EXODROMY BEYOND CONICALITY

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ABSTRACT. We show that compact subanalytic stratified spaces and algebraic stratifications of real varieties have finite exit-path ∞-categories, refining classical theorems of Lefschetz–Whitehead, Łojasiewicz, and Hironaka on the finiteness of the underlying homotopy types of these spaces. These stratifications are typically not conical; hence we cannot rely on the currently available exodromy equivalence between constructible sheaves on a stratified space, which requires conicality as a fundamental hypothesis. Building on ideas of Clausen and Ørsnes Jansen, we study the class of *exodromic* stratified spaces, for which the conclusion of the exodromy theorem holds. We prove two new fundamental properties of this class of stratified spaces: coarsenings of exodromic stratifications are exodromic, and every morphism between exodromic stratified spaces induces a functor between the associated exit path ∞-categories. As a consequence, we produce many new examples of exodromic stratified spaces, including: coarsenings of conical stratifications, locally finite subanalytic stratifications of real analytic spaces, and algebraic stratifications of real varieties. Our proofs are at the generality of stratified ∞-topoi, hence apply to even more general situations such as stratified topological stacks. Finally, we use the previously mentioned finiteness results to construct derived moduli stacks of constructible and perverse sheaves.

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0 INTRODUCTION

0.1 Motivation. Let (X, P) be a stratified space. MacPherson observed the following generalization of the monodromy equivalence: provided the stratification of X is sufficiently nice, the category of constructible sheaves of sets on (X, P) is equivalent to the category of functors from the *exit-path category* of (X, P) to **Set**. Treumann [38] provided the first general account of this phenomenon, and Treumann's result has since been generalized by Lurie [HA, Theorem A.9.3], Lejay [25], and Porta–Teyssier [32]. To contextualize the results of this paper, let us first recall the most general theorem of this form currently available. Write $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ for the ∞-category of hyperconstructible hypersheaves[1] of spaces on X. If the stratification of (X, P) is *conical* (see [32, Definition 2.1.9]) and the strata are locally weakly contractible, the *exodromy theorem*[2] [32, Theorem 5.4.1] provides an equivalence of ∞-categories

$$\Phi_{X,P} : \mathrm{Cons}_P^{\mathrm{hyp}}(X) \xrightarrow{\sim} \mathrm{Fun}(\Pi_\infty(X, P), \mathbf{Spc}) . \tag{0.1.1}$$

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[1] In this introduction, the reader can safely disregard the adjective "hyper". Hypersheaves are used in [21; 25; 32] to relax the geometric assumptions needed for the theorem.

[2] The term 'exodromy' was first introduced in [8] as a combination of 'monodromy' and 'exit-paths'.

Here $\Pi_\infty(X,P)$ is Lurie's exit-path ∞-category of (X,P), introduced in [HA, Definition A.6.2]. The objects of $\Pi_\infty(X,P)$ are the points of X. Roughly speaking, the 1-morphisms are *exit-paths* flowing from lower to higher strata (and once they exit a stratum are not allowed to return), the 2-morphisms are homotopies of exit-paths respecting stratifications, etc. The functor $\Phi_{X,P}$ carries a sheaf F to the functor informally described by sending a point $x \in X$ to the stalk F_x, and each exit-path $x \to y$ to a *specialization map* $F_x \to F_y$, together with higher coherences relating these data. Conicality has played an essential role in almost all exodromy theorems available in the literature. First, it is crucial in proving that the geometrically defined object $\Pi_\infty(X,P)$ is indeed an ∞-category [HA, Theorem A.6.4]. Second, it is used at various points in the proof of the equivalence (0.1.1). Many stratifications naturally arising in geometry *fail* to be conical: typical examples are general subanalytic stratifications of real analytic manifolds, such as those arising from the study of the Stokes phenomenon for algebraic differential equations [33]. Deep work of Thom, Mather, and Verdier among others on analytic stratified spaces has shown that conical (in fact, Whitney) refinements are always available [17]; however, it is sometimes essential to work with a fixed stratification. The purpose of this article is to generalize the exodromy equivalence to many naturally occurring *non-conically* stratified spaces, paying particular attention to the *conically refineable* situation.

0.2 Exodromic stratified spaces. To state the results of this paper, we need to briefly introduce the concept of an exit-path ∞-category without reference to any particular simplicial model. As highlighted by Ayala–Francis–Rozenblyum [6, Problem 0.0.9] and explained by Clausen–Ørsnes Jansen [14; 28; 29], one should be able to trade off the conicality of a stratified space (X,P) and Lurie's simplicial model for the exit-path ∞-category for the following three requirements of the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$:

0.2.1 Definition (cf. [14, Definition 3.5])**.** A stratified space $s\colon X \to P$ is *exodromic* if the following conditions are satisfied:

(1) The ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ is atomically generated.

(2) The subcategory $\mathrm{Cons}_P^{\mathrm{hyp}}(X) \subset \mathrm{Sh}^{\mathrm{hyp}}(X)$ is closed under both limits and colimits.

(3) The pullback functor $s^{*,\mathrm{hyp}}\colon\ \mathrm{Fun}(P,\mathbf{Spc}) \simeq \mathrm{Sh}^{\mathrm{hyp}}(P) \to \mathrm{Cons}_P^{\mathrm{hyp}}(X)$ preserves limits.

Let us comment these requirements. Concerning (1), note that the exodromy theorem guarantees that the ∞-category $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ can be written as an ∞-category of presheaves. *Atomic generation* is an intrinsic way to formulate this property: given a presentable ∞-category $\mathcal{C}$, an object $c \in \mathcal{C}$ is *atomic* if the functor

$$\mathrm{Map}_{\mathcal{C}}(c,-)\colon\ \mathcal{C} \to \mathbf{Spc}$$

preserves *all* colimits. Write $\mathcal{C}^{\mathrm{at}} \subset \mathcal{C}$ for the full subcategory spanned by the atomic objects. Then $\mathcal{C}$ is said to be *atomically generated* if the unique colimit-preserving extension

$$\mathrm{PSh}(\mathcal{C}^{\mathrm{at}}) \to \mathcal{C}$$

of $\mathcal{C}^{\mathrm{at}} \subset \mathcal{C}$ along the Yoneda embedding is an equivalence (see §1.1 for more background on this notion). In the setting of Definition 0.2.1, we write $\Pi_\infty(X,P)$ for the opposite of the full subcategory of $\mathrm{Cons}_P^{\mathrm{hyp}}(X)$ spanned by atomic objects. We refer to $\Pi_\infty(X,P)$ as the *exit-path ∞-category* of (X,P). The second feature is that the subcategory $\mathrm{Cons}_P^{\mathrm{hyp}}(X) \subset \mathrm{Sh}^{\mathrm{hyp}}(X)$ is closed under both limits and colimits. This is in some sense a *categorical regularity condition*, which is akin to but weaker than conicality: see [32, Corollary 5.4.4] for a proof in the conical setting, and see Definition 2.4.10 and Example 2.4.11 for other examples of regularity properties enjoyed by the conical situation. The third feature is that, by construction, the exit-path ∞-category of (X,P) is equipped with a functor to the stratifying poset P. Given conditions (1) and (2), condition (3) guarantees that the stratification of X equips $\Pi_\infty(X,P)$ with a functor $\Pi_\infty(X,P) \to P$; see Recollection 1.1.11.

0.3 The stability theorem. The analysis of the conical situation carried out in [32] shows that conically stratified spaces with locally weakly contractible strata are exodromic in the sense of Definition 0.2.1. However, the class of such stratified spaces does not have many stability properties; for example, a coarsening of a conical stratification need not be conical. As previously mentioned, in subanalytic geometry and real algebraic geometry conical refinements always exist, at least locally. The following is the main result of this paper, and in particular it implies that *every* subanalytic or real analytic stratified space is exodromic:

0.3.1 Theorem (stability properties of exodromic stratified spaces; Theorem 5.1.7)**.**

(1) Stability under pulling back to locally closed subposets: *If (X, P) is an exodromic stratified space, then for each locally closed subposet $S \subset P$, the stratified space $(X \times_P S, S)$ is exodromic and the induced functor*

$$\Pi_\infty(X \times_P S, S) \to \Pi_\infty(X, P) \times_P S$$

is an equivalence. As a consequence, the induced functor $\Pi_\infty(X, P) \to P$ is conservative.

(2) Functoriality: *The exodromy equivalence is functorial in all stratified maps between exodromic stratified spaces. That is, for every stratified map $f : (X, P) \to (Y, Q)$ between exodromic stratified spaces, under the exodromy equivalence the pullback functor*

$$f^{*,\mathrm{hyp}} : \mathrm{Cons}_Q^{\mathrm{hyp}}(Y) \to \mathrm{Cons}_P^{\mathrm{hyp}}(X)$$

is induced by a functor of exit-path ∞-categories

$$\Pi_\infty(X, P) \to \Pi_\infty(Y, Q) .$$

(3) Stability under coarsening and localization formula: *Let (X, R) be an exodromic stratified space and let $\phi : R \to P$ be a map of posets. Write W_P for the collection of morphisms in $\Pi_\infty(X, R)$ that the composite $\Pi_\infty(X, R) \to R \to P$ sends to equivalences. Then the stratified space (X, P) is exodromic and the natural functor $\Pi_\infty(X, R) \to \Pi_\infty(X, P)$ induces an equivalence*

$$\Pi_\infty(X, R)[W_P^{-1}] \simeq \Pi_\infty(X, P) .$$

(4) van Kampen: *The property of a stratified space being exodromic can be checked locally.*

(5) Stability of finiteness/compactness: *The property of an exit-path ∞-category being finite (resp., compact) is stable under pulling back to a locally closed subposet, is stable under coarsening, and can be checked on a finite open cover.*

Together, the items in Theorem 0.3.1 provide robust techniques to produce new examples of exodromic stratified spaces starting from conically stratified spaces. We will explain many new examples of stratified spaces momentarily. Before proceeding further, we comment on how Theorem 0.3.1 relates to existing results, and the our proof methods.

0.3.2 Existing Results. Item (1) was proven by Clausen–Ørsnes Jansen in a slightly different topological setting [14, Proposition 3.6-(1)], and by Ørsnes Jansen for topological stacks [29, Proposition 3.13-(1)]. Item (4) is an easy consequence of the theory, and, in the same settings, was previously observed by Clausen–Ørsnes Jansen [14, Proposition 3.6-(2)] and Ørsnes Jansen [29, Proposition 3.13-(2)]. Two early instances of (2) were proven in the conically stratified setting by Lurie [HA, Corollary A.9.4] in the case where $P = *$, and Ayala–Francis–Rozenblyum [6, Theorem 3.3.12] under some additional hypotheses on the stratifying posets. Recently, Ørsnes Jansen [29, Proposition 3.20] generalized the argument given by Ayala–Francis–Rozenblyum; however the hypotheses are still somewhat restrictive.

The first main contribution of Theorem 0.3.1 is that our results have no restrictions on the stratifiying posets. The second is that we prove functoriality of the exodromy equivalence in all maps of stratified spaces. This is a new result and may be somewhat surprising; with previous methods, even functoriality in the conical case was a nontrivial result, first proven in [32, Proposition 6.2.3]. The third is item (5) on the stability of finiteness/compactness; its proof requires a careful understanding of the localization formula from (3) and it generalizes classical finiteness results for homotopy types of real analytic manifolds. See Remark 0.4.4.

0.3.3 Methods. The final main contribution is that our result is actually even more general than Theorem 0.3.1. The point is that the conditions in Definition 0.2.1 only depend on the datum of the geometric morphism of ∞-topoi

$$\mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathrm{Fun}(P, \mathbf{Spc}) .$$

This is an example of a *stratified ∞-topos*, as introduced in the work of Barwick–Glasman–Haine [8]. Consequently, Definition 0.2.1 makes sense at the generality of stratified ∞-topoi. See §2, in particular Definition 2.2.10.

We prove Theorem 0.3.1 by proving its natural generalization to stratified ∞-topoi. See Theorem 3.0.1 for a precise statement. This generalization gives added flexibility; for example, it immediately applies to stratified topological stacks. It also subsumes all results of this form that we are aware of, for example, the stability results proven by Clausen–Ørsnes Jansen [14] and Ørsnes Jansen [29]. The topos-theoretic result has the added benefit of providing a common framework for the various contexts where exodromy was previously considered (e.g., sheaves vs. hypersheaves).

0.4 Applications of the stability theorem. We now state our main applications of Theorem 0.3.1. Since every conically stratified space with locally weakly contractible strata is exodromic and the class of conically stratified spaces with locally weakly contractible strata is stable under passing to open subsets, we deduce:

0.4.1 Corollary (Proposition 5.2.9)**.** *If a stratified space (X, P) locally admits a refinement by a conical stratification with locally weakly contractible strata, then (X, P) is exodromic.*

A theorem of Verdier guarantees that a locally finite subanalytic stratification of a real analytic space admits a refinement that is Whitney stratified [39, Théorème 2.2]. Since Whitney stratifications are conical [27; 36], a little more work on top of Theorem 0.3.1 shows:

0.4.2 Theorem (Theorem 5.3.9)**.** *Let (X, P) be a real analytic manifold equipped with a locally finite stratification by subanalytic subsets. Then:*

(1) *The stratified space (X, P) is exodromic.*

(2) *If X is compact, then the exit-path ∞-category $\Pi_\infty(X, P)$ is finite.*

0.4.3 Theorem (Theorem 5.3.13)**.** *Let X be an algebraic variety over $\mathbf{R}$ and let (X, P) be a finite stratification of X by Zariski locally closed subsets. Then:*

(1) *The stratified space (X, P) is is exodromic.*

(2) *The exit-path ∞-category $\Pi_\infty(X, P)$ is finite.*

0.4.4 Remark. Theorem 0.4.2-(2) and Theorem 0.4.3-(2) extend results of Lefschetz–Whitehead [24], Łojasiewicz [26], and Hironaka [22] on the finiteness of the underlying homotopy types of compact subanalytic spaces and real algebraic varieties.

As an application, we use Theorems 0.4.2 and 0.4.3 to prove representability results for moduli stacks of constructible and perverse sheaves. Let A be an animated commutative ring (i.e., simplicial commutative ring). Given a stratified space (X, P), we write $\mathbf{Cons}_P(X)$ for the derived prestack over A sending a derived affine A-scheme $\mathrm{Spec}(B)$ to the ∞-groupoid of hyperconstructible hypersheaves of B-modules on (X, P) with perfect stalks. See Recollection 5.4.5 and [32, §7.1]. Given a function $\mathfrak{p} \colon P \to \mathbf{Z}$, we write

$$^{\mathfrak{p}}\mathbf{Perv}_P(X) \subset \mathbf{Cons}_P(X)$$

for the derived subprestack of $\mathfrak{p}$-*perverse sheaves* on (X, P). See [32, §7.7] for details.

0.4.5 Theorem (Corollary 5.4.17)**.** *Let (X, P) be a stratified space and let A be an animated commutative ring. Assume one of the following conditions:*

(1) *(X, P) is a compact real analytic manifold equipped with a stratification by subanalytic subsets.*

(2) *(X, P) is an algebraic variety over $\mathbf{R}$ equipped with a finite stratification by Zariski locally closed subsets.*

Then the derived prestacks $\mathbf{Cons}_P(X)$ and $^{\mathfrak{p}}\mathbf{Perv}_P(X)$ are derived stacks that are locally geometric and locally of finite presentation.

0.5 Examples. We conclude the introduction with some examples of non-conical stratifications to which our results apply. First we demonstrate how to compute the exit-path ∞-category of a coarsening in a simple situation.

0.5.1 Example. Consider a circle stratified by a point, half-open interval, and open interval, as depicted on the right-hand side of Figure 1. This stratification of S^1 is not conical. However, the stratification of S^1 by two points and two half-open intervals appearing on the left-hand side of Figure 1 is a conical stratification that refines the stratification on the right-hand side. The exodromy theorem in the concical case shows that

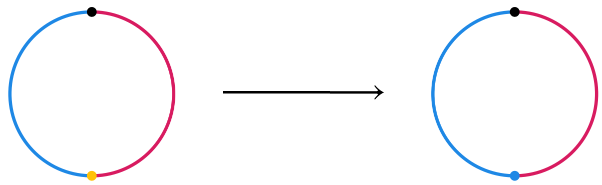
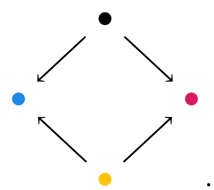


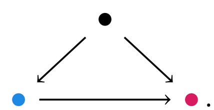
Figure 1. A non-conical stratification of S^1 is pictured on the right. On the left is a conical refinement of the right-hand stratification.

the exit-path ∞-category of the left-hand stratification of S^1 is equivalent to the poset

(0.5.2)



Thus Theorem 0.3.1-(3) implies that the exit-path ∞-category of the right-hand stratification is equivalent to the localization of the poset (0.5.2) at the morphism $\bullet \to \bullet$. This localization is simply the category given by a *noncommutative* triangle



0.5.3 Example (see Examples 5.3.5 and 5.3.10)**.** Favero and Huang [16] recently proved an exodromy result for certain non-conical stratifications naturally arising in mirror symmetry. Of particular interest are the *tree stratification* on a finite simplicial complex [16, §4.4] and the *Bondal–Ruan stratification* of the n-torus [10; 16, §5.2]. The tree stratification is a coarsening of the natural stratification on a finite simplicial complex, which is conical. Moreover, the Bondal–Ruan stratification is subanalytic. Thus Theorems 0.3.1 and 0.4.2 give an alternative perspective on Favero and Huang's exit-path description of constructible sheaves on these stratified spaces.

More examples arise naturally from the study of the Stokes phenomenon for algebraic differential equations. See [33] for more on this topic, as well as a systematic use of the results of this paper.

0.6 Linear overview. In § 1, we provide background on atomically generated ∞-categories, locally constant objects of ∞-topoi, and monodromy that we need for the rest of the paper. In § 2, be begin by recalling the theory of stratifications of ∞-topoi introduced in [8, §8.2] as well as constructible objects. We then explain what it means for a stratified ∞-topos to be *exodromic*, see Definition 2.2.10. We also prove a few basic results about the class of exodromic stratified ∞-topoi. In § 3, we prove a stability theorem for the class of exodromic stratified ∞-topoi, see Theorem 3.0.1. This is the main technical result of the paper, and implies the analogous result for stratified spaces stated in this introduction (Theorem 0.3.1). Section 4 explains when exodromy (with coefficients in the ∞-category of spaces) implies exodromy with coefficients in other presentable ∞-categories. The key takeaway is that exodromy with coefficients in a compactly assembled ∞-category is automatic (see Corollary 4.1.15). We need this result in order to prove our representability result for the derived moduli of constructible and perverse sheaves (Theorem 0.4.5). Section 5 is dedicated to applications of our stability theorem for exodromic stratified ∞-topoi (Theorem 3.0.1). We deduce Theorem 0.3.1, provide many natural examples of exodromic stratified spaces coming from geometry and topology, and prove all of the results stated in § 0.4. In Appendix A, we prove a number of categorical facts needed to control the localizations of exit-path ∞-categories we consider. Specifically, the results proven in Appendix A are needed to prove items (3) and (5) of Theorem 0.3.1. In Appendix B, we collect some background on open and closed subtopoi and recollements. We then explain the relationship between hypercompletion and recollements (see Proposition B.3.5). We need these results in a variety of places, for example, to ensure that the definition of a constructible object of a stratified ∞-topos recovers the more classical notion of a constructible (hyper)sheaf on a topological space.

