An Introduction to "Exodromy beyond Conicality"

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joint with

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Monodromy

Monodromy V1 If X is a locally simply connected top space, then there is an equivalence of categories

$$\underbrace{LC(X;Set)}_{\text{locally constant sheaves of sets on } X} \xrightarrow{\sim} Fun(\Pi_1(X), Set)$$

$$L \longmapsto [x \longmapsto L_x]$$

Monodromy V∞ If X is a locally weakly contractible top space, then there is an equivalence of ∞-categories

$$LC(X;\underbrace{Spc}_{\text{∞-category of spaces}}) \xrightarrow{\sim} Fun(\underbrace{\Pi_\infty(X)}_{\text{underlying homotopy type / ∞-groupoid}}, Spc)$$

Moreover, $LC(X;Spc) \subset Sh(X;Spc)$ is closed under lims + colims

Why sheaves of spaces?

- Spc is the universal choice of coefficients, and the result for sheaves of spaces implies the result for essentially arbitrary coefficients.

- For example, this implies that for a ring A,

$$D_{lc}(X;A) \simeq \mathrm{Fun}(\Pi_\infty(X), D(A)).$$

locally constant cohomology sheaves

- If you're not familiar with ∞-categories, replace

$$\mathrm{Spc} \rightsquigarrow \mathrm{Set}$$

and the story should still be interesting.

Stratified Spaces

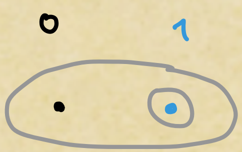
Def $(P, \leq)$ poset

The **Alexandroff topology** on P has

$$[U \subset P \text{ open}] \iff [p \in U \text{ and } q > p \Rightarrow q \in U]$$

U is upwards closed

Ex $P = \{0 < 1 < \cdots < n\}$

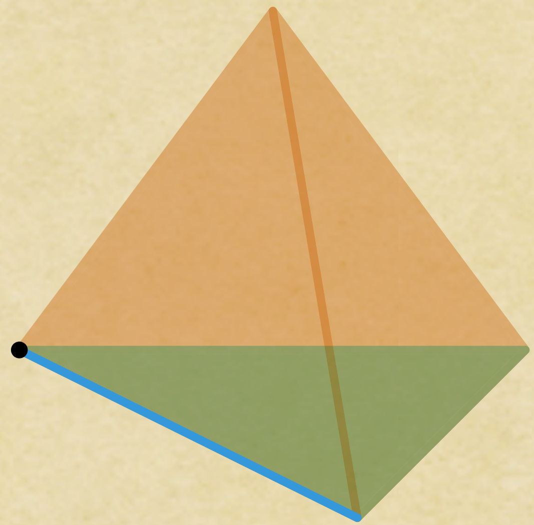


Def A **P-stratification** of a space X is a continuous map $s : X \longrightarrow P$.

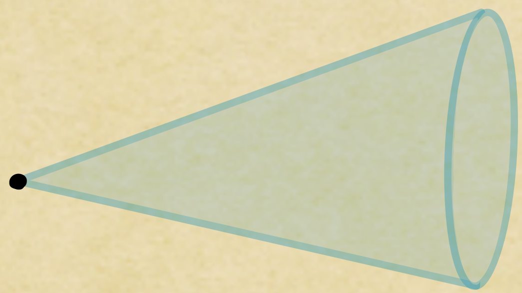
$>$ $p \in P$, the **pth-stratum** of X is $X_p := s^{-1}(p)$.

Ex $|\Delta^n| \longrightarrow \{0 < \cdots < n\}$

$0 < 1 < 2 < 3$



Ex $\mathrm{Cone}(S^1)$



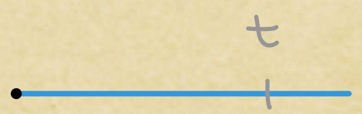
Constructible Sheaves

Def (X, P) stratified space

A sheaf $\mathcal{F}$ on X is **P-constructible** if for each $p \in P$, $\mathcal{F}|_{X_p}$ is locally constant.

Ntn $\mathrm{Cons}_P(X) \subset \mathrm{Sh}(X)$

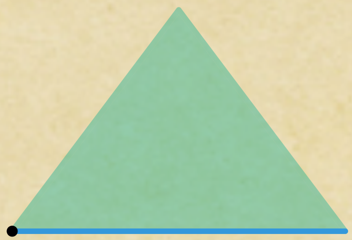
Q What are sheaves on $[0,1]$ locally constant on $(0,1]$?



