Profinite completions of products

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0 Introduction

Write $\mathbf{Spc}_\pi$ for the ∞-category of π-finite spaces. Given a set Σ of primes, write $\mathbf{Spc}_\Sigma \subset \mathbf{Spc}_\pi$ for the sull subcategory spanned by those π-finite spaces whose homotopy groups are Σ-groups. Write

$$(-)^\wedge_\Sigma \colon \operatorname{Pro}(\mathbf{Spc}) \to \operatorname{Pro}(\mathbf{Spc}_\Sigma)$$

for the Σ-completion functor, i.e., the left adjoint to the inclusion $\operatorname{Pro}(\mathbf{Spc}_\Sigma) \subset \operatorname{Pro}(\mathbf{Spc})$. One source of difficulty in profinite homotopy theory is that the Σ-completion functor does not preserve finite limits, or even finite products (see [SAG, Remark E.5.2.6; 2, Remark 3.10]). The purpose of this note is to explain two situations in which Σ-completion does preserve products.

The first is that if one is already in the setting of profinite homotopy theory, then Σ-completion preserves products:

0.1 Proposition (Corollary 2.9)**.** *Let Σ be a set of primes. Then the Σ-completion functor restricted to profinite spaces*

$$(-)^\wedge_\Sigma \colon \operatorname{Pro}(\mathbf{Spc}_\pi) \to \operatorname{Pro}(\mathbf{Spc}_\Sigma)$$

preserves products.

The second is that in the setting of étale homotopy theory, Σ-completion preserves finite products. Given a scheme X, write $\Pi^{\text{ét}}_\infty(X) \in \operatorname{Pro}(\mathbf{Spc})$ for the étale homotopy type of X.

0.2 Proposition (Corollary 2.15)**.** *Let Σ be a set of primes and let X and Y be qcqs schemes. Then the natural map of profinite spaces*

$$\left(\Pi^{\text{ét}}_\infty(X) \times \Pi^{\text{ét}}_\infty(Y)\right)^\wedge_\Sigma \to \Pi^{\text{ét}}_\infty(X)^\wedge_\Sigma \times \Pi^{\text{ét}}_\infty(Y)^\wedge_\Sigma$$

is an equivalence.

0.3 Remark. Proposition 0.2 fills a gap in the proof of [4, Theorem 5.3]. Chough's proof cites the false claim that profinite completion preserves finite limits. However, what Chough actually uses is Proposition 0.2 (with Σ the set of all primes). In particular, the conclusion of [4, Theorem 5.3] remains valid.

0.4 Proof Strategy. Propositions 0.1 and 0.2 turn out to be consequences of a more general result. To explain why this is the case, first note that since Σ-completion preserves cofitlered limits, to prove Proposition 0.1 it suffices to show that Σ-completion preserves finite products of π-finite spaces. This reduction is useful because π-finite spaces admit very nice presentations: every π-finite space can be written as the geometric realization (in $\mathbf{Spc}$) of a Kan complex with finitely many simplices in each dimension [SAG, Lemma E.1.6.5].

Similarly, to prove Proposition 0.2 we use that the étale homotopy type of a qcqs scheme admits a nice presentation. To see this, the first technical observation is that since protruncation preserves limits [5, Proposition 3.9] and profinite completion factors through protruncation, it suffices to replace the étale homotopy types on the left-hand side by their protruncations. Our work with Barwick and Glasman [1, Theorems 10.2.3 & 12.5.1] provides a description of the protruncated étale homotopy type as the protruncated classifying space of an explicit profinite category. Said differently, the protruncated étale homotopy type can be written as a geometric realization of a simplicial profinite space computed in the larger ∞-category of protruncated spaces (see Example A.9).

Hence we're done if we can prove the more general claim that Σ-completion preserves products of protruncated spaces that admit such presentations. This follows once we know that that geometric realizations preserve finite products in the ∞-categories of protruncated and Σ-profinite spaces (see Proposition 1.15 and Corollary 1.16). See Lemma 2.7 and Corollary 2.8 for the key categorical results that we use.

0.5 Linear Overview. Section 1 proves that geometric realizations preserve finite products in the ∞-categories of protruncated and profinite spaces. Section 2 proves Propositions 0.1 and 0.2. It is immediate from [1, Theorem 10.2.3] that the protruncated étale homotopy type can be written as the geometric realization of a simplicial profinite space. However, for ease of reference we have provided a detailed explanation of this fact in Appendix A.

0.6 Conventions. Throughout, we use the notational conventions of [5, §§1 & 3].

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1 Universality of colimits

In this section, we prove that geometric realizations are universal in the ∞-categories of protruncated and Σ-profinite spaces. A second goal of this section is to record a more general fact: colimits over diagrams that can be computed as finite colimits when valued in an n-category (see Definition 1.9) are universal in the ∞-categories of protruncated and Σ-profinite spaces (Proposition 1.15 and Corollary 1.16).

