Introduction/Overview of algebraic K-theory

Outline.

(1) Recollections on K_0 Easy to define, not general

(2) K-theory via group completion

(3) K-theory via Quillen's plus-construction Good to compute with, even less general

(4) Definition of connective/nonconnective K-theory as the universal additive/localizing invariant Elegant & general, harder to define

(5) Construction of connective K-theory via Waldhausen's $S_\bullet$-construction.

1. Recollections on K_0

Recall. The inclusion $\mathrm{Ab} \hookrightarrow \mathrm{CMon}$ admits a left adjoint $(-)^{gp} : \mathrm{CMon} \to \mathrm{Ab}$ called group completion.

$$M^{gp} \cong \frac{\mathbb{Z}[M]}{\langle [m+n]-[m]-[n] \rangle}.$$

Examples. $(\mathbb{N},+)^{gp} \cong \mathbb{Z}$, $(\mathbb{N},\cdot)^{gp} \cong 0$, $(\mathbb{N}\setminus\{0\},\cdot)^{gp} \cong \mathbb{Q}_{>0}$.

Notation. For an associative ring R, write $\mathrm{Proj}(R)$ for the category of finitely generated projective modules.

Definition. Let R be an (associative) ring. The Grothendieck group of R is the group completion

maximal subgroupoid

$$K_0(R) := (\pi_0 \mathrm{Proj}(R)^{\simeq}, \oplus)^{gp}$$

Eilenberg Swindle. If we replace $\mathrm{Proj}(R)$ with the category of countably generated / all projective R-modules, then the group completion is 0! ← Try as an exercise if you haven't seen this before

Takeaway. $K_0(R)$ only depends on the category of modules over R, hence is Morita-invariant.

Alternative formulation. Write $\mathrm{Perf}(R) := \underbrace{\mathrm{Mod}(R)}_{\text{`}D(R)\text{' } \infty\text{-category of } R\text{-module spectra}}^{\omega}$ (compact objects) for the stable ∞-category of perfect complexes over R. The map

$$K_0(R) \longrightarrow \frac{\mathbb{Z}[\pi_0 \mathrm{Perf}(R)^{\simeq}]}{\left\langle \begin{array}{l} [X]+[Z]=[Y] \text{ if there is} \\ \text{a cofiber seq. } X \to Y \to Z \end{array} \right\rangle}$$

$$P \longmapsto P[0]$$

is an isomorphism.

- K_0 is really an invariant of stable ∞-categories.
- $\mathrm{Perf}(R)$ is better than $\mathrm{Proj}(R)$:

  (1) We're allowed to consider more modules

  (2) We're splitting all SES's and not relying on the fact that every SES of projective modules splits.

  (3) Better / more generalizable category-theoretic extraction

$$R \rightsquigarrow \mathrm{Mod}(R) \rightsquigarrow \mathrm{Mod}(R)^{\omega}$$

Bad to gen. to nonconnective ring spectra

$$\left\{ R \rightsquigarrow \mathrm{Mod}(R) \rightsquigarrow \underbrace{\mathrm{Mod}(R)_{\geq 0}}_{\text{connective}} \rightsquigarrow \left(\begin{array}{c} \text{compact projective objects} \\ \text{of } \mathrm{Mod}(R)_{\geq 0} \end{array} \right) \right.$$

⚠ $\mathrm{Mod}(R)$ has no nonzero projectives!

- However, it is easier to define the K-theory spectrum from $\mathrm{Proj}(R)$, so we'll do that first.