0.7 Notational conventions. We use the following standard notation.

(1) We write $\mathbf{Cat}_\infty$ for the large ∞-category of small ∞-categories, and write $\mathbf{Spc} \subset \mathbf{Cat}_\infty$ for the full subcategory spanned by the spaces (i.e., ∞-groupoids or anima). We write $\mathbf{CAT}_\infty$ for the (very large) ∞-category of large ∞-categories.

(2) We write $\mathbf{Pr}^{\mathrm{R}}$ for the ∞-category of presentable ∞-categories and right adjoints and $\mathbf{Pr}^{\mathrm{L}}$ for the ∞-category of presentable ∞-categories and left adjoints.

(3) We write $\mathbf{RTop}_\infty$ for the ∞-category of ∞-topoi and *geometric morphisms*, i.e., right adjoints f_* whose left adjoint f^* is left exact. We write $\mathbf{LTop}_\infty$ for the ∞-category of ∞-topoi and left exact left adjoints.

(4) Given a small ∞-category $\mathcal{C}$, we write $\mathrm{PSh}(\mathcal{C}) \coloneqq \mathrm{Fun}(\mathcal{C}, \mathbf{Spc})$ for the ∞-category of presheaves of spaces on $\mathcal{C}$.

(5) For an integer $n \geq 0$, we write $[n]$ for the poset $\{0 < \cdots < n\}$

We later introduce notational conventions for (hyper)sheaves and constructibility; these are consistent with the notational conventions in our previous works [21; 32].

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1 Background on atomic generation, locally constant objects, and monodromy

In this section, we recall the necessary background on *atomically generated ∞-categories* (§ 1.1), tensor products of presentable ∞-categories (§ 1.2), and locally constant objects of ∞-topoi and monodromy (§ 1.3).

1.1 Recollections on atomic generation. We begin by recalling the background on atomically generated ∞-categories needed in this paper. In particular, we provide a useful way to check that a full subcategory of an atomically generated ∞-category is also atomically generated and compute its generators (Proposition 1.1.13). For more on this topic, we refer the reader to [Ker, Tag 03WR; HTT, §5.1.6; 14, §2.2]. We begin with some definitions.

1.1.1 Definition. Let $\mathcal{C}$ be a presentable ∞-category. An object $c \in \mathcal{C}$ is *atomic*[3] if the functor

$$\mathrm{Map}_{\mathcal{C}}(c, -)\colon \mathcal{C} \to \mathbf{Spc}$$

preserves colimits. We write $\mathcal{C}^{\mathrm{at}} \subset \mathcal{C}$ for the full subcategory spanned by the atomic objects.

1.1.2 Observation. The subcategory $\mathcal{C}^{\mathrm{at}} \subset \mathcal{C}$ is always small and idempotent complete. However, contrary to what happens to the full subcategory $\mathcal{C}^{\omega} \subset \mathcal{C}$ spanned by compact objects, the ∞-category $\mathcal{C}^{\mathrm{at}}$ typically does not have finite colimits.

1.1.3 Definition. Let $\mathcal{C}$ be a presentable ∞-category. We say that a small full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ *atomically generates* $\mathcal{C}$ if the unique colimit-preserving extension

$$\mathrm{PSh}(\mathcal{C}_0) \to \mathcal{C}$$

[3] Atomic objects are also referred to as *completely compact* objects [HTT, Definition 5.1.6.2].

of $\mathcal{C}_0 \subset \mathcal{C}$ along the Yoneda embedding is an equivalence. We say that $\mathcal{C}$ is *atomically generated* if there exists a small full subcategory $\mathcal{C}_0 \subset \mathcal{C}$ that atomically generates $\mathcal{C}$.

1.1.4 Remark. The unique colimit-preserving extension $\mathrm{PSh}(\mathcal{C}_0) \to \mathcal{C}$ of the inclusion $\mathcal{C}_0 \subset \mathcal{C}$ is left adjoint to the restricted Yoneda functor

$$y \colon \mathcal{C} \to \mathrm{PSh}(\mathcal{C}_0)\,, \qquad c \mapsto \mathrm{Map}_{\mathcal{C}}(-, c)\,.$$

Hence $\mathcal{C}_0$ atomically generates $\mathcal{C}$ if and only if y is an equivalence.

1.1.5 Example [HTT, Proposition 5.1.6.8]**.** Let $\mathcal{C}_0$ be a small ∞-category. Then, the atomic objects of $\mathrm{PSh}(\mathcal{C}_0)$ are the retracts of representable functors. In particular, the unique atomic object of **Spc** is the point $*$.

1.1.6 Observation. If $\mathcal{C}_0 \subset \mathcal{C}$ atomically generates $\mathcal{C}$, then $\mathcal{C}_0 \subset \mathcal{C}^{\mathrm{at}}$. Moreover, by [Ker, Tag 040X], the inclusion $\mathcal{C}_0 \subset \mathcal{C}^{\mathrm{at}}$ exhibits $\mathcal{C}^{\mathrm{at}}$ as the idempotent completion of $\mathcal{C}_0$. As a consequence, all of the functors given by extending the obvious inclusions along colimits

$$\begin{array}{ccc} \mathrm{PSh}(\mathcal{C}_0) & \longrightarrow & \mathrm{PSh}(\mathcal{C}^{\mathrm{at}}) \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

are equivalences. In particular, $\mathcal{C}^{\mathrm{at}}$ also atomically generates $\mathcal{C}$.

1.1.7 Definition. Let $L \colon \mathcal{D} \to \mathcal{C}$ be a left adjoint functor of ∞-categories. We say that L is *atomic* if the right adjoint $\mathcal{C} \to \mathcal{D}$ to L is also a left adjoint.

1.1.8 Observation. If $L \colon \mathcal{D} \to \mathcal{C}$ is an atomic functor between presentable ∞-categories, then L preserves atomic objects, i.e., $L(\mathcal{D}^{\mathrm{at}}) \subset \mathcal{C}^{\mathrm{at}}$. If R denotes the right adjoint to L, then the square

$$\begin{array}{ccc} \mathrm{PSh}(\mathcal{C}^{\mathrm{at}}) & \xrightarrow{L^*} & \mathrm{PSh}(\mathcal{D}^{\mathrm{at}}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[R]{} & \mathcal{D} \end{array}$$

commutes.

The converse is true if $\mathcal{C}$ and $\mathcal{D}$ are atomically generated:

1.1.9 Recollection [14, Lemma 2.6-(3)]**.** Let $L \colon \mathcal{D} \to \mathcal{C}$ be a left adjoint between atomically generated presentable ∞-categories. Then L is atomic if and only if L preserves atomic objects.

In this paper, we repeatedly use the fact that the ∞-category of atomically generated presentable ∞-categories and atomic functors is equivalent to the ∞-category of idempotent complete ∞-categories:

1.1.10 Notation. Write $\mathbf{Pr}^{\mathrm{L,at}} \subset \mathbf{Pr}^{\mathrm{L}}$ for the non-full subcategory with objects the atomically generated ∞-categories and morphisms atomic left adjoints. Write

$$\mathbf{Cat}_\infty^{\mathrm{idem}} \subset \mathbf{Cat}_\infty$$

for the full subcategory spanned by the idempotent complete ∞-categories.

1.1.11 Recollection [14, Proposition 2.7]**.** Consider the functor $\mathrm{PSh} \colon \mathbf{Cat}_\infty^{\mathrm{idem}} \to \mathbf{Pr}^{\mathrm{L}}$ sending a small idempotent complete ∞-category $\mathcal{C}_0$ to $\mathrm{PSh}(\mathcal{C}_0)$ with functoriality given by left Kan extension. This functor restricts to an equivalence

$$\mathrm{PSh} \colon \mathbf{Cat}_\infty^{\mathrm{idem}} \simeq \mathbf{Pr}^{\mathrm{L,at}}$$

with inverse given by

$$(-)^{\mathrm{at}} \colon \mathbf{Pr}^{\mathrm{L,at}} \simeq \mathbf{Cat}_\infty^{\mathrm{idem}}\,.$$

1.1.12 Notation. Let $\mathcal{C}$ be an atomically generated presentable ∞-category. To simplify notation later on, we write $\mathcal{C}^{\mathrm{ex}} := (\mathcal{C}^{\mathrm{at}})^{\mathrm{op}}$ for the opposite of the full subcategory of $\mathcal{C}$ spanned by the atomic objects. Thus there is a natural equivalence

$$\mathcal{C} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{ex}}, \mathbf{Spc}) .$$

The proof of Theorem 0.3.1 relies on the fact that a full subcategory of an atomically generated ∞-category that is closed under limits and colimits is also atomically generated:

1.1.13 Proposition. *Let $\mathcal{D}$ be an atomically generated presentable ∞-category and let $i\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be the inclusion of a full subcategory. If $\mathcal{C}$ is closed under both limits and colimits in $\mathcal{D}$, then:*

(1) *The ∞-category $\mathcal{C}$ is presentable and the inclusion $i\colon \mathcal{C} \hookrightarrow \mathcal{D}$ admits both a left adjoint $L\colon \mathcal{D} \to \mathcal{C}$ and a right adjoint $R\colon \mathcal{D} \to \mathcal{C}$.*

(2) *The ∞-category $\mathcal{C}$ is atomically generated by $L(\mathcal{D}^{\mathrm{at}})$.*

(3) *Let $W_L \subset \mathrm{Mor}(\mathcal{D})$ be the collection of L-equivalences. Let $W \subset W_L \cap \mathrm{Mor}(\mathcal{D}^{\mathrm{at}})$ be a subset of morphisms with the property that $\mathcal{C}$ coincides with the subcategory of W-local objects of $\mathcal{D}$. Then the functor*

$$L\colon \mathcal{D}^{\mathrm{at}} \to \mathcal{C}^{\mathrm{at}}$$

exhibits $\mathcal{C}^{\mathrm{at}}$ as the idempotent completion of the localization $\mathcal{D}^{\mathrm{at}}[W^{-1}]$.

Proof. Point (1) is a direct consequence of the reflection theorem of Ragimov–Schlank [34, Theorem 1.1].

To prove (2), first note that by Observation 1.1.8, the functor L preserves atomic objects. Hence [HTT, Proposition 5.1.6.10] implies that the functor

$$\mathrm{PSh}(L(\mathcal{D}^{\mathrm{at}})) \to \mathcal{C}$$

given by the left Kan extension of the inclusion $L(\mathcal{D}^{\mathrm{at}}) \subset \mathcal{C}$ along the Yoneda embedding is fully faithful. To complete the proof of (2), we need to show that this functor is also essentially surjective. Equivalently, we need to show that $L(\mathcal{D}^{\mathrm{at}})$ generates $\mathcal{C}$ under colimits. For this, let $c \in \mathcal{C}$. Since $\mathcal{D}$ is atomically generated, there exists a diagram $d_\bullet\colon A \to \mathcal{D}^{\mathrm{at}}$ and an equivalence

$$i(c) \simeq \operatorname*{colim}_{\alpha \in A} d_\alpha .$$

Applying the left adjoint L, we find that

$$c \simeq L(i(c)) \simeq \operatorname*{colim}_{\alpha \in A} L(d_\alpha) .$$

Thus $L(\mathcal{D}^{\mathrm{at}})$ generates $\mathcal{C}$ under colimits, as desired.

Now we prove (3). Combining (2) with Observation 1.1.6 shows that the inclusion $L(\mathcal{D}^{\mathrm{at}}) \subset \mathcal{C}^{\mathrm{at}}$ exhibits $\mathcal{C}^{\mathrm{at}}$ as the idempotent completion of $L(\mathcal{D}^{\mathrm{at}})$. Thus it suffices to prove that the functor

$$L\colon \mathcal{D}^{\mathrm{at}} \to L(\mathcal{D}^{\mathrm{at}}) \tag{1.1.14}$$

exhibits $L(\mathcal{D}^{\mathrm{at}})$ as the localization of $\mathcal{D}^{\mathrm{at}}$ at W. To see this, we apply the three criteria of [13, Proposition 7.1.11]. By definition, the functor (1.1.14) is essentially surjective. Moreover, upon passing to presheaves, precomposition with (1.1.14) is identified with $i\colon \mathcal{C} \hookrightarrow \mathcal{D}$ via the restricted Yoneda functor from Remark 1.1.4. Hence, precomposition with (1.1.14) is fully faithful with image contained in the full subcategory of presheaves $F \in \mathrm{PSh}(\mathcal{D}^{\mathrm{at}})$ that invert W. Via the restricted Yoneda functor, presheaves $F \in \mathrm{PSh}(\mathcal{D}^{\mathrm{at}})$ that invert W correspond to W-local objects of $\mathcal{D}$, that is objects of $\mathcal{C}$. Thus, [13, Proposition 7.1.11] applies and concludes the proof of (3). □

1.2 Sheaves with coefficients & tensor products of presentable ∞-categories. We now fix our conventions for sheaves with coefficients in a presentable ∞-category. For this, we make use of the tensor product of presentable ∞-categories; we refer the reader to [HA, §4.8.1] for a background.

1.2.1 Notation. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{E}$ be a presentable ∞-category. We write $\mathrm{Sh}(\mathcal{X};\mathcal{E})$ for the tensor product of presentable ∞-categories

$$\mathrm{Sh}(\mathcal{X};\mathcal{E}) := \mathcal{X} \otimes \mathcal{E} .$$

Since the tensor product $(-) \otimes \mathcal{E}$ defines a functor $\mathbf{Pr}^{\mathrm{R}} \to \mathbf{Pr}^{\mathrm{R}}$, the assignment $\mathcal{X} \mapsto \mathrm{Sh}(\mathcal{X}; \mathcal{E})$ defines a functor

$$\mathrm{Sh}(-; \mathcal{E}) \colon \mathbf{RTop}_\infty \to \mathbf{Pr}^{\mathrm{R}} .$$

1.2.2 Notation (sheaves on ∞-sites)**.** Let $(\mathcal{C}, \tau)$ be an ∞-site and $\mathcal{E}$ be a presentable ∞-category. We write

$$\mathrm{PSh}(\mathcal{C}; \mathcal{E}) \coloneqq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$$

for the ∞-category of $\mathcal{E}$-valued presheaves on $\mathcal{C}$. We also write

$$\mathrm{Sh}_\tau(\mathcal{C}; \mathcal{E}) \subset \mathrm{PSh}(\mathcal{C}; \mathcal{E})$$

for the full subcategory spanned by $\mathcal{E}$-valued presheaves that satisfy τ-descent. When $\mathcal{E} = \mathbf{Spc}$, we write

$$\mathrm{Sh}_\tau(\mathcal{C}) \coloneqq \mathrm{Sh}_\tau(\mathcal{C}; \mathbf{Spc}) .$$

1.2.3. The ∞-categories $\mathrm{PSh}(\mathcal{C}; \mathcal{E})$ and $\mathrm{Sh}_\tau(\mathcal{C}; \mathcal{E})$ are naturally identified with the tensor products of presentable ∞-categories $\mathrm{PSh}(\mathcal{C}) \otimes \mathcal{E}$ and $\mathrm{Sh}_\tau(\mathcal{C}) \otimes \mathcal{E}$ [SAG, Remark 1.3.1.6 & Proposition 1.3.1.7]. This justifies Notation 1.2.1.

1.2.4 (hypersheaves)**.** Let $(\mathcal{C}, \tau)$ be an ∞-site. In this paper, we often make use of the theory of *hypersheaves*. When $\mathcal{E}$ is the ∞-category of spaces, hypersheaves can be defined intrinsically in the ∞-topos $\mathrm{Sh}_\tau(\mathcal{C})$ as *hypercomplete objects*, that is, objects that are local with respect to ∞-connected maps. Hypersheaves thus form a full subcategory $\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) \subset \mathrm{Sh}_\tau(\mathcal{C})$. It is then possible to *define* hypersheaves with coefficients in $\mathcal{E}$ as the tensor product

$$\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}; \mathcal{E}) \coloneqq \mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) \otimes \mathcal{E} .$$

Each of the inclusions

$$\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C}) \qquad \text{and} \qquad \mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) \subset \mathrm{Sh}_\tau(\mathcal{C})$$

admits a left exact left adjoint adjoint. We refer the reader unfamiliar with hypercomplete objects and hypercompletion to [HTT, §§6.5.2–6.5.4] or [8, §3.11] for further reading on the subject.

1.2.5 Notation (sheaves on topological spaces)**.** Let X be a topological space and let $\mathcal{E}$ be a presentable ∞-category. We write $\mathrm{Open}(X)$ the poset of open subsets of X, ordered by inclusion. We regard $\mathrm{Open}(X)$ as a site with the covering families given by open covers. We write

$$\mathrm{Sh}(X; \mathcal{E}) \coloneqq \mathrm{Sh}(\mathrm{Open}(X); \mathcal{E}) \qquad \text{and} \qquad \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \coloneqq \mathrm{Sh}^{\mathrm{hyp}}(\mathrm{Open}(X); \mathcal{E}) .$$

1.2.6 Notation (functoriality)**.** Let $f_* \colon \mathcal{X} \to \mathcal{Y}$ be a geometric morphism. We write f^* for its left exact left adjoint. If f_* is an étale geometric morphism, we denote by $f_\sharp$ the left adjoint to f^*. Fix a presentable ∞-category $\mathcal{E}$. Then the functoriality of the tensor product in $\mathbf{Pr}^{\mathrm{L}}$ provides us with a colimit-preserving functor

$$f^* \otimes \mathcal{E} \colon \mathrm{Sh}(\mathcal{Y}; \mathcal{E}) \to \mathrm{Sh}(\mathcal{X}; \mathcal{E}) .$$

When there is no risk of confusion, we simply write f^* instead of $f^* \otimes \mathcal{E}$. Similarly, we write f_* for its right adjoint, and we apply a similar convention for $f_\sharp$.

1.3 Locally constant objects & monodromy. We now recall the basics of locally constant objects in ∞-topoi and monodromy. We also prove a few foundational results that we need later in the paper, but are not available elsewhere. For more background, we refer the reader to [HA, §A.1; 1, §3.1].

1.3.1 Notation (constant objects and global sections)**.** Let $\mathcal{X}$ be an ∞-topos. We write

$$\Gamma_{\mathcal{X},*} \colon \mathcal{X} \to \mathbf{Spc}$$

for the *global sections* functor given by

$$U \mapsto \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, U) .$$

The global sections functor admits a left exact left adjoint $\Gamma^*_{\mathcal{X}} \colon \mathbf{Spc} \to \mathcal{X}$ called the *constant sheaf functor*. If the ∞-topos $\mathcal{X}$ is clear from the context, we write Γ^* and Γ_* for $\Gamma^*_{\mathcal{X}}$ and $\Gamma_{\mathcal{X},*}$, respectively.

Given a presentable ∞-category $\mathcal{E}$, we say that an F object of $\mathrm{Sh}(\mathcal{X}; \mathcal{E})$ is *constant* if F lies in the image of the functor

$$\Gamma^* \otimes \mathcal{E} \colon \mathcal{E} \to \mathrm{Sh}(\mathcal{X}; \mathcal{E}) .$$

1.3.2. Note that **Spc** is the terminal ∞-topos [HTT, Proposition 6.3.4.1], so Γ_* is the unique geometric morphism $\mathcal{X} \to \mathbf{Spc}$.