The first observation is that finite colimits are universal in protruncated spaces.

1.1 Lemma. *Let $\mathcal{C}$ be an ∞-category with pullbacks and finite colimits. If finite colimits are universal in $\mathcal{C}$, then finite colimits are universal in* $\mathrm{Pro}(\mathcal{C})$*.*

Proof. By (the dual of) [HTT, Proposition 5.3.5.15], pullbacks, pushouts, and finite coproducts are computed 'levelwise' in $\mathrm{Pro}(\mathcal{C})$. Thus the assumption that finite colimits are universal in $\mathcal{C}$ implies the claim. □

1.2 Example. Finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc})$. For each integer $n \geq 0$, finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc}_{\leq n})$.

1.3 Recollection. A localization $L \colon \mathcal{C} \to \mathcal{D}$ is *locally cartesian* if for any cospan $X \to Z \leftarrow Y$ such that $X, Z \in \mathcal{D}$, the natural map $L(X \times_Z Y) \to X \times_Z L(Y)$ is an equivalence.

1.4 Example [5, Proposition 3.18]. For any set Σ of primes, the localization

$$(-)^{\wedge}_{\Sigma} \colon \operatorname{Pro}(\mathbf{Spc}_{<\infty}) \to \operatorname{Pro}(\mathbf{Spc}_{\Sigma})$$

is locally cartesian. However, $(-)^{\wedge}_{\Sigma}$ does not generally preserve finite products.

The following is immediate from the definitions:

1.5 Lemma. *Let $\mathcal{I}$ be an ∞-category, $\mathcal{C}$ an ∞-category with pullbacks and $\mathcal{I}$-shaped colimits, and let $L \colon \mathcal{C} \to \mathcal{D}$ be a locally cartesian localization. If $\mathcal{I}$-shaped colimits are universal in $\mathcal{C}$, then $\mathcal{I}$-shaped colimits are universal in $\mathcal{D}$.*

1.6 Example. Since the protruncation functor $\tau_{<\infty} \colon \operatorname{Pro}(\mathbf{Spc}) \to \operatorname{Pro}(\mathbf{Spc}_{<\infty})$ preserves limits [5, Proposition 3.9], Example 1.2 and Lemma 1.5 show that finite colimits are universal in $\operatorname{Pro}(\mathbf{Spc}_{<\infty})$.

Now we formulate the key property of the category $\mathbf{\Delta}^{\mathrm{op}}$ that we need.

1.7 Definition. Let $n \geq 0$ be an integer. A functor between ∞-categories $c \colon \mathcal{I} \to \mathcal{J}$ is *n-colimit-cofinal* if for every n-category $\mathcal{C}$ and functor $f \colon \mathcal{J} \to \mathcal{C}$, the following conditions are satisfied:

(1.7.1) The colimit $\operatorname{colim}_{\mathcal{J}} f$ exists if and only if the colimit $\operatorname{colim}_{\mathcal{I}} fc$ exists.

(1.7.2) If the colimit $\operatorname{colim}_{\mathcal{J}} f$ exists, then the natural map $\operatorname{colim}_{\mathcal{I}} fc \to \operatorname{colim}_{\mathcal{J}} f$ is an equivalence.

1.8 Example. For an integer $n \geq 0$, write $\mathbf{\Delta}_{\leq n} \subset \mathbf{\Delta}$ for the full subcategory spanned by those linearly ordered finite sets of cardinality $\leq n+1$. By [6, Proposition A.1], the inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\leq n} \subset \mathbf{\Delta}^{\mathrm{op}}$ is n-colimit-cofinal.

1.9 Definition. Let $\mathcal{I}$ be an ∞-category. We say that $\mathcal{I}$ is *almost finite* if for each integer $n \geq 0$, there exists a *finite* ∞-category $\mathcal{I}_n$ and an n-colimit-cofinal functor $c_n \colon \mathcal{I}_n \to \mathcal{I}$.

1.10 Example. If $\mathcal{I}$ is an ∞-category that admits a colimit-cofinal functor from a finite ∞-category, then $\mathcal{I}$ is almost finite.

1.11 Example. For each $n \geq 0$, the category $\mathbf{\Delta}^{\mathrm{op}}_{\leq n}$ is a finite ∞-category [3, Example 6.5.3]. Hence the category $\mathbf{\Delta}^{\mathrm{op}}$ is almost finite: the inclusion $\mathbf{\Delta}^{\mathrm{op}}_{\leq n} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is an n-colimit-cofinal functor from a finite ∞-category.