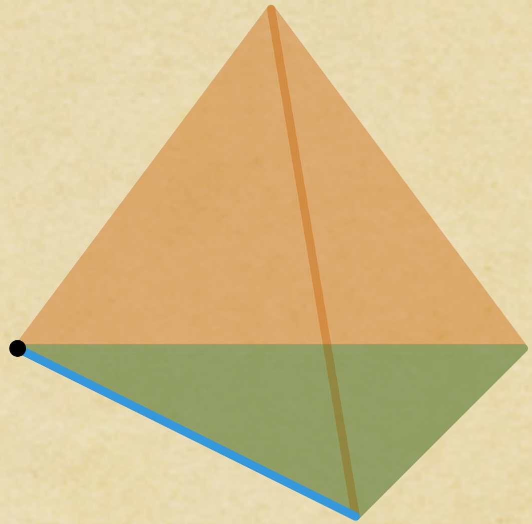
$$L_0 \xrightarrow[\text{map}]{\text{spec.}} L_t$$

A $\mathrm{Fun}(\bullet \longrightarrow \bullet,\ \mathrm{Spc})$

Other Examples

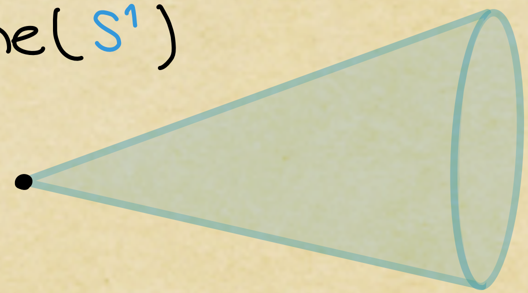


$\bullet \longrightarrow \bullet \longrightarrow \bullet$



$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$

Cone(S^1)



V $\qquad$ $W \circlearrowleft \mathbb{Z}$

map $V \longrightarrow W^{\mathbb{Z}}$

$\bullet \longrightarrow \bullet \circlearrowleft \mathbb{Z}$

Exodromy (MacPherson, Treumann, Lurie, Porta–Teyssier)

If $s : X \longrightarrow P$ is a conically stratified space with lwc strata, then there is an ∞-category

$$\mathrm{Exit}(X,P) : \text{the exit-path } \infty\text{-category of } (X,P)$$

along with an equivalence of ∞-categories

$$\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(\mathrm{Exit}(X,P), \mathrm{Spc}).$$

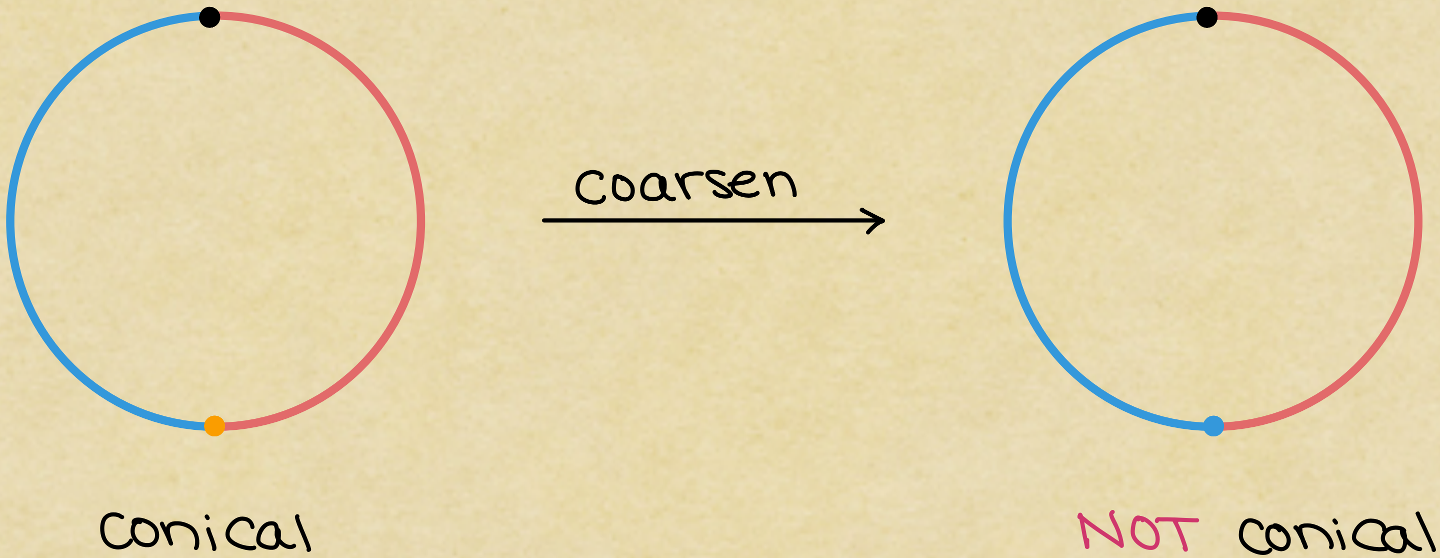
Moreover:

(1) $\mathrm{Cons}_P(X) \subset \mathrm{Sh}(X)$ is closed under limits & colimits.