1.3.3 Recollection (products of ∞-topoi)**.** The product in $\mathbf{RTop}_\infty$ is given by the *tensor product* in $\mathbf{Pr}^{\mathrm{R}}$; see [HA, Example 4.8.1.19; 3, Theorem 2.1.5]. In particular:

(1) If $f^* : \mathcal{X}' \to \mathcal{X}$ and $g^* : \mathcal{Y}' \to \mathcal{Y}$ are left exact left adjoints between ∞-topoi, then

$$f^* \otimes g^* : \mathcal{X}' \otimes \mathcal{Y}' \to \mathcal{X} \otimes \mathcal{Y}$$

is also a left exact left adjoint between ∞-topoi.

(2) The functor

$$\Gamma^*_{\mathcal{X}} \otimes \Gamma^*_{\mathcal{Y}} : \mathbf{Spc} \simeq \mathbf{Spc} \otimes \mathbf{Spc} \to \mathcal{X} \otimes \mathcal{Y}$$

is the constant sheaf functor.

1.3.4 Definition (locally constant objects)**.** Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{E}$ be a presentable ∞-category. An object $F \in \mathrm{Sh}(\mathcal{X};\mathcal{E})$ is *locally constant* if there exists an effective epimorphism $\coprod_{i\in I} U_i \twoheadrightarrow 1_{\mathcal{X}}$ such that for each $i \in I$, the image of F under the natural pullback functor

$$\mathrm{Sh}(\mathcal{X};\mathcal{E}) \to \mathrm{Sh}(\mathcal{X}_{/U_i};\mathcal{E})$$

is a constant object. We write

$$\mathrm{LC}(\mathcal{X};\mathcal{E}) \subset \mathrm{Sh}(\mathcal{X};\mathcal{E})$$

for the full subcategory spanned by the locally constant objects. When $\mathcal{E} = \mathbf{Spc}$, we simply write $\mathrm{LC}(\mathcal{X}) \subset \mathcal{X}$ for $\mathrm{LC}(\mathcal{X};\mathbf{Spc})$.

1.3.5 Observation. Given a geometric morphism of ∞-topoi $f_* : \mathcal{X} \to \mathcal{Y}$, the pullback functor $f^* : \mathcal{Y} \to \mathcal{X}$ carries $\mathrm{LC}(\mathcal{Y};\mathcal{E})$ to $\mathrm{LC}(\mathcal{X};\mathcal{E})$.

This recovers the usual notion of local constancy for (hyper)sheaves on topological spaces:

1.3.6 Example. Let X be a topological space and let $\mathcal{X}$ be either $\mathrm{Sh}(X)$ or $\mathrm{Sh}^{\mathrm{hyp}}(X)$. An object $F \in \mathrm{Sh}(\mathcal{X};\mathcal{E})$ is locally constant if and only if there exists an open cover $\{U_i\}_{i\in I}$ of X such that each restriction $F|_{U_i}$ is constant. See [25, Proposition 1.18].

1.3.7 Definition (monodromic ∞-topos)**.** We say that an ∞-topos $\mathcal{X}$ is *monodromic* or *locally of constant shape* if the constant sheaf functor $\Gamma^* : \mathbf{Spc} \to \mathcal{X}$ admits a left adjoint

$$\Gamma_\sharp : \mathcal{X} \to \mathbf{Spc}\,.$$

In this case, we write $\Pi_\infty(\mathcal{X}) \coloneqq \Gamma_\sharp(1_{\mathcal{X}})$ and call $\Pi_\infty(\mathcal{X})$ the *shape* of $\mathcal{X}$.

The following result of Lurie justifies the terminology in Definition 1.3.7:

1.3.8 Recollection (monodromy)**.** Let $\mathcal{X}$ be a monodromic ∞-topos. Then the full subcategory $\mathrm{LC}(\mathcal{X}) \subset \mathcal{X}$ is closed under limits and colimits. Moreover, there is a natural equivalence

$$\mathrm{LC}(\mathcal{X}) \simeq \mathrm{Fun}(\Pi_\infty(\mathcal{X}), \mathbf{Spc})$$

See [HA, Proposition A.1.6 & Theorem A.1.15]. Furthermore, for any presentable ∞-category $\mathcal{E}$, there is an equivalence

$$\mathrm{LC}(\mathcal{X}) \otimes \mathcal{E} \simeq \mathrm{LC}(\mathcal{X};\mathcal{E})\,.$$

See [1, Proposition 3.1.7]. In particular, $\mathrm{LC}(\mathcal{X};\mathcal{E}) \subset \mathrm{Sh}(\mathcal{X};\mathcal{E})$ is closed under limits and colimits.

1.3.9 Example (monodromy in topology)**.** Let X be a topological space.

(1) If X is locally weakly contractible, then the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(X)$ is monodromic. The functor

$$\Gamma_\sharp : \mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathbf{Spc}$$

is given by extending the functor sending an open $U \subset X$ to the underlying homotopy type of U along colimits. In particular $\Pi_\infty(\mathrm{Sh}^{\mathrm{hyp}}(X))$ coincides with the underlying homotopy type of X. See [21, Proposition 2.4].

(2) If X is paracompact and *locally of singular shape* in the sense of [HA, Definition A.4.15], then the ∞-topos $\mathrm{Sh}(X)$ is monodromic. Again, the functor $\Gamma_\sharp \colon \mathrm{Sh}(X) \to \mathbf{Spc}$ is given by extending the functor sending an open $U \subset X$ to the underlying homotopy type of U along colimits. In particular $\Pi_\infty(\mathrm{Sh}(X))$ coincides with the underlying homotopy type of X. See [HA, Theorem A.4.19].

An intriguing fact is that any ∞-topos étale over a monodromic ∞-topos is also monodromic:

1.3.10 Observation. Let $\mathcal{X}$ be a monodromic ∞-topos and let $U \in \mathcal{X}$. Then the slice ∞-topos $\mathcal{X}_{/U}$ is monodromic. To see this, note that the composite

$$\mathcal{X}_{/U} \xrightarrow{\text{forget}} \mathcal{X} \xrightarrow{\Gamma_{\mathcal{X},\sharp}} \mathbf{Spc}$$

is left adjoint to the constant sheaf functor. As a consequence, we see that

$$\Pi_\infty(\mathcal{X}_{/U}) = \Gamma_{\mathcal{X},\sharp}(U) .$$

We now explain the functoriality of the monodromy equivalence. To do so, we need the following lemma.

1.3.11 Lemma. *Let $K, L \in \mathbf{Spc}$ and let*

$$f^* \colon \mathrm{Fun}(L, \mathbf{Spc}) \to \mathrm{Fun}(K, \mathbf{Spc})$$

be a functor. The following are equivalent:

(1) *There exists a map of spaces $f \colon K \to L$ such that f^* is equivalent to the functor given by precomposition with f.*

(2) *The functor f^* preserves limits and colimits.*

(3) *The functor f^* is left exact and preserves colimits.*

Proof. Since every space is an idempotent complete ∞-category (see Lemma A.1.3), the equivalence (1) $\Leftrightarrow$ (2) follows from Recollection 1.1.11. Clearly (2) $\Rightarrow$ (3). For the remaining implication (3) $\Rightarrow$ (2), let f_* denote the right adjoint to f^*. By assumption, f_* is a geometric morphism. Note that by the straightening/unstraightening equivalences

$$\mathrm{Fun}(K, \mathbf{Spc}) \simeq \mathbf{Spc}_{/K} \qquad \text{and} \qquad \mathrm{Fun}(L, \mathbf{Spc}) \simeq \mathbf{Spc}_{/L} ,$$

the unique geometric morphisms to the terminal ∞-topos

$$\mathrm{Fun}(K, \mathbf{Spc}) \to \mathbf{Spc} \qquad \text{and} \qquad \mathrm{Fun}(L, \mathbf{Spc}) \to \mathbf{Spc}$$

are étale. Hence [HTT, Corollary 6.3.5.9] implies that f_* is an étale geometric morphism; in particular, f^* admits a left adjoint. □

1.3.12 Corollary. *Let $f_* \colon \mathcal{X} \to \mathcal{Y}$ be a geometric morphism between monodromic ∞-topoi. Then the functor*

$$f^* \colon \mathrm{LC}(\mathcal{X}) \to \mathrm{LC}(\mathcal{Y})$$

preserves limits and colimits.

Proof. Since $\mathcal{X}$ and $\mathcal{Y}$ are monodromic, $\mathrm{LC}(\mathcal{X}) \subset \mathcal{X}$ and $\mathrm{LC}(\mathcal{Y}) \subset \mathcal{Y}$ are closed under limits and colimits. The claim now follows from the monodromy equivalences for $\mathcal{X}$ and $\mathcal{Y}$ combined with Lemma 1.3.11. □

1.3.13 Notation. Write $\mathbf{RTop}_\infty^{\mathrm{mon}} \subset \mathbf{RTop}_\infty$ for the full subcategory spanned by the monodromic ∞-topoi.

1.3.14 Notation. Write $\mathbf{Pr}^{\mathrm{R,at}} \subset \mathbf{Pr}^{\mathrm{R}}$ for the (non-full) subcategory of $\mathbf{Pr}^{\mathrm{R}}$ with objects the atomically generated presentable ∞-categories and morphisms functors that are both left and right adjoints.

1.3.15. Note that the equivalence $\mathbf{Pr}^{\mathrm{L}} \simeq (\mathbf{Pr}^{\mathrm{R}})^{\mathrm{op}}$ given by passing to right adjoints restricts to an equivalence

$$\mathbf{Pr}^{\mathrm{L,at}} \simeq (\mathbf{Pr}^{\mathrm{R,at}})^{\mathrm{op}} .$$

1.3.16 Observation (functoriality of the shape)**.** The assignment $\mathcal{X} \mapsto \Pi_\infty(\mathcal{X})$ refines to a functor

$$\Pi_\infty \colon \mathbf{RTop}_\infty^{\mathrm{mon}} \to \mathbf{Spc} \subset \mathbf{Cat}_\infty^{\mathrm{idem}} .$$

Specifically, this functor is given by the composite

$$\mathbf{RTop}_\infty^{\mathrm{mon}} \xrightarrow{\mathrm{LC}(-)} (\mathbf{Pr}^{\mathrm{R,at}})^{\mathrm{op}} \simeq \mathbf{Pr}^{\mathrm{L,at}} \xrightarrow[\sim]{(-)^{\mathrm{ex}}} \mathbf{Cat}_\infty^{\mathrm{idem}} ,$$

where the left-hand functor sends $\mathcal{X}$ to the ∞-category $\mathrm{LC}(\mathcal{X})$ with functoriality given by pullback, and the right-hand functor sends an atomically generated ∞-category $\mathcal{C}$ to the ∞-category $\mathcal{C}^{\mathrm{ex}} = (\mathcal{C}^{\mathrm{at}})^{\mathrm{op}}$ given by the opposite of the subcategory of atomic objects.

We conclude with a Künneth formula for the shape of a product of monodromic ∞-topoi.

1.3.17 Recollection. The natural equivalence

$$\mathbf{Spc} \otimes \mathbf{Spc} \xrightarrow{\sim} \mathbf{Spc}$$

is induced by the functor

$$\begin{aligned} \mathbf{Spc} \times \mathbf{Spc} &\to \mathbf{Spc} \\ (K, L) &\mapsto K \times L . \end{aligned}$$

1.3.18 Observation. Let $\mathcal{X}$ and $\mathcal{Y}$ be monodromic ∞-topoi. Since the inclusions

$$\mathrm{LC}(\mathcal{X}) \hookrightarrow \mathcal{X} \qquad \text{and} \qquad \mathrm{LC}(\mathcal{Y}) \hookrightarrow \mathcal{Y}$$

are both left and right adjoints, the functor

$$\mathrm{LC}(\mathcal{X}) \otimes \mathrm{LC}(\mathcal{Y}) \to \mathcal{X} \otimes \mathcal{Y}$$

induced by the functoriality of the tensor product is also fully faithful and both a left and right adjoint.

1.3.19 Proposition (Künneth formula for monodromic ∞-topoi)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be monodromic ∞-topoi. Write $\Gamma_{\mathcal{X},\sharp} \colon \mathcal{X} \to \mathbf{Spc}$ and $\Gamma_{\mathcal{Y},\sharp} \colon \mathcal{Y} \to \mathbf{Spc}$ for the left adjoints to the constant sheaf functors. Then:*

(1) *The functor*

$$\Gamma_{\mathcal{X},\sharp} \otimes \Gamma_{\mathcal{Y},\sharp} \colon \mathcal{X} \otimes \mathcal{Y} \to \mathbf{Spc} \otimes \mathbf{Spc} \simeq \mathbf{Spc}$$

is left adjoint to the constant sheaf functor $\mathbf{Spc} \to \mathcal{X} \otimes \mathcal{Y}$. In particular, the ∞-topos $\mathcal{X} \otimes \mathcal{Y}$ is monodromic.

(2) *The natural map $\Pi_\infty(\mathcal{X} \otimes \mathcal{Y}) \to \Pi_\infty(\mathcal{X}) \times \Pi_\infty(\mathcal{Y})$ is an equivalence.*

(3) *The natural fully faithful functor*

$$\mathrm{LC}(\mathcal{X}) \otimes \mathrm{LC}(\mathcal{Y}) \to \mathcal{X} \otimes \mathcal{Y}$$

has image $\mathrm{LC}(\mathcal{X} \otimes \mathcal{Y})$.

Proof. For (1), note that by the functoriality of the tensor product of presentable ∞-categories, $\Gamma_{\mathcal{X},\sharp} \otimes \Gamma_{\mathcal{Y},\sharp}$ is left adjoint to the functor

$$\Gamma_{\mathcal{X}}^* \otimes \Gamma_{\mathcal{Y}}^* \colon \mathbf{Spc} \simeq \mathbf{Spc} \otimes \mathbf{Spc} \to \mathcal{X} \otimes \mathcal{Y} .$$

By Recollection 1.3.3-(2), $\Gamma_{\mathcal{X}}^* \otimes \Gamma_{\mathcal{Y}}^*$ is the constant sheaf functor; hence $\Gamma_{\mathcal{X},\sharp} \otimes \Gamma_{\mathcal{Y},\sharp}$ is left adjoint to the constant sheaf functor, as desired.

For (2), note that by Recollections 1.3.3 and 1.3.17, the functor

$$\Gamma_{\mathcal{X},\sharp} \otimes \Gamma_{\mathcal{Y},\sharp} \colon \mathcal{X} \otimes \mathcal{Y} \to \mathbf{Spc} \otimes \mathbf{Spc} \simeq \mathbf{Spc}$$

is induced by the functor

$$\begin{aligned} \mathcal{X} \times \mathcal{Y} &\to \mathbf{Spc} \\ (F, G) &\mapsto \Gamma_{\mathcal{X},\sharp}(F) \times \Gamma_{\mathcal{Y},\sharp}(G) . \end{aligned}$$

In particular, applying $\Gamma_{\mathcal{X},\sharp} \otimes \Gamma_{\mathcal{Y},\sharp}$ to the terminal object $1_{\mathcal{X}\otimes\mathcal{Y}} = 1_{\mathcal{X}} \otimes 1_{\mathcal{Y}}$, we have natural identifications

$$\begin{aligned}\Pi_\infty(\mathcal{X}\otimes\mathcal{Y}) &= (\Gamma_{\mathcal{X},\sharp}\otimes\Gamma_{\mathcal{Y},\sharp})(1_{\mathcal{X}}\otimes 1_{\mathcal{Y}}) \\ &= \Gamma_{\mathcal{X},\sharp}(1_{\mathcal{X}})\times\Gamma_{\mathcal{Y},\sharp}(1_{\mathcal{Y}}) \\ &= \Pi_\infty(\mathcal{X})\times\Pi_\infty(\mathcal{Y})\,.\end{aligned}$$

Item (3) is an immediate consequence of (2) and the formula

$$\mathrm{Fun}(\mathcal{C},\mathbf{Spc})\otimes\mathrm{Fun}(\mathcal{D},\mathbf{Spc})\simeq\mathrm{Fun}(\mathcal{C}\times\mathcal{D},\mathbf{Spc})\,. \qquad \square$$

2 Exit-path ∞-categories

In this section, we introduce *exodromic stratified* ∞*-topoi* and their *exit-path* ∞*-categories*. See Definition 2.2.10. These definitions are topos-theoretic generalizations of Clausen and Ørsnes Jansen's definition in the topological setting [14, Definition 3.5].

In §2.1, we start by reviewing the basics of the theory of stratified ∞-topoi introduced in [8]. In §2.2, we explain the basics of constructible objects in stratified ∞-topoi; we also define exodromic stratified ∞-topoi their exit-path ∞-categories. In §2.3, we discuss stratified morphisms that induce morphisms on the level of exit-path ∞-categories. In §2.4, we conclude with some results on the interaction between exodromic stratified ∞-topoi and hypercompletion.

2.1 Stratified ∞-topoi & stratified spaces. We now recall the theory of stratifications of ∞-topoi introduced in [8, §8.2]. This theory directly generalizes the theory of stratifications of topological spaces, but also applies to more general contexts such as stratifications of schemes and topological stacks. The starting point for the theory is the observation that hypersheaves on a poset P equipped with the Alexandroff topology are just functors out of P:

2.1.1 Recollection [4, Example A.11; 8, Example 3.12.15]**.** Let $\mathcal{E}$ be a presentable ∞-category and let P be a poset. Regard P as a topological space with the Alexandroff topology. Then there is a natural equivalence of ∞-categories

$$\mathrm{Fun}(P,\mathcal{E}) \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{hyp}}(P;\mathcal{E})\,.$$

2.1.2 Warning. It is necessary that we work with hypersheaves in Recollection 2.1.1; in general, $\mathrm{Sh}(P)$ is not hypercomplete. See [4, Example A.13].

2.1.3 Example [21, Lemma 5.21]**.** If P is a noetherian poset, then $\mathrm{Sh}(P)$ is hypercomplete, hence

$$\mathrm{Sh}(P)\simeq\mathrm{Fun}(P,\mathbf{Spc})\,.$$

2.1.4 Definition (stratified ∞-topos)**.** Let $\mathcal{X}$ be an ∞-topos and let P be a poset. A *P-stratification* of $\mathcal{X}$ is a geometric morphism

$$s_*\colon \mathcal{X}\to\mathrm{Fun}(P,\mathbf{Spc})\,.$$

To simplify notation, we often abusively denote a stratified ∞-topos by $(\mathcal{X},P)$.

Morphisms of stratified ∞-topoi are commutative squares. Here is the easiest way to formulate this.

2.1.5 Notation. We write **Poset** for the category of posets.

2.1.6 Definition. The *∞-category of stratified ∞-topoi* is the pullback

$$\begin{array}{ccc}\mathbf{StrTop}_\infty & \longrightarrow & \mathbf{Poset} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{Fun}(-,\mathbf{Spc})} \\ \mathrm{Fun}([1],\mathbf{RTop}_\infty) & \longrightarrow & \mathbf{RTop}_\infty\,.\end{array}$$

Here, the bottom horizontal functor sends a geometric morphism $s_*\colon\mathcal{X}\to\mathcal{P}$ to its target $\mathcal{P}$.