2. K-theory via group completion

Idea. Repeat Grothendieck's construction of $K_0(R)$ without truncating

$$(\pi_0(\mathrm{Proj}(R)^{\simeq}))^{gp} \rightsquigarrow (\underbrace{\mathrm{Proj}(R)^{\simeq}}_{E_\infty\text{-monoids}})^{gp}.$$

of monoids   of E_∞-monoids

Recollection on E_∞-Spaces & Spectra

Definition. Let C be an ∞-category with finite products. An E_∞-monoid in C is a functor $M : \mathrm{Set}_*^{fin} \longrightarrow C$ such that for all $n \geq 0$, the collapse maps

$$\{1,\dots,n\}_+ \longrightarrow \{i\}_+ \qquad j \longmapsto \begin{cases} i, & j = i \\ *, & \text{else} \end{cases}$$

induce an equivalence $M(\{1,\dots,n\}_+) \xrightarrow{\sim} \prod_{i=1}^{n} M(\{i\}_+)$

$n = 0 \Rightarrow M(*) = *$

$\Updownarrow$

$$\begin{tikzcd} S \vee T \arrow[r, "\text{collapse}"] \arrow[d, "\text{collapse}"'] & T \arrow[d] \\ S \arrow[r] & * \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \longmapsto \begin{tikzcd} M(S \vee T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M(T) \arrow[d] \\ M(S) \arrow[r] & M(*) \simeq * \end{tikzcd}$$

'excision'

> Underlying object: $M(\underbrace{\{1\}_+}_{S^0}) \in \mathrm{Spc}$

> $\mathrm{Mon}_{E_\infty}(C) \subset \mathrm{Fun}(\mathrm{Set}_*^{fin}, C)$ full subcategory of E_∞-monoids.

An E_∞-monoid M in Spc is an E_∞-group if $\pi_0 M$ with the induced monoid structure is a group.

> $\mathrm{Grp}_{E_\infty}(\mathrm{Spc}) \subset \mathrm{Mon}_{E_\infty}(\mathrm{Spc})$ full subcategory of E_∞-groups.

Recall. $Sp := \mathrm{Exc}_*(\underbrace{Spc_*^{fin}}_{\text{finite pointed spaces}}, Spc)$ $\quad F(*) \simeq *$

$$F\left(\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\lrcorner"] & Z \end{tikzcd}\right) \text{ is a pullback}$$

> $\Omega^\infty : Sp \longrightarrow Spc$ is evaluation at $S^0 = \{1\}_+$

> If X and Y are any pointed spaces, then

$$\begin{tikzcd} X \vee Y \arrow[r, "\text{collapse}"] \arrow[d, "\text{Collapse}"'] & Y \arrow[d] \\ X \arrow[r] & * \end{tikzcd} \quad \text{is a pushout}$$

- Taking X and Y to be finite pointed sets, we see that Ω^∞ factors as

$$\begin{tikzcd} Sp \arrow[r, "\text{rest.}"] \arrow[d, phantom, "\cap" rotate=90] & Grp_{E_\infty}(Spc) \arrow[r, "\text{forget}"] \arrow[d, phantom, "\cap" rotate=90] & Spc. \\ Fun(Spc_*^{fin}, Spc) \arrow[r] & Fun(Set_*^{fin}, Spc) & \end{tikzcd}$$

Recognition principle (Segal). $\Omega^\infty : Sp \longrightarrow Grp_{E_\infty}(Spc)$ restricts to an equivalence

$$\Omega^\infty : \underbrace{Sp_{\geq 0}}_{\text{connective}} \xrightarrow{\sim} Grp_{E_\infty}(Spc).$$

Point. To define the K-theory spectrum of R, we'll define an E_∞-group

Proposition. The inclusion $Grp_{E_\infty}(Spc) \hookrightarrow Mon_{E_\infty}(Spc)$ admits a left adjoint $(-)^{gp} : Mon_{E_\infty}(Spc) \longrightarrow Grp_{E_\infty}(Spc)$.

> One can prove this with the adjoint functor theorem.

⚠ M truncated $\not\Rightarrow$ M^{gp} truncated. This is actually good news for us!

Additive K-theory

'direct sum' K-theory, 'Segal' K-theory, Quillen's '$S^{-1}S$ construction'

Note. $(-)^{\simeq} : \mathrm{Cat}_\infty \longrightarrow \mathrm{Spc}$ is a right adjoint, hence preserves E_∞-monoids.