1.12 Recollection. A space X is *almost π-finite* if $\pi_0(X)$ is finite and all homotopy groups of X are finite. Since an almost π-finite space admits a presentation by a Kan complex with finitely many simplices in each dimension [SAG, Lemma E.1.6.5], the following implies that almost π-finite spaces are also almost finite:

1.13 Example. Let K be a simplicial set with finitely many simplices in each dimension and let $\mathcal{I}$ be the ∞-category presented by K. Then $\mathcal{I}$ is almost finite: we take $\mathcal{I}_n$ to be the ∞-category presented by the $(n+1)$-skeleton of $\operatorname{sk}_{n+1} K$ and $c_n \colon \mathcal{I}_n \to \mathcal{I}$ the functor induced by the inclusion $\operatorname{sk}_{n+1} K \subset K$.

1.14 Definition. Let $\mathcal{C}$ be an ∞-category with pullbacks. We say that *almost finite colimits are universal in $\mathcal{C}$* if for each almost finite ∞-category $\mathcal{I}$, the ∞-category $\mathcal{C}$ admits $\mathcal{I}$-shaped colimits and $\mathcal{I}$-shaped colimits are unviersal in $\mathcal{C}$.

The argument that Lurie gives in the proof of [SAG, Theorem E.6.3.1] shows that almost finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc}_\Sigma)$ (however, this result is only stated when Σ is the set of all primes). We also need to know that almost finite colimits are also universal in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$. The strategy is the same as Lurie's proof: we use that equivalences are checked on truncations to reduce to the case of finite colimits.

1.15 Proposition. *Almost finite colimits are universal in* $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$.

Moreover, Proposition 1.15 reproves [SAG, Theorem E.6.3.1]:

1.16 Corollary. *Let Σ be a set of primes. Then almost finite colimits are universal in* $\mathrm{Pro}(\mathbf{Spc}_\Sigma)$.

Proof of Corollary 1.16. Since the localization $(-)^\wedge_\Sigma \colon \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \to \mathrm{Pro}(\mathbf{Spc}_\Sigma)$ is locally cartesian [5, Proposition 3.18], this follows from Lemma 1.5 and Proposition 1.15. □

Proof of Proposition 1.15. Let $\mathcal{I}$ be an almost finite ∞-category, let $f \colon X \to Z$ be a morphism in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$, and let

$$g \colon \mathcal{I} \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})_{/Z}$$

be a diagram of protruncated spaces over Z. To prove the claim, it suffices to show that for each integer $n \geq 0$, the induced map

$$\tau_{\leq n}\left(\operatorname*{colim}_{i \in \mathcal{I}} X \times_Z g(i)\right) \to \tau_{\leq n}\left(X \times_Z \operatorname*{colim}_{i \in \mathcal{I}} g(i)\right)$$

is an equivalence in $\mathrm{Pro}(\mathbf{Spc}_{\leq n})$. Since $\mathcal{I}$ is almost finite, there exists a finite ∞-category $\mathcal{I}_{n+2}$ and $(n+2)$-colimit-cofinal functor $c_{n+2} \colon \mathcal{I}_{n+2} \to \mathcal{I}$. Consider the commutative diagram

$$\begin{tikzcd}
\operatorname*{colim}_{j \in \mathcal{I}_{n+2}} \tau_{\leq n}\left(X \times_Z gc_{n+2}(j)\right) \arrow[r, "\sim"] \arrow[d] & \tau_{\leq n}\left(\operatorname*{colim}_{j \in \mathcal{I}_{n+2}} X \times_Z gc_{n+2}(j)\right) \arrow[r] \arrow[d] & \tau_{\leq n}\left(X \times_Z \operatorname*{colim}_{j \in \mathcal{I}_{n+2}} gc_{n+2}(j)\right) \arrow[d] \\
\operatorname*{colim}_{i \in \mathcal{I}} \tau_{\leq n}\left(X \times_Z g(i)\right) \arrow[r, "\sim"] & \tau_{\leq n}\left(\operatorname*{colim}_{i \in \mathcal{I}} X \times_Z g(i)\right) \arrow[r] & \tau_{\leq n}\left(X \times_Z \operatorname*{colim}_{i \in \mathcal{I}} g(i)\right) .
\end{tikzcd}$$

(Here, the colimits in the leftmost column are computed in $\mathrm{Pro}(\mathbf{Spc}_{\leq n})$.) Since $\mathrm{Pro}(\mathbf{Spc}_{\leq n})$ is an $(n+1)$-category and $c_{n+2} \colon \mathcal{I}_{n+2} \to \mathcal{I}$ is $(n+2)$-colimit-cofinal, the leftmost vertical map is an equivalence. Thus the central vertical map is also an equivalence. Since $\mathcal{I}_{n+2}$ is finite and finite colimits are universal in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$ (Example 1.6), the top right-hand horizontal map is an equivalence.