2.1.7. Said differently, given stratified ∞-topoi $s_* \colon \mathcal{X} \to \operatorname{Fun}(P, \mathbf{Spc})$ and $t_* \colon \mathcal{Y} \to \operatorname{Fun}(Q, \mathbf{Spc})$, a *morphism of stratifed ∞-topoi* $(\mathcal{X}, P) \to (\mathcal{Y}, Q)$ consists of a commutative square of geometric morphisms

$$\begin{CD} \mathcal{X} @>{f_*}>> \mathcal{Y} \\ @V{s_*}VV @VV{t_*}V \\ \operatorname{Fun}(P, \mathbf{Spc}) @>>{\phi_*}> \operatorname{Fun}(Q, \mathbf{Spc}) \end{CD}$$

such that the pushforward functor ϕ_* is induced by a map of posets $\phi \colon P \to Q$ (equivalently, ϕ^* preserves limits). To simplify notation, we abusively denote a morphism of stratified ∞-topoi by $f_* \colon (\mathcal{X}, P) \to (\mathcal{Y}, Q)$.

It is often convenient to pull back a P-stratified ∞-topos to a locally closed subposet of P:

2.1.8 Recollection (locally closed subposets)**.** Let P be a poset. Then a subset $S \subset P$ is locally closed in the Alexandroff topology if and only if S is an *interval*: given $p, q \in S$ with $p \leq q$, S contains all $x \in P$ such that $p \leq x \leq q$.

2.1.9 Notation. Let $s_* \colon \mathcal{X} \to \operatorname{Fun}(P, \mathbf{Spc})$ be a stratified ∞-topos and let $i \colon S \hookrightarrow P$ be a locally closed subposet. We write $\mathcal{X}_S$ for the pullback

$$\begin{CD} \mathcal{X}_S @>{i_{S,*}}>> \mathcal{X} \\ @VVV @VV{s_*}V \\ \operatorname{Fun}(S, \mathbf{Spc}) @>>{i_*}> \operatorname{Fun}(P, \mathbf{Spc}) \,. \end{CD}$$

computed in $\mathbf{RTop}_\infty$. Observe that $i_{S,*}$ and i_* define a morphism of stratified ∞-topoi $(\mathcal{X}_S, S) \hookrightarrow (\mathcal{X}, P)$. For each $p \in P$, we call $\mathcal{X}_p \coloneqq \mathcal{X}_{\{p\}}$ the *p-th stratum* of $(\mathcal{X}, P)$.

2.1.10. Note that if $S \subset P$ is open, then $i_{S,*}$ is an open immersion of ∞-topoi and if $S \subset P$ is closed, then $i_{S,*}$ is a closed immersion of ∞-topoi. Hence for $S \subset P$ locally closed, $i_{S,*}$ is a locally closed immersion of ∞-topoi. (See Appendix B for background on locally closed immersions of ∞-topoi.)

2.1.11 Observation. Let $(f_*, \phi) \colon (\mathcal{X}, P) \to (\mathcal{Y}, Q)$ be a morphism of stratified ∞-topoi and let $T \subset Q$ be a locally closed subposet. Write $P_T \coloneqq \phi^{-1}(T)$, so that P_T is a locally closed subposet of P. Then we have a commutative cube of ∞-topoi and geometric morphisms

$$\begin{array}{ccccccc}
\mathcal{X}_{P_T} & & \xrightarrow{\;f_{T,*}\;} & & \mathcal{Y}_T & & \\
\downarrow & \searrow^{i_{P_T,*}} & & & \downarrow & \searrow^{i_{T,*}} & \\
& & \mathcal{X} & \xrightarrow{\;f_*\;} & & & \mathcal{Y} \\
\operatorname{Fun}(P_T, \mathbf{Spc}) & & \longrightarrow & & \operatorname{Fun}(T, \mathbf{Spc}) & & \downarrow \\
& \searrow & \downarrow & & & \searrow & \\
& & \operatorname{Fun}(P, \mathbf{Spc}) & & \xrightarrow{\;\phi_*\;} & & \operatorname{Fun}(Q, \mathbf{Spc})
\end{array}$$

In particular, the induced geometric morphism on pullbacks $f_{T,*} \colon \mathcal{X}_{P_T} \to \mathcal{Y}_T$ refines to a morphism of stratified ∞-topoi

$$(f_{T,*}, (\phi|_{P_T})_*) \colon (\mathcal{X}_{P_T}, P_T \to (\mathcal{Y}_T, T) \,.$$

In this paper, our main examples of stratified ∞-topoi come from stratified topological spaces.

2.1.12 Example (stratified ∞-topoi attached to stratified spaces)**.** Let $s \colon X \to P$ be a stratified space.

(1) Then

$$s_*^{\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathrm{Sh}^{\mathrm{hyp}}(P) \simeq \mathrm{Fun}(P, \mathbf{Spc})$$

is a P-stratified ∞-topos.

(2) If P is noetherian, then

$$s_* \colon \mathrm{Sh}(X) \to \mathrm{Sh}(P) \simeq \mathrm{Fun}(P, \mathbf{Spc})$$

is a P-stratified ∞-topos.

2.1.13 Example. Let $s \colon X \to P$ be a stratified topological stack in the sense of [29, Definition 3.1]. If P is noetherian, then $s_* \colon \mathrm{Sh}(X) \to \mathrm{Fun}(P, \mathbf{Spc})$ is a P-stratified ∞-topos.

2.1.14 Notation. Let (X, P) be a stratified space and $S \subset P$ a locally closed subposet. Write $X_S \coloneqq X \times_P S$. Then X_S is naturally an S-stratified space. Moreover, the inclusions $X_S \hookrightarrow X$ and $S \hookrightarrow P$ define a morphism of stratified spaces $i_S \colon (X, S) \hookrightarrow (X, P)$.

An important fact is that pulling back to a locally closed subposet commutes with taking (hyper)sheaves:

2.1.15 Lemma. *Let (X, P) be a stratified space and $S \subset P$ a locally closed subposet.*

(1) *The natural geometric morphism $\mathrm{Sh}^{\mathrm{hyp}}(X_S) \to \mathrm{Sh}^{\mathrm{hyp}}(X)_S$ is an equivalence.*

(2) *If P is noetherian, then the natural geometric morphism $\mathrm{Sh}(X_S) \to \mathrm{Sh}(X)_S$ is an equivalence.*

Proof. Immediate from Recollection 2.1.1, Example 2.1.3, Corollary B.1.10, Corollary B.3.9, and the definitions. □

Another useful fact is that in the noetherian setting, pulling back to strata is jointly conservative:

2.1.16 Lemma. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos. If the poset P is noetherian, then the pullback functors*

$$\left\{ i_p^* \colon \mathcal{X} \to \mathcal{X}_p \right\}_{p \in P}$$

are jointly conservative.

Proof. Let ϕ be a morphism in $\mathcal{X}$ such that for each $p \in P$, the morphism $i_p^*(\phi)$ is an equivalence; we need to show that ϕ is an equivalence. For each $p \in P$, write

$$P_{\geq p} \coloneqq \{ q \in P \mid q \geq p \} \qquad \text{and} \qquad P_{>p} \coloneqq P_{\geq p} \smallsetminus \{p\} .$$

Since the open subsets $\{P_{\geq p}\}_{p \in P}$ cover P, it suffices to show:

($*$) For each $p \in P$, the restriction $i_{P_{\geq p}}^*(\phi)$ is an equivalence in $\mathcal{X}_{P_{\geq p}}$.

We prove ($*$) by noetherian induction on $p \in P$. We need to show that if the restriction $i_{P_{\geq q}}^*(\phi)$ is an equivalence for each $q > p$, then $i_{P_{\geq p}}^*(\phi)$ is an equivalence. Note that

$$P_{\geq p} \smallsetminus \{p\} = P_{>p} = \bigcup_{q \in P_{>p}} P_{\geq q} .$$

Hence the inductive hypothesis implies that the restriction $i_{P_{>p}}^*(\phi)$ is an equivalence. By assumption $i_p^*(\phi)$ is also an equivalence. By recollement, the restriction functors

$$i_p^* \colon \mathcal{X}_{P_{\geq p}} \to \mathcal{X}_p \qquad \text{and} \qquad i_{P_{>p}}^* \colon \mathcal{X}_{P_{\geq p}} \to \mathcal{X}_{P_{>p}}$$

are jointly conservative, completing the proof. □

2.2 Constructible objects & exit-path ∞-categories. We now recall the basics of constructible objects of stratified ∞-topoi introduced in [8, §9.4]. We also define exit-path ∞-categories at this level of generality.

2.2.1 Definition (constructible objects)**.** Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $\mathcal{E}$ be a presentable ∞-category. An object $F \in \mathrm{Sh}(\mathcal{X};\mathcal{E})$ is *P-constructible* if for each $p \in P$, the restriction $i_p^*(F) \in \mathrm{Sh}(\mathcal{X}_p;\mathcal{E})$ is locally constant. We write

$$\mathrm{Cons}_P(\mathcal{X};\mathcal{E}) \subset \mathrm{Sh}(\mathcal{X};\mathcal{E})$$

for the full subcategory spanned by the P-constructible objects. If $\mathcal{E} = \mathbf{Spc}$, we simply write $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ for $\mathrm{Cons}_P(\mathcal{X};\mathbf{Spc})$.

2.2.2 Remark. Our terminology differs from the terminology used in [8, §9.4]. There, Barwick–Glasman–Haine use the term *formally constructible objects* for what we call constructible objects; their *constructible objects* are formally constructible objects that satisfy additional finiteness hypotheses. The reason for this is that [8] is mostly about ∞-topoi coming from algebraic geometry, where these finiteness hypotheses are necessary for a well-behaved theory.

2.2.3 Observation. Given a morphism of stratified ∞-topoi $f_* \colon (\mathcal{X}, P) \to (\mathcal{Y}, Q)$, the pullback functor $f^* \colon \mathcal{Y} \to \mathcal{X}$ carries $\mathrm{Cons}_Q(\mathcal{Y};\mathcal{E})$ to $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$.

It is often useful to write the ∞-category of constructible objects as a pullback:

2.2.4 Observation. The ∞-category $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ is the pullback

$$\begin{array}{ccc} \mathrm{Cons}_P(\mathcal{X};\mathcal{E}) & \longrightarrow & \prod_{p\in P} \mathrm{LC}(\mathcal{X}_p;\mathcal{E}) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{Sh}(\mathcal{X};\mathcal{E}) & \xrightarrow[\prod_p i_p^*]{} & \prod_{p\in P} \mathrm{Sh}(\mathcal{X}_p;\mathcal{E}) \end{array}$$

We use similar notation for constructible sheaves on stratified topological spaces.

2.2.5 Notation. Let (X, P) be a stratified topological space and let $\mathcal{E}$ be a presentable ∞-category.

(1) For the natural stratified ∞-topos $(\mathcal{X}, P) = (\mathrm{Sh}^{\mathrm{hyp}}(X), P)$, we write

$$\mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E}) \coloneqq \mathrm{Cons}_P(\mathcal{X};\mathcal{E})\,.$$

(2) If P is noetherian, then for the natural stratified ∞-topos $(\mathcal{X}, P) = (\mathrm{Sh}(X), P)$, we write

$$\mathrm{Cons}_P(X;\mathcal{E}) \coloneqq \mathrm{Cons}_P(\mathcal{X};\mathcal{E})\,.$$

Definition 2.2.1 recovers the usual notion of constructibility:

2.2.6 Observation. Let (X, P) be a stratified topological space and let $\mathcal{E}$ be a presentable ∞-category. In light of Example 1.3.6 and Lemma 2.1.15:

(1) An object $F \in \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$ is P-hyperconstructible in the sense of [21, Definition 5.2] if and only if F is P-constructible in the sense of Definition 2.2.1.

(2) Assume that P is noetherian. An object $F \in \mathrm{Sh}(X;\mathcal{E})$ is P-constructible in the sense of [21, Definition 5.2] if and only if F is P-constructible in the sense of Definition 2.2.1.

2.2.7 Example. Let P be a poset. Then every hypersheaf on P is P-constructible, i.e.,

$$\mathrm{Cons}_P^{\mathrm{hyp}}(P) = \mathrm{Sh}^{\mathrm{hyp}}(P)\,.$$

In light of Recollection 2.1.1, we deduce that $\mathrm{Cons}_P^{\mathrm{hyp}}(P) \simeq \mathrm{Fun}(P, \mathbf{Spc})$.

2.2.8 Convention. Let P be a poset. We almost always implicitly identify the ∞-categories $\mathrm{Sh}^{\mathrm{hyp}}(P;\mathcal{E})$, $\mathrm{Cons}_P^{\mathrm{hyp}}(P)$, and $\mathrm{Fun}(P,\mathcal{E})$.

For the next result, recall Notation 1.1.12.

2.2.9 Lemma. *For every poset P, we have natural equivalences*

$$\mathrm{Cons}_P^{\mathrm{hyp}}(P)^{\mathrm{ex}} \simeq \mathrm{Fun}(P, \mathbf{Spc})^{\mathrm{ex}} = P .$$

Proof. By Lemma A.1.3, P is idempotent complete. Hence the claim follows from Recollections 1.1.11 and 2.1.1. □

The following definition is a generalization of [14, Definition 3.5; 29, Definition 3.10]:

2.2.10 Definition (exodromic stratified ∞-topos & exit-path ∞-category)**.** A stratified ∞-topos

$$s_* : \mathcal{X} \to \mathrm{Fun}(P, \mathbf{Spc})$$

is *exodromic* if the following conditions are satisfied:

(1) The ∞-category $\mathrm{Cons}_P(\mathcal{X})$ is atomically generated.

(2) The subcategory $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ is closed under both limits and colimits.

(3) The pullback functor $s^* : \mathrm{Fun}(P, \mathbf{Spc}) \to \mathcal{X}$ preserves limits.

In this case we write

$$\Pi_\infty(\mathcal{X}, P) := \mathrm{Cons}_P(\mathcal{X})^{\mathrm{ex}}$$

for the *opposite* of the full subcategory of $\mathrm{Cons}_P(\mathcal{X})$ spanned by atomic objects (see Notation 1.1.12). We refer to $\Pi_\infty(\mathcal{X}, P)$ as the *exit-path ∞-category* of $(\mathcal{X}, P)$.

The importance of the last condition of Definition 2.2.10 is that it provides a functor from the exit-path ∞-category of $(\mathcal{X}, P)$ to the poset P.

2.2.11 Observation. Let $s_* : \mathcal{X} \to \mathrm{Fun}(P, \mathbf{Spc})$ be an exodromic stratified ∞-topos. Then the left adjoint

$$s_\sharp^{\mathrm{c}} : \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Fun}(P, \mathbf{Spc})$$

to s^* supplied by condition (3) of Definition 2.2.10 is atomic. By Observation 1.1.8 and Lemma 2.2.9, the functor $s_\sharp^{\mathrm{c}}$ restricts to a functor

$$s^{\mathrm{ex}} : \Pi_\infty(\mathcal{X}, P) \to P .$$

Now, some important examples.

2.2.12 Example. In light of Recollection 1.3.8, a trivially stratified ∞-topos $\Gamma_* : \mathcal{X} \to \mathbf{Spc}$ is exodromic if and only if $\mathcal{X}$ is monodromic in the sense of Definition 1.3.7.

2.2.13 Example (exodromy for conically stratified spaces)**.** Let (X, P) be a conically stratified topological space in the sense of [HA, Definition A.5.5].

(1) If the strata of (X, P) are locally weakly contractible, then the stratified ∞-topos $(\mathrm{Sh}^{\mathrm{hyp}}(X), P)$ is exodromic. Moreover, the exit-path ∞-category $\Pi_\infty(\mathrm{Sh}^{\mathrm{hyp}}(X), P)$ is given by Lurie's simplicial model for exit-paths $\mathrm{Sing}(X, P)$. See [32, Theorem 5.4.1].

(2) If P is noetherian and X is paracompact and locally of singular shape, then the stratified ∞-topos $(\mathrm{Sh}(X), P)$ is exodromic. Again, the exit-path ∞-category $\Pi_\infty(\mathrm{Sh}(X), P)$ is given by Lurie's simplicial model for exit-paths $\mathrm{Sing}(X, P)$. See [HA, Theorem A.9.3].

Ørsnes Jansen has also given incredible computations of exit-path ∞-categories of some important compactifications naturally arising in geometry:

2.2.14 Example (the work of Ørsnes Jansen [30; 28])**.**

(1) Let G be a connected reductive linear algebraic group defined over $\mathbf{Q}$ whose center is anisotropic over $\mathbf{Q}$. Let $\Gamma \subset G(\mathbf{Q})$ be a neat arithmetic subgroup. Write X for the symmetric space of maximal compact subgroups of $G(\mathbf{R})$ with Γ-action given by conjugation. Ørsnes Jansen showed that the ∞-topos of sheaves on the reductive Borel–Serre compactification $\Gamma\backslash\overline{X}^{\mathrm{RBS}}$ is exodromic and gave an explicit description of its exit-path ∞-category. See [30, Theorem 4.3].

(2) Let $g, n \geq 0$ be such that $2g - 2 + n > 0$. Write $\overline{\mathcal{M}}_{g,n}$ for the moduli stack of stable genus g nodal curves with n marked points (also called the *Deligne–Mumford–Knudsen compactification*). Write $\overline{\mathcal{M}}_{g,n}^{\mathrm{top}}$ for its underlying topological stack. The topological stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{top}}$ has a natural stratification by the poset of stable genus g dual graphs with n marked points. Ørsnes Jansen showed that the ∞-topos of sheaves on the topological stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{top}}$ is exodromic. Moreover, the exit-path ∞-category is equivalent to the opposite of the Charney–Lee category of stable genus g curves with n marked points [11; 12; 15]. See [28, Corollary 6.6 & Theorem 6.7].

Another feature of Definition 2.2.10 is that the inclusion of constructible objects admits both a left and right adjoint:

2.2.15 Notation (constructibilization)**.** Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Since $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ is closed under limits and colimits, [34, Theorem 1.1] implies that $\mathrm{Cons}_P(\mathcal{X})$ is presentable and the inclusion

$$i_{\mathcal{X},P} : \mathrm{Cons}_P(\mathcal{X}) \hookrightarrow \mathcal{X}$$

has both a left adjoint $\mathrm{L}_{\mathcal{X},P}$ and a right adjoint $\mathrm{R}_{\mathcal{X},P}$. We refer to these adjoints as the *left* and *right constructibilization functors*, respectively. In particular, $\mathrm{Cons}_P(\mathcal{X})$ is a localization of $\mathcal{X}$, and it coincides with the full subcategory of $\mathcal{X}$ spanned by $\mathrm{L}_{\mathcal{X},P}$-equivalences.

2.2.16 Example (equational criterion for constructibility)**.** Let (X, P) be a conically stratified topological space with locally weakly contractible strata. Then [32, Corollary 5.4.7] provides an explicit set of generating $\mathrm{L}_{X,P}$-equivalences in terms of conical charts. When $P = *$, we can take as a generating set all the inclusions $U \subset V$ between weakly contractible open subsets.