Definition. Write K^{add} for the composite

symm. mon. ∞-cats

$$K^{add} : \mathrm{Mon}_{E_\infty}(\mathrm{Cat}_\infty) \xrightarrow{(-)^{\simeq}} \mathrm{Mon}_{E_\infty}(\mathrm{Spc}) \xrightarrow{(-)^{gp}} \mathrm{Grp}_{E_\infty}(\mathrm{Spc}) \simeq \mathrm{Sp}_{\geq 0}.$$

$$(\mathcal{C}, \oplus) \longmapsto (\mathcal{C}^{\simeq})^{gp}$$

Note. Since $(-)^{\simeq}$ and $(-)^{gp}$ preserve filtered colims, so does K^{add}.

Definition. For a connective E_1-ring, write

'connective'

$$K^{cn}(R) := K^{add}(\underbrace{\mathrm{Proj}(R)}_{\text{retract of finite } \oplus \text{ of } R}) \in \mathrm{Sp}_{\geq 0}.$$

Lemma. Let R be a connective E_1-ring.

(1) $K_0(R) \cong \pi_0 K^{add}(R)$. define $K_*(R) := \pi_* K^{cn}(R)$

(2) $K_i(R) \xrightarrow{\sim} K_i(\tau_{\leq n} R)$ for $0 \leq i \leq n+1$

> This is an straightforward exercise in the definitions (using that adjoints are unique).

3. K-theory via Quillen's plus-construction

Idea. Want to give an explicit formula for group completion that lets us compute the K-theory space $\Omega^\infty K^{cn}(R)$. Steps:

(1) Construct an approximation $a : M_\infty \longrightarrow M^{gp}$ such that a is acyclic: a induces an iso. on reduced homology with coeffs. in any local system.

By Whitehead's Thm, just need to modify $\pi_1 \rightsquigarrow$ cellular description

(2) Invert acyclic morphisms $\rightsquigarrow$ the $\overbrace{\text{plus construction}}$ $M_\infty^+ \xrightarrow{\sim} M^{gp}$

$$\Omega^\infty K^{cn}(R) \simeq K_0(R) \times BGL_\infty(R)^+$$

Constructing the approximation

Key. For every $[P] \in \pi_0 \mathrm{Proj}(R)^{\simeq}$, there is a $[P']$ and an integer $n \geq 0$ such that $[P]+[P'] = [R^{\oplus n}] = n$

> To group complete, we just need to invert $[R] = 1$

Generalizing. Let $M \in \mathrm{Mon}_{E_\infty}(\mathrm{Spc})$ and let $x \in \pi_0(M)$ be such that for each $y \in \pi_0(M)$ there exists $n \geq 0$ and $y' \in \pi_0(M)$ such that $y + y' = nx$.

> Define

left M-module map, not a map of E_∞-monoids

$$\mathrm{tel}_x(M) := \mathrm{Colim}\left(M \xrightarrow{+x} M \xrightarrow{+x} M \longrightarrow \cdots\right)$$

Naive attempt to invert x

$\mathrm{tel}_x(M)$ inherits a left action of M

> Since x is invertible in M^{gp}, we obtain a map

$$a_x : \mathrm{tel}_x M \longrightarrow \mathrm{tel}_x(M^{gp}) \simeq M^{gp}.$$

proved a much more general result with Ore conditions

'Group completion Theorem' ($\overbrace{\text{McDuff-Segal}}$). In this situation, the comparison $a_x : \mathrm{tel}_x M \longrightarrow M^{gp}$ is acyclic.

Remark. The usual group completion theorem says that

$$\underbrace{H_*(M)[\pi_0(M)^{-1}] \xrightarrow{\sim} H_*(M^{gp})}_{\text{Both rings with the Pontryagin product}}.$$

Note that

$$H_*(\mathrm{tel}_x M) \cong H_*(M)[x^{-1}] \cong H_*(M)[\pi_0(M)^{-1}]$$

by the assumption on $x \in \pi_0(M)$. So this recovers the classical group completion theorem (for E_∞-monoids) under this assumption.