To complete the proof, it suffices to show that the rightmost vertical map is an equivalence. For this, consider the commutative square

$$\begin{tikzcd}
\operatorname*{colim}_{j \in \mathcal{I}_{n+2}} \tau_{\leq n+1}\, gc_{n+2}(j) \arrow[r, "\sim"] \arrow[d] & \tau_{\leq n+1}\left(\operatorname*{colim}_{j \in \mathcal{I}_{n+2}} gc_{n+2}(j)\right) \arrow[d] \\
\operatorname*{colim}_{i \in \mathcal{I}} \tau_{\leq n+1}\, g(i) \arrow[r, "\sim"] & \tau_{\leq n+1}\left(\operatorname*{colim}_{i \in \mathcal{I}} g(i)\right) ,
\end{tikzcd}$$

where the colimits in the left-hand column are computed in $\mathrm{Pro}(\mathbf{Spc}_{\leq n+1})$. Since $\mathrm{Pro}(\mathbf{Spc}_{\leq n+1})$ is an $(n+2)$-category and c_{n+2} is $(n+2)$-colimit-cofinal, the left-hand vertical map is an equivalence. Hence the right-hand vertical map is also an equivalence. As a consequence, the map

$$\operatorname*{colim}_{j \in \mathcal{J}_{n+2}} gc_{n+2}(j) \to \operatorname*{colim}_{i \in \mathcal{J}} g(i)$$

is n-connected. Thus the basechange

$$X \underset{Z}{\times} \operatorname*{colim}_{j \in \mathcal{J}_{n+2}} gc_{n+2}(j) \to X \underset{Z}{\times} \operatorname*{colim}_{i \in \mathcal{J}} g(i)$$

is also n-connected. Hence the map

$$\tau_{\leq n}\left(X \underset{Z}{\times} \operatorname*{colim}_{j \in \mathcal{J}_{n+2}} gc_{n+2}(j)\right) \to \tau_{\leq n}\left(X \underset{Z}{\times} \operatorname*{colim}_{i \in \mathcal{J}} g(i)\right)$$

is an equivalence, as desired. □

Note that the universality of geometric realizations implies that geometric realizations preserve finite products:

1.17 Lemma. *Let $\mathcal{J}$ be a sifted ∞-category and let $\mathcal{C}$ be an ∞-category with finite limits and $\mathcal{J}$-shaped colimits. If $\mathcal{J}$-shaped colimits are universal in $\mathcal{C}$, then the functor*

$$\operatorname{colim}_{\mathcal{J}} \colon \operatorname{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}$$

preserves finite products.

Proof. Let $X_\bullet, Y_\bullet \colon \mathcal{J} \to \mathcal{C}$ be functors. We have natural equivalences

$$\begin{aligned}
\operatorname*{colim}_{i \in \mathcal{J}} X_i \times Y_i &\xrightarrow{\sim} \operatorname*{colim}_{(i,j) \in \mathcal{J} \times \mathcal{J}} X_i \times Y_j && (\mathcal{J} \text{ is sifted}) \\
&\simeq \operatorname*{colim}_{i \in \mathcal{J}} \operatorname*{colim}_{j \in \mathcal{J}} (X_i \times Y_j) \\
&\xrightarrow{\sim} \operatorname*{colim}_{i \in \mathcal{J}} \left(X_i \times \operatorname*{colim}_{j \in \mathcal{J}} Y_j\right) && (\mathcal{J}\text{-shaped colimits are universal}) \\
&\xrightarrow{\sim} \left(\operatorname*{colim}_{i \in \mathcal{J}} X_i\right) \times \left(\operatorname*{colim}_{j \in \mathcal{J}} Y_j\right) && (\mathcal{J}\text{-shaped colimits are universal})\,.
\end{aligned}$$

□

1.18 Example. Let Σ be a set of primes. Proposition 1.15, Corollary 1.16, and Lemma 1.17 show that geometric realizations preserve finite products in the ∞-categories $\mathrm{Pro}(\mathbf{Spc}_\Sigma)$ and $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$.

2 Completions of products

The goal of this section is to prove Propositions 0.1 and 0.2. We do this by noting that in the setting of both of these results, the prospaces of interest can be written as geometric realizations of simplicial profinite spaces. We begin by axiomatizing the situation:

2.1 Definition. Let $\mathcal{C}$ be an ∞-category and $\mathcal{D} \subset \mathcal{C}$ a full subcategory. We say that an object $X \in \mathcal{C}$ *admits a $\mathcal{D}$-resolution* if there exists a simplicial object $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{D}$ and an equivalence

$$X \simeq \operatorname{colim}\left(\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{D} \hookrightarrow \mathcal{C} \right).$$

2.2 Notation. Write $\mathbf{Set}_{\mathrm{fin}}$ for the category of a finite sets and all maps.

2.3 Example. Let X be a space which is finite[1] or almost π-finite. Then:

(2.3.1) As an object of $\mathbf{Spc}$, the space X admits a $\mathbf{Set}_{\mathrm{fin}}$-resolution: If X is finite, then X can be written as the geometric realization of a simplicial set with finitely many nondegenerate simplices [12, Proposition 2.4].[2] If X is almost π-finite, then X can be written as the geometric realization of a Kan complex with finitely many simplices in each degree [SAG, Lemma E.1.6.5].