(2) $s^* : \mathrm{Fun}(P, \mathrm{Spc}) \simeq \mathrm{Sh}(P) \longrightarrow \mathrm{Sh}(X)$ preserves limits.

$\Longleftrightarrow$ induced by a functor $\mathrm{Exit}(X,P) \longrightarrow P$

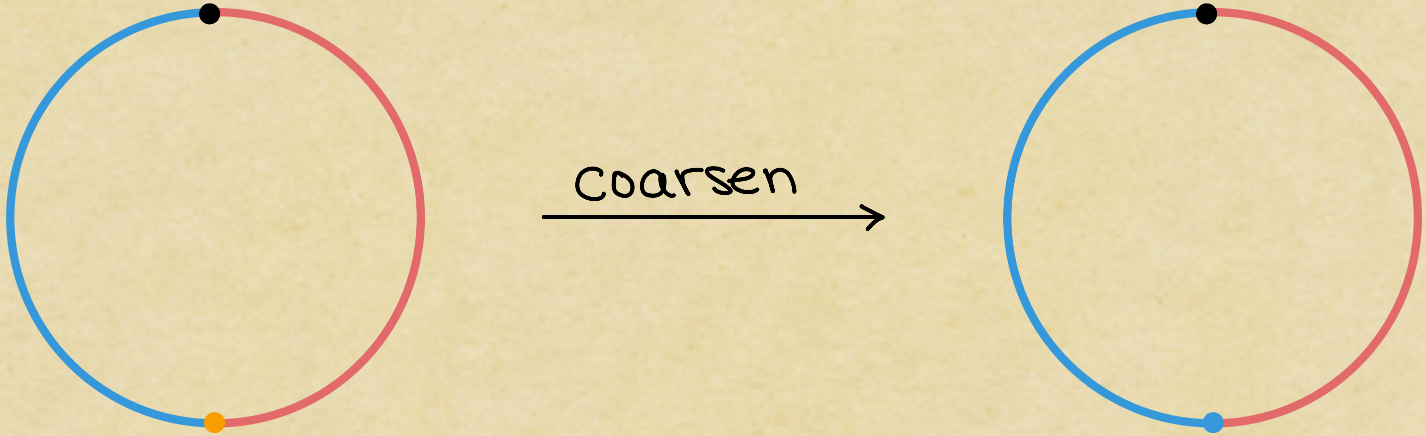
A simple example



Audience Question

(1) What is the exit-path category of the left strat space?

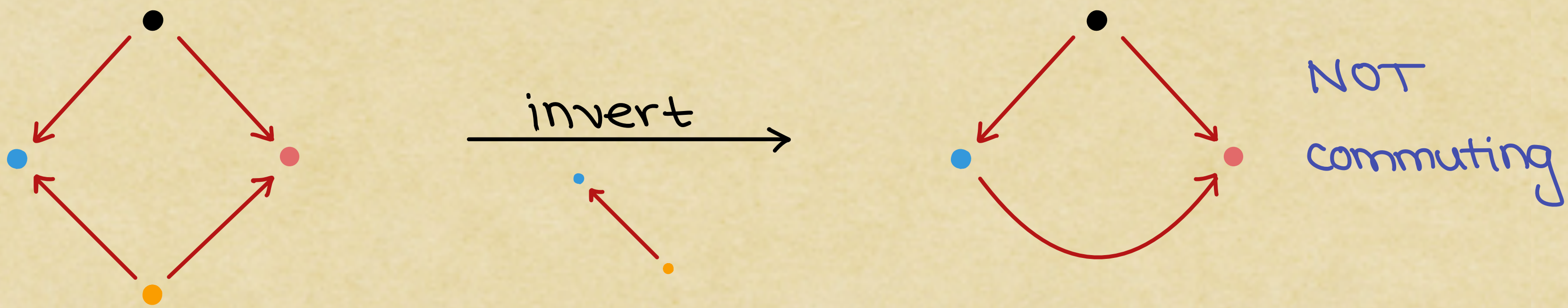
(2) What should be the exit-path category of the right strat space?



conical

NOT conical

Answer



Exit-path category

Exit-path category should be

Key Idea

Make the conclusion of the exodromy theorem into a definition.

Don't rely on a particular geometric model of $\mathrm{Exit}(X,P)$

[Ayala-Francis-Rozenblyum, Clausen-Ørsnes Jansen]

Atomic Generation

Q Given an ∞-category $\mathcal{C}$, when is $\mathcal{C} \simeq \mathrm{PSh}(\mathcal{C}_0)$?

for some $\mathcal{C}_0$

Def $\mathcal{C}$ ∞-category with colimits

$c \in \mathcal{C}$ is **atomic** if $\mathrm{Map}_{\mathcal{C}}(c,-) : \mathcal{C} \longrightarrow \mathrm{Spc}$ preserves colimits

> $\mathcal{C}^{at} \subset \mathcal{C}$ full subcat of atomic objects.