> The general group completion theorem can be stated along these lines if one chooses a (well-ordered) set of generators of $\pi_0(M)$.

Case of interest. R a connective E_1-ring. Write $\underbrace{\mathrm{sProj}(R) := \mathrm{tel}_1 \mathrm{Proj}(R)^{\simeq}}_{\text{'stable projective modules'}}$

$$\pi_0(\mathrm{sProj}(R)) = \mathrm{colim}\left(\pi_0 \mathrm{Proj}(R)^{\simeq} \xrightarrow{\oplus R} \pi_0 \mathrm{Proj}(R)^{\simeq} \xrightarrow{\oplus R} \cdots\right)$$

$$\cong \pi_0(\mathrm{Proj}(R)^{\simeq})^{gp} = K_0(R).$$

Given $P \in \mathrm{sProj}(R)$, we have

$$\Omega_P(\mathrm{sProj}(R)) \simeq \underset{n\to\infty}{\mathrm{colim}}\, \mathrm{Aut}_R(P \oplus R^{\oplus n})$$

Choosing $P' \in \mathrm{Proj}(R)$ such that $P \oplus P' \cong R^{\oplus r}$ and cofinality shows that

$$\Omega_P(\mathrm{sProj}(R)) \simeq \underset{k\to\infty}{\mathrm{colim}}\, \mathrm{Aut}_R(R^{\oplus k}) =: GL_\infty(R).$$

Takeaway. There is a non-natural equivalence

$$\mathrm{sProj}(R) \overset{\text{only as spaces}}{\simeq} K_0(R) \times BGL_\infty(R).$$

and an acyclic morphism $\mathrm{sProj}(R) \xrightarrow{a} \Omega^\infty K^{cn}(R) := (\mathrm{Proj}(R)^{\simeq})^{gp}$.

Inverting acyclic morphisms

Lemma. $f: X \to Y$ is acyclic iff f is an ~~epimorphism~~ effective epimorphism in Spc: the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, equal] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

is a pushout in Spc

Proposition. The class of acyclic morphisms in Spc is closed under:

(1) Composition

(2) Base Change

(3) Cobase Change

(4) Colimits

(5) Products

Moreover:

(6) gf + f acyclic $\Rightarrow$ g acyclic

Lemma. Acyclic maps form the left class of a factorization system on Spc.

> Every morphism f factors uniquely as $f = h \circ a$ where a is acyclic and h is right orthogonal to acyclic morphisms.

Definition. The plus construction $(-)^+ : \mathrm{Spc} \to \mathrm{Spc}$ is the Bousfield localization associated to this factorization system.

> $X \to X^+$ is an acyclic map and X^+ is local with respect to acyclic maps

Definition.

(1) A group G is hypoabelian if G has no nontrivial $\overbrace{\text{perfect}}^{P=[P,P]}$ subgroups.

(2) A space Y is hypoabelian if for all $y \in Y$, $\pi_1(Y,y)$ is hypoabelian.

Note. Since $K_1(R)$ is abelian, $\Omega^\infty K^{cn}(R)$ is hypoabelian.

Proposition. A space Y is local with respect to acyclic maps if and only if Y is hypoabelian.

> $\mathrm{Spc}^{hypo} \underset{\perp}{\overset{(-)^+}{\leftrightarrows}} \mathrm{Spc}$, $\pi_1(X^+, x) \cong \pi_1(X,x) / \left(\begin{smallmatrix}\text{maximal} \\ \text{perfect sub.}\end{smallmatrix}\right)$.

Exercise. Construct X^+ from X by only adding $\underbrace{\text{2-cells}}_{\text{kill perfect subs of } \pi_1}$ and $\underbrace{\text{3-cells}}_{\text{fix } H_*}$.