(2.3.2) Since the Yoneda embedding $\mathbf{Spc} \hookrightarrow \mathrm{Pro}(\mathbf{Spc})$ preserves colimits, when regarded as an object of $\mathrm{Pro}(\mathbf{Spc})$, the space X admits a $\mathbf{Set}_{\mathrm{fin}}$-resolution.

(2.3.3) Since the protruncation functor $\tau_{<\infty} : \mathrm{Pro}(\mathbf{Spc}) \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})$ preserves colimits, the protruncated space $\tau_{<\infty}(X)$ admits a $\mathbf{Set}_{\mathrm{fin}}$-resolution.

(2.3.4) Since the profinite completion functor $(-)^\wedge_\pi : \mathrm{Pro}(\mathbf{Spc}) \hookrightarrow \mathrm{Pro}(\mathbf{Spc}_\pi)$ preserves colimits, the profinite space $X^\wedge_\pi$ admits a $\mathbf{Set}_{\mathrm{fin}}$-resolution.

Algebraic geometry also gives rise to many examples of protruncated spaces admitting profinite resolutions.

2.4 Notation (shapes)**.** Given an ∞-topos $\boldsymbol{X}$, we write $\Pi_\infty(\boldsymbol{X}) \in \mathrm{Pro}(\mathbf{Spc})$ for the *shape* of $\boldsymbol{X}$. We write $\Pi_{<\infty}(\boldsymbol{X})$ for the protruncation of $\Pi_\infty(\boldsymbol{X})$.

2.5. Recall that an ∞-topos $\boldsymbol{X}$ is *spectral* in the sense of [1, Definition 9.2.1] if $\boldsymbol{X}$ is bounded coherent and the ∞-category $\mathrm{Pt}(\boldsymbol{X})$ of points of $\boldsymbol{X}$ has the property that every endomorphism of an object of $\mathrm{Pt}(\boldsymbol{X})$ is an equivalence. The most important example of a spectral ∞-topos is the étale ∞-topos of a qcqs scheme [1, Example 9.2.4]. Example A.9 explains why the protruncated shape $\Pi_{<\infty}(\boldsymbol{X})$ of a spectral ∞-topos admits a natural $\mathrm{Pro}(\mathbf{Spc}_\pi)$-resolution.

The next few results are the key categorical input we need. To state them, we axiomatize the abstract properties of the subcategory $\mathrm{Pro}(\mathbf{Spc}_\pi) \subset \mathrm{Pro}(\mathbf{Spc}_{<\infty})$.

2.6 Notation. Let $\mathcal{C}$ be an ∞-category with geometric realizations and finite products, and let $L : \mathcal{C} \to \mathcal{D}$ be a localization. Assume that geometric realizations preserve finite products in both $\mathcal{C}$ and $\mathcal{D}$. Write

$$\mathcal{C}_{|\mathcal{D}|} \subset \mathcal{C}$$

for the smallest full subcategory containing $\mathcal{D} \subset \mathcal{C}$ and closed under geometric realizations and retracts.

[1] I.e., in the smallest subcategory of $\mathbf{Spc}$ containing the point and closed under finite coproducts and pushouts. Equivalently, a space X is finite if and only if X is represented by a finite CW complex.

[2] In fact, every finite space is equivalent to the classifying space of a finite poset; see [HTT, Proposition 4.1.1.3(3) & Variant 4.2.3.16; Ker, Tag 02MU; 10, Theorem 1].

2.7 Lemma. *In the setting of Notation 2.6, let $X_\bullet, Y_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ be simplicial objects with colimits X and Y. Assume that for each $n \geq 0$, the natural map*

$$L(X_n \times Y_n) \to L(X_n) \times L(Y_n)$$

is an equivalence. Then the natural map

$$L(X \times Y) \to L(X) \times L(Y)$$

is an equivalence.

Proof. We compute

$$\begin{aligned}
L(X \times Y) &\simeq L\left(\operatorname*{colim}_{m \in \mathbf{\Delta}^{\mathrm{op}}} X_m \times \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} Y_n\right) & \\
&\simeq L\left(\operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} X_n \times Y_n\right) & &\text{(geometric realizations preserve finite products in } \mathcal{C}\text{)} \\
&\simeq \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} L(X_n \times Y_n) & \\
&\xrightarrow{\sim} \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} L(X_n) \times L(Y_n) & &\text{(assumption)} \\
&\simeq \operatorname*{colim}_{m \in \mathbf{\Delta}^{\mathrm{op}}} L(X_m) \times \operatorname*{colim}_{n \in \mathbf{\Delta}^{\mathrm{op}}} L(Y_n) & &\text{(geometric realizations preserve finite products in } \mathcal{D}\text{)} \\
&\simeq L(X) \times L(Y)\,. &
\end{aligned}$$

□

2.8 Corollary. *In the setting of Notation 2.6:*

(1) *The full subcategory of $\mathcal{C} \times \mathcal{C}$ spanned by those objects (X, Y) such that the natural map*

$$L(X \times Y) \to L(X) \times L(Y)$$

is an equivalence is closed under geometric realizations and retracts.