A very important fact is that exodromic stratified ∞-topoi are automatically monodromic:

2.2.17 Lemma (exodromy implies monodromy)**.** *Let $s_* : \mathcal{X} \to \mathrm{Fun}(P, \mathbf{Spc})$ be an exodromic stratified ∞-topos. Then:*

(1) *The ∞-topos $\mathcal{X}$ is monodromic.*

(2) *The full subcategory $\mathrm{LC}(\mathcal{X}) \subset \mathrm{Cons}_P(\mathcal{X})$ is closed under limits and colimits.*

Proof. First we prove (1). In light of Recollection 1.3.8, we need to show that the constant sheaf functor $\Gamma^* : \mathbf{Spc} \to \mathcal{X}$ preserves limits. Note that Γ^* factors as a composite

$$\mathbf{Spc} \longrightarrow \mathrm{Fun}(P, \mathbf{Spc}) \xrightarrow{s^*} \mathrm{Cons}_P(\mathcal{X}) \hookrightarrow \mathcal{X} ,$$

where the left-most functor is the constant functor. The constant functor $\mathbf{Spc} \to \mathrm{Fun}(P, \mathbf{Spc})$ preserves limits, and by assumption both s^* and the inclusion $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ preserve limits. Hence Γ^* preserves limits, as desired.

For (2), note that both $\mathrm{LC}(\mathcal{X})$ and $\mathrm{Cons}_P(\mathcal{X})$ are closed under limits and colimits in $\mathcal{X}$. □

For later use, let us record the following:

2.2.18 Corollary. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $\mathcal{E}$ be a presentable ∞-category. If $(\mathcal{X}, P)$ is exodromic, then the terminal object of $\mathrm{Sh}(\mathcal{X}; \mathcal{E})$ is constant (hence P-constructible).*

Proof. By Lemma 2.2.17, we know that $\Gamma^* : \mathbf{Spc} \to \mathcal{X}$ is both a left and right adjoint. By the functoriality of the tensor product, the induced functor $\Gamma^* \otimes \mathcal{E}$ is also both a left and a right adjoint. In particular, $\Gamma^* \otimes \mathcal{E}$ preserves the terminal object; hence the terminal object of $\mathrm{Sh}(\mathcal{X}; \mathcal{E})$ is constant. □

2.3 Exodromic morphisms. We now discuss the functoriality of exit-path ∞-categories. The main point of this subsection is that given a morphism $f_* : (\mathcal{X}, P) \to (\mathcal{Y}, Q)$ between exodromic stratified ∞-topoi, it is not *a priori* clear if f_* induces a functor

$$\Pi_\infty(\mathcal{X}, P) \to \Pi_\infty(\mathcal{Y}, Q)$$

on exit-path ∞-categories.

2.3.1 Observation (constructible $*$-pushforward)**.** Let $f_* : (\mathcal{X}, P) \to (\mathcal{Y}, Q)$ be a morphism between exodromic stratified ∞-topoi. Since the functor $f^* : \mathcal{Y} \to \mathcal{X}$ preserves colimits, we deduce that

$$f^* : \mathrm{Cons}_Q(\mathcal{Y}) \to \mathrm{Cons}_P(\mathcal{X})$$

preserves colimits as well. In particular, it admits a right adjoint

$$f_*^{\mathrm{c}} : \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_Q(\mathcal{Y}) .$$

Unraveling the definitions, we see that f_*^{c} is related to the pushforward functor f_* by the formula

$$f_*^{\mathrm{c}} = \mathrm{R}_{\mathcal{Y},Q} \circ f_* \circ i_{\mathcal{X},P} ,$$

where $\mathrm{R}_{\mathcal{Y},Q}$ is the right constructibilization functor of Notation 2.2.15. In particular, if f_* takes P-constructible objects to Q-constructible objects, then $f_*^{\mathrm{c}} \simeq f_*$.

The following is a generalization of [14, Definition 3.5-(3)]:

2.3.2 Definition. Let $f_* : (\mathcal{X}, P) \to (\mathcal{Y}, Q)$ be a morphism between exodromic stratified ∞-topoi. We say that f_* *is exodromic* if the left adjoint

$$f^* : \mathrm{Cons}_Q(\mathcal{Y}) \to \mathrm{Cons}_P(\mathcal{X})$$

also preserves limits. In this case, we denote its left adjoint by

$$f_\sharp^{\mathrm{c}} : \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_Q(\mathcal{Y}) .$$

As a consequence of the equivalence $\mathbf{Cat}_\infty^{\mathrm{idem}} \simeq \mathbf{Pr}^{\mathrm{L,at}}$ of Recollection 1.1.11, the functor $f_\sharp^{\mathrm{c}}$ restricts to a functor

$$f^{\mathrm{ex}} : \Pi_\infty(\mathcal{X}, P) \to \Pi_\infty(\mathcal{Y}, Q) .$$

The following are two important examples of exodromic morphisms:

2.3.3 Example. Let $\phi : P \to Q$ be a map of posets. Equip both P and Q with the identity stratifications. Then Recollection 2.1.1 shows that the morphism of stratified ∞-topoi

$$\phi_* : (\mathrm{Fun}(P, \mathbf{Spc}), P) \to (\mathrm{Fun}(Q, \mathbf{Spc}), Q)$$

is exodromic.

2.3.4 Example. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. If $\mathcal{X}$ and $\mathcal{Y}$ are monodromic, then Corollary 1.3.12 shows that the morphism of trivially stratified ∞-topoi

$$f_* : (\mathcal{X}, *) \to (\mathcal{Y}, *)$$

is exodromic.

In fact, we will see that Definition 2.3.2 is superfluous: one of the goals of § 3 is to show that *every* morphism between exodromic stratified ∞-topoi is exodromic. However, this is not obvious; see Theorem 3.2.3 for details.

We conclude this subsection with a few useful observations about exodromic morphisms.

2.3.5 Observation. Let $f_* : (\mathcal{X}, P) \to (\mathcal{Y}, Q)$ be a morphism of stratified ∞-topoi. Assume the following:

(1) $(\mathcal{X}, P)$ and $(\mathcal{Y}, Q)$ are exodromic.

(2) $f^* : \mathcal{Y} \to \mathcal{X}$ admits a left adjoint $f_\sharp : \mathcal{X} \to \mathcal{Y}$.

Then f_* is exodromic. Moreover, the functors

$$f_\sharp^{\mathrm{c}} : \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_Q(\mathcal{Y}) \qquad \text{and} \qquad f_\sharp : \mathcal{X} \to \mathcal{Y}$$

are related by the formula

$$f_\sharp^{\mathrm{c}} \simeq \mathrm{L}_{\mathcal{Y},Q} \circ f_\sharp \circ i_{\mathcal{X},P} ,$$

where $\mathrm{L}_{\mathcal{Y},Q}$ is the left constructibilization functor of Notation 2.2.15. In particular, if $f_\sharp$ carries P-constructible objects to Q-constructible objects, then there is a canonical identification $f_\sharp^{\mathrm{c}} \simeq f_\sharp$.

2.3.6 Observation (pullback functoriality)**.** Let $f_* \colon (\mathcal{X}, P) \to (\mathcal{Y}, Q)$ be a morphism between exodromic stratified ∞-topoi. If f_* is exodromic, then Observation 1.1.8 yields a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\Pi_\infty(\mathcal{Y}, Q), \mathbf{Spc}) & \xrightarrow{-\circ f^{\mathrm{ex}}} & \mathrm{Fun}(\Pi_\infty(\mathcal{X}, P), \mathbf{Spc}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Cons}_Q(\mathcal{Y}) & \xrightarrow[f^*]{} & \mathrm{Cons}_P(\mathcal{X}) , \end{array}$$

where the vertical equivalences exhibit the exit-path ∞-categories $\Pi_\infty(\mathcal{Y}, Q)$ and $\Pi_\infty(\mathcal{X}, P)$ as the opposites of the subcategories of atomic objects of the targets.

2.3.7 Observation ($\sharp$-pushforward functoriality)**.** As a consequence of Observation 2.3.6, there is also a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\Pi_\infty(\mathcal{X}, P), \mathbf{Spc}) & \xrightarrow{f^{\mathrm{ex}}_!} & \mathrm{Fun}(\Pi_\infty(\mathcal{Y}, Q), \mathbf{Spc}) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Cons}_P(\mathcal{X}) & \xrightarrow[f^{\mathrm{c}}_\sharp]{} & \mathrm{Cons}_Q(\mathcal{Y}) , \end{array}$$

where $f^{\mathrm{ex}}_!$ denotes left Kan extension along f^{ex}. Since left Kan extension commutes with the Yoneda embedding, we also deduce that there is a commutative square

$$\begin{array}{ccc} \Pi_\infty(\mathcal{X}, P)^{\mathrm{op}} & \xrightarrow{f^{\mathrm{ex,op}}} & \Pi_\infty(\mathcal{Y}, Q)^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathrm{Cons}_P(\mathcal{X}) & \xrightarrow[f^{\mathrm{c}}_\sharp]{} & \mathrm{Cons}_Q(\mathcal{Y}) , \end{array}$$

where the vertical functors are the inclusions of the subcategories of atomic objects.

2.4 Exodromy & hypercompletion. Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. The goal of this subsection is to show that the hypercompletion $\mathcal{X}^{\mathrm{hyp}}$ with the induced stratification is also exodromic, the ∞-categories $\mathrm{Cons}_P(\mathcal{X})$ and $\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$ coincide, and the exit-path ∞-categories $\Pi_\infty(\mathcal{X}, P)$ and $\Pi_\infty(\mathcal{X}^{\mathrm{hyp}}, P)$ coincide. We do not accomplish this in complete generality, however, we prove that this the case under an additional assumption on $(\mathcal{X}, P)$; see Definition 2.4.10 and Proposition 2.4.14. This assumption is satisfied, for example, when P is noetherian and $\mathcal{X}$ is the ∞-topos of sheaves associated to a conically stratified space for which exodromy is already known.

2.4.1 Notation. Let $s_* \colon \mathcal{X} \to \mathrm{Fun}(P, \mathbf{Spc})$ be a stratified ∞-topos. Then the composite

$$\mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X} \xrightarrow{s_*} \mathrm{Fun}(P, \mathbf{Spc})$$

defines a P-stratification of $\mathcal{X}^{\mathrm{hyp}}$. We always regard the hypercompletion of a stratified ∞-topos with this induced stratification. Also note that since the ∞-topos $\mathrm{Fun}(P, \mathbf{Spc})$ is hypercomplete, the stratification

$$\mathcal{X}^{\mathrm{hyp}} \to \mathrm{Fun}(P, \mathbf{Spc})$$

coincides with the geometric morphism s^{hyp}_* obtained by applying the hypercompletion functor to the stratification s_*.

We start by showing that if $(\mathcal{X}, P)$ is exodromic, then every P-constructible object of $\mathcal{X}$ is hypercomplete. For this, we need a few lemmas.

2.4.2 Lemma. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and $S \subset P$ a locally closed subposet. Then the natural geometric morphism*

$$(\mathcal{X}_S)^{\mathrm{hyp}} \to (\mathcal{X}^{\mathrm{hyp}})_S$$

is an equivalence of S-stratified ∞-topoi.

Proof. This is a special case of Proposition B.3.8. □

2.4.3 Lemma. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi and let $f^* \colon \mathcal{Y} \to \mathcal{X}$ be a functor that preserves both limits and colimits. Let $G \in \mathcal{Y}$.*

(1) *If G is hypercomplete, then $f^*(G)$ is hypercomplete.*

(2) *If G is the limit of its Postnikov tower, then $f^*(G)$ is the limit of its Postnikov tower.*

Proof. Item (1) is the content of [HA, Lemma A.2.6]. For (2), note that since f^* is a left exact left adjoint, [HTT, Proposition 5.5.6.28] shows that for each $n \geq 0$, we have

$$f^* \tau_{\leq n}^{\mathcal{Y}} \simeq \tau_{\leq n}^{\mathcal{X}} f^* .$$

Since G is the limit of its Postnikov tower and f^* preserves limits, we see that

$$\begin{aligned} f^*(G) &\simeq f^*\left(\lim_{n \in \mathbf{N}^{\mathrm{op}}} \tau_{\leq n}^{\mathcal{Y}}(G)\right) \\ &\simeq \lim_{n \in \mathbf{N}^{\mathrm{op}}} f^* \tau_{\leq n}^{\mathcal{Y}}(G) \\ &\simeq \lim_{n \in \mathbf{N}^{\mathrm{op}}} \tau_{\leq n}^{\mathcal{X}} f^*(G) . \end{aligned}$$

□

2.4.4 Corollary. *Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos.*

(1) *If $F \in \mathrm{Cons}_P(\mathcal{X})$, then F is the limit of its Postnikov tower in $\mathcal{X}$. In particular, we have*

$$\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}^{\mathrm{hyp}} .$$

(2) *We have*

$$\mathrm{Cons}_P(\mathcal{X}) \subset \mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$$

as full subcategories of $\mathcal{X}^{\mathrm{hyp}}$.

(3) *The functor $s^* \colon \mathrm{Fun}(P, \mathbf{Spc}) \to \mathcal{X}$ factors through $\mathcal{X}^{\mathrm{hyp}}$.*

(4) *The constant sheaf functor $\Gamma^* \colon \mathbf{Spc} \to \mathcal{X}$ factors through $\mathcal{X}^{\mathrm{hyp}} \subset \mathcal{X}$.*

Proof. Note that since $(\mathcal{X}, P)$ is exodromic, the ∞-category $\mathrm{Cons}_P(\mathcal{X})$ is an ∞-topos and the inclusion $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ preserves limits and colimits. Hence (1) is a special case of Lemma 2.4.3-(2). For (2), note that the inclusion $(\mathcal{X}^{\mathrm{hyp}}, P) \hookrightarrow (\mathcal{X}, P)$ is a morphism of stratified ∞-topoi. Hence the hypercompletion functor $\mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$ carries $\mathrm{Cons}_P(\mathcal{X})$ to $\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$. By (1), every object of $\mathrm{Cons}_P(\mathcal{X})$ is already hypercomplete, hence

$$\mathrm{Cons}_P(\mathcal{X}) \subset \mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$$

as full subcategories of $\mathcal{X}^{\mathrm{hyp}}$.

Item (3) is an immediate consequence of item (1) and the fact that s^* factors through $\mathrm{Cons}_P(\mathcal{X})$. Item (4) is immediate from (3) and the fact that Γ^* factors as the composite

$$\mathbf{Spc} \longrightarrow \mathrm{Fun}(P, \mathbf{Spc}) \xrightarrow{s^*} \mathrm{Cons}_P(\mathcal{X}) \hookrightarrow \mathcal{X} ,$$

where the left-most functor is the constant functor. □

2.4.5 Observation. Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Corollary 2.4.4 implies that the left constructibilization functor $\mathrm{L}_{\mathcal{X},P} \colon \mathcal{X} \to \mathrm{Cons}_P(\mathcal{X})$ factors as a the composite of hypercompletion $\mathcal{X} \to \mathcal{X}^{\mathrm{hyp}}$ with a localization

$$\mathrm{L}_{\mathcal{X},P}^{\mathrm{hyp}} \colon \mathcal{X}^{\mathrm{hyp}} \to \mathrm{Cons}_P(\mathcal{X}) .$$

In turn, $\mathrm{Cons}_P(\mathcal{X})$ can be identified with the full subcategory of $\mathcal{X}^{\mathrm{hyp}}$ spanned by objects that are local with respect to $\mathrm{L}_{\mathcal{X},P}^{\mathrm{hyp}}$-equivalences. Hence a morphism $\phi \colon F \to G$ in $\mathcal{X}$ is an $\mathrm{L}_{\mathcal{X},P}$-equivalence if and only if its hypercompletion ϕ^{hyp} is an $\mathrm{L}_{\mathcal{X},P}^{\mathrm{hyp}}$-equivalence.

Our next goal is to show that if $\mathcal{X}$ is monodromic, then $\mathcal{X}^{\mathrm{hyp}}$ is also monodromic and $\mathrm{LC}(\mathcal{X}) = \mathrm{LC}(\mathcal{X}^{\mathrm{hyp}})$. For this, we need the following lemma.

2.4.6 Lemma. *Let $\mathcal{X}$ be an ∞-topos and write $i_* : \mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ for the inclusion.*

(1) *A morphism $f : U \to V$ in $\mathcal{X}^{\mathrm{hyp}}$ is an effective epimorphism if and only if $i_*(f)$ is an effective epimorphism in $\mathcal{X}$.*

(2) *The functor $i_* : \mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ preserves coproducts.*

(3) *Given an effective epimorphism $\coprod_{\alpha \in A} U_\alpha \to 1_{\mathcal{X}^{\mathrm{hyp}}}$ in $\mathcal{X}^{\mathrm{hyp}}$, the induced map*

$$\coprod_{\alpha \in A} i_*(U_\alpha) \to 1_{\mathcal{X}}$$

is an effective epimorphism in $\mathcal{X}$.

Proof. For (1), first assume that $i_*(f)$ is an effective epimorphism. Then since i^* preserves effective epimorphisms and i_* is fully faithful, $f \simeq i^* i_*(f)$ is also an effective epimorphism. Conversely, assume that f is an effective epimorphism. Note that

$$\tau_{\leq 0}^{\mathcal{X}} i_*(f) = \tau_{\leq 0}^{\mathcal{X}^{\mathrm{hyp}}}(f) .$$

Since the property of a morphism being an effective epimorphism only depends on the 0-truncation [HTT, Proposition 7.2.1.14] and f is an effective epimorphism, we deduce that $i_*(f)$ is an effective epimorphism.

Item (2) is the content of [SAG, Lemma D.6.7.2]. Finally, (3) is immediate from (1) and (2). □

2.4.7 Lemma. *Let $\mathcal{X}$ be an ∞-topos. If every constant object of $\mathcal{X}$ is hypercomplete, then:*

(1) *For each $U \in \mathcal{X}$, every constant object of $\mathcal{X}_{/U}$ is hypercomplete.*

(2) *Every locally constant object of $\mathcal{X}$ is hypercomplete.*

(3) *The inclusion $\mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ carries $\mathrm{LC}(\mathcal{X}^{\mathrm{hyp}})$ to $\mathrm{LC}(\mathcal{X})$.*

(4) *We have $\mathrm{LC}(\mathcal{X}^{\mathrm{hyp}}) = \mathrm{LC}(\mathcal{X})$ as full subcategories of $\mathcal{X}$.*

Proof. For (1), write $p^* : \mathcal{X} \to \mathcal{X}_{/U}$ for the pullback functor. Observe that the constant sheaf functor $\mathbf{Spc} \to \mathcal{X}_{/U}$ factors as a composite

$$\mathbf{Spc} \xrightarrow{\Gamma^*} \mathcal{X} \xrightarrow{p^*} \mathcal{X}_{/U} .$$

Since the pullback functor p^* is both a left and a right adjoint, Lemma 2.4.3 shows that p^* preserves hypercompleteness. Hence the claim follows from the assumption that every constant object of $\mathcal{X}$ is hypercomplete.

For (2), let $L \in \mathrm{LC}(\mathcal{X})$ and choose an effective epimorphism $\coprod_{\alpha \in A} U_\alpha \twoheadrightarrow 1_{\mathcal{X}}$ such that for each $\alpha \in A$, the pullback $L \times U_\alpha$ is a constant object of $\mathcal{X}_{/U_\alpha}$. Then by (1), for each $\alpha \in A$, the object $L \times U_\alpha \in \mathcal{X}_{/U_\alpha}$ is hypercomplete. The claim now follows from the fact that hypercompleteness is a local property [HTT, Remark 6.5.2.22].