↳ How Quillen originally constructed X^+

Example. R a connective $\mathbb{E}_1$-ring. Then

$$\underset{\substack{\rotatebox{90}{$\simeq$} \\ K_0(R)\times BGL(R)^+}}{a^+ : \mathrm{sProj}(R)^+} \xrightarrow{\sim} \Omega^\infty K^{cn}(R).$$

Since $\pi_1(\mathrm{sProj}(R)) \cong GL_\infty(\pi_0 R)$ and the maximal perfect in $GL_\infty(A)$ is the subgroup $E(A)$ of elementary matrices, $K_1(R) \cong \underbrace{GL_\infty(\pi_0 R)/E(\pi_0 R)}_{\text{Classical def. of } K_1(R)}$.

Summary of Section. For a connective $\mathbb{E}_1$-ring R, there is a non-natural identification

only as spaces

$$\Omega^\infty K^{cn}(R) \simeq \underbrace{K_0(R) \times BGL_\infty(R)^+}_{\text{Quillen's original def of higher K-theory}}.$$

4. K-theory as the universal additive/localizing invariant

Motivations. In §1 we saw we could define $K_0(R)$ in 2 ways:

(1) Add inverses to $\pi_0 \underbrace{\mathrm{Proj}(R)}_{\text{All SES's split}}^{\simeq}$

(2) Split cofiber seqs. in $\mathrm{Perf}(R) = \mathrm{Mod}(R)^{\omega}$, then add inverses.

Take for granted for a moment. Connective K-theory K^{cn} makes sense for stable ∞-categories. ← Construction will come later (§5)

Goal. Give a universal characterization of K^{cn} after Blumberg-Gepner-Tabuada.

Idea. K_0 splits exact sequences of perfect complexes. The K-theory functor should split exact sequences of stable ∞-categories.

More motivation. We also want a nonconnective variant K of K-theory.

> Given a scheme X with closed subscheme $Z \subset X$ and $U := X \setminus Z$, there is a 'localization sequence'

Notion of exact seq.

$$\underset{\uparrow \text{ supported on } Z}{\mathrm{QCoh}_Z(X)} \hookrightarrow \mathrm{QCoh}(X) \longrightarrow \mathrm{QCoh}(U)$$

expressing $\mathrm{QCoh}(U)$ as the Verdier quotient $\mathrm{QCoh}(X)/\mathrm{QCoh}_Z(X)$.

> On K^{cn}, this is almost a fiber sequence, but $K_0(X) \to K_0(U)$ need not be surjective ⇝ Should have negative K-groups

Working Context.

> Cat_∞^{st} : ∞-category of small stable ∞-categories, exact functors

> $\mathrm{Cat}_\infty^{perf} \subset \mathrm{Cat}_\infty^{st}$: subcategory of idempotent complete ∞-categories

- Localization $\mathrm{Idem}: \mathrm{Cat}_\infty^{st} \longrightarrow \mathrm{Cat}_\infty^{perf}$.
- R E_1-ring, then $\mathrm{Perf}(R) \in \mathrm{Cat}_\infty^{perf}$

> $\mathrm{Cat}_\infty^{perf} \underset{\mathrm{Ind}}{\overset{(-)^\omega}{\rightleftarrows}}$ (compactly gen. presentable stable ∞-cats. & left adjoints that preserve compact objs.) $\subset \mathrm{LPr}^{st}$

$\otimes$ representing biexact functors

unit: $\mathrm{Sp}^\omega = \mathrm{Sp}^{fin}$

Lurie $\otimes$ rep. $C \times D \longrightarrow E$ preserving colims sep. in each variable

unit: Sp

⚠ $\mathrm{Cat}_\infty^{perf}$ is not stable.

Definition. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathrm{Cat}_\infty^{perf}$ is exact if:

(1) It is a $\underbrace{\text{cofiber sequence}}_{C \simeq \mathrm{Idem}(\underbrace{B/A}_{\text{Verdier quotient}})}$ a property since the space of equivs $gf \simeq 0$ is contractible or empty

(2) f is fully faithful.

