite $Y^{\mathrm{perf}} \to X^{\mathrm{perf}} \to X$ factors as

$$Y^{\mathrm{perf}} \to Y_n \to X$$

and the composite $Y_n \to X \xrightarrow{f} Y$ is given by Frob_Y. Since Frob_Y is a universal homeomorphism, it follows that the localization $\mathbf{Sch}^{\mathrm{ft}}_k[\mathrm{UH}^{-1}]$ is equivalent to the localization $\mathbf{Sch}^{\mathrm{ft}}_k[\mathrm{Frob}^{-1}]$ at absolute Frobenii.

It remains to show that $\mathbf{Sch}^{\mathrm{awn,tft}}_k$ is the localization of $\mathbf{Sch}^{\mathrm{ft}}_k$ at at absolute Frobenii. For this note that we may also describe the category $\mathbf{Sch}^{\mathrm{ft}}_k[\mathrm{Frob}^{-1}]$ as follows. The objects are finite type k-schemes and for two objects X and Y, the set of morphisms $X \to Y$ is given by

$$\mathrm{colim}\left(\mathrm{Hom}_k(X, Y) \xrightarrow{\mathrm{Frob}^*} \mathrm{Hom}_k(X_1, Y) \xrightarrow{\mathrm{Frob}^*} \mathrm{Hom}_k(X_2, Y) \xrightarrow{\mathrm{Frob}^*} \cdots \right).$$

With this description it is easy to see that $\mathbf{Sch}^{\mathrm{awn,tft}}_k$ has the correct universal property. □

2. The étale reconstruction property

The purpose of this section is to formulate what it means for a field k to have the property that absolutely weakly normal schemes topologically of finite type over k can be reconstructed from their étale topoi. In order to do this, in §2.1, we begin by setting some topos-theoretic notations and recalling a bit of background from topos theory. Schemes topologically of finite type over a field are necessarily *Jacobson*. In §2.2, we recall the necessary background on Jacobson schemes. A particularly important point is that every morphism between schemes topologically of finite type over a field carries closed points to closed points. However, not every geometric morphism between the étale topoi of such schemes has this property.

In §2.3, we study the class of geometric morphisms that send closed points to closed points, which we called *pinned* geometric morphisms. We show that for schemes X and Y topologically of finite type over k, the groupoid of pinned geometric morphisms $X_{\text{ét}} \to Y_{\text{ét}}$ over $\text{Spec}(k)_{\text{ét}}$ is equivalent to a set. This rigidity property is fundamental; we need it to make descent arguments in §4. We also use this rigidity property to formulate what it means for a field to *satisfy étale reconstruction*, see Proposition 2.28 and Definition 2.29. In §2.4, we record some immediate consequences of the étale reconstruction property (see Proposition 2.36).
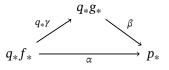
2.1   Notation and recollections on topoi. We make use of the following notation throughout the rest of the paper.

2.1 Notation. We write **RTop** for the $(2, 1)$-category with objects topoi, 1-morphisms (right adjoints in) geometric morphisms, and 2-morphisms natural isomorphisms.

We also have a relative notion of the $(2, 1)$-category of topoi sliced over a particular topos.

2.2 Notation. Let $\mathcal{B}$ be a topos. We write $\mathbf{RTop}_{\mathcal{B}}$ for the $(2, 1)$-category of topoi over $\mathcal{B}$ defined in the following manner:

(2.2.1) An object of $\mathbf{RTop}_{\mathcal{B}}$ is a geometric morphism of topoi $p_* \colon \mathcal{X} \to \mathcal{B}$.

(2.2.2) A 1-morphism from $p_* \colon \mathcal{X} \to \mathcal{B}$ to $q_* \colon \mathcal{Y} \to \mathcal{B}$ is a pair (f_*, α) where $f_* \colon \mathcal{X} \to \mathcal{Y}$ is a geometric morphism and α is a natural isomorphism $\alpha \colon q_* f_* \xrightarrow{\sim} p_*$.

(2.2.3) A 2-morphism from $(f_*, \alpha) \colon \mathcal{X} \to \mathcal{Y}$ to $(g_*, \beta) \colon \mathcal{X} \to \mathcal{Y}$ is a natural isomorphism $\gamma \colon f_* \xrightarrow{\sim} g_*$ such that the triangle of natural isomorphisms

$$
\begin{array}{ccc}
 & q_* g_* & \\
\nearrow^{q_*\gamma} & & \searrow^{\beta} \\
q_* f_* & \xrightarrow{\quad\alpha\quad} & p_*
\end{array}
$$

commutes.

2.3 Notation. Let $\mathcal{B}$ be a topos. Given geometric morphisms $p_* \colon \mathcal{X} \to \mathcal{B}$ and $q_* \colon \mathcal{Y} \to \mathcal{B}$, we write $\text{Hom}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$ for the Hom *groupoid* in $\mathbf{RTop}_{\mathcal{B}}$. If $\mathcal{B} = S_{\text{ét}}$ is the étale topos of a scheme S, we simply write

$$\mathbf{RTop}_S \coloneqq \mathbf{RTop}_{S_{\text{ét}}} \qquad \text{and} \qquad \text{Hom}_S(\mathcal{X}, \mathcal{Y}) \coloneqq \text{Hom}_{S_{\text{ét}}}(\mathcal{X}, \mathcal{Y}) \,.$$

Moreover, if $S = \text{Spec}(A)$ is affine, we write

$$\mathbf{RTop}_A \coloneqq \mathbf{RTop}_{\text{Spec}(A)_{\text{ét}}} \qquad \text{and} \qquad \text{Hom}_A(\mathcal{X}, \mathcal{Y}) \coloneqq \text{Hom}_{\text{Spec}(A)_{\text{ét}}}(\mathcal{X}, \mathcal{Y}) \,.$$

2.4 Recollection (the underlying space of a topos). Let $\mathcal{X}$ be a topos. We write $\text{Open}(\mathcal{X})$ for the poset of isomorphism classes of subobjects of the terminal object of $\mathcal{X}$. The poset $\text{Open}(\mathcal{X})$ is a locale. We write $|\mathcal{X}|$ for the topological space associated to the locale $\text{Open}(\mathcal{X})$, i.e., the topological space of points of $\text{Open}(\mathcal{X})$.

If $f_* \colon \mathcal{X} \to \mathcal{Y}$ is a geometric morphism, the pullback functor f^* preserves subobjects of the terminal object, thus restricts to a morphism of locales

$$f^* \colon \text{Open}(\mathcal{X}) \to \text{Open}(\mathcal{Y}) \,.$$

We write $|f_*| \colon |\mathcal{X}| \to |\mathcal{Y}|$ for the induced map of underlying spaces.

2.5 Example. Let X be a scheme. Then there is a natural isomorphism from the locale of open subsets of the underlying space $|X|$ to the locale $\mathrm{Open}(X_{\mathrm{ét}})$. Since $|X|$ is sober, we deduce that there is a natural isomorphism of topological spaces $|X_{\mathrm{ét}}| \xrightarrow{\sim} |X|$.

We also recall some background about categories of points of topoi.

2.6 Notation (category of points). Let $\mathcal{X}$ be a topos. We write $\mathrm{Pt}(\mathcal{X})$ for the *category of points* of $\mathcal{X}$. Objects of $\mathrm{Pt}(\mathcal{X})$ are left exact left adjoints $x^* : \mathcal{X} \to \mathbf{Set}$ and morphisms are natural transformations. The assignment $\mathcal{X} \mapsto \mathrm{Pt}(\mathcal{X})$ defines a functor

$$\mathrm{Pt} : \mathbf{RTop} \to \mathbf{Cat}$$

from the $(2,1)$-category of topoi to the $(2,1)$-category of categories. The functoriality is given by sending a geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ to the functor

$$\mathrm{Pt}(f_*) : \mathrm{Pt}(\mathcal{X}) \to \mathrm{Pt}(\mathcal{Y}), \quad x^* \mapsto x^* f^*$$

with the obvious functoriality in natural transformations.

2.7 Example [SGA 4$_{\mathrm{II}}$, Exposé VIII, Théorème 7.9]. Let X be a qcqs scheme. The Grothendieck School computed the category of points of $X_{\mathrm{ét}}$. The objects of $\mathrm{Pt}(X_{\mathrm{ét}})$ are geometric points $\bar{x} \to X$. Given geometric points $\bar{x} \to X$ and $\bar{y} \to X$, the set of morphisms $\bar{x} \to \bar{y}$ is the set

$$\mathrm{Hom}_{\mathrm{Pt}(X_{\mathrm{ét}})}(\bar{x}, \bar{y}) = \mathrm{Hom}_X(\mathrm{Spec}(\mathcal{O}_{X,y}^{\mathrm{sh}}), \mathrm{Spec}(\mathcal{O}_{X,x}^{\mathrm{sh}}))$$

of morphisms of X-schemes from the strictly henselian local scheme associated to $\bar{y}$ to that of $\bar{x}$. Composition in $\mathrm{Pt}(X_{\mathrm{ét}})$ is given by composition of morphisms of X-schemes.
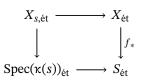
2.8. An important consequences of Example 2.7 are that for each geometric point $\bar{x} \to X$ with image $x \in X$, the set of endomorphisms of $\bar{x}$ in $\mathrm{Pt}(X_{\mathrm{ét}})$ is isomorphic to the absolute Galois group $\mathrm{Gal}(\kappa(\bar{x})/\kappa(x))$. In particular, every endomorphism of $\bar{x}$ in $\mathrm{Pt}(X_{\mathrm{ét}})$ is an automorphism.

A few times throughout this paper, we need the following observation. In general, the functor

$$(-)_{\mathrm{ét}} : \mathbf{Sch} \to \mathbf{RTop}$$

does not preserve fiber products (see [18, Remark 2.6]). However, it does preserve pullbacks along points:

2.9 Lemma. *Let $f : X \to S$ be a morphism between qcqs schemes and $s \in S$. Then the square of étale topoi*

$$
\begin{array}{ccc}
X_{s,\mathrm{ét}} & \longrightarrow & X_{\mathrm{ét}} \\
\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathrm{Spec}(\kappa(s))_{\mathrm{ét}} & \longrightarrow & S_{\mathrm{ét}}
\end{array}
$$

is a pullback square in **RTop**.

Proof. The same argument as in [12, Proposition 2.3], replacing the strictly henselian local ring by the local ring $\mathcal{O}_{S,s}$ and using that $\mathrm{Spec}(\mathcal{O}_{S,s})$ is isomorphic to the limit of the cofiltered diagram of affine open subschemes of S containing s. $\qquad\square$

2.2 Background on Jacobson schemes. We now recall the basics about Jacobson schemes. Being Jacobson is a topological property of schemes. The idea is that 'the closed points see everything about the topology of a Jacobson space'.

2.10 Notation. Given a topological space T, we write $T_{\mathrm{cl}} \subset T$ for the subset of closed points. For a scheme X, we write $X_{\mathrm{cl}} := |X|_{\mathrm{cl}}$.

2.11 Recollection. A topological space T is *Jacobson* if every closed subset $Z \subset T$ is the closure of $Z \cap T_{\mathrm{cl}}$. We say that a scheme X is *Jacobson* if the underlying topological space $|X|$ is Jacobson. See [9, Remark 3.34] for a number of equivalent characterizations of Jacobson spaces.

2.12 Lemma. *If $f : X \to S$ is a morphism of schemes that is locally topologically of finite type and S is Jacobson, then X is also Jacobson.*

Proof. Factor f as a composite of a universal homeomorphsm $X \to X'$ followed by a morphism $X' \to S$ locally of finite type. Since S is Jacobson, by [STK, Tag 02J5], the scheme X' is Jacobson. Since $|X|$ is homeomorphic to $|X'|$, the scheme X is also Jacobson. $\qquad\square$

2.13 Example. If X is a scheme locally topologically of finite type over a field, then X is Jacobson.

As with many topological properties of schemes, there are more algebro-geometric characterizations of Jacobson schemes. To state them, we need the following definition.

2.14 Definition. Let X be a scheme. We say that $x \in X$ is a *finite type point* if the natural morphism $\mathrm{Spec}(\kappa(x)) \to X$ is of finite type. We write $X_{\mathrm{ft\text{-}pts}} \subset |X|$ for the subset of finite type points.

2.15 Proposition [STK, Tag 01TB]**.** *Let X be a scheme. The following are equivalent:*

(2.15.1) *The scheme X is Jacobson.*

(2.15.2) *The set $X_{\mathrm{ft\text{-}pts}}$ of finite type points of X is the set X_{cl} of closed points of X.*

(2.15.3) *Every morphism $X' \to X$ locally of finite type carries closed points of X' to closed points of X.*

(2.15.4) *For every morphism $f : X' \to X$ locally of finite type and closed point $x' \in X'$, the point $f(x')$ is closed and the field extension $\kappa(x') \supset \kappa(f(x'))$ is finite.*

To conclude this subsection, we prove that morphisms from reduced Jacobson schemes are determined by their values on closed points. For this, we need the following lemma.

2.16 Lemma. *Let T be a Jacobson topological space and let $S \subset T$ be a locally closed subspace. If S contains all of the closed points of T, then $S = T$.*

Proof. The claim is immediate from the fact that the map

$$\{\text{locally closed subsets of } T\} \to \{\text{locally closed subsets of } T_{\mathrm{cl}}\}$$
$$W \mapsto W \cap T_{\mathrm{cl}}$$

is bijective [STK, Tag 005Z]. $\qquad\square$

2.17 Corollary. *Let X be a reduced Jacobson scheme and let $Z \subset X$ be a locally closed subscheme. If Z contains all of the closed points of X, then $Z = X$ as schemes.*

Proof. By Lemma 2.16, we see that $|Z| = |X|$ as topological spaces. Since X is reduced, we deduce that $Z = X$ as schemes. $\qquad\square$

2.18 Lemma. *Let X be a reduced Jacobson scheme, and for each closed point $x \in X_{\mathrm{cl}}$ choose a field extension $L_x \supset \kappa(x)$. Then the induced morphism*

$$\coprod_{x \in X_{\mathrm{cl}}} \mathrm{Spec}(L_x) \to X$$

is an epimorphism in the category of schemes.

Proof. Note that for each field extension $L \supset k$, the induced morphism of schemes $\mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is an epimorphism. So it suffices to prove the claim when $L_x = \kappa(x)$ for each $x \in X_{\mathrm{cl}}$. Let $f, g : X \to Y$ be morphisms that become equal after precomposition with the natural morphism

$$c_X : \coprod_{x \in X_{\mathrm{cl}}} \mathrm{Spec}(\kappa(x)) \to X \,.$$

Consider the equalizer subscheme $\mathrm{eq}(f, g) \subset X$, which is locally closed. By assumption, $\mathrm{eq}(f, g)$ contains all of the closed points of X. Since X is Jacobson and reduced, Corollary 2.17 shows that $\mathrm{eq}(f, g) = X$ as schemes. Hence $f = g$. $\qquad\square$

2.3   Pinned morphisms and the étale reconstruction property. Let k be a field, and let X and Y be schemes locally topologically of finite type over k. In light of Proposition 2.15, every morphism of k-schemes $X \to Y$ carries closed points to closed points. Not every geometric morphism $X_{\text{ét}} \to Y_{\text{ét}}$ over $\text{Spec}(k)_{\text{ét}}$ has this property (see Example 2.30). Hence, in our reconstruction results, we impose this restriction on the morphisms of topoi.

2.19   Definition (pinned morphisms).

(2.19.1) A map of topological spaces $f \colon S \to T$ is *pinned* if f carries closed points of S to closed points of T.

(2.19.2) A morphism of schemes $f \colon X \to Y$ is *pinned* if the induced map on underlying topological spaces $|f| \colon |X| \to |Y|$ is pinned.

(2.19.3) A geometric morphism of topoi $f_* \colon \mathcal{X} \to \mathcal{Y}$ is *pinned* if the induced map of underlying topological spaces $|f_*| \colon |\mathcal{X}| \to |\mathcal{Y}|$ is pinned.

2.20   Notation. Let $\mathcal{B}$ be a topos. We write $\mathbf{RTop}_{\mathcal{B}}^{\text{pin}} \subset \mathbf{RTop}_{\mathcal{B}}$ for the subcategory with all objects, 1-morphisms the *pinned* geometric morphisms, and 2-morphisms all natural isomorphisms. Given topoi $\mathcal{X}, \mathcal{Y} \in \mathbf{RTop}_{\mathcal{B}}$, write

$$\text{Hom}_{\mathcal{B}}^{\text{pin}}(\mathcal{X}, \mathcal{Y}) \subset \text{Hom}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$$

for the full subgroupoid spanned by the pinned geometric morphisms. When $\mathcal{B}$ is the étale topos of a scheme, we make the same notational simplifications as in Notation 2.3.

2.21   Remark (Voevodsky's terminology). Voevodsky uses the term 'admissible' rather than 'pinned', see [28, p. 513]. We find the term 'pinned' more evocative: the requirement that the geometric morphism carries closed points to closed points 'pins down' the morphism.

2.22   Proposition (rigidity of pinned morphisms). *Let S be a scheme and let X and Y be schemes locally topologically of finite type over S.*

(2.22.1) *Let $(f_*, \alpha) \colon X_{\text{ét}} \to Y_{\text{ét}}$ be a pinned geometric morphism over $S_{\text{ét}}$. Then (f_*, α) has no nontrivial automorphisms in $\text{Hom}_S(X_{\text{ét}}, Y_{\text{ét}})$.*

(2.22.2) *The groupoid $\text{Hom}_S^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$ is equivalent to a set.*

Proof. Item (2.22.2) is simply a reformulation of (2.22.1), so we just need to prove (2.22.1).