(2) *If $X, Y \in \mathcal{C}_{|\mathcal{D}|}$, then the natural map $L(X \times Y) \to L(X) \times L(Y)$ is an equivalence.*

(3) *If $X, Y \in \mathcal{C}$ admit $\mathcal{D}$-resolutions, then the natural map $L(X \times Y) \to L(X) \times L(Y)$ is an equivalence.*

Proof. Item (1) follows from Lemma 2.7 and the fact that equivalences are closed under retracts. Now (2) is an immediate consequence of (1) and the definition of $\mathcal{C}_{|\mathcal{D}|}$ as the closure of $\mathcal{D} \subset \mathcal{C}$ under geometric realizations and retracts. Finally, (3) is a special case of (2). □

In the rest of this section, we record consequences of Corollary 2.8.

2.9 Corollary. *Let Σ be a set of prime numbers. Then the Σ-completion functor*

$$(-)^{\wedge}_{\Sigma} \colon \operatorname{Pro}(\mathbf{Spc}_{\pi}) \to \operatorname{Pro}(\mathbf{Spc}_{\Sigma})$$

preserves products.

Proof. Since Σ-completion preserves cofiltered limits and the terminal object, it suffices to show that Σ-completion preserves binary products of profinite spaces. Again because Σ-completion preserves cofiltered limits, we are reduced to showing that if X and Y are π-finite spaces, then the natural map

$$(X \times Y)^{\wedge}_{\Sigma} \to X^{\wedge}_{\Sigma} \times Y^{\wedge}_{\Sigma}$$

is an equivalence. Since π-finite spaces admit $\mathbf{Set}_{\mathrm{fin}}$-resolutions (Example 2.3) and geometric realizations preserve finite products in $\mathrm{Pro}(\mathbf{Spc}_{\pi})$ and $\mathrm{Pro}(\mathbf{Spc}_{\Sigma})$ (Example 1.18), the claim follows from Corollary 2.8. □

2.10 Warning. The functor $(-)^{\wedge}_{\Sigma} \colon \mathrm{Pro}(\mathbf{Spc}_{\pi}) \to \mathrm{Pro}(\mathbf{Spc}_{\Sigma})$ does not generally preserve pullbacks, or even loop objects.

We now introduce a slight enlargement of the subcategory of protruncated spaces admitting a $\mathrm{Pro}(\mathbf{Spc}_{\pi})$-resolution on which profinite completion preserves finite products.

2.11 Notation. Let

$$\mathrm{Pro}(\mathbf{Spc}_{<\infty})' \subset \mathrm{Pro}(\mathbf{Spc}_{<\infty})$$

denote the smallest full subcategory containing $\mathrm{Pro}(\mathbf{Spc}_{\pi})$ and closed under geometric realizations, retracts, and cofiltered limits.

2.12 Observation (procompact spaces)**.** Write $\mathbf{Spc}^{\omega} \subset \mathbf{Spc}$ for the full subcategory spanned by the compact objects.[3] Then by Example 2.3, the image of

$$\tau_{<\infty} \colon \mathrm{Pro}(\mathbf{Spc}^{\omega}) \to \mathrm{Pro}(\mathbf{Spc}_{<\infty})$$

is contained in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})'$.

2.13 Proposition. *Let Σ be a set of primes and let $X, Y \in \mathrm{Pro}(\mathbf{Spc}_{<\infty})'$. Then the natural map*

$$(X \times Y)^{\wedge}_{\Sigma} \to X^{\wedge}_{\Sigma} \times Y^{\wedge}_{\Sigma}$$

is an equivalence.

Proof. By Corollary 2.9, the Σ-completion functor

$$(-)^{\wedge}_{\Sigma} \colon \mathrm{Pro}(\mathbf{Spc}_{\pi}) \to \mathrm{Pro}(\mathbf{Spc}_{\Sigma})$$

preserves products. Hence it suffices to prove the claim in the special case where Σ is the set of all primes. For this, note that cofiltered limits preserve finite products in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$ and $\mathrm{Pro}(\mathbf{Spc}_{\pi})$; moreover, the profinite completion functor

$$(-)^{\wedge}_{\pi} \colon \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \to \mathrm{Pro}(\mathbf{Spc}_{\pi})$$

preserves cofiltered limits. Thus it suffices to treat the case where X and Y are in the smallest full subcategory of $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$ containing $\mathrm{Pro}(\mathbf{Spc}_{\pi})$ and closed under geometric realizations and retracts. In this case, since geometric realizations preserve finite products in $\mathrm{Pro}(\mathbf{Spc}_{<\infty})$ and $\mathrm{Pro}(\mathbf{Spc}_{\pi})$ (Example 1.18), Corollary 2.8 shows that the natural map

$$(X \times Y)^{\wedge}_{\pi} \to X^{\wedge}_{\pi} \times Y^{\wedge}_{\pi}$$

is an equivalence. □

[3] Recall that a space X is compact if and only if X is a retract of a finite space. In more classical terminology, a space X is compact if and only if X is represented by a *fintiely dominated* CW complex.