(1) + (2) $\Longleftrightarrow$ fiber & cofiber sequence in $\mathrm{Cat}_\infty^{perf}$

An exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is split exact if in addition f and g admit right adjoints.

Definition. Let D be a stable ∞-category and $F : \mathrm{Cat}_\infty^{\mathrm{perf}} \to D$ a functor. We say that:

(add) F is an additive invariant if $F(0) \simeq 0$ and F sends split exact sequences in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ to cofiber sequences in D.

(loc) F is a localizing invariant if $F(0) \simeq 0$ and F sends exact sequences in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ to cofiber sequences in D.

(fin) F is finitary if F preserves filtered colimits.

Definition / Theorem (Blumberg-Gepner-Tabuada). The ∞-category of noncommutative additive/localizing motives is the universal finitary additive/localizing invariant

$$\begin{tikzcd} \mathcal{U}^{\mathrm{add}} : \mathrm{Cat}_\infty^{\mathrm{perf}} \arrow[r] \arrow[dr] & \mathrm{Mot}^{\mathrm{add}} \arrow[d, dashed, "\exists!"] \\ & D \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}^{\mathrm{loc}} : \mathrm{Cat}_\infty^{\mathrm{perf}} \arrow[r] \arrow[dr] & \mathrm{Mot}^{\mathrm{loc}} \arrow[d, dashed, "\exists!"] \\ & D \end{tikzcd}$$

The symmetric monoidal structure on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ descends to symmetric monoidal structures on $\mathrm{Mot}^{\mathrm{add}}$, $\mathrm{Mot}^{\mathrm{loc}}$, $\mathcal{U}^{\mathrm{add}}$, $\mathcal{U}^{\mathrm{loc}}$.

Definition (BGT). $C \in \mathrm{Cat}_\infty^{\mathrm{perf}}$. The connective / nonconnective K-theory of C is the mapping spectrum

$$K^{cn}(C) := \mathrm{Map}(\underbrace{\mathcal{U}^{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}})}_{\otimes \text{ unit}}, \mathcal{U}^{\mathrm{add}}(C)),$$

$$K(C) := \mathrm{Map}(\overbrace{\mathcal{U}^{\mathrm{loc}}(\mathrm{Sp}^{\mathrm{fin}})}, \mathcal{U}^{\mathrm{loc}}(C)).$$

> Since localizing invariants are additive, we get a functor $\mathrm{Mot}^{\mathrm{add}} \to \mathrm{Mot}^{\mathrm{loc}}$.

$\rightsquigarrow K^{cn} \longrightarrow K$.

Proposition. $K^{cn} \xrightarrow{\sim} \tau_{\geq 0} K$.

Other localizing invariants. $\underbrace{\text{Topological Hochschild/cyclic homology}}_{\text{Nat's talk}}$

$\rightsquigarrow$ the source of the cyclotomic trace $tr_{cyc}: K \longrightarrow TC$

$$\begin{tikzcd}
\mathrm{Cat}_\infty^{perf} \arrow[rr, "\mathcal{U}^{loc}"] \arrow[dr, "THH"'] & & \mathrm{Mot}^{loc} \arrow[dl, dotted, "\exists!"] \\
& \underbrace{\mathrm{CycSp}}_{\text{Cyclotomic Spectra}} &
\end{tikzcd}$$

$$K(C) := \mathrm{Map}_{\mathrm{Mot}^{loc}}(\mathcal{U}^{loc}(\mathbb{S}), \mathcal{U}^{loc}(C))$$

$$\Big\downarrow tr_{cyc} \text{ by functoriality}$$

$$\simeq \mathrm{Map}_{\mathrm{CycSp}}(THH(\mathbb{S}), THH(C))$$

$$TC(C) := \mathrm{Map}_{\mathrm{CycSp}}(\mathbb{S}, THH(C))$$

Since $THH(\mathbb{S}) = \underbrace{\mathbb{S} \otimes \mathbb{S}}_{\mathbb{S} \otimes \mathbb{S}} \simeq \mathbb{S}$.