We first prove the claim under the additional assumption that S is the spectrum of a field k. Note that, by descent, we can assume that k is separably closed. Let $\phi \colon f^* \xrightarrow{\sim} f^*$ be an automorphism. The claim follows if we can show that for every *closed* geometric point $x \to X_{\text{ét}}$ the composite $x^*\phi \colon x^*f^* \xrightarrow{\sim} x^*f^*$ is the identity. This is clear because a map in $X_{\text{ét}}$ is the identity if and only if it is the identity on stalks at closed points. Since f_* is pinned, the composite $f_* \circ x_*$ also defines a closed geometric point of $Y_{\text{ét}}$. By Example 2.7, we may assume that $f_* \circ x_*$ is induced by a morphism of k-schemes $y \colon \text{Spec}(\bar{k}) \to Y$ and $x^*\phi$ defines an automorphism of y^*. Let $y_0 \in Y$ denote the image of y. Again by Example 2.7, the group of automorphisms of y^* is isomorphic to the absolute Galois group $G_{\kappa(y_0)}$ of $\kappa(y_0)$. Since y_0 is a closed point and k is separably closed it follows that $G_{\kappa(y_0)} = \{\text{id}\}$ and thus $x^*\phi = \text{id}$.

For the case of a general base scheme S, fix an automorphism $\phi \colon f^* \to f^*$ in $\text{Hom}_S^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$. Using Lemma 2.9, we see that pulling back ϕ along any point $s \colon \text{Spec}(\kappa(s)) \to S$ induces an automorphism $s^*\phi$ of s^*f^* in the groupoid $\text{Hom}_{\kappa(s)}^{\text{pin}}(X_{s,\text{ét}}, Y_{s,\text{ét}})$, which we have already shown to be equivalent to a set. In particular $s^*\phi = \text{id}$. This in particular implies that ϕ is the identity stalk-wise, so $\phi = \text{id}$. $\square$

2.23   Convention. Let k be a field and let X and Y be schemes locally topologically of finite type over k. Then using Proposition 2.22, we tacitly regard the groupoid $\text{Hom}_k^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$ as a set, i.e., we identify $\text{Hom}_k^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$ with the set of *isomorphism classes* of geometric morphisms $X_{\text{ét}} \to Y_{\text{ét}}$ over $\text{Spec}(k)_{\text{ét}}$.

We are almost ready to formulate what it means for a field k to have the property that absolutely weakly normal schemes topologically of finite type over k can be reconstructed from their étale topoi. First we show that every morphism between schemes *topologically* of finite type is a pinned.

2.24 Lemma. *Let S be a Jacobson scheme and let X and Y be schemes locally topologically of finite type over S. Then every morphism of S-schemes $f \colon X \to Y$ is pinned.*

Proof. We work locally on the target to assume that X and S are affine. Let $s \colon X \to X'$ and $t \colon Y \to Y'$ be universal homeomorphisms to finite type S-schemes. Then tf factors as

$$X \xrightarrow{\Gamma_f} X \times_S Y \xrightarrow{s \times t} X' \times_S Y' \xrightarrow{\mathrm{pr}_2} Y'.$$

Since $X \to S$ is separated, the graph Γ_f is a closed immersion; in particular, Γ_f is pinned. Since s and t are universal homeomorphisms, $s \times t$ is a universal homeomorphism; in particular, $s \times t$ is pinned. Since pr_2 is a morphism of schemes locally of finite type over S, Lemma 2.12 and Proposition 2.15 show that pr_2 is pinned. It follows that tf is pinned. Since t is a universal homeomorphism, this also implies that f is pinned. $\square$

There are a number of equivalent formulations of what it means for a field k to have the property that absolutely weakly normal schemes topologically of finite type over k can be reconstructed from their étale topoi. We prove their equivalence before stating the definition.

2.25 Observation. Let S be a scheme. Absolute weak normalization induces an adjunction

$$(\mathbf{Sch}^{\mathrm{awn}})_{/S^{\mathrm{awn}}} \xrightleftharpoons[(-)^{\mathrm{awn}}]{} \mathbf{Sch}_{/S}.$$

The left adjoint sends an absolutely weakly normal scheme X over S^{awn} to the composite

$$X \to S^{\mathrm{awn}} \to S$$

of the structure morphism with the counit. In particular, if X is an absolutely weakly normal S-scheme and Y is an S-scheme, there is a natural bijection

$$\mathrm{Hom}_S(X, Y) \cong \mathrm{Hom}_{S^{\mathrm{awn}}}(X, Y^{\mathrm{awn}}).$$

2.26 Lemma. *Let k be a field and let X and Y be schemes locally topologically of finite type over k. The following are equivalent:*

(2.26.1) *The natural map $\mathrm{Hom}_k(X^{\mathrm{awn}}, Y) \to \mathrm{Hom}_k^{\mathrm{pin}}(X^{\mathrm{awn}}_{\text{ét}}, Y_{\text{ét}})$ is bijective.*

(2.26.2) *The natural map $\mathrm{Hom}_{k^{\mathrm{awn}}}(X^{\mathrm{awn}}, Y^{\mathrm{awn}}) \to \mathrm{Hom}_{k^{\mathrm{awn}}}^{\mathrm{pin}}(X^{\mathrm{awn}}_{\text{ét}}, Y^{\mathrm{awn}}_{\text{ét}})$ is bijective.*

Proof. Let us first comment that since X and Y are locally topologically of finite type over k, Lemma 1.27 shows that both X^{awn} and Y^{awn} are locally topologically of finite type over k. Hence by Proposition 2.22, the right-hand sides are sets, so the statements actually make sense.

To prove the claim, notice that the counits $Y^{\mathrm{awn}} \to Y$ and $\mathrm{Spec}(k^{\mathrm{awn}}) \to \mathrm{Spec}(k)$ induce a commutative square

(2.27)

$$\begin{array}{ccc}
\mathrm{Hom}_{k^{\mathrm{awn}}}(X^{\mathrm{awn}}, Y^{\mathrm{awn}}) & \longrightarrow & \mathrm{Hom}_{k^{\mathrm{awn}}}^{\mathrm{pin}}(X^{\mathrm{awn}}_{\text{ét}}, Y^{\mathrm{awn}}_{\text{ét}}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_k(X^{\mathrm{awn}}, Y) & \longrightarrow & \mathrm{Hom}_k^{\mathrm{pin}}(X^{\mathrm{awn}}_{\text{ét}}, Y_{\text{ét}}).
\end{array}$$

By Observation 2.25, the left-hand vertical map is bijective. By the topological invariance of the étale topos, for any scheme S, the counit $S^{\mathrm{awn}} \to S$ induces an equivalence of étale topoi $S^{\mathrm{awn}}_{\text{ét}} \xrightarrow{\sim} S_{\text{ét}}$. Hence the right-hand vertical map is bijective. Thus the top map in (2.27) is bijective if and only if the bottom map is bijective, as claimed. $\square$

For the next result, recall Notations 1.30 and 2.20.

2.28 Proposition. *Let k be a field. The following conditions are equivalent:*

(2.28.1) *The functor $(-)_{\text{ét}} \colon \mathbf{Sch}_k^{\mathrm{awn,tft}} \to \mathbf{RTop}_k^{\mathrm{pin}}$ is fully faithful.*

(2.28.2) *The functor $(-)_{\text{ét}} \colon \mathbf{Sch}_k^{\mathrm{ft}}[\mathrm{UH}^{-1}] \to \mathbf{RTop}_k^{\mathrm{pin}}$ is fully faithful.*

(2.28.3) *For all finite type k-schemes X and Y, the natural map*

$$\operatorname{Hom}_k(X^{\mathrm{awn}}, Y) \to \operatorname{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, Y_{\text{ét}})$$

is bijective.

(2.28.4) *For all schemes X and Y topologically of finite type over k, the natural map*

$$\operatorname{Hom}_k(X^{\mathrm{awn}}, Y) \to \operatorname{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, Y_{\text{ét}})$$

is bijective.

(2.28.5) *For all schemes X and Y topologically of finite type over k with X absolutely weakly normal, the natural map*

$$\operatorname{Hom}_k(X, Y) \to \operatorname{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}})$$

is bijective.

(2.28.6) *For all schemes X and Y topologically of finite type over k, the natural map*

$$\operatorname{Hom}_{k^{\mathrm{awn}}}(X^{\mathrm{awn}}, Y^{\mathrm{awn}}) \to \operatorname{Hom}_{k^{\mathrm{awn}}}^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, Y_{\text{ét}}^{\mathrm{awn}})$$

is bijective.

Proof. It is an immediate consequence of Lemma 1.32 that (2.28.1), (2.28.2), and (2.28.3) are equivalent. Clearly (2.28.4) $\Rightarrow$ (2.28.3). To see that (2.28.3) $\Rightarrow$ (2.28.4), let X and Y be schemes topologically of finite type over k. Then there exist universal homeomorphisms $X \to X'$ and $Y \to Y'$ where X' and Y' are of finite type over k. By assumption (2.28.3), the natural map

$$\operatorname{Hom}_k((X')^{\mathrm{awn}}, Y') \to \operatorname{Hom}_k^{\mathrm{pin}}((X')_{\text{ét}}^{\mathrm{awn}}, Y'_{\text{ét}})$$

is bijective. By Lemma 2.26, this is equivalent to the claim that the natural map

$$\operatorname{Hom}_{k^{\mathrm{awn}}}((X')^{\mathrm{awn}}, (Y')^{\mathrm{awn}}) \to \operatorname{Hom}_{k^{\mathrm{awn}}}^{\mathrm{pin}}((X')_{\text{ét}}^{\mathrm{awn}}, (Y')_{\text{ét}}^{\mathrm{awn}})$$

is bijective. Since the universal homeomorphisms $X \to X'$ and $Y \to Y'$ induce isomorphisms on absolute weak normalizations, we deduce that the natural map

$$\operatorname{Hom}_{k^{\mathrm{awn}}}(X^{\mathrm{awn}}, Y^{\mathrm{awn}}) \to \operatorname{Hom}_{k^{\mathrm{awn}}}^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, Y_{\text{ét}}^{\mathrm{awn}})$$

is bijective. Applying Lemma 2.26 again, we deduce that the natural map

$$\operatorname{Hom}_k(X^{\mathrm{awn}}, Y) \to \operatorname{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, Y_{\text{ét}})$$

is bijective, as desired.

The equivalence (2.28.4) $\Leftrightarrow$ (2.28.5) is immediate from the fact that the absolute weak normalization of a scheme topologically of finite type over k is also topologically of finite type over k (Lemma 1.27). The equivalence (2.28.4) $\Leftrightarrow$ (2.28.6) is immediate from Lemma 2.26. $\qquad\square$

The following is the main definition of this section.

2.29 Definition (étale reconstruction)**.** Let k be a field. We say that k *satisfies étale reconstruction* if the equivalent conditions of Proposition 2.28 are satisfied.

2.30 Example. The restriction to pinned morphisms in the étale reconstruction property is necessary. To see this, note that for a field of characteristic 0 the map of Galois groups $G_{k(t)} \to G_k$ admits a continuous section. This section induces a morphism $\operatorname{Spec}(k)_{\text{ét}} \to \operatorname{Spec}(k(t))_{\text{ét}}$ in $\mathbf{RTop}_k$. Composing this with the geometric morphism induced by the morphism $\operatorname{Spec}(k(t)) \to \mathbf{P}_k^1$ picking out the generic point, we get a morphism

$$\operatorname{Spec}(k)_{\text{ét}} \to \mathbf{P}_{k,\text{ét}}^1$$

in $\mathbf{RTop}_k$ whose underlying map on topological spaces hits the generic point. Thus this geometric morphism is not induced be a morphism of schemes $\operatorname{Spec}(k) \to \mathbf{P}_k^1$.

2.31 Example. If k is an algebraically closed field of characteristic 0, then k does *not* satisfy étale reconstruction. If C and C' are smooth proper curves over k of the same genus, then the étale topoi of C and C' are equivalent (over the terminal topos $\mathrm{Spec}(k)_{\text{ét}} \simeq \mathbf{Set}$). See [7].

In characteristic 0, étale reconstruction has another reformulation.

2.32 Corollary. *Let k be a field of characteristic 0. Then the following are equivalent:*

(2.32.1) *The field k satisfies étale reconstruction.*

(2.32.2) *For all schemes finite type k-schemes X and Y with X seminormal, the natural map*

$$\mathrm{Hom}_k(X, Y) \to \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}})$$

is bijective.

Proof. Immediate from Lemma 1.29 and the fact that in characteristic 0, absolute weak normalization and seminormalization coincide (Corollary 1.15). □

2.4 Consequences of étale reconstruction. We conclude this section by recording some immediate consequences of the étale reconstruction property (see Proposition 2.36). First we fix some notation.

2.33 Notation. Let $\mathcal{B}$ be a topos. Given $\mathcal{X}, \mathcal{Y} \in \mathbf{RTop}_{\mathcal{B}}$, write

$$\mathrm{Equiv}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y}) \subset \mathrm{Hom}_{\mathcal{B}}(\mathcal{X}, \mathcal{Y})$$

for the full subgroupoid spanned by the equivalences of topoi $\mathcal{X} \xrightarrow{\sim} \mathcal{Y}$ over $\mathcal{B}$. When $\mathcal{B}$ is the étale topos of a scheme, we make the same notational simplifications as in Notation 2.3.

2.34 Convention. Let k be a field and X and Y schemes topologically of finite type over k. Then

$$\mathrm{Equiv}_k(X_{\text{ét}}, Y_{\text{ét}}) \subset \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}}) \,.$$

Following Convention 2.23, we tacitly identify the groupoid $\mathrm{Equiv}_k(X_{\text{ét}}, Y_{\text{ét}})$ with the set of *isomorphism classes* of equivalences $X_{\text{ét}} \xrightarrow{\sim} Y_{\text{ét}}$ over $\mathrm{Spec}(k)_{\text{ét}}$.

2.35 Lemma. *Let X and Y be topologically noetherian schemes. Then a geometric morphism $f_* \colon X_{\text{ét}} \to Y_{\text{ét}}$ is an equivalence if and only if $\mathrm{Pt}(f_*) \colon \mathrm{Pt}(X_{\text{ét}}) \to \mathrm{Pt}(Y_{\text{ét}})$ is an equivalence.*

Proof. Since X and Y are topologically noetherian schemes, the topoi $X_{\text{ét}}$ and $Y_{\text{ét}}$ are coherent; moreover, Corollary A.9 shows that every geometric morphism $f_* \colon X_{\text{ét}} \to Y_{\text{ét}}$ is coherent. The claim is now an immediate consequence of the Makkai–Reyes Conceptual Completeness Theorem (see [19, Corollary 2.3.5; 20, Theorem 7.1.8] or [SAG, Theorem A.9.0.6]). □

We now state the main result of this subsection.

2.36 Proposition. *Let k be a field that satisfies étale reconstruction and let X and Y be absolutely weakly normal schemes topologically of finite type over k. Then:*

(2.36.1) *The natural map $\mathrm{Isom}_k(X, Y) \to \mathrm{Equiv}_k(X_{\text{ét}}, Y_{\text{ét}})$ is bijective.*

(2.36.2) *A morphism of k-schemes $f \colon X \to Y$ is an isomorphism if and only if $\mathrm{Pt}(f_*)$ is an equivalence of categories.*

Proof. Item (2.36.1) is immediate from formulation (2.28.1) of the étale reconstruction property. For (2.36.2), note that our assumptions guarantee that X and Y are topologically noetherian. Hence the claim follows from (2.36.1) and Lemma 2.35. □

2.37 Remark (reconstruction from condensed categories of points)**.** Let $\mathcal{X}$ be a coherent topos. The work of Barwick–Glasman–Haine [2] and Lurie [19] provides two natural refinements of the category $\mathrm{Pt}(\mathcal{X})$ to a *condensed* category (in the sense of condensed mathematics [23]). Let k be a field satisfying étale reconstruction. As a consequence of Proposition 2.36, [2, Theorem 9.3.1 & Proposition 13.5.2], and [19, Remark 2.2.9 & Theorem 4.3.3], the functor sending an absolutely weakly normal scheme X topologically of finite type over k to either of these condensed categories of points of $X_{\text{ét}}$ is fully faithful.

3. PROOF OF INJECTIVITY

Let S be a qcqs Jacobson scheme and let X and Y be schemes topologically of finite type over S. In this section, we prove that if X is reduced and all fibers of $X \to S$ are reduced, then the natural map

$$(3.1) \qquad\qquad \mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_S^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}})$$

is injective. See Corollary 3.10. This is the 'easy part' of the reconstruction theorem; in particular, it requires minimal conditions on the base scheme S.

By taking fibers over S, it is straightforward to reduce to the case where S is a field. The crux of the argument is that since morphisms from reduced schemes topologically of finite type over a field are determined by their values on closed points (Lemma 2.18), we can reduce the claim that (3.1) is injective to the case where X is the spectrum of a finite extension of k.

We start by observing that reconstructing a morphism of schemes from a morphism of étale topoi is easy if the target is a finite separable field extension of k.