For (3), let $L \in \mathrm{LC}(\mathcal{X}^{\mathrm{hyp}})$; we wish to show that $L \in \mathrm{LC}(\mathcal{X})$. Choose an effective epimorphism

$$\phi : \coprod_{\alpha \in A} U_\alpha \twoheadrightarrow 1_{\mathcal{X}^{\mathrm{hyp}}} = 1_{\mathcal{X}}$$

in $\mathcal{X}^{\mathrm{hyp}}$ such that for each $\alpha \in A$, the pullback

$$L \times U_\alpha \in (\mathcal{X}^{\mathrm{hyp}})_{/U_\alpha} = (\mathcal{X}_{/U_\alpha})^{\mathrm{hyp}}$$

is constant. By Lemma 2.4.6-(3) the effective epimorphism $\phi : \coprod_{\alpha \in A} U_\alpha \twoheadrightarrow 1_{\mathcal{X}}$ in $\mathcal{X}^{\mathrm{hyp}}$ is also an effective epimorphism in the larger ∞-topos $\mathcal{X}$. Hence it suffices to show that each $L \times U_\alpha$ is also a constant object of the larger ∞-topos $\mathcal{X}_{/U_\alpha}$. For this, note that by (1), every constant object of $\mathcal{X}_{/U_\alpha}$ is hypercomplete.

Item (4) is immediate from items (2) and (3). □

2.4.8 Proposition. *Let $\mathcal{X}$ be a monodromic ∞-topos. Then:*

(1) *The composite*

$$\mathcal{X}^{\mathrm{hyp}} \overset{i_*}{\hookrightarrow} \mathcal{X} \xrightarrow{\Gamma_\sharp} \mathbf{Spc}$$

is left adjoint to the constant hypersheaf functor $\mathbf{Spc} \to \mathcal{X}^{\mathrm{hyp}}$. *In particular,* $\mathcal{X}^{\mathrm{hyp}}$ *is monodromic.*

(2) *The inclusion* $\mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ *carries* $\mathrm{LC}(\mathcal{X}^{\mathrm{hyp}})$ *to* $\mathrm{LC}(\mathcal{X})$. *Moreover, we have* $\mathrm{LC}(\mathcal{X}^{\mathrm{hyp}}) = \mathrm{LC}(\mathcal{X})$ *as full subcategories of* $\mathcal{X}$.

(3) *The natural map* $\Pi_\infty(\mathcal{X}^{\mathrm{hyp}}) \to \Pi_\infty(\mathcal{X})$ *is an equivalence.*

Proof. For (1), note that since $\Gamma^* : \mathbf{Spc} \to \mathcal{X}$ factors through $\mathcal{X}^{\mathrm{hyp}}$, for $F \in \mathcal{X}^{\mathrm{hyp}}$ and $K \in \mathbf{Spc}$, we have natural equivalences

$$\begin{aligned}\mathrm{Map}_{\mathbf{Spc}}(\Gamma_\sharp i_*(F), K) &\simeq \mathrm{Map}_{\mathcal{X}}(i_*(F), \Gamma^*(K)) \\ &\simeq \mathrm{Map}_{\mathcal{X}^{\mathrm{hyp}}}(F, \Gamma^*(K))\ .\end{aligned}$$

Item (2) is a special case of Lemma 2.4.7. Finally, by (2), the pullback functor $\mathrm{LC}(\mathcal{X}) \to \mathrm{LC}(\mathcal{X}^{\mathrm{hyp}})$ is an equivalence (in fact, the identity). Hence (3) follows from the definition of the shape. □

2.4.9 Warning. Let $\mathcal{X}$ be a monodromic ∞-topos and $F \in \mathcal{X}$. If the hypercompletion of F is a locally constant object of $\mathcal{X}^{\mathrm{hyp}}$, then it is not necessarily the case that F is a locally constant object of $\mathcal{X}$.

Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Corollary 2.4.4-(2) shows that $\mathrm{Cons}_P(\mathcal{X}) \subset \mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$. In the case of a trivial stratification, we have just seen that this inclusion is an equality. For a general stratification, we do not know if this holds; we offer the following simple sufficient condition for this to hold. This condition covers many concrete cases of interest.

2.4.10 Definition. Let $(\mathcal{X}, P)$ be a stratified ∞-topos. We say that $(\mathcal{X}, P)$ is *weakly conical* if for every locally closed subset $S \subset P$, the functor

$$i_{S,*} : \mathcal{X}_S \to \mathcal{X}$$

takes $\mathrm{Cons}_S(\mathcal{X})$ to $\mathrm{Cons}_P(\mathcal{X})$.

This definition is motivated by the following:

2.4.11 Example. Let (X, P) be a conically stratified space with locally weakly contractible strata. Then $(\mathrm{Sh}^{\mathrm{hyp}}(X), P)$ is weakly conical by [32, Proposition 6.8.1]; this ultimately relies on [32, Lemma 5.3.4], which is the hard step needed to prove the exodromy equivalence in the conical setting. On the other hand, consider the non-conical stratification of a circle pictured on the right-hand side of Figure 1: in this case, the pushforward of a constant sheaf on the open stratum is not hyperconstructible with respect to the given stratification. Thus, this property is a special feature of the conical situation.

2.4.12 Lemma. *Let* $(\mathcal{X}, P)$ *be a weakly conical exodromic stratified* ∞*-topos. Let* $\phi : F_1 \to F_2$ *be a* $\mathrm{L}_{\mathcal{X},P}$*-equivalence (see Notation 2.2.15). Then for every locally closed subset* $S \subset P$*, the morphism* $i_S^*(\phi)$ *is an* $\mathrm{L}_{\mathcal{X}_S,S}$*-equivalence.*

Proof. We have to show that for all $G \in \mathrm{Cons}_S(\mathcal{X}_S)$, the map $i_S^*(\phi)$ induces an equivalence

$$\mathrm{Map}_{\mathcal{X}_S}(i_S^*(F_2), G) \to \mathrm{Map}_{\mathcal{X}_S}(i_S^*(F_1), G)\ .$$

By adjunction, this follows immediately from the fact that ϕ is a P-equivalence and that $i_{S,*}(G) \in \mathrm{Cons}_P(\mathcal{X})$. □

2.4.13 Lemma. *Let* $(\mathcal{X}, P)$ *be an exodromic stratified* ∞*-topos. If the inclusion* $\mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ *carries* $\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$ *to* $\mathrm{Cons}_P(\mathcal{X})$*, then:*

(1) *We have* $\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}}) = \mathrm{Cons}_P(\mathcal{X})$ *as full subcategories of* $\mathcal{X}$.

(2) *The stratified* ∞*-topos* $(\mathcal{X}^{\mathrm{hyp}}, P)$ *is exodromic.*

(3) *The natural functor* $\Pi_\infty(\mathcal{X}^{\mathrm{hyp}}, P) \to \Pi_\infty(\mathcal{X}, P)$ *is an equivalence of* ∞*-categories.*

Proof. Since $(\mathcal{X}, P)$ is exodromic, Corollary 2.4.4-(2) guarantees that

$$\mathrm{Cons}_P(\mathcal{X}) \subset \mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}}) .$$

Our assumption guarantees that this inclusion is an equality.

For (2), note that by (1) and the assumption that $(\mathcal{X}, P)$ is exodromic, the ∞-category $\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$ is atomically generated. In light of Corollary 2.4.4-(3), all that remains to be shown is that the full subcategory

$$\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}}) \subset \mathcal{X}^{\mathrm{hyp}}$$

is closed under limits and colimits. Again by (1), we have $\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}}) = \mathrm{Cons}_P(\mathcal{X})$. Moreover, since $(\mathcal{X}, P)$ is exodromic, $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ is closed under limits and colimits. The claim now follows from the fact that $\mathcal{X}^{\mathrm{hyp}}$ is a localization of $\mathcal{X}$.

Item (3) is immediate from items (1) and (2) and the definition of the exit-path ∞-category of an exodromic stratified ∞-topos. □

The following is the main result of this subsection.

2.4.14 Proposition. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos. Assume that P is noetherian and that $(\mathcal{X}, P)$ is both exodromic and weakly conical. Then:*

(1) *The inclusion $\mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ carries $\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$ to $\mathrm{Cons}_P(\mathcal{X})$.*

(2) *The stratified ∞-topos $(\mathcal{X}^{\mathrm{hyp}}, P)$ is exodromic.*

(3) *The natural functor $\Pi_\infty(\mathcal{X}^{\mathrm{hyp}}, P) \to \Pi_\infty(\mathcal{X}, P)$ is an equivalence of ∞-categories.*

Proof. First note that by Lemma 2.4.13, it suffices to prove (1). Since $(\mathcal{X}, P)$ is exodromic, Corollary 2.4.4-(2) guarantees that

$$\mathrm{Cons}_P(\mathcal{X}) \subset \mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}}) .$$

We prove the other inclusion by noetherian induction, observing that the case $P = *$ has already been dealt with in Proposition 2.4.8-(1). Fix $F \in \mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$ and $p \in P$. Set $Q \coloneqq P_{\geq p}$. Then Q is an open subset of P; in particular i_Q^* preserves hypercomplete objects. Thus,

$$i_Q^*(F) \simeq i_Q^{*,\mathrm{hyp}}(F) \in \mathrm{Cons}_P(\mathcal{X}_Q^{\mathrm{hyp}}) .$$

In other words, we can assume without loss of generality that p is a minimal element of P.

Now set $S \coloneqq P_{>p}$. Again, S is an open subset of P. Moreover, $\mathcal{X}^{\mathrm{hyp}}$ is the recollement of $\mathcal{X}_p^{\mathrm{hyp}}$ and $\mathcal{X}_S^{\mathrm{hyp}}$. In particular, for each $F \in \mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$, there is a pullback square

$$\begin{array}{ccc} F & \longrightarrow & i_{p,*} i_p^{*,\mathrm{hyp}}(F) \\ \big\downarrow & \lrcorner & \big\downarrow \\ i_{S,*} i_S^*(F) & \longrightarrow & i_{p,*} i_p^{*,\mathrm{hyp}} i_{S,*} i_S^*(F) . \end{array} \tag{2.4.15}$$

Thanks to Observation 2.4.5, it is enough to prove that for every $\mathrm{L}_{\mathcal{X},P}^{\mathrm{hyp}}$-equivalence $\phi \colon G_1 \to G_2$ in $\mathcal{X}^{\mathrm{hyp}}$, the object F is ϕ-local. By virtue of the pullback square (2.4.15), it suffices to prove that the other three terms are ϕ-local. The inductive hypothesis guarantees that

$$i_S^*(F) \simeq i_S^{*,\mathrm{hyp}}(F)$$

belongs to $\mathrm{Cons}_S(\mathcal{X}_S)$. Since $(\mathcal{X}, P)$ is weakly conical, it follows that $i_{S,*} i_S^*(F) \in \mathrm{Cons}_P(\mathcal{X})$; in particular, $i_{S,*} i_S^*(F)$ is ϕ-local. As for the other two terms, first recall from Observation 2.4.5 that ϕ, seen as a morphism in $\mathcal{X}$, is an $\mathrm{L}_{\mathcal{X},P}$-equivalence. In particular, Lemma 2.4.12 guarantees that $i_p^*(\phi)$ is an $\mathrm{L}_{\mathcal{X}_p}$-equivalence. Applying Observation 2.4.5 once more, we deduce that

$$i_p^{*,\mathrm{hyp}}(\phi) \simeq (i_p^*(\phi))^{\mathrm{hyp}}$$

is an $\mathrm{L}^{\mathrm{hyp}}_{\mathcal{X}_p}$-equivalence as well. Thus, it immediately follows from adjunction that $i_{p,*}i_p^{*,\mathrm{hyp}}(F)$ is ϕ-local. To conclude, observe that since $i_{S,*}i_S^*(F) \in \mathrm{Cons}_P(\mathcal{X})$, then

$$i_p^* i_{S,*} i_S^*(F) \in \mathrm{LC}(\mathcal{X}_p) = \mathrm{LC}(\mathcal{X}_p^{\mathrm{hyp}}) .$$

In particular we have

$$i_p^{*,\mathrm{hyp}} i_{S,*} i_S^*(F) = i_p^* i_{S,*} i_S^*(F) ,$$

and the conclusion follows. □

We conclude with a question about generalizing Proposition 2.4.14.

2.4.16 Question. Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Does the inclusion $\mathcal{X}^{\mathrm{hyp}} \hookrightarrow \mathcal{X}$ carry $\mathrm{Cons}_P(\mathcal{X}^{\mathrm{hyp}})$ to $\mathrm{Cons}_P(\mathcal{X})$? (If so, then $(\mathcal{X}^{\mathrm{hyp}}, P)$ is exodromic and $\Pi_\infty(\mathcal{X}^{\mathrm{hyp}}, P) \simeq \Pi_\infty(\mathcal{X}, P)$.)

3 Stability properties of exodromic stratified ∞-topoi

The goal of this section is to prove the following 'stability theorem' for the class of exodromic stratified ∞-topoi:

3.0.1 Theorem (stability properties of exodromic stratified ∞-topoi)**.**

(1) Stability under pulling back to locally closed subposets: *If $(\mathcal{X}, P)$ is an exodromic stratified ∞-topos, then for each locally closed subposet $S \subset P$, the stratified ∞-topos $(\mathcal{X}_S, S)$ is exodromic and the induced functor*

$$\Pi_\infty(\mathcal{X}_S, S) \to \Pi_\infty(\mathcal{X}, P) \times_P S$$

is an equivalence. In particular, the induced functor $\Pi_\infty(\mathcal{X}, P) \to P$ is conservative. See Corollary 3.1.17.

(2) Every morphism between exodromic stratified ∞-topoi is exodromic. *See Theorem 3.2.3.*

(3) Stability under coarsening and localization formula: *Let $(\mathcal{X}, R)$ be an exodromic stratified ∞-topos and let $\phi\colon R \to P$ be a map of posets. Write W_P for the collection of morphisms in $\Pi_\infty(\mathcal{X}, R)$ that the composite $\Pi_\infty(\mathcal{X}, R) \to R \to P$ sends to equivalences. Then the stratified ∞-topos $(\mathcal{X}, P)$ is exodromic and the natural functor $\Pi_\infty(\mathcal{X}, R) \to \Pi_\infty(\mathcal{X}, P)$ induces an equivalence*

$$\Pi_\infty(\mathcal{X}, R)[W_P^{-1}] \simeq \Pi_\infty(\mathcal{X}, P)$$

See Theorem 3.3.5.

(4) van Kampen: *Existence of exit-path ∞-categories can be checked by descent. See Proposition 3.4.2 for a precise formulation.*

(5) Künneth formula: *Let $(\mathcal{X}, P)$ and $(\mathcal{Y}, Q)$ be exodromic stratified ∞-topoi. If P and Q are noetherian, then the stratified ∞-topos $(\mathcal{X} \otimes \mathcal{Y}, P \times Q)$ is exodromic and there are natural equivalences of ∞-categories*

$$\mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_Q(\mathcal{Y}) \simeq \mathrm{Cons}_{P\times Q}(\mathcal{X} \otimes \mathcal{Y})$$

and

$$\Pi_\infty(\mathcal{X} \otimes \mathcal{Y}, P \times Q) \simeq \Pi_\infty(\mathcal{X}, P) \times \Pi_\infty(\mathcal{Y}, Q) .$$

See Proposition 3.5.5.

(6) Stability of finiteness/compactness: *The property of an exit-path ∞-category being finite (resp., compact) is stable under pulling back to a locally closed subposet, is stable under coarsening, and can be checked on a finite cover. See §3.6 for a precise formulation.*

Subsection 3.1 proves (1), §3.2 proves (2), §3.3 proves (3), §3.4 proves (4), §3.5 proves (5), and §3.6 proves (6). Before moving on, we also pose two question related to Theorem 3.0.1. First:

3.0.2 Question. Can one prove the Künneth formula without the extra noetherian hypothesis?

Second, as noted earlier (see Observation 1.3.10), if $\mathcal{X}$ is a monodromic ∞-topos and $U \in \mathcal{X}$, then the slice ∞-topos $\mathcal{X}_{/U}$ is also monodromic. We have not listed the analogous stability property for exodromic ∞-topoi in Theorem 3.0.1; we do not know if it is true. Thus we ask:

3.0.3 Question. Let $(\mathcal{X}, P)$ be a stratified ∞-topos and $U \in \mathcal{X}$. Then composing the natural geometric morphism $\mathcal{X}_{/U} \to \mathcal{X}$ with the stratification of $\mathcal{X}$ gives $\mathcal{X}_{/U}$ a natural P-stratification. Is the stratified ∞-topos $(\mathcal{X}_{/U}, P)$ exodromic?

3.1 Stability under pulling back to locally closed subposets. Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. The purpose of this subsection is to show that for each locally closed subposet $S \subset P$, the stratified ∞-topos $(\mathcal{X}_S, S)$ is exodromic, the inclusion $i_{S,*} \colon (\mathcal{X}_S, S) \hookrightarrow (\mathcal{X}, P)$ is exodromic, and the natural functor

$$\Pi_\infty(\mathcal{X}_S, S) \to \Pi_\infty(\mathcal{X}, P) \times_P S$$

is an equivalence (see Corollary 3.1.17). This result generalizes [14, Proposition 3.6-(2); 29, Proposition 3.13-(1)] to the setting of exodromic stratified ∞-topoi; the proof is essentially the same as theirs, just adapted to our more general setting. A key step is to show that both constructible objects and functors out of exit-path ∞-categories satisfy *recollement*. We refer the reader to [HA, §A.8; SAG, §7.2; 2, §6.1; 35, §2] for background on recollements.

We start by proving a general recollement result for ∞-categories of functors out of an ∞-category with a functor to a poset.

3.1.1 Notation. Let $F \colon \mathcal{C} \to P$ be a functor from an ∞-category to a poset. Given a full subposet $S \subset P$, we write $\mathcal{C}_S \coloneqq \mathcal{C} \times_P S$.

3.1.2 Observation. In the setting of Notation 3.1.1, note that since the inclusion $S \subset P$ is fully faithful, its basechange $\mathcal{C}_S \to \mathcal{C}$ is fully faithful with image those objects lying over S.

3.1.3 Proposition. *Let $F \colon \mathcal{C} \to P$ be a functor from an ∞-category to a poset, and let $Z \subset P$ be a closed subposet with open complement $U = P \smallsetminus Z$. Write $i \colon \mathcal{C}_Z \hookrightarrow \mathcal{C}$ and $j \colon \mathcal{C}_U \hookrightarrow \mathcal{C}$ for the inclusions. Then the restriction functors*

$$i^* \colon \mathrm{Fun}(\mathcal{C}, \mathbf{Spc}) \to \mathrm{Fun}(\mathcal{C}_Z, \mathbf{Spc}) \qquad \textit{and} \qquad j^* \colon \mathrm{Fun}(\mathcal{C}, \mathbf{Spc}) \to \mathrm{Fun}(\mathcal{C}_U, \mathbf{Spc})$$

exhibit $\mathrm{Fun}(\mathcal{C}, \mathbf{Spc})$ *as the recollement of* $\mathrm{Fun}(\mathcal{C}_Z, \mathbf{Spc})$ *and* $\mathrm{Fun}(\mathcal{C}_U, \mathbf{Spc})$.