2.14. In particular, if X and Y are protruncated spaces that admit $\mathrm{Pro}(\mathbf{Spc}_\pi)$-resolutions, then the natural map $(X \times Y)^\wedge_\Sigma \to X^\wedge_\Sigma \times Y^\wedge_\Sigma$ is an equivalence.

Since protruncation preserves finite products, Proposition 2.13 and Example A.9 show:

2.15 Corollary. *Let Σ be a set of primes and let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be spectral ∞-topoi. Then the natural map*

$$(\Pi_\infty(\boldsymbol{X}) \times \Pi_\infty(\boldsymbol{Y}))^\wedge_\Sigma \to \Pi_\infty(\boldsymbol{X})^\wedge_\Sigma \times \Pi_\infty(\boldsymbol{Y})^\wedge_\Sigma$$

is an equivalence.

A Classifying prospaces via geometric realizations

The purpose of this appendix is to explain why the protruncated shape of a spectral ∞-topos (e.g., the étale ∞-topos of a qcqs scheme) admits a presentation as a geometric realization of a simplicial profinite space (Example A.9). Using the description of the protruncated shape of a spectral ∞-topos as a protruncated classifying prospace given by Barwick–Glasman–Haine [1, Theorem 10.2.3], this is an exercise in the definitions. For the ease of the reader, we spell out the details here.

We make use of the description of ∞-categories as simplicial spaces.

A.1 Recollection (∞-categories as simplicial spaces)**.** The nerve functor

$$\begin{aligned} \mathrm{N} \colon \mathbf{Cat}_\infty &\to \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathbf{Spc}) \\ \mathcal{C} &\mapsto [I \mapsto \mathrm{Fun}(I, \mathcal{C})^\simeq] \end{aligned}$$

is fully faithful [HA, Proposition A.7.10; SAG, §A.8.2; 7; 8, §1; 11]. One can explicitly identify its image; objects in the image of this embedding are often called *complete Segal spaces* or *categories internal to spaces*. Under this embedding, the subcategory $\mathbf{Spc} \subset \mathbf{Cat}_\infty$ corresponds to the constant functors $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathbf{Spc}$. Moreover, the localization $\mathrm{B} \colon \mathbf{Cat}_\infty \to \mathbf{Spc}$ is given by geometric realization.

A.2 Notation. Let $\mathcal{C}$ be an ∞-category with finite limits. We write

$$\mathrm{Cat}(\mathcal{C}) \subset \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{C})$$

for the full subcategory spanned by the *categories internal to* $\mathcal{C}$. See [1, Definition 13.1.1; 9, Proposition 3.2.7] for the definition.

A.3 Notation. Write $\mathbf{Cat}_{<\infty} \subset \mathbf{Cat}_\infty$ for the full subcategory spanned by those ∞-categories $\mathcal{C}$ for which there exists an integer $n \geq 0$ such that $\mathcal{C}$ is an n-category. Write $\mathbf{Cat}_{\infty,\pi} \subset \mathbf{Cat}_{<\infty}$ for the full subcategory spanned by those ∞-categories with the property that there are finitely many objects up to equivalence and all mapping spaces are π-finite.

A.4 Observation. The nerve $\mathrm{N} \colon \mathbf{Cat}_\infty \xrightarrow{\sim} \mathrm{Cat}(\mathbf{Spc})$ restricts to equivalences

$$\mathbf{Cat}_{<\infty} \xrightarrow{\sim} \mathrm{Cat}(\mathbf{Spc}_{<\infty}) \qquad \text{and} \qquad \mathbf{Cat}_{\infty,\pi} \xrightarrow{\sim} \mathrm{Cat}(\mathbf{Spc}_\pi)\,.$$

In order to describe protruncated classifying spaces via geometric realizations, it is useful to describe pro-∞-categories as category objects in prospaces.