3.2 Recollection. Let k be a field, let $L \supset k$ be a finite separable field extension, and write $Y = \mathrm{Spec}(L)$. We conflate notation and also write $Y \in \mathrm{Spec}(k)_{\text{ét}}$ for the associated representable sheaf. Then there is a natural equivalence $Y_{\text{ét}} \xrightarrow{\sim} (\mathrm{Spec}(k)_{\text{ét}})_{/Y}$ of topoi over $\mathrm{Spec}(k)_{\text{ét}}$. Let $q_* : \mathrm{Spec}(L)_{\text{ét}} \to \mathrm{Spec}(k)_{\text{ét}}$ denote the natural geometric morphism. In this case, the geometric morphisms into $Y_{\text{ét}}$ over k are easy to compute. Indeed by [SGA 4$_\mathrm{I}$, Exposé IV, Proposition 5.12], for any k-scheme $p : X \to \mathrm{Spec}(k)$, the functor

$$\mathrm{Hom}_k(X_{\text{ét}}, \mathrm{Spec}(L)_{\text{ét}}) \to \mathrm{Hom}_k(X, \mathrm{Spec}(L))$$

that sends a geometric morphism (t_*, α) to the morphism of schemes given by the composite

$$X \cong t^*(Y) \xrightarrow{t^*(\Delta)} t^*(Y \times_{\mathrm{Spec}(k)} Y) = t^* q^*(Y) \cong p^*(Y) = X \times_{\mathrm{Spec}(k)} Y \xrightarrow{\mathrm{pr}_2} Y$$

is an equivalence of groupoids. Furthermore it is easy to check that this isomorphism defines an inverse to the obvious map

$$\mathrm{Hom}_k(X, \mathrm{Spec}(L)) \to \mathrm{Hom}_k(X_{\text{ét}}, \mathrm{Spec}(L)_{\text{ét}}) \,.$$

3.3 Recollection (closed subtopoi). Let $\mathcal{X}$ be a topos and $U \in \mathrm{Open}(\mathcal{X})$ a subterminal object. Recall that the *closed complement* $\mathcal{Z} \subset \mathcal{X}$ of the open subtopos $\mathcal{X}_{/U}$ is the full subcategory of $\mathcal{X}$ spanned by those $\mathcal{F} \in \mathcal{X}$ such that the projection $\mathrm{pr}_2 : \mathcal{F} \times U \to U$ is an isomorphism. Furthermore if $e_* : \mathcal{W} \to \mathcal{X}$ is a geometric morphism such that $e^*(U) = \varnothing$ is the intial object, then there is a unique factorization

$$\mathcal{W} \xrightarrow{e_*} \mathcal{X}$$
$$e'_* \searrow \qquad \uparrow i_*$$
$$\mathcal{Z}$$

3.4 Remark. Let X be a scheme and let $Z \subset X$ a closed subscheme with open complement $U \subset X$. Then the closed complement of the open subtopos $U_{\text{ét}} \subset X_{\text{ét}}$ is equivalent to the topos $Z_{\text{ét}}$. (Note that this statement is true for any closed subscheme structure on $|Z|$.) Furthermore a geometric morphism $e_* : W_{\text{ét}} \to X_{\text{ét}}$ factors through the closed complement $Z_{\text{ét}}$ if and only if the map of topological spaces $|e_*| : |W| \to |X|$ factors through $|Z| \subset |X|$.

3.5 Lemma. *Let k be a perfect field and X a scheme locally topologically of finite type over k. Then for any closed point $x \in X$, the field extension $\kappa(x) \supset k$ is finite.*

Proof. By definition, there is a universal homeomorphism of k-schemes $f : X \to X'$ where X' is locally of finite type over k. Then $f(x) \in X'$ is closed and Proposition 2.15 shows that the field extension $\kappa(f(x)) \supset k$ is finite. Since k is perfect, the finite extension $\kappa(f(x))$ is also perfect. Since f is a universal homeomorphism, the induced extension on residue fields $\kappa(f(x)) \supset \kappa(x)$ is a universal homeomorphism. Since $\kappa(f(x))$ is perfect, [STK, Tag 0CNC] shows that the field extension $\kappa(f(x)) \supset \kappa(x)$ is an isomorphism. $\square$

3.6 Proposition. *Let k be a perfect field, $L \supset k$ an algebraic field extension, and Y a scheme topologically of finite type over k. Then the natural map*

$$Y(L) \to \mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(L)_{\mathrm{ét}}, Y_{\mathrm{ét}})$$

is bijective.

Proof. Consider the commutative triangle

$$Y(L) \longrightarrow \mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(L)_{\mathrm{ét}}, Y_{\mathrm{ét}})$$
$$\{\text{closed points of } Y\}$$

It suffices to see that the horizontal map becomes a bijection after taking fibers over each closed point $y_0 \in Y$. By Remark 3.4, the map on fibers is the natural map

$$\mathrm{Hom}_k(\mathrm{Spec}(L), \mathrm{Spec}(\kappa(y_0))) \to \mathrm{Hom}_k(\mathrm{Spec}(L)_{\mathrm{ét}}, \mathrm{Spec}(\kappa(y_0))_{\mathrm{ét}}) \,.$$

By Lemma 3.5, the residue field $\kappa(y_0)$ is a finite field extension of k; thus the claim follows from Recollection 3.2. $\square$

3.7 Observation. Let S be a scheme and $f, g \colon X \to Y$ morphisms of S-schemes. Let $T \to S$ be a faithfully flat map. Then by fpqc descent, $f = g$ if and only if $f_T = g_T$ as morphisms of T-schemes $X_T \to Y_T$.

In particular, since every extension of fields $K \supset k$ is faithfully flat, given morphisms of k-schemes $f, g \colon X \to Y$, we have $f = g$ if and only if $f_K = g_K$.
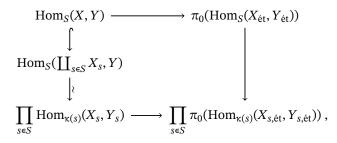
We now prove injectivity in general.

3.8 Notation. Given a groupoid $\mathcal{G}$, we write $\pi_0(\mathcal{G})$ for the set of isomorphism classes of objects of $\mathcal{G}$.

3.9 Theorem. *Let S be a qcqs scheme and let X and Y be schemes topologically of finite type over S. If X is reduced and all fibers of X over S are reduced, then the natural map*

$$\mathrm{Hom}_S(X, Y) \to \pi_0(\mathrm{Hom}_S(X_{\mathrm{ét}}, Y_{\mathrm{ét}}))$$

is injective.

Proof. Since X is reduced, the natural morphism $\coprod_{s \in S} X_s \to X$ is an epimorphism. Notice that we have a commutative square

$$\begin{array}{ccc}
\mathrm{Hom}_S(X, Y) & \longrightarrow & \pi_0(\mathrm{Hom}_S(X_{\mathrm{ét}}, Y_{\mathrm{ét}})) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_S(\coprod_{s \in S} X_s, Y) & & \\
\downarrow \wr & & \\
\prod_{s \in S} \mathrm{Hom}_{\kappa(s)}(X_s, Y_s) & \longrightarrow & \prod_{s \in S} \pi_0(\mathrm{Hom}_{\kappa(s)}(X_{s,\mathrm{ét}}, Y_{s,\mathrm{ét}})) \,,
\end{array}$$

where the right-hand vertical map is given by pulling back along the morphisms $\mathrm{Spec}(\kappa(s))_{\mathrm{ét}} \to S_{\mathrm{ét}}$, using Lemma 2.9. Since the left-hand vertical map is injective, to show that the top horizontal map is injective, it suffices to show that the bottom horizontal map is injective. Since all fibers of X over S are reduced, we may reduce to the case where $S = \mathrm{Spec}(k)$ is the spectrum of a field.

By Observation 3.7, we may furthermore assume that k is perfect. Since X is reduced and topologically of finite type over k, the natural morphism

$$\coprod_{x \in X_{\mathrm{cl}}} \mathrm{Spec}(\kappa(x)) \to X$$

is an epimorphism (Lemma 2.18). Thus Lemma 3.5 shows that we can reduce to the case where $X = \mathrm{Spec}(L)$ for some finite field extension $L \supset k$. In this case the claim follows from Proposition 3.6.   □

If S is furthermore Jacobson, Proposition 2.22 and Lemma 2.24 imply the following variant:

3.10 Corollary. *Let S be a qcqs Jacobson scheme and let X and Y be schemes topologically of finite type over S. If X is reduced and all fibers of X over S are reduced, then the natural map*

$$\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_S^{\mathrm{pin}}(X_{\text{ét}}, Y_{\text{ét}})$$

is injective.

3.11 Remark. Corollary 3.10 shows that in the equivalent formulations of étale reconstruction in Proposition 2.28, we can replace the word 'bijective' by 'surjective' and these statements are still equivalent to étale reconstruction.

4.   REDUCTION TO REGULAR SOURCE AND TARGET $\mathbf{G}_{\mathrm{m}}$

Let k be a field. The goal of this section is to prove that k satisfies étale reconstruction in the sense of Definition 2.29 if and only if for each affine, regular, connected, finite type k-scheme X such that $X(k) \neq \varnothing$, the natural map

$$\mathrm{Hom}_k(X^{\mathrm{awn}}, \mathbf{G}_{\mathrm{m}}) \to \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, \mathbf{G}_{\mathrm{m},\text{ét}})$$

is bijective. We accomplish this as follows. In §4.1, we prove that for a fixed k-scheme Y, the functor

$$X \mapsto \mathrm{Hom}_k(X^{\mathrm{awn}}, Y)$$

satisfies descent for Voevodsky's *h-topology* [29; 30]. In §4.2, we prove that the étale topos also satisfies h-descent. This implies that the functor

$$X \mapsto \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, Y_{\text{ét}})$$

is an h-sheaf (Corollary 4.12). By the theory of alterations, every scheme of finite type over a field admits an h-hypercover by regular finite type schemes; this allows us to reduce reconstruction to the case where X an affine, regular, finite type k-scheme.

Since one can extract the Zariski topological space of X from the étale topos $X_{\text{ét}}$, in order to reconstruct X, it suffices to reconstruct the structure sheaf $\mathcal{O}_X$, i.e., to prove reconstruction when $Y = \mathbf{A}_k^1$. With a little more work, we show that it actually suffices to treat the case $Y = \mathbf{G}_{\mathrm{m}}$ and X is connected and has a rational point. Subsection 4.3 makes this informal argument precise. There we also show the more refined statement that k satisfies étale reconstruction if and only if for each a pinned geometric morphism $\phi_* : X_{\text{ét}}^{\mathrm{awn}} \to \mathbf{G}_{\mathrm{m},\text{ét}}$, there is a morphism of k-schemes $X^{\mathrm{awn}} \to \mathbf{G}_{\mathrm{m}}$ that agrees with ϕ_* on $\bar{k}$-points.

We note that the idea that it suffices to prove reconstruction for X affine and $Y = \mathbf{G}_{\mathrm{m}}$ appeared in Voevodsky's work [28].

4.1   h-descent for morphisms from absolutely weakly normal schemes. The purpose of this subsection is to prove the following:

4.1 Theorem. *Let S be a scheme and let Y be an S-scheme. Then the presheaf* $\mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ *given by*

$$X \mapsto \mathrm{Hom}_S(X^{\mathrm{awn}}, Y)$$

is an h-sheaf.

For finite type schemes over a noetherian scheme of characteristic 0 this was proven in [30, Theorem 3.2.9]. The general statement is an easy consequence of the case $Y = \mathbf{A}_S^1$:

4.2 Proposition [STK, Tag 0EVU]. *Let S be a scheme. Then the presheaf* $\mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ *given by*

$$X \mapsto \mathrm{Hom}_S(X^{\mathrm{awn}}, \mathbf{A}_S^1)$$

is an h-sheaf.

Proof of Theorem 4.1. We want to show that for any h-cover $W \to X$, the natural map

$$\mathrm{Hom}_S(X^{\mathrm{awn}}, Y) \to \lim \left(\mathrm{Hom}_S(W^{\mathrm{awn}}, Y) \rightrightarrows \mathrm{Hom}_S((W \times_X W)^{\mathrm{awn}}, Y) \right)$$

is an isomorphism. Using Lemma 1.18, we may work Zariski locally on Y and therefore assume that Y and S are both affine. Furthermore, since absolute weak normalization preserves Zariski covers (Lemma 1.18), the functor

$$X \mapsto \mathrm{Hom}_S(X^{\mathrm{awn}}, Y)$$

is a Zariski sheaf. Hence we may assume that $X = \mathrm{Spec}(A)$ and $W = \mathrm{Spec}(B)$ are affine. Now Proposition 4.2 implies that the natural ring homomorphism

$$A^{\mathrm{awn}} \to \lim \left(B^{\mathrm{awn}} \rightrightarrows (B \otimes_A B)^{\mathrm{awn}} \right)$$

is an isomorphism, which implies our claim. $\qquad\square$

4.3 Example. Let S be a scheme. Then the presheaf $\mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ given by $X \mapsto \mathrm{Hom}_S(X^{\mathrm{awn}}, \mathbf{G}_\mathrm{m})$ is an h-sheaf.

4.2 h-descent for the étale topos. We now prove that the étale topos satisfies h-descent. We make use of the following general result.

4.4 Notation. We write $\mathbf{Cat}$ for the $(2, 1)$-category with objects locally small categories, 1-morphisms functors, and 2-morphisms natural isomorphisms. Write $\mathbf{Gpd} \subset \mathbf{Cat}$ for the full subcategory spanned by the groupoids. We write $\mathbf{LTop}$ for the $(2, 1)$-category with objects topoi, 1-morphisms left exact left adjoints, and 2-morphisms natural isomorphisms.

4.5 Recollection. The inclusion $\mathbf{LTop} \to \mathbf{Cat}$ preserves limits. See [HTT, Proposition 6.3.2.3] for the proof of this result in the setting of ∞-topoi. In the setting of ordinary topoi, the proof is verbatim the same.

4.6 Proposition. *Let $\mathcal{C}^\bullet : \Delta_+ \to \mathbf{Cat}$ be an augmented cosimplicial category. Let $G : \mathcal{C}^{-1} \to \mathcal{C}^0$ denote the agumentation. Assume that:*

(4.6.1) *The category $\mathcal{C}^{-1}$ admits limits of G-split equalizer diagrams, and those equalizers are preserved by G.*

(4.6.2) *For every morphism $\alpha : [m] \to [n]$ in Δ_+, the square*

$$
\begin{array}{ccc}
\mathcal{C}^m & \xrightarrow{\ d^0\ } & \mathcal{C}^{m+1} \\
{\scriptstyle \alpha^*}\downarrow & & \downarrow{\scriptstyle ([0]\star\alpha)^*} \\
\mathcal{C}^n & \xrightarrow[\ d^0\ ]{} & \mathcal{C}^{n+1}
\end{array}
$$

is horizontally right adjointable. (In particular, this requires that the coface functors d^0 be left adjoints.)

Then the natural functor $\theta : \mathcal{C}^{-1} \to \lim_{[n]\in\Delta} \mathcal{C}^n$ admits a fully faithful right adjoint. Moreover, if G is conservative, then θ is an equivalence of categories.

Proof. The ∞-categorical version of this statement is the content of [HA, Corollary 4.7.5.3]. Since the inclusion $\mathbf{Cat} \hookrightarrow \mathbf{Cat}_\infty$ of ordinary categories into ∞-categories is a right adjoint, the statement here is an immediate consequence of the ∞-categorical statement. $\qquad\square$

4.7 Remark. Proposition 4.6 is also a consequence of old results of Beck (ubpublished) and Bénabou–Roubaud [3]. See [16, p. 270]. However, these works do not give the explicit statement we use; Lurie's result is the ∞-categorical generalization of the exact result we use.

4.8 Proposition. *The functor $(-)_{\mathrm{ét}} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{LTop}$ with pullback functoriality is an h-sheaf.*

Proof. Since the assignment $X \mapsto X_{\mathrm{ét}}$ is a Zariski sheaf, all that remains to be verified is that $X \mapsto X_{\mathrm{ét}}$ satisfies descent for proper surjections. Let $f : X \to Y$ be a proper surjective morphism of schemes; we need to show that the induced augmented cosimplicial diagram of categories

(4.9) $$Y_{\mathrm{ét}} \xrightarrow{\ f^*\ } X_{\mathrm{ét}} \rightrightarrows (X \times_Y X)_{\mathrm{ét}} \substack{\longrightarrow\\[-0.6em]\longrightarrow\\[-0.6em]\longrightarrow} \cdots$$

is a limit diagram. We prove this by showing that (4.9) satisfies the hypotheses of Proposition 4.6. For (4.6.1), note that as a pullback functor, f^* is left exact. Moreover, since f is surjective, the functor f^* is conservative, so all that remains to be shown is (4.6.2).

For (4.6.2), notice that relevant square of categories is given by applying $(-)_{\text{ét}}$ to the pullback square of schemes

$$
\begin{array}{ccc}
X \times_Y X^{\times_Y[n]} & \xrightarrow{\text{pr}_2} & X^{\times_Y[n]} \\
\text{id}_X \times \alpha \downarrow & & \downarrow \alpha \\
X \times_Y X^{\times_Y[m]} & \xrightarrow[\text{pr}_2]{} & X^{\times_Y[m]} \, .
\end{array}
$$

Here the vertical morphisms are induced by $\alpha : [m] \to [n]$. Also notice that since f is proper and proper morphisms are stable under pullback, for each $\ell \geq -1$, the projection morphism

$$\text{pr}_2 : X \times_Y X^{\times_Y[\ell]} \to X^{\times_Y[\ell]}$$

is proper. By the proper basechange theorem for étale sheaves of *sets* [SGA 4$_{\text{II}}$, Exposé XII, Théorème 5.1(i)], the induced square of étale topoi

$$
\begin{array}{ccc}
(X^{\times_Y[m]})_{\text{ét}} & \xrightarrow{\text{pr}_2^*} & (X \times_Y X^{\times_Y[m]})_{\text{ét}} \\
(\text{id}_X \times \alpha)^* \downarrow & & \downarrow \alpha^* \\
(X \times_Y X^{\times_Y[n]})_{\text{ét}} & \xrightarrow[\text{pr}_2^*]{} & (X^{\times_Y[n]})_{\text{ét}} \, .
\end{array}
$$

is horizontally right adjointable, as desired. $\qquad\square$

4.10 Remark. Rydh has proven that the quasicompact étale site and quasicompact separated étale site both satisfy h-descent [22, Corollaries 5.15 & 5.16]; one can use this to show that the étale topos satisfies h-descent. However, our proof method is quite different and, given Proposition 4.6, more direct.