Proof. Note that since every object of $\mathcal{C}$ belongs to either $\mathcal{C}_U$ or $\mathcal{C}_Z$ and equivalences in $\mathrm{Fun}(\mathcal{C}, \mathbf{Spc})$ are detected pointwise, the functors j^* and i^* are jointly conservative. Hence the only nontrivial point to check is that the composite $j^* i_*$ is constant with value the terminal object of $\mathrm{Fun}(\mathcal{C}_U, \mathbf{Spc})$.

For this, consider the pullback square of ∞-categories

$$\begin{array}{ccc} \varnothing & \xrightarrow{\ b\ } & \mathcal{C}_Z \\ {\scriptstyle a}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle i} \\ \mathcal{C}_U & \underset{j}{\hookrightarrow} & \mathcal{C} \, . \end{array}$$

Since i is a right fibration (Lemma A.2.6), i is a *proper* functor in the sense of [14, Definition 2.22]. (See also [HTT, §4.1.2; 13, §4.4].) Hence proper basechange [14, Theorem 2.27] implies that the exchange transformation

$$j^* i_* \to a_* b^*$$

is an equivalence. To complete the proof, notice that the functor

$$b^* \colon \mathrm{Fun}(\mathcal{C}_Z, \mathbf{Spc}) \to \mathrm{Fun}(\varnothing, \mathbf{Spc}) \simeq *$$

is the unique functor and the functor $a_* \colon * \to \mathrm{Fun}(\mathcal{C}_U, \mathbf{Spc})$ picks out the terminal object. □

We now turn to showing that constructible objects satisfy recollement. Let us introduce a special class of coefficients we are interested in:

3.1.4 Definition. We say that a presentable ∞-category $\mathcal{E}$ is *compatible with recollements* if for every recollement datum of ∞-topoi

$$i^* \colon \mathcal{X} \to \mathcal{Z} \qquad \text{and} \qquad j^* \colon \mathcal{X} \to \mathcal{U} \, ,$$

the induced functors

$$i^* \otimes \mathcal{E} : \mathcal{X} \otimes \mathcal{E} \to \mathcal{Z} \otimes \mathcal{E} \qquad \text{and} \qquad j^* \otimes \mathcal{E} : \mathcal{X} \otimes \mathcal{E} \to \mathcal{U} \otimes \mathcal{E}$$

exhibit $\mathcal{X} \otimes \mathcal{E}$ as the recollement of $\mathcal{Z} \otimes \mathcal{E}$ and $\mathcal{U} \otimes \mathcal{E}$.

3.1.5 Recollection. It follows respectively from [19, Corollary 2.18 and Proposition 2.26] that if $\mathcal{E}$ is either compactly generated or stable, then it is compatible with recollements.

3.1.6 Observation (see Recollection B.1.6 and Proposition B.1.8)**.** Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $Z \subset P$ be a closed subposet with open complement $U = P \smallsetminus Z$. Then the functors

$$i_Z^* : \mathcal{X} \to \mathcal{X}_Z \qquad \text{and} \qquad i_U^* : \mathcal{X} \to \mathcal{X}_U$$

exhibit $\mathcal{X}$ as the recollement of $\mathcal{X}_Z$ and $\mathcal{X}_U$.

3.1.7 Lemma. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $Z \subset P$ be a closed subposet with open complement $U = P \smallsetminus Z$. Let $\mathcal{E}$ be a presentable ∞-category. Assume that $\mathcal{E}$ is compatible with recollements and that the terminal object in $\mathrm{Sh}(\mathcal{X}; \mathcal{E})$ is P-constructible. Then:*

(1) *If $F \in \mathrm{Cons}_U(\mathcal{X}_U; \mathcal{E})$, then $i_{U,!}(F) \in \mathrm{Cons}_P(\mathcal{X}; \mathcal{E})$.*

(2) *If $G \in \mathrm{Cons}_Z(\mathcal{X}_Z; \mathcal{E})$, then $i_{Z,*}(G) \in \mathrm{Cons}_P(\mathcal{X}; \mathcal{E})$.*

(3) *The composite $i_U^* i_{Z,*} : \mathrm{Cons}_Z(\mathcal{X}_Z; \mathcal{E}) \to \mathrm{Cons}_U(\mathcal{X}_U; \mathcal{E})$ is constant with value the terminal object.*

(4) *The functors*

$$i_Z^* : \mathrm{Cons}_P(\mathcal{X}; \mathcal{E}) \to \mathrm{Cons}_Z(\mathcal{X}_Z; \mathcal{E}) \qquad \textit{and} \qquad i_U^* : \mathrm{Cons}_P(\mathcal{X}; \mathcal{E}) \to \mathrm{Cons}_U(\mathcal{X}_U; \mathcal{E})$$

are jointly conservative.

Proof. All of these claims essentially follow from Recollection 3.1.5. For (1), note that since $i_U^* i_{U,!}(F) \simeq F$, it suffices to show that $i_Z^* i_{U,!}(F)$ is locally constant on $\mathcal{X}_Z$. By recollement, the functor

$$i_Z^* i_{U,!} : \mathrm{Sh}(\mathcal{X}_U; \mathcal{E}) \to \mathrm{Sh}(\mathcal{X}_Z; \mathcal{E})$$

is constant with value the initial object, which is U-constructible. For (2), note that since $i_Z^* i_{Z,*}(G) \simeq G$, it suffices to show that $i_U^* i_{Z,*}(G)$ is U-constructible on $\mathcal{X}_U$. Again by recollement, the functor

$$i_U^* i_{Z,*} : \mathrm{Sh}(\mathcal{X}_Z; \mathcal{E}) \to \mathrm{Sh}(\mathcal{X}_U; \mathcal{E})$$

is constant with value the terminal object. Since $i_U^* : \mathrm{Sh}(\mathcal{X}; \mathcal{E}) \to \mathrm{Sh}(\mathcal{X}_U; \mathcal{E})$ is a right adjoint and since the terminal object in $\mathrm{Sh}(\mathcal{X}; \mathcal{E})$ is P-constructible by assumption, it follows that the terminal object in $\mathrm{Sh}(\mathcal{X}_U; \mathcal{E})$ is U-constructible. In particular, $i_U^* i_{Z,*}$ carries $\mathrm{Cons}_Z(\mathcal{X}_Z; \mathcal{E})$ to $\mathrm{Cons}_U(\mathcal{X}_U; \mathcal{E})$, thus proving at the same time (2) and (3). Item (4) is immediate from recollement. $\square$

3.1.8 Lemma. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $U \subset P$ be an open subposet. Let $\mathcal{E}$ be a presentable ∞-category. Assume that $\mathcal{E}$ is compatible with recollements and that the terminal object of $\mathrm{Sh}(\mathcal{X}; \mathcal{E})$ is P-constructible. Then:*

(1) *Write $\varnothing$ for the initial object of $\mathrm{Cons}_Z(\mathcal{X}_Z; \mathcal{E})$ and set*

$$\ker(i_Z^*) := \{X \in \mathrm{Cons}_P(\mathcal{X}) \mid i_Z^*(X) \simeq \varnothing\}.$$

Then the induced functor

$$i_{U,!} : \mathrm{Cons}_U(\mathcal{X}_U) \hookrightarrow \ker(i_Z^*)$$

is an equivalence.

(2) *Write $*$ for the terminal object of $\mathrm{Cons}_U(\mathcal{X}_U; \mathcal{E})$ and set*

$$\ker(i_U^*) := \{X \in \mathrm{Cons}_P(\mathcal{X}; \mathcal{E}) \mid i_U^*(X) \simeq *\}.$$

Then the induced functor

$$i_{Z,*} : \mathrm{Cons}_Z(\mathcal{X}_Z; \mathcal{E}) \hookrightarrow \ker(i_U^*)$$

is an equivalence

Proof. In both cases, it suffices to check essential surjectivity. So let $X \in \ker(i_Z^*)$ and consider the counit $c\colon i_{U,!}i_U^*(X) \to X$ in $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$. By Lemma 3.1.7-(4), it suffices to show that $i^*(c)$ and $i_Z^*(c)$ are equivalences. The former follows from the full faithfulness of $i_{U,!}$, and the latter follows from the definition of $\ker(i_Z^*)$. For (2), the same argument applies, starting with the unit $u\colon F \to i_{Z,*}i_Z^*(F)$ in place of the counit. □

3.1.9 Lemma. *Let $(\mathcal{X},P)$ be a stratified ∞-topos and let $Z \subset P$ be a closed subposet with open complement $U = P \smallsetminus Z$. Let $\mathcal{E}$ be a presentable ∞-category. Assume that $\mathcal{E}$ is compatible with recollements and that the terminal object of $\mathrm{Sh}(\mathcal{X};\mathcal{E})$ is P-constructible. Then:*

(1) *If $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ is presentable, then $\mathrm{Cons}_P(\mathcal{X}_Z;\mathcal{E})$ and $\mathrm{Cons}_P(\mathcal{X}_U;\mathcal{E})$ are also presentable.*

(2) *If $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ is closed under colimits in $\mathrm{Sh}(\mathcal{X};\mathcal{E})$, then the functor $i_U^*\colon \mathrm{Cons}_P(\mathcal{X};\mathcal{E}) \to \mathrm{Cons}_U(\mathcal{X}_U;\mathcal{E})$ preserves colimits.*

(3) *If $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ is closed under finite limits in $\mathrm{Sh}(\mathcal{X};\mathcal{E})$, then the functor $i_Z^*\colon \mathrm{Cons}_P(\mathcal{X};\mathcal{E}) \to \mathrm{Cons}_Z(\mathcal{X}_Z;\mathcal{E})$ is left exact.*

(4) *If $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ is presentable and closed under colimits and finite limits in $\mathrm{Sh}(\mathcal{X};\mathcal{E})$, then the functors i_Z^* and i_U^* exhibit $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ as the recollement of $\mathrm{Cons}_Z(\mathcal{X}_Z;\mathcal{E})$ and $\mathrm{Cons}_U(\mathcal{X}_U;\mathcal{E})$.*

Proof. For (1), notice that Lemma 3.1.7-(2) implies that $\mathrm{Cons}_Z(\mathcal{X}_Z;\mathcal{E})$ is a localization of $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$. Moreover, Lemma 3.1.8-(2) immediately implies that $\mathrm{Cons}_Z(\mathcal{X};\mathcal{E})$ is closed under weakly contractible colimits inside $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$; in particular

$$\mathrm{Cons}_Z(\mathcal{X};\mathcal{E}) \subset \mathrm{Cons}_P(\mathcal{X};\mathcal{E})$$

is closed under filtered colimits. Thus, the ∞-categorical reflection theorem [34, Theorem 1.1] implies that $\mathrm{Cons}_Z(X;\mathcal{E})$ is presentable. Then, Lemma 3.1.8-(1) implies that $\mathrm{Cons}_U(\mathcal{X}_U;\mathcal{E})$ is presentable.

Item (2) follows from the given assumption, the full faithfulness of $\mathrm{Cons}_U(\mathcal{X}_U;\mathcal{E})$ inside of $\mathrm{Sh}(\mathcal{X}_U;\mathcal{E})$, and the fact that $i_U^*\colon \mathrm{Sh}(\mathcal{X};\mathcal{E}) \to \mathrm{Sh}(\mathcal{X}_U;\mathcal{E})$ preserves colimits and preserve constructible objects. A similar argument shows (3) as well.

We are left to prove (4). In virtue of Lemma 3.1.7, all we are left to do is to check that i_U^* admits a right adjoint and that i_Z^* is left exact. The first statement follows from (1), (2), and the adjoint functor theorem, while the second follows directly from (3). □

In what follows, we will need to use the fact that given an open immersion of ∞-topoi $j_*\colon \mathcal{U} \hookrightarrow \mathcal{Y}$, the ∞-topos $\mathcal{U}$ is naturally identified with the slice $\mathcal{Y}_{/j_!(1)}$. Hence we recall some basic results about slice ∞-categories.

3.1.10 Recollection. Let $i\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be a fully faithful functor of ∞-categories and let $c \in \mathcal{C}$. Then:

(1) The induced functor $i\colon \mathcal{C}_{/c} \to \mathcal{D}_{/i(c)}$ is fully faithful.

(2) If $i\colon \mathcal{C} \hookrightarrow \mathcal{D}$ admits a left adjoint $L\colon \mathcal{D} \to \mathcal{C}$, then $i\colon \mathcal{C}_{/c} \to \mathcal{D}_{/i(c)}$ admits a left adjoint given by the induced functor

$$L\colon \mathcal{D}_{/i(c)} \to \mathcal{C}_{/Li(c)} \simeq \mathcal{C}_{/c}\ .$$

(3) If $i\colon \mathcal{C} \hookrightarrow \mathcal{D}$ admits a right adjoint $R\colon \mathcal{D} \to \mathcal{C}$, then $i\colon \mathcal{C}_{/c} \to \mathcal{D}_{/i(c)}$ admits a right adjoint given by the induced functor

$$R\colon \mathcal{D}_{/i(c)} \to \mathcal{C}_{/Ri(c)} \simeq \mathcal{C}_{/c}\ .$$

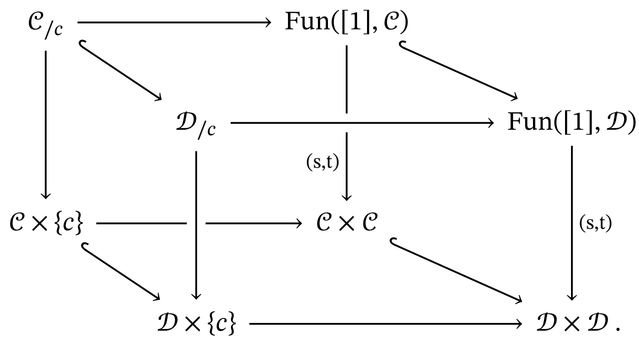
See [HTT, Proposition 5.2.5.1].

3.1.11 Lemma. *Let $\mathcal{D}$ be an ∞-category, $\mathcal{C} \subset \mathcal{D}$ a full subcategory, and $c \in \mathcal{C}$. Then the natural square*

$$\begin{array}{ccc} \mathcal{C}_{/c} & \hookrightarrow & \mathcal{D}_{/c} \\ \downarrow & & \downarrow \\ \mathcal{C} & \hookrightarrow & \mathcal{D} \end{array}$$

is a pullback square of ∞-categories. Here the vertical functors are the forgetful functors.

Proof. Consider the commutative cube



By definition, the front and back vertical faces are pullbacks. Since $\mathcal{C} \subset \mathcal{D}$ is a full subcategory, the right-hand vertical face is a pullback. Hence the left-hand vertical face is also a pullback. □

Let us now give an alternative description of constructible objects in a stratified ∞-topos obtained by pulling back to an open subposet.

3.1.12 Lemma. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $U \subset P$ be an open subposet. Then:*

(1) *The square*

$$\begin{array}{ccc} \mathrm{Cons}_U(\mathcal{X}_U) & \hookrightarrow & \mathcal{X}_U \\ {\scriptstyle i_{U,!}}\downarrow & & \downarrow{\scriptstyle i_{U,!}} \\ \mathrm{Cons}_P(\mathcal{X}) & \hookrightarrow & \mathcal{X} \end{array}$$

is a pullback square of ∞-categories.

(2) *There is a commutative square*

$$\begin{array}{ccc} \mathrm{Cons}_U(\mathcal{X}_U) & \hookrightarrow & \mathcal{X}_U \\ {\scriptstyle i_{U,!}}\downarrow{\scriptstyle \wr} & & {\scriptstyle \wr}\downarrow{\scriptstyle i_{U,!}} \\ \mathrm{Cons}_P(\mathcal{X})_{/i_{U,!}(1)} & \hookrightarrow & \mathcal{X}_{/i_{U,!}(1)} \end{array}$$

where the vertical functors are equivalences and the horizontal functors are the natural inclusions.

Proof. For (1), note that it suffices to show that the fully faithful functor

$$i_{U,!} : \mathrm{Cons}_U(\mathcal{X}_U) \hookrightarrow \mathrm{Cons}_P(\mathcal{X}) \cap i_{U,!}(\mathcal{X}_U)$$

is essentially surjective. For this, let $G \in \mathcal{X}_U$ be such that $i_{U,!}(G)$ is P-constructible. Write $Z \coloneqq P \smallsetminus U$. Then $i_Z^* i_{U,!}(G) = \varnothing$ and $i_U^* i_{U,!}(G)$ is U-constructible. Hence $G \in \mathrm{Cons}_U(\mathcal{X}_U)$, completing the proof.

For (2), note that $i_{U,*} : \mathcal{X}_U \hookrightarrow \mathcal{X}$ is an open immersion of ∞-topoi, the exceptional left adjoint $i_{U,!} : \mathcal{X}_U \hookrightarrow \mathcal{X}$ induces an equivalence $\mathcal{X}_U \xrightarrow{\sim} \mathcal{X}_{/i_{U,!}(1)}$ fitting into a commutative triangle

$$\begin{array}{ccc} \mathcal{X}_U & \xrightarrow{\sim} & \mathcal{X}_{/i_{U,!}(1)} \\ & {\scriptstyle i_{U,!}}\searrow \quad \swarrow{\scriptstyle \text{forget}} & \\ & \mathcal{X} & \end{array} .$$

Since $i_{U,!}(1) \in \mathrm{Cons}_P(\mathcal{X})$, the claim follows from item (1) combined with Lemma 3.1.11. □

3.1.13 Proposition (recollement). *Let $s_* : \mathcal{X} \to \mathrm{Fun}(P, \mathbf{Spc})$ be an exodromic stratified ∞-topos and let $Z \subset P$ be a closed subposet with open complement $U = P \smallsetminus Z$. Then:*

(1) *The functors*
$$i_Z^* \colon \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_Z(\mathcal{X}_Z) \qquad \textit{and} \qquad i_U^* \colon \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_U(\mathcal{X}_U)$$
exhibit $\mathrm{Cons}_P(\mathcal{X})$ *as the recollement of* $\mathrm{Cons}_Z(\mathcal{X}_Z)$ *and* $\mathrm{Cons}_U(\mathcal{X}_U)$*.*

(2) *The stratified* ∞*-topos* $(\mathcal{X}_U, U)$ *is exodromic, the morphism* $i_{U,*} \colon (\mathcal{X}_U, U) \hookrightarrow (\mathcal{X}, P)$ *is exodromic, and the induced functor*
$$\Pi_\infty(X_U, U) \to \Pi_\infty(X, P)_U$$
is an equivalence.

(3) *The stratified* ∞*-topos* $(\mathcal{X}_Z, Z)$ *is exodromic, the morphism* $i_{Z,*} \colon (\mathcal{X}_Z, Z) \hookrightarrow (\mathcal{X}, P)$ *is exodromic, and the induced functor*
$$\Pi_\infty(X_Z, Z) \to \Pi_\infty(X, P)_Z$$
is an equivalence.