Ex By Yoneda, $\mathrm{PSh}(\mathcal{C}_0)^{at} \supset \mathcal{C}_0$.

Def An ∞-category $\mathcal{C}$ with colimits is **atomically generated**

if the unique colimit-preserving extension

$$\mathrm{PSh}(\mathcal{C}^{at}) \longrightarrow \mathcal{C}$$

of $\mathcal{C}^{at} \subset \mathcal{C}$ is an equivalence.

Def A stratified space $s: X \longrightarrow P$ is **exodromic** if:

(1) $\mathrm{Cons}_P(X) \subset \mathrm{Sh}(X)$ is closed under limits & colimits

(2) $\mathrm{Cons}_P(X)$ is atomically generated.

(3) $s^*: \mathrm{Fun}(P, \mathrm{Spc}) \longrightarrow \mathrm{Sh}(X)$ preserves limits.

Write

$$\boxed{\mathrm{Exit}(X,P) := \left(\mathrm{Cons}_P(X)^{at}\right)^{op}.}$$

Then by definition

$$\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(\mathrm{Exit}(X,P), \mathrm{Spc}).$$

The Stability Theorem

Thm (H.-Porta-Teyssier)

(1) **Stability under pulling back to locally closed subposets**

If (X,P) is exodromic and $S \subset P$ is $\underbrace{\text{locally closed}}$, then $(X \times_P S, S)$ is exodromic and

$p < q < r$ and $p, r \in S$
$\Rightarrow q \in S$

$$\mathrm{Exit}(X \times_P S, S) \xrightarrow{\sim} \mathrm{Exit}(X,P) \underset{P}{\times} S.$$

As a consequence, the induced functor

$$\mathrm{Exit}(X,P) \longrightarrow P$$

is conservative.

(2) **Functoriality** $(X,P) \longmapsto \mathrm{Exit}(X,P)$ is functorial in all maps of exodromic stratified spaces.

The exodromy equivalence is functorial in all maps of exodromic stratified spaces.

(3) Stability under coarsening If (X,R) is exodromic, then for any map of posets $\phi : R \longrightarrow P$, the stratified space (X,P) is exodromic.

Write W_P for those morphisms in $\mathrm{Exit}(X,R)$ that

$$\mathrm{Exit}(X,R) \longrightarrow R \xrightarrow{\phi} P$$

sends to identities. Then

$$\mathrm{Exit}(X,R)[W_P^{-1}] \xrightarrow{\sim} \mathrm{Exit}(X,P).$$

(4) van Kampen The property of being exodromic can be checked locally + colimit formula for $\mathrm{Exit}(-,-)$.

(5) Stability of finiteness/compactness the property of an exit-path ∞-category being finite/compact:

- can be checked on a finite cover
- is stable under pulling back to locally closed subsets
- is stable under coarsening

Methods We actually prove everything for stratified ∞-topoi.

The results then apply to stratified top. stacks

Examples

Cor If (X,P) locally admits a refinement by a conical stratification with locally weakly contractible strata, then (X,P) is exodromic.

> Deep theorems of Thom, Mather, and Verdier show that such refinements are often available in geometric situations

Ex If (X,P) admits a refinement by a triangulation, then (X,P) is exodromic.

Thm If (X,P) is a real analytic manifold with a locally finite stratification by subanalytic subsets, then:

(1) (X,P) is exodromic

(2) X compact $\implies$ $\mathrm{Exit}(X,P)$ is finite

Thm If (X,P) is a stratification of the $\mathbb{R}$-points of an $\mathbb{R}$-variety by Zariski locally closed subsets, then:

(1) (X,P) is exodromic

(2) $\mathrm{Exit}(X,P)$ is finite

Refine results of Lefschetz–Whitehead, Łojasiewicz, & Hironaka

Applications

Application 1 (HPT) Representability results for derived moduli of constructible and perverse sheaves.

> Crucially uses the finiteness of exit-paths!

Application 2 (PT, using App 1) Hall algebras

Application 3 (PT, using App 1) A new approach to Stokes data using exit-paths + representability of the derived moduli of Stokes data

> Their results generalize hard theorems of Sabbah.

Beautiful Computations

Ørsnes Jansen computed exit-path ∞-categories for:

(1) Reductive Borel-Serre compactifications

(2) $\underbrace{\overline{\mathcal{M}}_{g,n}}$ Stratified by Stable genus g dual graphs w/ n marked points

moduli of Stable genus g nodal curves w/ n marked points