4.11 Remark (arc-descent for the étale topos)**.** One can also show that the the étale topos satisfies descent for the finer *arc-topology* introduced by Bhatt and Mathew [4]. Given Proposition 4.8, the proof of this is exactly the same as the proof of the analogous result for the derived ∞-category of constructible étale sheaves [4, Theorem 5.13].

4.12 Corollary. *Let k be a field and let Y be a k-scheme. Then the functor $\textbf{Sch}_k^{\text{op}} \to \textbf{Gpd}$ given by*

$$X \mapsto \text{Hom}_k^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$$

is an h-sheaf. Furthermore, when restricted to schemes topologically of finite type over k, it is an h-sheaf of sets.

Proof. By Proposition 4.8, it follows that $\text{Hom}_k((-)_{\text{ét}}, Y_{\text{ét}})$ is an h-sheaf of groupoids. Since $\text{Hom}_k^{\text{pin}}((-)_{\text{ét}}, Y_{\text{ét}})$ is a subfunctor of the former, we only need to show that a geometric morphism $f_* : X_{\text{ét}} \to Y_{\text{ét}}$ is pinned if there is an h-cover $p : W \to X$ such that $(f \circ p)_*$ is pinned. For this it suffices to see that p is surjective on closed points. This holds since the fiber over every closed point is nonempty an qcqs and therefore has a closed point. The 'furthermore' part is a consequence of Proposition 2.22. $\qquad\square$

4.3 Reduction to the regular case. In this subsection, we show that a field k satisfies étale reconstruction if and only if for each affine, regular, connected, finite type k-scheme X with a rational point and pinned geometric morphism $\phi_* : X_{\text{ét}}^{\text{awn}} \to \textbf{G}_{\text{m,ét}}$, there exists a morphism of k-schemes $X^{\text{awn}} \to \textbf{G}_{\text{m,ét}}$ that agrees with ϕ_* on $\bar{k}$-points.

We begin with a technical proposition needed in our proof. To prove this, we state some facts about the étale topos proven in Appendix A.

4.13. Let X be a scheme. We write $\acute{\textbf{E}}\textbf{t}_X^{\text{qcs}} \subset \acute{\textbf{E}}\textbf{t}_X$ for the full subcategory spanned by those $p : X' \to X$ such that p is quasicompact and separated. If X is quasiseparated, then restriction along the inclusion defines an equivalence of categories

$$\text{Sh}_{\text{ét}}(\acute{\textbf{E}}\textbf{t}_X) \xrightarrow{\sim} \text{Sh}_{\text{ét}}(\acute{\textbf{E}}\textbf{t}_X^{\text{qcs}})$$

See Lemma A.4. Moreover, if X and Y are topologically noetherian schemes, then for every geometric morphism $f_* \colon X_{\text{ét}} \to Y_{\text{ét}}$, the functor f^* sends $\mathbf{\acute{E}t}_Y^{\text{qcs}}$ to $\mathbf{\acute{E}t}_X^{\text{qcs}}$. Hence f_* is induced by the morphism of sites $f^* \colon \mathbf{\acute{E}t}_Y^{\text{qcs}} \to \mathbf{\acute{E}t}_X^{\text{qcs}}$. See Corollary A.9.

The following is a slight generalization of [28, Proposition 2.3]. For convenience of the reader we also spell out a proof.

4.14 Proposition. *Let k be a field and let X and Y be schemes topologically of finite type over k; also assume that X is reduced. Let $(f_*, \alpha) \colon X_{\text{ét}} \to Y_{\text{ét}}$ be a pinned geometric morphism of topoi over $\operatorname{Spec}(k)_{\text{ét}}$. Assume that for each $V \in \mathbf{\acute{E}t}_Y^{\text{qcs}}$, there exists a morphism of k-schemes $\phi_V \colon f^*(V) \to V$ such that for any morphism of k-schemes $x \colon \operatorname{Spec}(\bar{k}) \to f^*(V)$ we have*

$$(f|_V)_* \circ x_* \cong \phi_{V,*} \circ x_*$$

in $\operatorname{Hom}_k^{\text{pin}}(\operatorname{Spec}(\bar{k})_{\text{ét}}, Y_{\text{ét}})$. Then $f_* \cong \phi_{Y,*}$ in $\operatorname{Hom}_k^{\text{pin}}(X_{\text{ét}}, Y_{\text{ét}})$.

Proof. First note that, as just explained, $f^*(V)$ is representable by an étale X-scheme. Now observe that by assumption the square

(4.15)
$$
\begin{array}{ccc}
f^*(V) & \xrightarrow{\phi_V} & V \\
\downarrow & & \downarrow \\
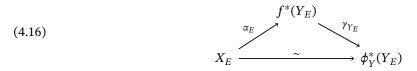X & \xrightarrow{\phi_Y} & Y
\end{array}
$$

commutes after precomposing with any geometric point $\operatorname{Spec}(\bar{k}) \to f^*(V)$. Since X is reduced, $f^*(V)$ is also reduced; hence it follows that the square (4.15) commutes. The universal property of the pullback induces a morphism of étale X-schemes
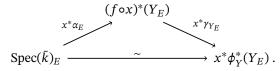
$$\gamma_V \colon f^*(V) \to \phi_Y^*(V) .$$

In particular, γ_V is quasicompact and étale. The same reasoning shows that $(\gamma_V)_{V \in \mathbf{\acute{E}t}_Y^{\text{qcs}}}$ defines a natural transformation $\gamma \colon f^* \to \phi_Y^*$.

We claim that each γ_V is an isomorphism. For this it suffices to see that γ_V induces an isomorphism on geometric fibers over closed points of X. Indeed, because X is Jacobson, this implies that γ_V is an étale surjective monomorphism and thus an isomorphism by [STK, Tag 06NC]. But for a morphism $x \colon \operatorname{Spec}(\bar{k}) \to X$ we have by assumption $x^* f^* \cong (x \circ \phi_Y)^*$ and therefore γ induces an endomorphism of the stalk functor $(x \circ \phi_Y)^*$. But by the description of the category of points given recalled in Example 2.7, any endomorphism of a point of the étale topos is an isomorphism. In particular, γ is an isomorphism.

To conclude the proof, we only need to check that the isomorphism γ is compatible with α, i.e., that γ defines an 2-isomorphism in $\mathbf{LTop}_k$. Explicitly we have to check that for any finite separable field extension E of k the triangle of X-schemes

(4.16)
$$
\begin{array}{ccc}
 & f^*(Y_E) & \\
{}^{\alpha_E}\nearrow & & \searrow^{\gamma_{Y_E}} \\
X_E & \xrightarrow{\ \sim\ } & \phi_Y^*(Y_E)
\end{array}
$$

commutes. It again suffices to check that for any geometric point $x \colon \operatorname{Spec}(\bar{k}) \to X$ whose image is a closed point of Y, the triangle (4.16) commutes after applying x^*. The induced triangle reads

$$
\begin{array}{ccc}
 & (f \circ x)^*(Y_E) & \\
{}^{x^* \alpha_E}\nearrow & & \searrow^{x^* \gamma_{Y_E}} \\
\operatorname{Spec}(\bar{k})_E & \xrightarrow{\ \sim\ } & x^* \phi_Y^*(Y_E) .
\end{array}
$$

Therefore, it suffices to see that the natural isomorphism $x^* \gamma$ defines a natural isomorphism in $\mathbf{LTop}_k$. For this we apply Lemma 4.17 below to the geometric morphism $f_* \circ x_*$, the map of schemes $\phi_Y \circ x$, the natural
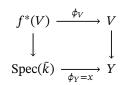
isomorphism $\gamma_x : x^* \circ f^* \xRightarrow{\sim} x^* \phi_Y^*$ that we are given by assumption, and the collection of maps

$$\tilde{\phi}_V : x^* f^*(V) \longrightarrow f^*(V) \xrightarrow{\phi_V} V \,,$$

where the left-hand map is the projection from the pullback. For this we have to verify that the assumptions of Lemma 4.17 below are satisfied. The only nontrivial point that we have to verify is that there are natural isomorphisms $x^* f|_V^* \cong \tilde{\phi}_V^*$ for any $V \in \acute{\mathbf{E}}\mathbf{t}_Y^{\mathrm{qcs}}$. But $x^* f^*(V)$ is just a disjoint union of spectra of separably closed fields, and by assumption we have such a natural isomorphism for every connected component, giving us the desired natural transformation in general. Thus Lemma 4.17 implies that $x^* \gamma = \gamma_x$. In particular $x^* \gamma$ is a natural isomorphism in $\mathbf{LTop}_k$, because γ_x is by assumption. $\qquad\square$
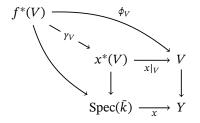
4.17 Lemma. *Let k be a field and let Y be scheme topologically of finite type over k. Let $f_* : \mathrm{Spec}(\bar{k})_{\acute{e}t} \to Y_{\acute{e}t}$ be a pinned geometric morphism, $x : \mathrm{Spec}(\bar{k}) \to Y$ a morphism of k-schemes and $\gamma : x_* \xRightarrow{\sim} f_*$ an isomorphism in $\mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(\bar{k})_{\acute{e}t}, Y_{\acute{e}t})$. Assume that we are given a natural collection of maps $(\phi_V : f^*(V) \to V)_{V \in \acute{\mathbf{E}}\mathbf{t}_Y^{\mathrm{qcs}}}$ such that:*

(4.17.1) We have $\phi_Y = x$.

(4.17.2) We have natural isomorphisms $f|_V^* \xRightarrow{\sim} \phi_V^*$ in $\mathrm{Hom}_k^{\mathrm{pin}}(f^*(V)_{\acute{e}t}, V_{\acute{e}t})$.

Then for each $V \in \acute{\mathbf{E}}\mathbf{t}_Y^{\mathrm{qcs}}$, the map $\tilde{\gamma}_V : f^*(V) \to x^*(V)$ induced by the naturality square

$$
\begin{array}{ccc}
f^*(V) & \xrightarrow{\phi_V} & V \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\bar{k}) & \xrightarrow{\phi_Y = x} & Y
\end{array}
$$

and the universal property of the pullback agrees with $\gamma_V : f^*(V) \xRightarrow{\sim} x^*(V)$.

Proof. By the universal property of the pullback we only have to check that the diagram of schemes

commutes. The only non-obvious equation we have to check for this is $\phi_V = x|_V \circ \gamma_V$. For this, by Proposition 3.6 it suffices to show that the associated diagram in $\mathbf{RTop}_k$ commutes.

By assumption (4.17.2) on the maps ϕ_V, it suffices to provide a natural isomorphism

$$\delta : f|_V^* \xRightarrow{\sim} \gamma_V^* \circ x|_V^*$$

in $\mathbf{LTop}_k$. For any quasicompact separated étale morphism $T \to V$, the commutative square

$$
\begin{array}{ccc}
f^*(T) & \xrightarrow[\sim]{\gamma_T} & x^*(T) \\
\downarrow & & \downarrow \\
f^*(V) & \xrightarrow[\gamma_V]{\sim} & x^*(V)
\end{array}
$$

induces an isomorphism

$$\delta_T : f^*(T) \to f^*(V) \times_{x^*(V)} x^*(T) \,.$$

Finally, it is easy to check that the morphisms $(\delta_T)_{T \in \acute{\mathbf{E}}\mathbf{t}_V^{\mathrm{qcs}}}$ define a natural isomorphism in $\mathbf{LTop}_k$, because γ does. $\qquad\square$

We are now ready to prove the main result of this section. Our proof makes use of the h-descent results of §§4.1 and 4.2 and the following observation.

4.18 Theorem (reduction to regular source and target $\mathbf{G}_{\mathrm{m}}$). *Let k be a field. Then the following statements are equivalent:*

(4.18.1) *The field k satisfies étale reconstruction in the sense of Definition 2.29. That is, for all finite type k-schemes X and Y, the natural map*

$$\mathrm{Hom}_k(X^{\mathrm{awn}}, Y) \to \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, Y_{\text{ét}})$$

is bijective.

(4.18.2) *For all finite type k-schemes X and Y, such that X is affine, regular, connected, and $X(k) \neq \varnothing$, the natural map*

$$\mathrm{Hom}_k(X^{\mathrm{awn}}, Y) \to \mathrm{Hom}_k^{\mathrm{pin}}(X_{\text{ét}}^{\mathrm{awn}}, Y_{\text{ét}})$$

is bijective.

(4.18.3) *Let X and Y be finite type k-schemes, such that X is affine, regular, connected, and $X(k) \neq \varnothing$. Then for any pinned geometric morphism $\phi_* \colon X_{\text{ét}} \to Y_{\text{ét}}$ over $\mathrm{Spec}(k)_{\text{ét}}$, there exists a morphism of k-schemes $f \colon X^{\mathrm{awn}} \to Y$ such that the induced map $f(\bar{k}) \colon X^{\mathrm{awn}}(\bar{k}) \to Y(\bar{k})$ agrees with $\phi_*(\bar{k})$.*

(4.18.4) *Let X be an affine, regular, connected, finite type k-scheme such that $X(k) \neq \varnothing$. Then for any pinned geometric morphism $\phi_* \colon X_{\text{ét}}^{\mathrm{awn}} \to \mathbf{A}_{\text{ét}}^1$ over $\mathrm{Spec}(k)_{\text{ét}}$, there exists a morphism of k-schemes $f \colon X^{\mathrm{awn}} \to \mathbf{A}^1$ such that the induced map*

$$f(\bar{k}) \colon X^{\mathrm{awn}}(\bar{k}) \to \bar{k}$$

agrees with $\phi_*(\bar{k})$.

(4.18.5) *Let X be an affine, regular, connected, finite type k-scheme such that $X(k) \neq \varnothing$. Then for any pinned geometric morphism $\phi_* \colon X_{\text{ét}}^{\mathrm{awn}} \to \mathbf{G}_{\mathrm{m},\text{ét}}$ over $\mathrm{Spec}(k)_{\text{ét}}$, there exists a morphism of k-schemes $f \colon X^{\mathrm{awn}} \to \mathbf{G}_{\mathrm{m}}$ such that the induced map*

$$f(\bar{k}) \colon X^{\mathrm{awn}}(\bar{k}) \to \bar{k}^{\times}$$

agrees with $\phi_*(\bar{k})$.

Proof. Clearly (4.18.1) $\Rightarrow$ (4.18.2). To see that (4.18.2) $\Rightarrow$ (4.18.1), note that by Theorem 4.1 and Corollary 4.12 both the source and target of the map in (4.18.1) satisfy h-descent in X. Thus we may work h-locally and by the theory of alterations [14, Theorem 1.1; 15, Exposé IX, Théorème 1.1; 13, Theorem 4.4; 26, Theorem 1.2.5] we may reduce to the case where X is regular. By further working Zariski locally, we may reduce to the case where X is affine. Passing to a finite extension of k if necessary, we may assume that X is connected and $X(k) \neq \varnothing$. This completes the proof that (4.18.2) $\Rightarrow$ (4.18.1).

Clearly (4.18.2) implies (4.18.3) and the converse follows from Corollary 3.10 combined with Proposition 4.14. Clearly (4.18.3) implies (4.18.4) and (4.18.5). Since we can cover $\mathbf{A}^1$ by two copies of $\mathbf{G}_{\mathrm{m}}$, it follows that if we manage to construct the desired morphism of k-schemes over both copies of $\mathbf{G}_{\mathrm{m}}$, we can reconstruct it over $\mathbf{A}^1$: since the two maps agree on $X(\bar{k})$ by assumption, they will automatically agree over the intersection. Thus (4.18.5) implies (4.18.4).

To finish the proof it thus suffices to show that (4.18.4) $\Rightarrow$ (4.18.3). We may work Zariski locally on both the source and target to assume that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine. Let $\phi_* \colon X_{\text{ét}} \to Y_{\text{ét}}$ be a pinned geometric morphism. Define a map of sets

$$\psi \colon B = \mathrm{Hom}_k(Y, \mathbf{A}^1) \longrightarrow \mathrm{Hom}_k(X^{\mathrm{awn}}, \mathbf{A}^1) = A^{\mathrm{awn}}$$

by sending $b \colon Y \to \mathbf{A}_k^1$ to the unique element $\psi(b) \colon X^{\mathrm{awn}} \to \mathbf{A}^1$ such that $\psi(b)(\bar{k}) = b_* \circ \phi_*(\bar{k})$. Indeed, we assumed that such an element $\psi(b)$ exists and it is clearly unique. We claim that $\psi \colon B \to A^{\mathrm{awn}}$ is a ring

homomorphism. In other words we have to see that the square

$$
\begin{CD}
\mathrm{Hom}_k(Y, \mathbf{A}^1 \times \mathbf{A}^1) @>{\mathrm{mult}_*}>> \mathrm{Hom}_k(Y, \mathbf{A}^1) \\
@V{\psi \times \psi}VV @VV{\psi}V \\
\mathrm{Hom}_k(X^{\mathrm{awn}}, \mathbf{A}^1 \times \mathbf{A}^1) @>>{\mathrm{mult}_*}> \mathrm{Hom}_k(X^{\mathrm{awn}}, \mathbf{A}^1)
\end{CD}
$$

where mult : $\mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$ is the multiplication map, commutes and similarly for the addition map. Both cases are exactly the same, so we just explain the case of the multiplication map.