5. K-theory via the $\underbrace{S_\bullet\text{-construction}}_{\text{Waldhausen}}$

Goal. Give a construction of connective K-theory by splitting all sequences in C

subposet of (i,j) with $i \le j$

Notation. $Ar[n] := Fun([1],[n]) \subset [n] \times [n]$

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & (0,n) \arrow[d] \\
 & (1,1) \arrow[r] & (1,2) \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & (1,n) \arrow[d] \\
 & & \ddots \arrow[r] & \vdots \arrow[r] \arrow[d] & \vdots \arrow[d] \\
 & & & (n-1,n-1) \arrow[r] & (n-1,n) \arrow[d] \\
 & & & & (n,n)
\end{tikzcd}$$

Definition. Let C be a stable ∞-category. For each $n \ge 0$, write

$$S_n(C) \subset Fun(Ar[n], C)$$

for the full subcat of those diagrams

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \cdots \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_n \arrow[d] \\
 & 0 \arrow[r] & X_2/X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \cdots \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X_n/X_1 \arrow[d] \\
 & & \ddots \arrow[r] & \vdots \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \vdots \arrow[d] \\
 & & & 0 \arrow[r] & X_n/X_{n-1} \arrow[d] \\
 & & & & 0
\end{tikzcd}$$

exact (= bicartesian) square

> $[n] \mapsto S_n(C)$ defines a simplicial stable ∞-category.

Observations.

> $S_0(C) = \{0 \text{ objects of } C\} \simeq *$

> Restriction along the 'top row' $\{1 < \cdots < n\} \hookrightarrow Ar[n]$

$$i \longmapsto (0,i)$$

defines an equivalence

$$S_n(C) \xrightarrow{\sim} \mathrm{Fun}(\{1 < \cdots < n\}, C).$$

In particular, $S_n(C)$ is stable.

$S_{\bullet+1}(C)^{\simeq} \in \mathrm{Fun}(\Delta^{op}, \mathrm{Cat}_\infty)$ is the complete Segal space rep. C.

Definition. Let C be a stable ∞-category. The K-theory space of C is the loopspace

$$K^{cn}(C) := \Omega \underbrace{|S_\bullet(C)^{\simeq}|}_{\text{connected!}} := S_0(C) \underset{|S_\bullet(C)^{\simeq}|}{\times} S_0(C).$$

Note. $K^{cn}(C)$ naturally has the structure of an E_∞-monoid: $(C, \oplus) \in \mathrm{Mon}_{E_\infty}(\mathrm{Cat}_\infty)$ and $S_\bullet(-)$, $(-)^{\simeq} : \mathrm{Cat}_\infty \to \mathrm{Spc}$, $|-| : \mathrm{Fun}(\Delta^{op}, \mathrm{Spc}) \to \mathrm{Spc}$ all commute with finite products.

Better. $K^{cn}(C)$ is an E_∞-group:

$$\pi_0 K^{cn}(C) \cong \frac{\mathbb{Z}[\pi_0 C^{\simeq}]}{\left\langle \begin{array}{l} [X]+[Z]=[Y] \text{ if there is a cofib.} \\ \text{sequence } X \to Y \to Z \end{array} \right\rangle} \quad \text{so } [\Sigma X] = -[X].$$

> We'll write $K^{cn}(C) \in \mathrm{Sp}_{\geq 0}$ for the associated connective spectrum.

Consistency with previous definition of K^{cn}:

(1) If $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{Perf}}$, then

$$\mathrm{Map}(\mathcal{U}^{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}}), \mathcal{U}^{\mathrm{add}}(\mathcal{C})) \simeq \Omega|S_\bullet(\mathcal{C})^{\simeq}|.$$

(2) If R is a connective E_1-ring, then $K^{cn}(\mathrm{Perf}(R)) \simeq \underset{(\mathrm{Proj}(R)^{\simeq})^{gp}}{\underset{\shortparallel}{K^{cn}(R)}}$.