A.5 Observation. By [HTT, Proposition 5.3.5.11; 1, Proposition 13.1.12], the composite

$$\mathbf{Cat}_{<\infty} \xrightarrow[\mathrm{N}]{\sim} \mathrm{Cat}(\mathbf{Spc}_{<\infty}) \hookrightarrow \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{<\infty}))$$

extends along cofiltered limits to a fully faithful right adjoint

$$\mathrm{N} \colon \mathrm{Pro}(\mathbf{Cat}_{<\infty}) \hookrightarrow \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{<\infty}))\,.$$

This functor restricts to a fully faithful right adjoint

$$\mathrm{N} \colon \mathrm{Pro}(\mathbf{Cat}_{\infty,\pi}) \hookrightarrow \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{\pi}))\,.$$

A.6 Remark. We do not know if the embedding $\mathrm{N} \colon \mathbf{Cat}_{\infty} \hookrightarrow \mathbf{Cat}(\mathrm{Pro}(\mathbf{Spc}))$ extends along cofiltered limits to a fully faithful functor $\mathrm{Pro}(\mathbf{Cat}_{\infty}) \to \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}))$.

A.7 Observation. It is immediate from the definitions that the following diagram of fully faithful right adjoints commutes

$$\begin{tikzcd}
\mathrm{Pro}(\mathbf{Spc}_{\pi}) \arrow[r, shift left, "\mathrm{B}^{\wedge}_{\pi}"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, hook] & \mathrm{Pro}(\mathbf{Cat}_{\infty,\pi}) \arrow[l, shift left, hook] \arrow[r, shift right, hook', "\mathrm{N}"'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, hook] & \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{\pi})) \arrow[l, shift right] \arrow[r, shift right, hook'] \arrow[r, phantom, "\perp"] \arrow[d, shift left, hook] & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathbf{Spc}_{\pi})) \arrow[l, shift right] \arrow[d, shift left, hook] \\
\mathrm{Pro}(\mathbf{Spc}_{<\infty}) \arrow[r, shift left, "\mathrm{B}_{<\infty}"] \arrow[r, phantom, "\perp"] \arrow[u, shift left, "(-)^{\wedge}_{\pi}"] \arrow[u, phantom, "\dashv"] & \mathrm{Pro}(\mathbf{Cat}_{<\infty}) \arrow[l, shift left, hook] \arrow[r, shift right, hook', "\mathrm{N}"'] \arrow[r, phantom, "\perp"] \arrow[u, shift left] \arrow[u, phantom, "\dashv"] & \mathrm{Cat}(\mathrm{Pro}(\mathbf{Spc}_{<\infty})) \arrow[l, shift right] \arrow[r, shift right, hook'] \arrow[r, phantom, "\perp"] \arrow[u, shift left] \arrow[u, phantom, "\dashv"] & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathbf{Spc}_{<\infty}))\,. \arrow[l, shift right] \arrow[u, shift left, "(-)^{\wedge}_{\pi}\circ -"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

The long composite left adjoints

$$\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathbf{Spc}_{<\infty})) \to \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \qquad \text{and} \qquad \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Pro}(\mathbf{Spc}_{\pi})) \to \mathrm{Pro}(\mathbf{Spc}_{\pi})\,.$$

are simply the colimit functors. Since the diagram of left adjoints also commutes we deduce:

A.8 Corollary. *Let $\mathcal{C} \in \mathrm{Pro}(\mathbf{Cat}_{\infty,\pi})$ be a profinite ∞-category. Then there are natural equivalences*

$$\mathrm{B}_{<\infty}(\mathcal{C}) \simeq \operatorname{colim}\left(\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{N}(\mathcal{C})} \mathrm{Pro}(\mathbf{Spc}_{\pi}) \hookrightarrow \mathrm{Pro}(\mathbf{Spc}_{<\infty}) \right)$$

and

$$\mathrm{B}^{\wedge}_{\pi}(\mathcal{C}) \simeq \operatorname{colim}\left(\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{N}(\mathcal{C})} \mathrm{Pro}(\mathbf{Spc}_{\pi}) \right)\,.$$

A.9 Example. Let $\boldsymbol{X}$ be a spectral ∞-topos. Through [1, Theorem 9.3.1], Barwick–Glasman–Haine refined the the ∞-category $\mathrm{Pt}(\boldsymbol{X})$ of points of $\boldsymbol{X}$ to a profinite ∞-category

$$\widehat{\Pi}_{(\infty,1)}(\boldsymbol{X}) \in \mathrm{Pro}(\mathbf{Cat}_{\infty,\pi})$$

called the *stratified shape* of $\boldsymbol{X}$. In [1, Theorem 10.2.3] they show that there is a natural equivalence

$$\Pi_{<\infty}(\boldsymbol{X}) \xrightarrow{\sim} \mathrm{B}_{<\infty}\left(\widehat{\Pi}_{(\infty,1)}(\boldsymbol{X})\right)\,.$$

That is, the protruncated shape of $\boldsymbol{X}$ can be recovered as the protruncated classifying space of the stratified shape $\widehat{\Pi}_{(\infty,1)}(\boldsymbol{X})$. Hence Corollary A.8 shows that the protruncated shape $\Pi_{<\infty}(\boldsymbol{X})$ admits a natural $\mathrm{Pro}(\mathbf{Spc}_{\pi})$-resolution in the sense of Definition 2.1.

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