For this, by Proposition 2.15 and Lemma 2.18 we may assume that $X = \mathrm{Spec}(\bar{k})$. Now observe that if we show that the square

(4.19)
$$
\begin{CD}
\mathrm{Hom}_k(Y, \mathbf{A}^1 \times \mathbf{A}^1) @>>> \mathrm{Hom}_k^{\mathrm{pin}}(Y_{\mathrm{\acute{e}t}}, (\mathbf{A}^1 \times \mathbf{A}^1)_{\mathrm{\acute{e}t}}) \\
@V{\psi \times \psi}VV @VV{- \circ \phi_*}V \\
\mathrm{Hom}_k(\mathrm{Spec}(\bar{k}), \mathbf{A}^1 \times \mathbf{A}^1) @>{\sim}>> \mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(\bar{k})_{\mathrm{\acute{e}t}}, (\mathbf{A}^1 \times \mathbf{A}^1)_{\mathrm{\acute{e}t}})
\end{CD}
$$

commutes, then the claim easily follows from Corollary 3.10. A consequence of Proposition 3.6 the canonical map

$$
\mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(\bar{k})_{\mathrm{\acute{e}t}}, (\mathbf{A}^1 \times \mathbf{A}^1)_{\mathrm{\acute{e}t}}) \to \mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(\bar{k})_{\mathrm{\acute{e}t}}, \mathbf{A}^1_{\mathrm{\acute{e}t}}) \times \mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(\bar{k})_{\mathrm{\acute{e}t}}, \mathbf{A}^1_{\mathrm{\acute{e}t}})
$$

induced by the projections is an isomorphism. Thus it suffices to see that the square (4.19) commutes after composing with the two projections $\mathbf{A}^1 \times \mathbf{A}^1 \to \mathbf{A}^1$. This is immediate from the definition of ψ.

It follows that ψ is indeed a map of rings and therefore induces a morphism of schemes $f : X^{\mathrm{awn}} \to Y$. All that is left to show is that for any morphism of k-schemes $x : \mathrm{Spec}(\bar{k}) \to X^{\mathrm{awn}}$. we have $\phi_* \circ x_* \cong f_* \circ x_*$. In other words, we have to show that the square

$$
\begin{CD}
\mathrm{Hom}_k(\mathrm{Spec}(\bar{k}), X^{\mathrm{awn}}) @>{\sim}>> \mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(\bar{k})_{\mathrm{\acute{e}t}}, X_{\mathrm{\acute{e}t}}) \\
@V{f \circ -}VV @VV{\phi_* \circ -}V \\
\mathrm{Hom}_k(\mathrm{Spec}(\bar{k}), Y) @>{\sim}>> \mathrm{Hom}_k^{\mathrm{pin}}(\mathrm{Spec}(\bar{k})_{\mathrm{\acute{e}t}}, Y_{\mathrm{\acute{e}t}})
\end{CD}
$$

commutes. Note that since Y is affine it suffices to prove this after composing with any morphism of k-schemes $b : Y \to \mathbf{A}^1$. Then the claim is clear from the construction of ψ. □

4.20 Remark. For the proof of Theorem 0.1, it is not necessary to use the theory alterations. Indeed, using that any finite type k-scheme admits a finite surjection from its normalization, the proof of Theorem 4.18 also shows that it suffices to prove the conclusion of (4.18.5) for normal schemes. But by Proposition 5.22 the proof given in §6 works more generally for normal schemes over finitely generated fields.

5. AMENABLE FIELDS

In §4, we proved that a field k satisfies étale reconstruction if and only if for each affine, regular, connected, finite type k-scheme X with a rational point and pinned geometric morphism $\phi_* : X^{\mathrm{awn}}_{\mathrm{\acute{e}t}} \to \mathbf{G}_{\mathrm{m,\acute{e}t}}$, there exists a morphism of k-schemes $X^{\mathrm{awn}} \to \mathbf{G}_{\mathrm{m,\acute{e}t}}$ that agrees with ϕ_* on $\bar{k}$-points. In §6, we will check this criterion for infinite fields that are finitely over their prime fields. However, the argument we give works over more general fields. The idea is to relate both sides to the étale cohomology group

$$
\mathrm{H}^1_{\mathrm{\acute{e}t}}(X; \mathbf{G}_{\mathrm{m}}) \cong \mathrm{Pic}(X)
$$

and prove that the desired map is surjective by an an obstruction-theoretic argument. The purpose of this section is to introduce the class of fields that for which our argument works; we call these *amenable* fields.

In order to formulate the condition of amenability properly, we need to avoid some bad behavior that happens at the characteristic. To do this, in §5.1, we introduce variants of divisibility, profinite completion,

Tate modules, and étale cohomology with coefficients in $\hat{\mathbf{Z}}(1)$ that are all away from a prime. In §5.2, we introduce amenable fields and show that finitely generated fields are amenable.

5.1 Prime-to-p Kummer theory. We begin by introducing prime-to-p divisibility.

5.1 Notation. Let p be a prime number or 1. We write $\mathbf{N}_{p'} \subset \mathbf{N}$ for the set of positive integers n such that $\gcd(n, p) = 1$. Throughout, we regard $\mathbf{N}_{p'}$ as a poset ordered by divisibility.

5.2 Definition. Let p be a prime number or 1, and let A be an abelian group. We say that an element $a \in A$ is *p'-divisible* (or *prime-to-p-divisible*) if for each $n \in \mathbf{N}_{p'}$, there exists an element $b \in A$ such that $nb = a$. We say that A is *p'-divisible* if every element of A is p'-divisible.

5.3 Observation. If $p = 1$, then a p'-divisible abelian group is exactly a *divisible* abelian group.

5.4 Notation. Let A be an abelian group (or group scheme).

(5.4.1) For each integer $n \geq 1$, write $A[n] \subset A$ for the kernel of multiplication by n, i.e., the *n-torsion* in A.

(5.4.2) Let p be a prime number or 1. The *p'-Tate module* of A is the limit

$$\mathrm{T}_{p'}(A) := \lim_{n \in (\mathbf{N}_{p'})^{\mathrm{op}}} A[n] \,.$$

For m dividing n, the transition map $A[n] \to A[m]$ is given by multiplication by n/m.

5.5 Observation. Let p be a prime number or 1. If an abelian group A has no nonzero p'-divisible torsion elements, then $\mathrm{T}_{p'}(A) = 0$.

5.6 Lemma. *Let p be a prime number or 1 and let A be an abelian group. Then:*

(5.6.1) *For each $n \in \mathbf{N}_{p'}$, the natural homomorphism $A \to A[1/p]$ induces an isomorphism*

$$A[n] \xrightarrow{\sim} (A[1/p])[n]$$

on n-torsion subgroups.

(5.6.2) *The natural homomorphism $A \to A[1/p]$ induces an isomorphism*

$$\mathrm{T}_{p'}(A) \xrightarrow{\sim} \mathrm{T}_{p'}(A[1/p]) \,.$$

Proof. For (5.6.1), observe that since $\mathbf{Z}[1/p]$ is flat over $\mathbf{Z}$, we have

$$(A[n])[1/p] \cong A[n] \otimes_{\mathbf{Z}} \mathbf{Z}[1/p] \cong (A[1/p])[n] \,.$$

So it suffices to show that p acts invertibly on $A[n]$. For this, note that $A[n]$ is a $\mathbf{Z}/n$-module and since p is coprime to n, the class of p in $\mathbf{Z}/n$ is a unit. Item (5.6.2) is an immediate consequence of (5.8.1) and the definition of the p'-Tate module. $\square$

Now we study completions of abelian groups away from a prime.

5.7 Notation. Let p be a prime number or 1. For an abelian group A, the *p'-completion* of A is the limit

$$A^{\wedge}_{p'} := \lim_{n \in (\mathbf{N}_{p'})^{\mathrm{op}}} A/n \,,$$

where the transition maps are the quotient maps. That is, $A^{\wedge}_{p'}$ is the maximal prime-to-p quotient of the profinite completion of A. We write

$$\hat{\mathbf{Z}}_{p'} := \mathbf{Z}^{\wedge}_{p'} \cong \prod_{\substack{\text{primes} \\ \ell \neq p}} \mathbf{Z}_{\ell} \,.$$

5.8 Lemma. *Let p be a prime number or 1 and let A be an abelian group. Then:*

(5.8.1) *For each $n \in \mathbf{N}_{p'}$, the natural homomorphism $A \to A[1/p]$ induces an isomorphism $A/n \xrightarrow{\sim} A[1/p]/n$.*

(5.8.2) *The natural homomorphism $A \to A[1/p]$ induces an isomorphism $A^{\wedge}_{p'} \xrightarrow{\sim} A[1/p]^{\wedge}_{p'}$.*

Proof. For (5.8.1), observe that

$$A[1/p]/n \cong (A/n) \otimes_{\mathbf{Z}} \mathbf{Z}[1/p] \cong (A/n)[1/p] \,,$$

so it suffices to show that p acts invertibly on A/n. For this, note that A/n is a $\mathbf{Z}/n$-module and since p is coprime to n, the class of p in $\mathbf{Z}/n$ is a unit. Item (5.8.2) is an immediate consequence of (5.8.1) and the definition of p'-completion. □

5.9. Let k be a field of exponential characteristic p and let X be a k-scheme. Then for any $n \in \mathbf{N}_{p'}$, the Kummer sequence provides a natural short exact sequence

(5.10) $$0 \longrightarrow \mathcal{O}^{\times}(X)/n \longrightarrow \mathrm{H}^1_{\text{ét}}(X; \boldsymbol{\mu}_n) \longrightarrow \mathrm{Pic}(X)[n] \longrightarrow 0 \,.$$

Here, $\boldsymbol{\mu}_n$ is the étale sheaf of n-th roots of unity.

5.11 Definition (the p'-Kummer map). Let k be a field of exponential characteristic p and let X be a k-scheme. Write

$$\mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1)) \coloneqq \lim_{n \in (\mathbf{N}_{p'})^{\text{op}}} \mathrm{H}^1_{\text{ét}}(X; \boldsymbol{\mu}_n) \,.$$

We then define the p'-*Kummer map*

$$\mathrm{Kum} : \ \mathcal{O}^{\times}(X)^{\wedge}_{p'} \longrightarrow \mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1))$$

as the limit of the maps $\mathcal{O}^{\times}(X)/n \to \mathrm{H}^1_{\text{ét}}(X; \boldsymbol{\mu}_n)$ for $n \in \mathbf{N}_{p'}$, coming from the Kummer sequence.

5.12 Observation (functoriality of $\mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1))$). Let k be a field of exponential characteristic p and let X and Y be k-schemes. Then for each positive integer n coprime to p, the sheaves $\boldsymbol{\mu}_{n,X}$ and $\boldsymbol{\mu}_{n,Y}$ are the pullbacks of $\boldsymbol{\mu}_{n,\mathrm{Spec}(k)}$. As a consequence, for any geometric morphism $f_* : X_{\text{ét}} \to Y_{\text{ét}}$ over $\mathrm{Spec}(k)_{\text{ét}}$ (not necessarily induced by a morphism of schemes), we have $f^*(\boldsymbol{\mu}_{n,Y}) \cong \boldsymbol{\mu}_{n,X}$. Hence for each $i \geq 0$, the assignment $X \mapsto \mathrm{H}^i_{\text{ét}}(X; \boldsymbol{\mu}_n)$ is functorial in geometric morphisms of étale topoi over $\mathrm{Spec}(k)_{\text{ét}}$. Thus the assignment

$$X \mapsto \mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1))$$

is also functorial in geometric morphisms of étale topoi over $\mathrm{Spec}(k)_{\text{ét}}$.

5.13 Observation. Let k be a field of exponential characteristic p and let X be a k-scheme. For m dividing n, the transition map $\mathcal{O}^{\times}(X)/n \to \mathcal{O}^{\times}(X)/m$ is surjective. Hence the cofiltered system defining the p'-completion of $\mathcal{O}^{\times}(X)$ is a Mittag-Leffler system. So [STK, Tag 0598] shows that taking limits over the sequences (5.10), we obtain an exact sequence

(5.14) $$0 \longrightarrow \mathcal{O}^{\times}(X)^{\wedge}_{p'} \xrightarrow{\ \mathrm{Kum}\ } \mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1)) \longrightarrow \mathrm{T}_{p'}(\mathrm{Pic}(X)) \longrightarrow 0 \,.$$

In particular, the p'-Kummer map is an isomorphism if and only if $\mathrm{T}_{p'}(\mathrm{Pic}(X)) = 0$.

5.2 Amenability. The following is the main definition of this section.

5.15 Definition. Let k be a field of exponential characteristic p. We say that k is *amenable* if the following conditions hold:

(5.15.1) For each torus T over k, the abelian group $T(k)$ has no nonzero p'-divisible elements.

(5.15.2) For each regular finite type k-scheme X, we have $\mathrm{T}_{p'}(\mathrm{Pic}(X)) = 0$.

For future use, we record the following consequences of Observations 5.5 and 5.13:

5.16 Observation. Let k be of exponential characteristic p.

(5.16.1) If condition (5.15.1) is satisfied, then for any torus T over k, the Kummer map

$$T(k) \to \lim_{n \in (\mathbf{N}_{p'})^{\text{op}}} \mathrm{H}^1_{\text{ét}}(k; T[n])$$

is injective.

(5.16.2) If for each regular finite type k-scheme X, the group $\mathrm{Pic}(X)$ has no p'-divisible torsion elements, then condition (5.15.2) is satisfied.

(5.16.3) Condition (5.15.2) is satisfied if and only if for each regular finite type k-scheme X, the p'-Kummer map $\mathrm{Kum} \colon \mathcal{O}^{\times}(X)^{\wedge}_{p'} \to \mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1))$ is an isomorphism.

We now deduce analogous facts about absolute weak normalizations.

5.17 Lemma. *Let k be a field of exponential characteristic p and let X be a qcqs seminormal k-scheme. Then the natural homomorphism $\mathcal{O}^{\times}(X) \to \mathcal{O}^{\times}(X^{\mathrm{awn}})$ induces an isomorphism*

$$\mathcal{O}^{\times}(X)[1/p] \xrightarrow{\sim} \mathcal{O}^{\times}(X^{\mathrm{awn}}) .$$

Proof. If $p = 1$, then since X is seminormal, $X^{\mathrm{awn}} = X$ and there is nothing to show. If $p > 1$, then X^{awn} is the perfection of X, thus applying [STK, Tag 01Z0] we deduce that

$$\mathcal{O}^{\times}(X^{\mathrm{awn}}) \cong \mathrm{colim}\left(\mathcal{O}^{\times}(X) \xrightarrow{\cdot p} \mathcal{O}^{\times}(X) \xrightarrow{\cdot p} \cdots \right)$$
$$= \mathcal{O}^{\times}(X)[1/p] . \qquad \square$$

5.18 Lemma (perfection and the Picard group [5, Lemma 3.5]). *Let X be a qcqs $\mathbf{F}_p$-scheme. Then pullback along the counit $X^{\mathrm{perf}} \to X$ induces an isomorphism*

$$\mathrm{Pic}(X)[1/p] \xrightarrow{\sim} \mathrm{Pic}(X^{\mathrm{perf}}) .$$

5.19 Lemma. *Let k be an amenable field and let X be a regular finite type k-scheme. Then:*

(5.19.1) *Pullback along the counit $X^{\mathrm{awn}} \to X$ induces an isomorphism*

$$\mathrm{T}_{p'}(\mathrm{Pic}(X)) \xrightarrow{\sim} \mathrm{T}_{p'}(\mathrm{Pic}(X^{\mathrm{awn}})) .$$

(5.19.2) *The p'-Kummer map $\mathrm{Kum} \colon \mathcal{O}^{\times}(X^{\mathrm{awn}})^{\wedge}_{p'} \to \mathrm{H}^1_{\text{ét}}(X^{\mathrm{awn}}; \widehat{\mathbf{Z}}_{p'}(1))$ is an isomorphism.*

(5.19.3) *We have $\mathrm{T}_{p'}(\mathrm{Pic}(X^{\mathrm{awn}})) = 0$.*

Proof. For (5.19.1), note that if $p = 1$, then $X^{\mathrm{awn}} = X$. Hence Lemma 5.18 shows that with no conditions on p, pullback along the counit induces an isomorphism

$$\mathrm{Pic}(X)[1/p] \xrightarrow{\sim} \mathrm{Pic}(X^{\mathrm{awn}}) .$$

By Lemma 5.6 we see that pullback along the counit induces isomorphisms

$$\mathrm{T}_{p'}(\mathrm{Pic}(X)) \xrightarrow{\sim} \mathrm{T}_{p'}(\mathrm{Pic}(X)[1/p]) \xrightarrow{\sim} \mathrm{T}_{p'}(\mathrm{Pic}(X^{\mathrm{awn}})) .$$

To conclude, note that Observation 5.13 shows that (5.19.2) $\Leftrightarrow$ (5.19.3). Moreover, Observation 5.16 shows that $\mathrm{T}_{p'}(\mathrm{Pic}(X)) = 0$; so item (5.19.1) implies (5.19.3). $\square$

The property of being amenable is stable under finite field extensions:

5.20 Lemma. *Let k be an amenable field and let $L \supset k$ be a finite field extension. Then L is also amenable.*

Proof. If X is regular and of finite type over L, it is also regular and of finite type over k. Thus, we only need to check the condition on tori. If $L \supset k$ is separable, then this follows by noting that for any torus T over L, the Weil restriction $\mathrm{Res}_{L/k}(T)$ is a torus over k with the property that

$$\mathrm{Res}_{L/k}(T)(k) = T(L) .$$

In the case that $L \supset k$ is finite purely inseparable, $\mathrm{Res}_{L/k}$ does not carry tori to tori, so we cannot reason in the same way. In this case, note that by Galois descent, it is enough to check that our statement is true for the torus $\mathbf{G}_{\mathrm{m}}$ over L. Since $L \supset k$ is finite purely inseparable, there is some $n > 0$ such that $L^{p^n} \subset k$. If $a \in L^{\times}$ is p'-divisible, then so is $a^{p^n} \in k$. By our assumption on k, this implies that $a^{p^n} = 1$. Thus $a = 1$, so the statement holds for tori. $\square$

5.21 Definition. We say that a field k is *finitely generated* if k is a finitely generated field extension of its prime field.

We now show that finitely generated fields are amenable. We first recall the the following fundamental result on the finite generation of Picard groups.

5.22 Proposition [11, Proposition 6.1]. *Let k be a finitely generated field and let X be normal finite type k-scheme. Then $\mathrm{Pic}(X)$ is finitely generated.*

5.23 Lemma. *Let L be a field of exponential characteristic p. If for any torus T over k, the abelian group $T(L)$ has no nonzero p'-divisible elements, then the function field $L(t)$ has the same property.*

Proof. It suffices to show the statement for the standard torus $\mathbf{G}_{\mathrm{m},L(t)}$, so assume that $f \in L(t)^{\times}$ is p'-divisible. We then immediately see that f is constant, and by the assumption on L, this shows that $f = 1$. □

5.24 Proposition. *If k is a finitely generated field, then k is amenable.*

Proof. Given Proposition 5.22, we only have to show that tori over k have no p'-divisible points. Since any finitely generated field is a finite extension of a purely transcendental extension of its prime field, by Lemma 5.20, it is enough to show that if L has the property that no L-torus has p'-divisible elements, then $L(t)$ has the same property. This is the content of Lemma 5.23. □

6. Reconstruction for amenable discretely valued fields

The goal of this section is to prove the main result of this paper: if k is an amenable field that admits a nontrivial discrete valuation, then k satisfies étale reconstruction (Theorem 6.24). In particular, we deduce that infinite finitely generated fields satisfy étale reconstruction (Corollary 6.25).

In §6.1 we provide a 'cohomological' criterion for amenable fields to satisfy étale reconstruction (Proposition 6.12). Subsection 6.2 proves some preparatory lemmas needed to check this criterion in the presence of a discrete valuation. In §6.3, we prove the main result.

6.1 A cohomological criterion for reconstructing amenable fields.
To state our cohomological criterion for étale reconstruction, we begin by fixing some notation.

6.1 Notation (the canonical class in $\mathrm{H}^1_{\text{ét}}(\mathbf{G}_{\mathrm{m}}; \widehat{\mathbf{Z}}_{p'}(1))$). Let k be a field of exponential characteristic p. For each positive integer n coprime to p, note that the Kummer sequence provides a canonical class

$$c_{1,n} \in \mathrm{H}^1_{\text{ét}}(\mathbf{G}_{\mathrm{m}}; \boldsymbol{\mu}_n) \,.$$

Note that for m dividing n, the homomorphism $\boldsymbol{\mu}_n \to \boldsymbol{\mu}_m$ given by raising to the n/m-th power induces a homomorphism

$$\mathrm{H}^1_{\text{ét}}(\mathbf{G}_{\mathrm{m}}; \boldsymbol{\mu}_n) \to \mathrm{H}^1_{\text{ét}}(\mathbf{G}_{\mathrm{m}}; \boldsymbol{\mu}_m)$$

that sends $c_{1,n}$ to $c_{1,m}$. We write c_1 for the class

$$(c_{1,n})_{n \in (\mathbf{N}_{p'})^{\text{op}}} \in \mathrm{H}^1_{\text{ét}}(\mathbf{G}_{\mathrm{m}}; \widehat{\mathbf{Z}}_{p'}(1)) \,.$$

6.2 Notation. Let k be a field of exponential characteristic p and let X be a k-scheme. Following Voevodsky we write

$$\mathcal{O}^{\times}_{\text{top}}(X) := \mathrm{Hom}^{\text{pin}}_k(X_{\text{ét}}, \mathbf{G}_{\mathrm{m},\text{ét}}) \,.$$

Define a map χ'_{top} by the assignment

$$\chi'_{\text{top}}: \; \mathcal{O}^{\times}_{\text{top}}(X) \to \mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1))$$
$$f^* \mapsto f^*(c_1) \,.$$

Here, note that the class $f^*(c_1)$ is well-defined by the functoriality of the assignment $X \mapsto \mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1))$ in geometric morphisms of étale topoi over $\mathrm{Spec}(k)_{\text{ét}}$ explained in Observation 5.12.

6.3. By construction, the square

(6.4)
$$
\begin{array}{ccc}
\mathcal{O}^{\times}(X) & \xrightarrow{\;\mathrm{can}\;} & \mathcal{O}^{\times}(X)_{p'}^{\wedge} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Kum}} \\
\mathcal{O}_{\mathrm{top}}^{\times}(X) & \xrightarrow[\chi'_{\mathrm{top}}]{} & \mathrm{H}^1_{\mathrm{\acute{e}t}}(X; \widehat{\mathbf{Z}}_{p'}(1))
\end{array}
$$

commutes.

6.5 Construction (χ_{top})**.** Let k be an amenable field of exponential characteristic p and let X be a regular finite type k-scheme. Recall from Lemma 5.19 that the p'-Kummer map

$$
\mathrm{Kum}:\ \mathcal{O}^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge} \longrightarrow \mathrm{H}^1_{\mathrm{\acute{e}t}}(X^{\mathrm{awn}}; \widehat{\mathbf{Z}}_{p'}(1))
$$

is an isomorphism. We write χ_{top} for the composite

$$
\mathcal{O}_{\mathrm{top}}^{\times}(X^{\mathrm{awn}}) \xrightarrow{\;\chi'_{\mathrm{top}}\;} \mathrm{H}^1_{\mathrm{\acute{e}t}}(X^{\mathrm{awn}}; \widehat{\mathbf{Z}}_{p'}(1)) \xrightarrow[\sim]{\;\mathrm{Kum}^{-1}\;} \mathcal{O}^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge}\ .
$$

Our goal is to show that if the image of χ_{top} is contained in the image of the natural homomorphsim $\mathcal{O}^{\times}(X^{\mathrm{awn}}) \to \mathcal{O}^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge}$, then k satisfies étale reconstruction. To prove this criterion, we need a construction and a couple of observations.

6.6 Construction. Let k be a field of exponential characteristic p, let Y be a k^{awn}-scheme, and let $E \supset k^{\mathrm{awn}}$ be a finite field extension. For any morphism of k^{awn}-schemes $x:\ \mathrm{Spec}(E) \to Y$, we obtain a pullback map

$$
x^*:\ \mathrm{H}^1_{\mathrm{\acute{e}t}}(Y; \widehat{\mathbf{Z}}_{p'}(1)) \longrightarrow \mathrm{H}^1_{\mathrm{\acute{e}t}}(\mathrm{Spec}(E); \widehat{\mathbf{Z}}_{p'}(1)) \cong (E^{\times})_{p'}^{\wedge}\ .
$$

Thus for any fixed class $\xi \in \mathrm{H}^1_{\mathrm{\acute{e}t}}(Y; \widehat{\mathbf{Z}}_{p'}(1))$ we obtain a map

$$
Y(E) \to (E^{\times})_{p'}^{\wedge}\ , \qquad x \mapsto x^*(\xi)\ .
$$

Passing to the colimit over all finite extensions of k^{awn}, we obtain a map

$$
\psi_{\xi}:\ Y(\bar{k}) \to \operatorname*{colim}_{E}(E^{\times})_{p'}^{\wedge}\ .
$$

We write

$$
\widehat{\mathrm{ev}}(-,-):\ Y(\bar{k}) \times \mathrm{H}^1_{\mathrm{\acute{e}t}}(Y; \widehat{\mathbf{Z}}_{p'}(1)) \to \operatorname*{colim}_{E}(E^{\times})_{p'}^{\wedge}
$$

for the pairing corresponding to the assignment that sends a class $\xi \in \mathrm{H}^1_{\mathrm{\acute{e}t}}(Y; \widehat{\mathbf{Z}}_{p'}(1))$ to the map ψ_{ξ}.

6.7 Notation. Let k be an amenable field of exponential characteristic p and let X be a regular finite type k-scheme. We also write $\widehat{\mathrm{ev}}$ for the composite

(6.8)
$$
X^{\mathrm{awn}}(\bar{k}) \times \mathcal{O}^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge} \xrightarrow[\sim]{\;\mathrm{id} \times \mathrm{Kum}\;} X^{\mathrm{awn}}(\bar{k}) \times \mathrm{H}^1_{\mathrm{\acute{e}t}}(X^{\mathrm{awn}}; \widehat{\mathbf{Z}}_{p'}(1)) \xrightarrow{\;\widehat{\mathrm{ev}}\;} \operatorname*{colim}_{E}(E^{\times})_{p'}^{\wedge}\ .
$$

6.9 Observation. Let k be an amenable field of exponential characteristic p. Then for any separable field extension $E \supset k$, the canonical map

$$
E^{\times} \to (E^{\times})_{p'}^{\wedge}
$$

is injective because it may be identified with the Kummer map. Since $(E^{\mathrm{awn}})^{\times} = E^{\times}[1/p]$ and p is invertible on the right hand side, we also deduce that the natural map

$$
(E^{\mathrm{awn}})^{\times} \to (E^{\mathrm{awn},\times})_{p'}^{\wedge}
$$

is injective. Passing to the colimit over all finite separable field extensions, we therefore obtain an injective map

$$
i:\ \bar{k}^{\times} \to \operatorname*{colim}_{E}(E^{\mathrm{awn},\times})_{p'}^{\wedge}\ .
$$

6.10 Observation. If $Y = \mathbf{G}_{\mathrm{m}}$ in Construction 6.6 and $c_1 \in \mathrm{H}^1_{\text{ét}}(\mathbf{G}_{\mathrm{m}}; \widehat{\mathbf{Z}}_{p'}(1))$ is the canonical class, then the induced map

$$\psi_{c_1} = \widehat{\mathrm{ev}}(-, c_1) : \bar{k}^\times = \mathbf{G}_{\mathrm{m}}(\bar{k}) \to \operatorname*{colim}_E (E^\times)^\wedge_{p'}$$
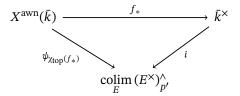
is the canonical map i.

6.11 Observation. For any pinned geometric morphism $f_* : X_{\text{ét}} \to Y_{\text{ét}}$ between schemes topologically of finite type over k^{awn}, by construction, the square

$$
\begin{array}{ccc}
X(\bar{k}) \times \mathrm{H}^1_{\text{ét}}(Y; \widehat{\mathbf{Z}}_{p'}(1)) & \xrightarrow{\ f_* \times \mathrm{id}\ } & Y(\bar{k}) \times \mathrm{H}^1_{\text{ét}}(Y; \widehat{\mathbf{Z}}_{p'}(1)) \\
{\scriptstyle \mathrm{id} \times f^*} \downarrow & & \downarrow {\scriptstyle \widehat{\mathrm{ev}}} \\
X(\bar{k}) \times \mathrm{H}^1_{\text{ét}}(X; \widehat{\mathbf{Z}}_{p'}(1)) & \xrightarrow[\ \widehat{\mathrm{ev}}\ ]{} & \operatorname*{colim}_E (E^\times)^\wedge_{p'}
\end{array}
$$

commutes. In particular, taking $Y = \mathbf{G}_{\mathrm{m}}$ and restricting to

$$X^{\mathrm{awn}}(\bar{k}) \times \{c_1\} \subset X^{\mathrm{awn}}(\bar{k}) \times \mathrm{H}^1_{\text{ét}}(\mathbf{G}_{\mathrm{m}}; \widehat{\mathbf{Z}}_{p'}(1)),$$

it follows that for any $f_* \in \mathcal{O}^\times_{\mathrm{top}}(X^{\mathrm{awn}})$ the triangle

$$
\begin{array}{ccc}
X^{\mathrm{awn}}(\bar{k}) & \xrightarrow{\quad\quad f_* \quad\quad} & \bar{k}^\times \\
& {\scriptstyle \psi_{\chi_{\mathrm{top}}(f_*)}} \searrow \quad \swarrow {\scriptstyle i} & \\
& \operatorname*{colim}_E (E^\times)^\wedge_{p'} &
\end{array}
$$

commutes.

6.12 Proposition. *For an amenable field k of exponential characteristic p, the following are equivalent:*

(6.12.1) *The field k satisfies étale reconstruction.*

(6.12.2) *For each affine, regular, connected, finite type k-scheme X with $X(k) \neq \varnothing$, we have*

$$\mathrm{im}\left(\chi_{\mathrm{top}} : \mathcal{O}^\times_{\mathrm{top}}(X^{\mathrm{awn}}) \to \mathcal{O}^\times(X^{\mathrm{awn}})^\wedge_{p'}\right) \subset \mathrm{im}\left(\mathrm{can} : \mathcal{O}^\times(X^{\mathrm{awn}}) \to \mathcal{O}^\times(X^{\mathrm{awn}})^\wedge_{p'}\right).$$

Proof. To see that (6.12.1) $\Rightarrow$ (6.12.2), note that if k satisfies étale reconstruction, then for each regular finite type k-scheme X, the natural map

$$\mathcal{O}^\times(X^{\mathrm{awn}}) \to \mathcal{O}^\times_{\mathrm{top}}(X^{\mathrm{awn}})$$

is an isomorphism. So the commutativity of the square (6.4) shows that $\mathrm{im}(\chi_{\mathrm{top}}) \subset \mathrm{im}(\mathrm{can})$.

For the implication (6.12.2) $\Rightarrow$ (6.12.1), choose a separable closure $\bar{k}$ of k^{awn}, and let $\phi_* \in \mathcal{O}^\times_{\mathrm{top}}(X^{\mathrm{awn}})$ be a pinned geometric morphism $X^{\mathrm{awn}}_{\text{ét}} \to \mathbf{G}_{\mathrm{m,ét}}$ over k. Theorem 4.18 shows that it suffices to construct an element $f \in \mathcal{O}^\times(X^{\mathrm{awn}})$ that agrees with ϕ_* on $X^{\mathrm{awn}}(\bar{k})$. By assumption, there exists an element $f \in \mathcal{O}^\times(X^{\mathrm{awn}})$ mapping to $\chi_{\mathrm{top}}(\phi_*)$ under the natural map

$$\mathrm{can} : \mathcal{O}^\times(X^{\mathrm{awn}}) \to \mathcal{O}^\times(X^{\mathrm{awn}})^\wedge_{p'}.$$

It follows from Observation 6.11 that for any $\bar{x} \in X^{\mathrm{awn}}(\bar{k})$ we have

$$i(f(\bar{x})) = \widehat{\mathrm{ev}}(\bar{x}, \mathrm{can}(f)) = \widehat{\mathrm{ev}}(\bar{x}, \chi_{\mathrm{top}}(\phi_*)) = \psi_{\chi_{\mathrm{top}}(\phi_*)}(\bar{x}) = i(\phi_*(\bar{x})).$$

Because k is amenable, i is injective (Observation 6.9), so $f(\bar{x}) = \phi_*(\bar{x})$, as desired. $\square$

6.2   Finite generation & units. The rest of this section is dedicated to verifying the hypotheses of Proposition 6.12 when k is amenable and admits a nontrivial discrete valuation. Our argument relies on understanding the kernel of the homomorphism $\mathcal{O}^\times(X^{\mathrm{awn}}) \to (k^{\mathrm{awn}})^\times$ induced by a rational point $x \in X(k)$. It will be useful to know that this kernel is finitely generated and free; this subsection proves this.

6.13 Recollection. A field extension $K \supset k$ is *regular* if it is separable and k is algebraically closed in K.

6.14 Example [9, Proposition 5.51]**.** Let k be a field and let X be an integral k-scheme. Then the field extension $\mathrm{K}(X) \supset k$ is regular if and only if X is geometrically integral over k.

6.15 Lemma. *Let $A \hookrightarrow B$ be an injective ring homomorphism. If A is integrally closed in B, then the abelian group $B^\times/A^\times$ is torsion-free.*

Proof. Let $b \in B^\times$ be an element such that $b^n = a$ for some $a \in A^\times$. Then the polynomial $t^n - a \in A[t]$ is an integral equation for b. Since A is integrally closed in B, we deduce that $b \in A$. Moreover, since b^n is a unit in A, we deduce that b is also a unit in A. Hence the class of b is the identity in $B^\times/A^\times$.   □

6.16 Proposition. *Let k be a field and $K \supset k$ a finitely generated regular field extension. Let $R \subset K$ be a k-subalgebra that is finitely generated as a k-algebra. Then:*

(6.16.1) *The abelian group $R^\times/k^\times$ is finitely generated and free.*

(6.16.2) *For any k-algebra augmentation $\varepsilon \colon R \to k$, there is an isomorphism $\ker(\varepsilon^\times) \cong R^\times/k^\times$. In particular, $\ker(\varepsilon^\times)$ is finitely generated and free.*

Proof. For (6.16.1), the claim that $R^\times/k^\times$ is finitely generated is proven in [17, Chapter 2, Corollary 7.3]. Hence it suffices to show that the group $R^\times/k^\times$ is torsion-free. Notice that the incusion $R \subset K$ induces an injection $R^\times/k^\times \hookrightarrow K^\times/k^\times$. By Lemma 6.15, the group $K^\times/k^\times$ is torsion-free; hence the subgroup $R^\times/k^\times$ is also torsion-free.

For (6.16.2), consider the short exact sequence of abelian groups

$$1 \longrightarrow \ker(\varepsilon^\times) \longhookrightarrow R^\times \xrightarrow{\ \varepsilon^\times\ } k^\times \longrightarrow 1 \,.$$

Since ε is an augmentation, $\varepsilon^\times \colon R^\times \to k^\times$ admits a section. The splitting lemma completes the proof.   □

6.17 Notation. Let k be a field and let X be a k-scheme with fixed rational point $x \in X(k)$. Denote by $\mathcal{O}_1^\times(X) \subset \mathcal{O}^\times(X)$ the subgroup spanned by those f such that $f(x) = 1$. That is, $\mathcal{O}_1^\times(X)$ is the kernel of the map $\mathcal{O}^\times(X) \to k^\times$ induced by the k-point x. By functoriality, the k-point x induces a k^{awn}-point x^{awn} of X^{awn}. We also write $\mathcal{O}_1^\times(X^{\mathrm{awn}}) \subset \mathcal{O}^\times(X^{\mathrm{awn}})$ for the subgroup spanned by those f such that $f(x^{\mathrm{awn}}) = 1$.

6.18 Corollary. *Let k be a field and let X be an affine, geometrically integral, finite type k-scheme. Then for each $x \in X(k)$, the abelian group $\mathcal{O}_1^\times(X)$ is finitely generated and free.*

Proof. Since X is geometrically integral and of finite type over k, Example 6.14 shows that the field extension $\mathrm{K}(X) \supset k$ is finitely generated and regular. Since the k-subalgebra $\mathcal{O}(X) \subset \mathrm{K}(X)$ is finitely generated, the claim now follows from Proposition 6.16.   □

6.19 Lemma. *Let k be a field of exponential characteristic p and let X be an affine, regular, connected, finite type k-scheme with $X(k) \neq \varnothing$. Then for each $x \in X(k)$, the $\mathbf{Z}[1/p]$-module $\mathcal{O}_1^\times(X^{\mathrm{awn}})$ is finitely generated and free.*

Proof. First notice that since X is a connected k-scheme that admits a rational point, X is geometrically connected [STK, Tag 04KV]. Since X is regular and of finite type over k, we deduce that X is also geometrically irreducible.

If $p = 1$, then $X = X^{\mathrm{awn}}$ and X is smooth over k. Since X is geometrically irreducible, [STK, Tag 056T] shows that X is geometrically integral over k. Thus Corollary 6.18 shows that $\mathcal{O}_1^\times(X)$ is a finitely generated free abelian group. If $p > 1$, then since the natural morphism $(X_{k^{\mathrm{perf}}})_{\mathrm{red}} \to X$ is a universal homeomorphism, Lemma 5.17 shows that

$$\mathcal{O}_1^\times(X^{\mathrm{awn}}) \cong \mathcal{O}_1^\times((X_{k^{\mathrm{perf}}})_{\mathrm{red}})[1/p] \,.$$

Now note that since X is geometrically irreducible, $(X_{k^{\mathrm{perf}}})_{\mathrm{red}}$ is also geometrically irreducible. Moreover, since k^{perf} is perfect and $(X_{k^{\mathrm{perf}}})_{\mathrm{red}}$ is reduced, $(X_{k^{\mathrm{perf}}})_{\mathrm{red}}$ is also geometrically reduced. Hence $(X_{k^{\mathrm{perf}}})_{\mathrm{red}}$ is geometrically integral. To conclude, note that Corollary 6.18 shows that $\mathcal{O}_1^{\times}((X_{k^{\mathrm{perf}}})_{\mathrm{red}})$ is a finitely generated free abelian group.   □

6.3 Amenable discretely valued fields satisfy étale reconstruction.
We now verify that the hypothesis of Proposition 6.12 holds for amenable fields that admit a nontrivial discrete valuation. This is a slight generalization of Voevodsky's argument in [28, Proposition 3.4]. For this, we need two preparatory results.

6.20. Let p be a prime number or 1. Recall that by Lemma 5.8, we have $\widehat{\mathbf{Z}}_{p'} \cong \mathbf{Z}[1/p]_{p'}^{\wedge}$. Also note that the natural homomorphism $\mathbf{Z}[1/p] \to \mathbf{Z}[1/p]_{p'}^{\wedge}$ is injective. Hence for the rest of this section, we tacitly regard $\mathbf{Z}[1/p]$ as a subgroup of $\widehat{\mathbf{Z}}_{p'}$.

6.21 Lemma. *Let p be a prime number or 1, and let $\alpha \in \widehat{\mathbf{Z}}_{p'}$. If there exists a nonzero element $n \in \mathbf{Z}[1/p]$ such that $n\alpha \in \mathbf{Z}[1/p] \subset \widehat{\mathbf{Z}}_{p'}$, then $\alpha \in \mathbf{Z}[1/p]$.*

Proof. For $\ell \neq p$, write $\mathrm{pr}_\ell : \widehat{\mathbf{Z}}_{p'} \to \mathbf{Z}_\ell$ for the projection. By assumption,

$$\alpha \in \mathbf{Q} \subset \widehat{\mathbf{Z}}_{p'} \otimes \mathbf{Q} \qquad \text{and} \qquad \mathrm{pr}_\ell(\alpha) \in \mathbf{Z}_\ell \subset \mathbf{Q}_\ell \ .$$

Thus the ℓ-adic valuation of α is greater than or equal to 0. So $\alpha = m/p^k$ for some $m \in \mathbf{Z}$ and $k \in \mathbf{N}$.   □

The following result is the only part that seems to break over a finite field:

6.22 Proposition. *Let k be an amenable field of exponential characteristic p. Let X be an affine, regular, connected, finite type k-scheme with a fixed rational point $x \in X(k)$. Choose a basis $g_1, \dots, g_m$ for the finitely generated free $\mathbf{Z}[1/p]$-module $\mathcal{O}_1^{\times}(X^{\mathrm{awn}})$. Let*

$$\xi = g_1^{a_1} \cdots g_m^{a_m} \in \mathcal{O}_1^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge}$$

where $a_1, \dots, a_m \in \widehat{\mathbf{Z}}_{p'}$, be such that $\widehat{\mathrm{ev}}(-, \xi)$ factors through

$$i : \bar{k}^{\times} \to \mathrm{colim}_E (E^{\times})_{p'}^{\wedge} \ .$$

If k admits a nontrivial discrete valuation, then $\xi \in \mathcal{O}_1^{\times}(X^{\mathrm{awn}})$.

Proof. We note that the discrete valuation on k extends to a nontrivial $\mathbf{Z}[1/p]$-valued valuation on k^{awn}. We make the following claim:

6.23 Claim. *For any $1 \leq d \leq m$, we can find a finite extension $E \supset k^{\mathrm{awn}}$, points $x_1, \dots, x_d \in X^{\mathrm{awn}}(E)$ and an extension of the valuation on k^{awn} to a valuation $\nu : E^{\times} \to \mathbf{Z}[1/p]$ such that the determinant of the $\mathbf{Z}[1/p]$-matrix*

$$A(d) = (\nu(g_s(x_t)))_{s,t}$$

is nonzero.

If we can prove Claim 6.23 we are done. To see this, note that since $\mathcal{O}_1^{\times}(X^{\mathrm{awn}})$ is a finitely generated free $\mathbf{Z}[1/p]$-module, to prove that $\xi \in \mathcal{O}_1^{\times}(X^{\mathrm{awn}})$, we equivalently need to show that $a_1, \dots, a_m \in \mathbf{Z}[1/p]$. Write $\hat{\nu} : (E^{\times})_{p'}^{\wedge} \to \widehat{\mathbf{Z}}_{p'}$ for the p'-completion of ν. Note that for $d = m$ we have

$$A(m) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \hat{\nu}(\widehat{\mathrm{ev}}(x_1, \xi)) \\ \vdots \\ \hat{\nu}(\widehat{\mathrm{ev}}(x_m, \xi)) \end{pmatrix} .$$

Since $\hat{\nu}(\widehat{\mathrm{ev}}(x_d, \xi)) \in \mathbf{Z}[1/p]$ by assumption, by multiplying with the adjugate of the $\mathbf{Z}[1/p]$-matrix $A(m)$, we see that each a_j has the property that

$$\det(A(m))a_j \in \mathbf{Z}[1/p] \ .$$

Since $\det(A(m)) \in \mathbf{Z}[1/p]$ is nonzero, Lemma 6.21 implies that $a_1, \dots, a_m \in \mathbf{Z}[1/p]$, as desired.   □

Proof of Claim 6.23. We proceed by induction on d. For the base case $d = 1$, we claim that there is some finite extension $E \supset k^{\mathrm{awn}}$, and some $x \in X^{\mathrm{awn}}(E)$ such that $\nu(g_1(x)) \neq 0$. Pick a nontrivial extension of the valuation of k^{awn} to E, and consider the induced map $g_1 : X^{\mathrm{awn}} \to \mathbf{G}_{\mathrm{m}}$. Since g_1 is not constant, g_1 is an open map. Since the complement of $g_1(X^{\mathrm{awn}})$ is a finite set of closed points, we can pick some $x \in X^{\mathrm{awn}}(E)$ for some extension $E \supset k^{\mathrm{awn}}$ such that $\nu(g_1(x)) \neq 0$, .

For the induction step, assume that the statement holds for d. Consider the matrix $A(d+1) = (\nu(g_s(x_t)))_{s,t}$ for some points $x_1, \ldots, x_{d+1}$ yet to be chosen, and expand $\det(A(d + 1))$ formally in minors as

$$\det(A(d + 1)) = \sum_{s=1}^{d+1} (-1)^{s+1} \nu(g_s(x_1)) M_s \,,$$

where M_s is the determinant we obtain by removing the row and column containing $g_s(x_1)$. By induction, there exist a finite extension $E' \supset k^{\mathrm{awn}}$ and points $x_2, \ldots, x_{d+1} \in X^{\mathrm{awn}}(E')$ such that $M_1 \neq 0$. We now consider the function

$$g' := g_1^{\pm M_1} \cdots g_{d+1}^{\pm M_{d+1}} \in \mathcal{O}_1^{\times}(X^{\mathrm{awn}}) \,.$$

Since the g_j are linearly independent and $M_1 \neq 0$, the function g' is not constant. Thus, g' is open, so there is a finite extension $E \supset E'$ and a $x_1 \in X^{\mathrm{awn}}(E')$ such that $\nu(g'(x_1)) \neq 0$ and the claim follows. $\qquad\square$

The following is the main result of this paper:

6.24 Theorem. *If k is an amenable field that admits a nontrivial discrete valuation, then k satisfies étale reconstruction.*

Proof. Let p denote the exponential characteristic of k and choose a separable closure $\bar{k} \supset k^{\mathrm{awn}}$. By Proposition 6.12, we need to show that for each affine, regular, connected, finite type k-scheme X with a fixed rational point $x \in X(k)$, we have

$$\mathrm{im}\left(\chi_{\mathrm{top}} : \mathcal{O}_{\mathrm{top}}^{\times}(X^{\mathrm{awn}}) \to \mathcal{O}^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge}\right) \subset \mathrm{im}\left(\mathrm{can} : \mathcal{O}^{\times}(X^{\mathrm{awn}}) \to \mathcal{O}^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge}\right) \,.$$

Let $\phi_* \in \mathcal{O}_{\mathrm{top}}^{\times}(X^{\mathrm{awn}})$ and write $\xi := \chi_{\mathrm{top}}(\phi_*)$. After scaling ϕ_*, we can, without loss of generality assume that $\phi_*(x^{\mathrm{awn}}) = 1$. Thus ξ lies in the group $\mathcal{O}_1^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge}$.

We note that the pairing

$$\widehat{\mathrm{ev}} : X^{\mathrm{awn}}(\bar{k}) \times \mathcal{O}_1^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge} \to \mathrm{colim}_E (E^{\times})_{p'}^{\wedge}$$

defined by (6.8) extends the usual evaluation map

$$X^{\mathrm{awn}}(\bar{k}) \times \mathcal{O}_1^{\times}(X^{\mathrm{awn}}) \longrightarrow \bar{k}^{\times} \overset{i}{\hookrightarrow} \mathrm{colim}_E (E^{\times})_{p'}^{\wedge} \,.$$

Next, we observe that, by construction, $\widehat{\mathrm{ev}}(-, \xi) = \psi_{\chi_{\mathrm{top}}(\phi_*)}$. In particular, since ϕ_* is pinned, $\widehat{\mathrm{ev}}(-, \xi)$ factors through $i : \bar{k}^{\times} \hookrightarrow \mathrm{colim}_E(E^{\times})_{p'}^{\wedge}$. Thus by Proposition 6.22, ξ is in the image of the natural homomorphism

$$\mathrm{can} : \mathcal{O}_1^{\times}(X^{\mathrm{awn}}) \to \mathcal{O}_1^{\times}(X^{\mathrm{awn}})_{p'}^{\wedge} \,,$$

as desired. $\qquad\square$

6.25 Corollary. *If k is an infinite field that is also finitely generated, then k satisfies étale reconstruction.*

Proof. Since k is finitely generated, Proposition 5.24 shows that k is amenable. Since k is also infinite, k admits a nontrivial discrete valuation. Hence k satisfies the hypotheses of Theorem 6.24. $\qquad\square$

APPENDIX A. MORPHISMS OF ÉTALE SITES AND ÉTALE TOPOI

Given a scheme X, write $\mathbf{Ét}_X^{\mathrm{qcs}} \subset \mathbf{Ét}_X$ for the subsite of the étale site spanned by the quasicompact separated morphisms. If X is quasiseparated, then $\mathbf{Ét}_X^{\mathrm{qcs}}$ is a basis for the étale topology (see Lemma A.4). In this appendix, we verify that if X and Y are topologically noetherian schemes, then every geometric morphism $f_* : X_{\mathrm{ét}} \to Y_{\mathrm{ét}}$ is induced by a morphism of sites $\mathbf{Ét}_Y^{\mathrm{qcs}} \to \mathbf{Ét}_X^{\mathrm{qcs}}$. See Corollary A.9. This allows us to repackage the étale reconstruction property in site-theoretic terms (see Corollary A.15). In [28], Voevodsky exclusively works with quasicompact separated étale sites; however, Grothendieck's conjecture [10, p. 7] is specifically about étale topoi. So Corollary A.9 also verifies that Voevodsky's site-theoretic result is equivalent to the topos-theoretic statement appearing in Grothendieck's letter.

We start by introducing a few bases for the étale site. Before doing so, we remind the reader that quasicompactness and quasiseparatedness are *topological* properties of schemes and morphisms of schemes.

A.1 Recollection (quasicompactness and quasiseparatedness). A map of topological spaces $f : S \to T$ is *quasicompact* (resp., *quasiseparated*) if for every quasicompact (resp., quasiseparated) open $V \subset T$, the preimage $f^{-1}(V) \subset S$ is quasicompact (resp., quasiseparated). A morphism of schemes $f : X \to Y$ is *quasicompact* (resp., *quasiseparated*) if the induced map on underlying topological spaces $|f| : |X| \to |Y|$ is quasicompact (resp., quasiseparated).

If S is a noetherian space, then every subset of S is quasicompact and quasiseparated. In particular, in this case, every map $f : S \to T$ to an arbitrary topological space T is qcqs.

A.2 Notation (variants of the étale site). Let X be a scheme. We write

$$\mathbf{Ét}_X^{\mathrm{sep}} \subset \mathbf{Ét}_X , \qquad \mathbf{Ét}_X^{\mathrm{qcs}} \subset \mathbf{Ét}_X , \qquad \text{and} \qquad \mathbf{Ét}_X^{\mathrm{aff}} \subset \mathbf{Ét}_X$$

for the full subcategories spanned by those $p : X' \to X$ such that p is separated, p is quasicompact and separated, and X' is affine over $\mathrm{Spec}(\mathbf{Z})$, respectively.

A.3 Observation. Let X be a scheme.

(A.3.1) By cancellation, every morphism in $\mathbf{Ét}_X^{\mathrm{sep}}$ is separated and every morphism in $\mathbf{Ét}_X^{\mathrm{qcs}}$ is quasicompact and separated. Moreover, since separated and quasicompact morphisms are stable under base-change, the full subcategories

$$\mathbf{Ét}_X^{\mathrm{qcs}} \subset \mathbf{Ét}_X^{\mathrm{sep}} \subset \mathbf{Ét}_X$$

are closed under finite limits.

(A.3.2) Since affine schemes are separated and every morphism with separated source is separated, we have

$$\mathbf{Ét}_X^{\mathrm{aff}} \subset \mathbf{Ét}_X^{\mathrm{sep}} .$$

(A.3.3) Assume that X is quasiseparated. Since every morphism from a quasicompact scheme to a quasiseparated scheme is quasicompact, we have

$$\mathbf{Ét}_X^{\mathrm{aff}} \subset \mathbf{Ét}_X^{\mathrm{qcs}} .$$

(A.3.4) If X is qcqs, then every object of $\mathbf{Ét}_X^{\mathrm{qcs}}$ is qcqs.

A.4 Lemma. *Let X be a scheme. Then:*

(A.4.1) *The full subcategories*

$$\mathbf{Ét}_X^{\mathrm{aff}} \subset \mathbf{Ét}_X^{\mathrm{sep}} \subset \mathbf{Ét}_X$$

are both bases for the étale topology. Hence restriction along the inclusions define equivalences of topoi

$$\mathrm{Sh}_{\mathrm{ét}}(\mathbf{Ét}_X) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{ét}}(\mathbf{Ét}_X^{\mathrm{sep}}) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{ét}}(\mathbf{Ét}_X^{\mathrm{aff}}) .$$

The inverses are given by right Kan extension along the inclusions.

(A.4.2) *Assume that X is quasiseparated. Then $\mathbf{Ét}_X^{\mathrm{qcs}} \subset \mathbf{Ét}_X$ is a basis for the étale topology. Hence restriction along the inclusion defines an equivalences of topoi*

$$\mathrm{Sh}_{\mathrm{ét}}(\mathbf{Ét}_X) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{ét}}(\mathbf{Ét}_X^{\mathrm{qcs}}) .$$

The inverse is given by right Kan extension along the inclusion.

Proof. First note that by [19, Propositions B.6.4 & B.6.6], the statements about equivalences of topoi follow from the statements about bases. To prove (A.4.1), first note that it suffices to prove that $\acute{E}t_X^{aff} \subset \acute{E}t_X$ is a basis. This is immediate from the fact that Zariski covers are étale covers. For (A.4.2) note that since X is quasiseparated, we have

$$\acute{E}t_X^{aff} \subset \acute{E}t_X^{qcs} .$$

Hence the claim is a consequence of (A.4.1).   □

In the remainder of this subsection, we explain why every *coherent* geometric morphism between the étale topoi of qcqs schemes is induced by a morphism between quasicompact separated étale sites.

A.5 Recollection (coherent topoi). Let $\mathcal{X}$ be a topos. An object $X \in \mathcal{X}$ is *quasicompact* if for every epimorphism

$$\coprod_{i \in I} X_i \twoheadrightarrow X$$

there exists a finite subset $J \subset I$ such that $\coprod_{j \in J} X_j \to X$ is an epimorphism. An object $X \in \mathcal{X}$ is *quasiseparated* if for all diagrams $Y \to X \leftarrow Y'$ with Y and Y' quasicompact, the pullback $Y \times_X Y'$ is quasicompact. Finally, an object $X \in \mathcal{X}$ is *coherent* if X is both quasicompact and quasiseparated. A geometric morphism $f_* \colon \mathcal{X} \to \mathcal{Y}$ of topoi is *coherent* if for every coherent object $Y \in \mathcal{Y}$, the pullback $f^*(Y)$ is coherent. See [SGA 4_{II}, Exposé VI; 19, §C.5] for more on coherent topoi.

A.6 Example. Let X be a qcqs scheme. Then the étale topos $X_{\acute{e}t}$ is coherent. Moreover, [SGA 4_{II}, Exposé IX, Proposition 2.7] shows that an étale sheaf $\mathcal{F} \in X_{\acute{e}t}$ is coherent if and only if $\mathcal{F}$ is constructible.

A.7 Lemma. *Let X and Y be qcqs schemes and let $f_* \colon X_{\acute{e}t} \to Y_{\acute{e}t}$ be a geometric morphism. The following are equivalent:*

(A.7.1) *The geometric morphism f_* is coherent.*

(A.7.2) *The induced map of topological spaces $|f_*| \colon |X| \to |Y|$ is quasicompact.*

Proof. If f_* is coherent, then by restricting the morphism f_* to subobjects of the terminal object of $Y_{\acute{e}t}$ we deduce that induced map $|f_*|$ is quasicompact. Conversely, assume that $|f_*|$ is quasicompact. Since X and Y are qcqs, in light of Example A.6 we need to show that for each constructible sheaf $\mathcal{F} \in Y_{\acute{e}t}$, the pullback $f^*(\mathcal{F})$ is constructible. By assumption, there is a finite stratification $\{Y_i\}_{i \in I}$ of Y where each $Y_i \subset Y$ is qcqs and locally closed such that on each stratum, $\mathcal{F}|_{Y_i}$ is finite locally constant. Since $|f_*|$ is quasicompact, the stratification pulls back to a stratification $\{|f_*|^{-1}(Y_i)\}_{i \in I}$ of X by qcqs locally closed subschemes such that on each stratum, $f^*(\mathcal{F})$ is finite locally constant.   □

A.8 Theorem. *Let X and Y be qcqs schemes and let $f_* \colon X_{\acute{e}t} \to Y_{\acute{e}t}$ be a coherent geometric morphism. Then for each $V \in \acute{E}t_Y^{qcs}$, the étale sheaf $f^*(V)$ is representable by an object of $\acute{E}t_X^{qcs}$. Hence f_* is induced by the morphism of sites*

$$f^* \colon \acute{E}t_Y^{qcs} \to \acute{E}t_X^{qcs} .$$

Proof. The diagonal $V \to V \times_Y V$ is a clopen immersion; let V' be the complementary clopen subscheme. Since f_* is coherent, by Lemma A.7 the sheaves $f^*(V)$ and $f^*(V')$ are constructible. Hence $f^*(V)$ and $f^*(V')$ are represented by an étale algebraic spaces $U \to X$ and $U' \to X$. Furthermore, pulling back the isomorphism $V \sqcup V' \xrightarrow{\sim} V \times_Y V$, we obtain an isomorphism

$$U \sqcup U' \xrightarrow{\sim} U \times_X U .$$

Thus the diagonal of U is a closed immersion, equivalently, $U \to X$ is separated. Since an étale algebraic space is also locally quasi-finite, it follows from [STK, Tag 03XX] that U is a scheme and $U \to X$ is quasicompact and separated, as desired.   □

A.9 Corollary. *Let X and Y be schemes. If X is topologically noetherian and Y is qcqs, then:*

(A.9.1) *Every geometric morphism $f_* \colon X_{\acute{e}t} \to Y_{\acute{e}t}$ is coherent.*

(A.9.2) *Every geometric morphism $f_* \colon X_{\acute{e}t} \to Y_{\acute{e}t}$ is induced by a morphism of quasicompact separated étale sites $\acute{E}t_Y^{qcs} \to \acute{E}t_X^{qcs}$.*

Proof. By Theorem A.8, it suffices to prove (A.9.1). For this, note that by Lemma A.7, it suffices to shows that $|f_*| \colon |X| \to |Y|$ is quasicompact. Since $|X|$ is noetherian, this is automatic. $\qquad\square$

We conclude by reformulating the étale reconstruction property (see Proposition 2.28 and Definition 2.29) in terms of sites.

A.10 Notation. Let k be a field and let X and Y be k-schemes. Assume that X is topologically noetherian and Y is qcqs. We write $\mathrm{Hom}_k(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}, \mathbf{\acute{E}t}_X^{\mathrm{qcs}})$ for the groupoid of morphisms $f^{-1} \colon \mathbf{\acute{E}t}_Y^{\mathrm{qcs}} \to \mathbf{\acute{E}t}_X^{\mathrm{qcs}}$ in the $(2,1)$-category of sites under the étale site of $\mathrm{Spec}(k)$. We write

$$\mathrm{Hom}_k^{\mathrm{pin}}(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}, \mathbf{\acute{E}t}_X^{\mathrm{qcs}}) \subset \mathrm{Hom}_k(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}, \mathbf{\acute{E}t}_X^{\mathrm{qcs}})$$

for the full subgroupoid spanned by those morphisms of sites $f^{-1} \colon \mathbf{\acute{E}t}_Y^{\mathrm{qcs}} \to \mathbf{\acute{E}t}_X^{\mathrm{qcs}}$ such that the induced geometric morphism

$$X_{\mathrm{\acute{e}t}} \simeq \mathrm{Sh}_{\mathrm{\acute{e}t}}(\mathbf{\acute{E}t}_X^{\mathrm{qcs}}) \xrightarrow{\ f_*\ } \mathrm{Sh}_{\mathrm{\acute{e}t}}(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}) \simeq Y_{\mathrm{\acute{e}t}}$$

is a pinned geometric morphism in the sense of Definition 2.19.

A.11 Corollary. *Let k be a field and let X and Y be k-schemes. Assume that X is topologically noetherian and Y is qcqs. Then the natural functors*

$$\mathrm{Hom}_k(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}, \mathbf{\acute{E}t}_X^{\mathrm{qcs}}) \to \mathrm{Hom}_k(X_{\mathrm{\acute{e}t}}, Y_{\mathrm{\acute{e}t}}) \qquad and \qquad \mathrm{Hom}_k^{\mathrm{pin}}(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}, \mathbf{\acute{E}t}_X^{\mathrm{qcs}}) \to \mathrm{Hom}_k^{\mathrm{pin}}(X_{\mathrm{\acute{e}t}}, Y_{\mathrm{\acute{e}t}})$$

given by $f^{-1} \mapsto f_*$ are equivalences of groupoids.

Proof. Immediate from Corollary A.9 and the definitions. $\qquad\square$

A.12. In light of Proposition 2.22 and Corollary A.11, if X and Y are topologically of finite type over k, then the groupoid $\mathrm{Hom}_k^{\mathrm{pin}}(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}, \mathbf{\acute{E}t}_X^{\mathrm{qcs}})$ is equivalent to a set.

A.13 Corollary. *Let k be a field and let X and Y be schemes topologically of finite type over k. Then the following are equivalent:*

(A.13.1) *The natural map*

$$\mathrm{Hom}_k(X, Y) \to \mathrm{Hom}_k^{\mathrm{pin}}(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}, \mathbf{\acute{E}t}_X^{\mathrm{qcs}}), \qquad f \mapsto f^{-1}$$

is bijective.

(A.13.2) *The natural map $\mathrm{Hom}_k(X, Y) \to \mathrm{Hom}_k^{\mathrm{pin}}(X_{\mathrm{\acute{e}t}}, Y_{\mathrm{\acute{e}t}})$ is bijective.*

A.14 Remark. Corollary A.13 shows that Voevodsky's site-theoretic result [28, Theorem 3.1] is equivalent to the following statement: for every finitely generated field k of characteristic 0 and finite type k-schemes X and Y with X normal, the natural map

$$\mathrm{Hom}_k(X, Y) \to \mathrm{Hom}_k^{\mathrm{pin}}(X_{\mathrm{\acute{e}t}}, Y_{\mathrm{\acute{e}t}})$$

is bijective.

A.15 Corollary. *Let k be a field. Then k satisfies étale reconstruction if and only if for all schemes X and Y topologically of finite type over k with X absolutely weakly normal, the natural map*

$$\mathrm{Hom}_k(X, Y) \to \mathrm{Hom}_k^{\mathrm{pin}}(\mathbf{\acute{E}t}_Y^{\mathrm{qcs}}, \mathbf{\acute{E}t}_X^{\mathrm{qcs}})$$

is bijective.

REFERENCES

STK The Stacks Project Authors, *Stacks project*, stacks.math.columbia.edu, 2024.

HTT J. Lurie, *Higher topos theory*, Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, vol. 170, pp. xviii+925, ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558, math.ias.edu/~lurie/papers/HTT.pdf.

HA   _____, *Higher algebra*, Sep. 2017, math.ias.edu/~lurie/papers/HA.pdf.

SAG _____, *Spectral algebraic geometry*, Feb. 2018, math.ias.edu/~lurie/papers/SAG-rootfile.pdf.

Ker _____, *Kerodon*, kerodon.net, 2024.

SGA 4$_\mathrm{I}$ *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Séminaire de Géométrie Algébrique du Bois Marie 1963–64 (SGA 4). Dirigé par M. Artin, A. Grothendieck, J.-L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint–Donat. Lecture Notes in Mathematics, Vol. 269. Berlin: Springer-Verlag, 1963–64, pp. xix+525.

SGA 4$_\mathrm{II}$ *Théorie des topos et cohomologie étale des schémas. Tome 2*, Séminaire de Géométrie Algébrique du Bois Marie 1963–64 (SGA 4). Dirigé par M. Artin, A. Grothendieck, J.-L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint–Donat. Lecture Notes in Mathematics, Vol. 270. Berlin: Springer-Verlag, 1963–64, iv+ 418.

1. C. Barwick, *Topological rigidification of schemes*, Dec. 2010, arXiv:1012.1889.

2. C. Barwick, S. Glasman, and P. J. Haine, *Exodromy*, Aug. 2020, arXiv:1807.03281v7.

3. J. Bénabou and J. Roubaud, *Monades et descente*, C. R. Acad. Sci. Paris Sér. A-B, vol. 270, A96–A98, 1970.

4. B. Bhatt and A. Mathew, *The arc-topology*, Duke Math. J., vol. 170, no. 9, pp. 1899–1988, 2021. DOI: 10.1215/00127094-2020-0088, arXiv:1807.04725.

5. B. Bhatt and P. Scholze, *Projectivity of the Witt vector affine Grassmannian*, Invent. Math., vol. 209, no. 2, pp. 329–423, 2017. DOI: 10.1007/s00222-016-0710-4, arXiv:1507.06490.

6. F. Borceux, *Handbook of categorical algebra. 1*, Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994, vol. 50, pp. xvi+345, ISBN: 0-521-44178-1, Basic category theory.

7. D. Clausen, *Answer to Math Overflow Question 38466: When do two elliptic curves have equivalent small étale toposes?* Sep. 2010, MO:38466.

8. D. L. Costa, *Seminormality and projective modules*, in *Paul Dubreil and Marie-Paule Malliavin Algebra Seminar, 34th Year (Paris, 1981)*, Lecture Notes in Math. Vol. 924, Springer, Berlin-New York, 1982, pp. 400–412.

9. U. Görtz and T. Wedhorn, *Algebraic geometry I. Schemes—with examples and exercises*, Second, Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, 2020, pp. vii+625, ISBN: 978-3-658-30732-5; 978-3-658-30733-2. DOI: 10.1007/978-3-658-30733-2.

10. A. Grothendieck, *Brief an G. Faltings*, in *Geometric Galois actions, 1*, London Math. Soc. Lecture Note Ser. Vol. 242, Cambridge: Cambridge Univ. Press, 1997, pp. 49–58, With an English translation on pp. 285–293.

11. R. Guralnick, D. B. Jaffe, W. Raskind, and R. Wiegand, *On the Picard group: Torsion and the kernel induced by a faithfully flat map*, J. Algebra, vol. 183, no. 2, pp. 420–455, 1996. DOI: 10.1006/jabr.1996.0228.

12. P. J. Haine, T. Holzschuh, and S. Wolf, *The fundamental fiber sequence in étale homotopy theory*, Int. Math. Res. Not. IMRN, no. 1, pp. 175–196, 2024. DOI: 10.1093/imrn/rnad018, arXiv:2209.03476.

13. M. Hoyois, S. Kelly, and P. A. Østvær, *The motivic Steenrod algebra in positive characteristic*, J. Eur. Math. Soc. (JEMS), vol. 19, no. 12, pp. 3813–3849, 2017. DOI: 10.4171/JEMS/754, arXiv:1305.5690.

14. L. Illusie, *On Gabber's refined uniformization*, Mar. 2009, Talks at the University of Tokyo, Jan. 17, 22, 31, Feb. 7, 2008. imo.universite-paris-saclay.fr/~luc.illusie/refined_uniformization3.pdf.

15. L. Illusie, Y. Laszlo, and F. Orgogozo, Eds., *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*. Société Mathématique de France, Paris, 2014, i–xxiv and 1–625, arXiv:1207.3648.

16. G. Janelidze and W. Tholen, *Facets of descent. I*, Appl. Categ. Structures, vol. 2, no. 3, pp. 245–281, 1994. DOI: 10.1007/BF00878100.

17. S. Lang, *Fundamentals of Diophantine geometry*. Springer-Verlag, New York, 1983, pp. xviii+370, ISBN: 0-387-90837-4. DOI: 10.1007/978-1-4757-1810-2.

18. Q. Lu and W. Zheng, *Duality and nearby cycles over general bases*, Duke Math. J., vol. 168, no. 16, pp. 3135–3213, 2019. DOI: 10.1215/00127094-2019-0057, arXiv:1712.10216.

19. J. Lurie, *Ultracategories*, 2018, math.ias.edu/~lurie/papers/Conceptual.pdf.

20. M. Makkai and G. E. Reyes, *First order categorical logic*, Lecture Notes in Mathematics, Vol. 611. Springer-Verlag, Berlin-New York, 1977, pp. viii+301, ISBN: 3-540-08439-8, Model-theoretical methods in the theory of topoi and related categories.

21. Z. Mao, *Answer to math Overflow Question 470168: Non-noetherian schemes with noetherian underlying space (in the Zariski topology)*, Apr. 2024, `MO:470168`.

22. D. Rydh, *Submersions and effective descent of étale morphisms*, Bull. Soc. Math. France, vol. 138, no. 2, pp. 181–230, 2010. DOI: `10.24033/bsmf.2588`, `arXiv:0710.2488`.

23. P. Scholze, *Lectures on condensed mathematics*, Apr. 2019, Lecture notes available at `www.math.uni-bonn.de/people/scholze/Condensed.pdf`.

24. J. Stix, *Projective anabelian curves in positive characteristic and descent theory for log-étale covers*, Bonner Mathematische Schriften. Universität Bonn, Mathematisches Institut, Bonn, 2002, vol. 354, pp. xviii+118, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2002.

25. R. G. Swan, *On seminormality*, J. Algebra, vol. 67, no. 1, pp. 210–229, 1980. DOI: `10.1016/0021-8693(80)90318-X`.

26. M. Temkin, *Tame distillation and desingularization by p-alterations*, Ann. of Math. (2), vol. 186, no. 1, pp. 97–126, 2017. DOI: `10.4007/annals.2017.186.1.3`, `arXiv:1508.06255`.

27. C. Traverso, *Seminormality and Picard group*, Ann. Scuola Norm. Sup. Pisa (3), vol. 24, pp. 585–595, 1970.

28. V. A. Voevodskiĭ, *Étale topologies of schemes over fields of finite type over* **Q**, Izv. Akad. Nauk SSSR Ser. Mat., vol. 54, no. 6, pp. 1155–1167, 1990. DOI: `10.1070/IM1991v037n03ABEH002156`.

29. V. Voevodsky, *Homology of schemes and covariant motives*. ProQuest LLC, Ann Arbor, MI, 1992, p. 64, Thesis (Ph.D.)–Harvard University.

30. _____, *Homology of schemes*, Selecta Math. (N.S.), vol. 2, no. 1, pp. 111–153, 1996. DOI: `10.1007/BF01587941`.