Proof. Since the terminal object of $\mathcal{X}$ is P-constructible, (1) follows directly from Lemma 3.1.9-(4). For (2), let us first prove that $\mathrm{Cons}_U(\mathcal{X}_U)$ is closed under limits and colimits in $\mathcal{X}_U$. By Lemma 3.1.12-(2), we have a commutative square
$$\begin{array}{ccc} \mathrm{Cons}_U(\mathcal{X}_U) & \hookrightarrow & \mathcal{X}_U \\ {\scriptstyle i_{U,!}}\downarrow\wr & & \wr\downarrow{\scriptstyle i_{U,!}} \\ \mathrm{Cons}_P(\mathcal{X})_{/i_{U,!}(1)} & \hookrightarrow & \mathcal{X}_{/i_{U,!}(1)} \end{array}$$
where the vertical functors are equivalences. Since $(\mathcal{X}, P)$ is exodromic, the inclusion $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ admits both a left and right adjoint. Hence Recollection 3.1.10 shows that the inclusion $\mathrm{Cons}_U(\mathcal{X}_U) \subset \mathcal{X}_U$ admits both a left and right adjoint. Write $s_{U,*} \colon \mathcal{X}_U \to \mathrm{Fun}(U, \mathbf{Spc})$ for the induced stratification and $j \colon U \hookrightarrow P$ for the inclusion. All we are left to show is that the ∞-category $\mathrm{Cons}_U(\mathcal{X}_U)$ is atomically generated by $\Pi_\infty(\mathcal{X}, P)_U$ and that the pullback functor $s_U^* \colon \mathrm{Fun}(U, \mathbf{Spc}) \to \mathrm{Cons}_U(\mathcal{X}_U)$ preserves limits. To see that $\mathrm{Cons}_U(\mathcal{X}_U)$ is atomically generated by $\Pi_\infty(\mathcal{X}, P)_U$, notice that since $i_Z^* i_{U,!}(1) = \varnothing$ and $i_U^* i_{U,!}(1) = 1$, the fully faithful functor
$$i_{U,!} \colon \mathrm{Cons}_U(\mathcal{X}_U) \hookrightarrow \mathrm{Cons}_P(\mathcal{X}) \simeq \mathrm{Fun}(\Pi_\infty(\mathcal{X}, P), \mathbf{Spc})$$
has image those functors $F \colon \Pi_\infty(\mathcal{X}, P) \to \mathbf{Spc}$ such that the composite
$$\Pi_\infty(\mathcal{X}, P)_Z \hookrightarrow \Pi_\infty(\mathcal{X}, P) \xrightarrow{F} \mathbf{Spc}$$
is constant with value the initial object. Now note that this full subcategory coincides with the image of the fully faithful functor
$$\mathrm{Fun}(\Pi_\infty(\mathcal{X}, P)_U, \mathbf{Spc}) \hookrightarrow \mathrm{Fun}(\Pi_\infty(\mathcal{X}, P), \mathbf{Spc})$$
given by left Kan extension along the inclusion $\Pi_\infty(\mathcal{X}, P)_U \hookrightarrow \Pi_\infty(\mathcal{X}, P)$.

To see that $s_U^* \colon \mathrm{Fun}(U, \mathbf{Spc}) \to \mathrm{Cons}_U(\mathcal{X}_U)$ preserves limits, notice that we have a commutative square
$$\begin{array}{ccc} \mathrm{Fun}(P, \mathbf{Spc}) & \xrightarrow{j^*} & \mathrm{Fun}(U, \mathbf{Spc}) \\ {\scriptstyle s^*}\downarrow & & \downarrow{\scriptstyle s_U^*} \\ \mathrm{Cons}_P(\mathcal{X}) & \xrightarrow[i_U^*]{} & \mathrm{Cons}_U(\mathcal{X}_U) \,. \end{array}$$
Since j_* is fully faithful, we see that there are equivalences
$$s_U^* \simeq s_U^* j^* j_* \simeq i_U^* s^* j_* \,.$$
Since the functors i_U^*, s^*, and j_* all preserve limits, we deduce that s_U^* preserves limits, as desired.

For (3), recall from Lemma 3.1.8-(1) that
$$\mathrm{Cons}_Z(\mathcal{X}_Z) \simeq \ker\left(i_U^* \colon \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_U(\mathcal{X}_U)\right) . \tag{3.1.14}$$

Since $(\mathcal{X}, P)$ is exodromic by assumption, $(\mathcal{X}_U, U)$ is exodromic by (2), and i_U^* preserves limits and colimits, we deduce that $\mathrm{Cons}_Z(\mathcal{X}_Z) \subset \mathcal{X}_Z$ is closed under limits and colimits. Proposition 3.1.3 and the identification (3.1.14) show that the ∞-category $\mathrm{Cons}_Z(\mathcal{X}_Z)$ is atomically generated by $\Pi_\infty(\mathcal{X}, P)_Z$ and the functor $i_Z^* \colon \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_Z(\mathcal{X})$ preserves limits and colimits.

Write $s_{Z,*} \colon \mathcal{X}_Z \to \mathrm{Fun}(Z, \mathbf{Spc})$ for the induced stratification and $i \colon Z \hookrightarrow P$ for the inclusion. All that remains to be shown is that the pullback functor $s_Z^* \colon \mathrm{Fun}(Z, \mathbf{Spc}) \to \mathrm{Cons}_Z(\mathcal{X}_Z)$ preserves limits. For this, notice that we have a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(P, \mathbf{Spc}) & \xrightarrow{i^*} & \mathrm{Fun}(Z, \mathbf{Spc}) \\ {\scriptstyle s^*}\downarrow & & \downarrow{\scriptstyle s_Z^*} \\ \mathrm{Cons}_P(\mathcal{X}) & \xrightarrow[i_Z^*]{} & \mathrm{Cons}_Z(\mathcal{X}_Z) . \end{array}$$

Since i_* is fully faithful, we see that there are equivalences

$$s_Z^* \simeq s_Z^* i^* i_* \simeq i_Z^* s^* i_* .$$

Since the functors i_Z^*, s^*, and i_* all preserve limits, we deduce that s_Z^* preserves limits, as desired. □

3.1.15. In the setting of Proposition 3.1.13, the recollement takes the following form:

$$\mathrm{Cons}_Z(\mathcal{X}_Z) \;\underset{i_{Z,*}}{\overset{i^{\mathrm{c}}_{Z,\sharp}}{\rightleftarrows}}\!\!\xleftarrow{\;i_Z^*\;}\; \mathrm{Cons}_P(\mathcal{X}) \;\underset{i^{\mathrm{c}}_{U,*}}{\overset{i_{U,!}}{\leftleftarrows}}\!\!\xrightarrow{\;i_U^*\;}\; \mathrm{Cons}_U(\mathcal{X}_U) .$$

Here the functors $i_{Z,*}$, i_Z^*, $i_{U,!}$, and i_U^* agree with the ones at the level of the ∞-topoi $\mathcal{X}_Z$, $\mathcal{X}_U$, and $\mathcal{X}$. The functor $i^{\mathrm{c}}_{U,*}$ does not necessarily agree with the pushforward $i_{U,*} \colon \mathcal{X}_U \hookrightarrow \mathcal{X}$, and the functor $i^{\mathrm{c}}_{Z,\sharp}$ is 'extra' in the sense that it does not come for free from the theory of recollements.

For the next result, we need the following useful characterization of when a functor of exit-path ∞-categories is fully faithful in terms of the constructible pushforwards:

3.1.16 Lemma. *Let $f_* \colon (\mathcal{X}, P) \to (\mathcal{Y}, Q)$ be a morphism between exodromic stratified ∞-topoi. If f_* is exodromic, then the following are equivalent:*

(1) *The functor $f^{\mathrm{ex}} \colon \Pi_\infty(\mathcal{X}, P) \to \Pi_\infty(\mathcal{Y}, Q)$ is fully faithful.*

(2) *The functor $f^{\mathrm{c}}_\sharp \colon \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_Q(\mathcal{Y})$ is fully faithful.*

(3) *The functor $f^{\mathrm{c}}_* \colon \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_Q(\mathcal{Y})$ is fully faithful.*

Proof. Immediate from the fact that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is fully faithful if and only if either of the functors

$$F_!, F_* \colon \mathrm{Fun}(\mathcal{C}, \mathbf{Spc}) \to \mathrm{Fun}(\mathcal{D}, \mathbf{Spc})$$

given by left or right Kan extension along F is fully faithful. □

By writing a locally closed immersion of posets as the composite of a closed immersion and an open immersion, we deduce the main result of this subsection:

3.1.17 Corollary (stability under pulling back to locally closed subposets)**.** *Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos and let $S \subset P$ be a locally closed subposet. Then:*

(1) *The stratified ∞-topos $(\mathcal{X}_S, S)$ is exodromic and the morphism of stratified ∞-topoi $i_{S,*} \colon (\mathcal{X}_S, S) \hookrightarrow (\mathcal{X}, P)$ is exodromic.*

(2) *The ∞-topos $\mathcal{X}_S$ is monodromic.*

(3) *The natural functor $\Pi_\infty(\mathcal{X}_S, S) \to \Pi_\infty(\mathcal{X}, P)_S$ is an equivalence.*

(4) *The functors $i^{\mathrm{c}}_{S,\sharp}, i^{\mathrm{c}}_{S,*} \colon \mathrm{Cons}_S(\mathcal{X}_S) \to \mathrm{Cons}_P(\mathcal{X})$ are both fully faithful.*

(5) *The natural functor* $\Pi_\infty(\mathcal{X}, P) \to P$ *is conservative.*

Proof. Choose an open subposet $U \subset P$ containing S such that S is closed in U. For (1), apply Proposition 3.1.13-(2) to both the open inclusion $U \subset P$ and closed inclusion $S \subset U$. Item (2) follows from (1) and Lemma 2.2.17. For (3), applying Proposition 3.1.13-(3) to the closed inclusion $S \subset U$ and the open inclusion $U \subset P$, we see that there are equivalences

$$\begin{aligned}\Pi_\infty(\mathcal{X}_S, S) &\xrightarrow{\sim} \Pi_\infty(\mathcal{X}_U, U) \times_U S \\ &\xrightarrow{\sim} (\Pi_\infty(\mathcal{X}, P) \times_P U) \times_U S \\ &\simeq \Pi_\infty(\mathcal{X}, P)_S \,.\end{aligned}$$

By Observation 3.1.2, the natural functor $\Pi_\infty(\mathcal{X}, P)_S \to \Pi_\infty(\mathcal{X}, P)$ is fully faithful; hence Lemma 3.1.16 shows that (4) follows from (3). For (5), note that by Recollection A.1.1, we need to show that each fiber $\Pi_\infty(\mathcal{X}, P)_p$ is an ∞-groupoid. Since each $p \in P$ is locally closed, item (1) shows that

$$\Pi_\infty(\mathcal{X}, P)_p \simeq \Pi_\infty(\mathcal{X}_p, \{p\}) \,.$$

The conclusion now follows from the fact that $\Pi_\infty(\mathcal{X}_p, \{p\})$ is an ∞-groupoid (Recollection 1.3.8). □

We conclude by recording a few consequences of Corollary 3.1.17. First, we can describe the objects of the exit-path ∞-category.

3.1.18 Observation (the objects of $\Pi_\infty(\mathcal{X}, P)$)**.** Let $(\mathcal{X}, P)$ be an exodromic stratified space. Corollary 3.1.17 implies that there is a natural identification

$$\Pi_\infty(\mathcal{X}, P)^{\simeq} \simeq \coprod_{p \in P} \Pi_\infty(\mathcal{X}_p)$$

between the maximal sub-∞-groupoid of $\Pi_\infty(\mathcal{X}, P)$ and the coproduct of the shapes of the ∞-topoi $\mathcal{X}_p$.

Second, equivalences of constructible objects can be checked by pulling back to strata:

3.1.19 Corollary. *Let* $(\mathcal{X}, P)$ *be an exodromic stratified* ∞*-topos and let* $\{S_\alpha\}_{\alpha \in A}$ *be a collection of locally closed subposets of* P *such that* $\bigcup_{\alpha \in A} S_\alpha = P$*. Then the restriction functors*

$$\left\{ i_{S_\alpha}^* : \operatorname{Cons}_P(\mathcal{X}) \to \operatorname{Cons}_{S_\alpha}(\mathcal{X}_{S_\alpha}) \right\}_{\alpha \in A}$$

are jointly conservative.

Proof. Since each $p \in P$ is locally closed, by further restricting to the strata, it suffices to show that the restriction functors

$$\left\{ i_p^* : \operatorname{Cons}_P(\mathcal{X}) \to \operatorname{LC}(\mathcal{X}_p) \right\}_{p \in P}$$

are jointly conservative. By Corollary 3.1.17, the stratified ∞-topos $(\mathcal{X}_p, \{p\})$ is exodromic and the inclusion $i_{p,*} : (\mathcal{X}_p, \{p\}) \hookrightarrow (\mathcal{X}, P)$ is exodromic. Hence the claim follows from the identification of the restriction functor $i_p^* : \operatorname{Cons}_P(\mathcal{X}) \to \operatorname{LC}(\mathcal{X}_p)$ with the functor

$$\operatorname{Fun}(\Pi_\infty(\mathcal{X}, P), \mathbf{Spc}) \to \operatorname{Fun}(\Pi_\infty(\mathcal{X}_p), \mathbf{Spc})$$

given by precomposition with the inclusion $\Pi_\infty(\mathcal{X}_p) \simeq \Pi_\infty(\mathcal{X}, P)_p \hookrightarrow \Pi_\infty(\mathcal{X}, P)$. □

Finally, the ∞-category of constructible objects with arbitrary presentable coefficients is still presentable:

3.1.20 Lemma. *Let* $(\mathcal{X}, P)$ *be a stratified* ∞*-topos and let* $\mathcal{E}$ *be a presentable* ∞*-category. If for each* $p \in P$*, the stratum* $\mathcal{X}_p$ *is monodromic, then the the* ∞*-category* $\operatorname{Cons}_P(\mathcal{X}; \mathcal{E})$ *is presentable and closed under colimits in* $\operatorname{Sh}(\mathcal{X}; \mathcal{E})$*.*

Proof. By definition, $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ fits into a pullback square of ∞-categories

$$\begin{array}{ccc} \mathrm{Cons}_P(\mathcal{X};\mathcal{E}) & \longrightarrow & \prod_{p\in P} \mathrm{LC}(\mathcal{X}_p;\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Sh}(\mathcal{X};\mathcal{E}) & \xrightarrow[\prod_p i_p^*]{} & \prod_{p\in P} \mathrm{Sh}(\mathcal{X}_p;\mathcal{E}) \end{array}$$

Since each $\mathcal{X}_p$ is monodromic, by Recollection 1.3.8, $\mathrm{LC}(\mathcal{X}_p;\mathcal{E})$ is presentable and closed under limits and colimits in $\mathrm{Sh}(\mathcal{X}_p;\mathcal{E})$. The fact that the forgetful functor $\mathbf{Pr}^{\mathrm{L}} \to \mathbf{CAT}_\infty$ preserves limits [HTT, Proposition 5.5.3.13] completes the proof. ☐

3.1.21 Corollary. *Let $(\mathcal{X}, P)$ be an exodromic stratified topos. Then for any presentable ∞-category $\mathcal{E}$, the ∞-category $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ is presentable and closed under colimits in $\mathrm{Sh}(\mathcal{X};\mathcal{E})$.*

Proof. Combine Corollary 3.1.17 and Lemma 3.1.20. ☐

3.2 All morphisms are exodromic. We now use Corollary 3.1.17 to show that *every* morphsim between exodromic stratified ∞-topoi is exodromic. We start by proving this in the special case where the target is trivially stratified.

3.2.1 Lemma. *Let $f_* : (\mathcal{X}, P) \to (\mathcal{Y}, *)$ be a morphism of stratified ∞-topoi, where the target is trivially stratified. If the stratified ∞-topoi $(\mathcal{X}, P)$ and $(\mathcal{Y}, *)$ are exodromic, then the morphism f_* is exodromic.*

Proof. Since $(\mathcal{X}, P)$ is exodromic, Lemma 2.2.17-(1) shows that the trivially stratified ∞-topos $(\mathcal{X}, *)$ is exodromic. The morphism f_* factors as a composite

$$(\mathcal{X}, P) \longrightarrow (\mathcal{X}, *) \longrightarrow (\mathcal{Y}, *) .$$

By Lemma 2.2.17-(2), the left-hand morphism is exodromic, and by Example 2.3.4 the right-hand morphism is exodromic. Hence the composite is exodromic. ☐

For the following result, we introduce the following variant of Notation 2.1.9.

3.2.2 Notation. Let $(\mathcal{X}, R)$ be a stratified ∞-topos and $\phi : R \to P$ be a map of posets. Given $p \in P$, we write $R_p \coloneqq \phi^{-1}(p)$ for the full subposet of R given by the fiber of ϕ over p. Note that $\mathcal{X}_p = \mathcal{X}_{R_p}$. Hence the stratum $\mathcal{X}_p$ is naturally a R_p-stratified ∞-topos and the geometric morphism $i_{p,*} : \mathcal{X}_p \hookrightarrow \mathcal{X}$ defines a morphism of stratified ∞-topoi $(\mathcal{X}_p, R_p) \hookrightarrow (\mathcal{X}, R)$.

3.2.3 Theorem (all morphisms are exodromic). *Let $f_* : (\mathcal{X}, P) \to (\mathcal{Y}, Q)$ be a morphism between exodromic stratified ∞-topoi. Then f_* is exodromic.*

Proof. By Corollary 3.1.19, the functors

$$\left\{ i_{P_q}^* : \mathrm{Cons}_P(\mathcal{X}) \to \mathrm{Cons}_{P_q}(\mathcal{X}_q) \right\}_{q\in Q}$$

are jointly conservative. Moreover, since the subposet $P_q \subset P$ is locally closed, by Corollary 3.1.17-(1) these functors also preserve limits and colimits. Hence it suffices to show that for each $q \in Q$, the composite $i_{P_q}^* f^*$ preserves limits and colimits.

As in Observation 2.1.11, write $f_q : (\mathcal{X}_q, P_q) \to (\mathcal{Y}_q, \{q\})$ for the induced morphism of stratified ∞-topoi. Note that we have a commutative square

$$\begin{array}{ccc} \mathrm{Cons}_Q(\mathcal{Y}) & \xrightarrow{f^*} & \mathrm{Cons}_P(\mathcal{X}) \\ {\scriptstyle i_q^*}\downarrow & & \downarrow{\scriptstyle i_{P_q}^*} \\ \mathrm{LC}(\mathcal{Y}_q) & \xrightarrow[f_q^*]{} & \mathrm{Cons}_{P_q}(\mathcal{X}_q) . \end{array}$$

Again by Corollary 3.1.17-(1), the functor i_q^* preserves limits and colimits. To complete the proof, note that by Corollary 3.1.17-(1) the stratified ∞-topoi $(\mathcal{X}_q, P_q)$ and $(\mathcal{Y}_q, \{q\})$ are exodromic; hence Lemma 3.2.1 shows that functor f_q^* preserves limits and colimits. Thus $f_q^* i_q^*$ preserves limits and colimits. □

We can now cleanly state the functoriality of exit-path ∞-categories. For this, recall Notation 1.3.14 and Definition 2.1.6.

3.2.4 Notation. Write $\mathbf{StrTop}_\infty^{\mathrm{ex}} \subset \mathbf{StrTop}_\infty$ for the full subcategory spanned by the exodromic stratified ∞-topoi.

3.2.5 Observation (functoriality of exit-path ∞-categories)**.** The assignment $(\mathcal{X}, P) \mapsto \Pi_\infty(\mathcal{X}, P)$ refines to a functor

$$\Pi_\infty(-,-) \colon \mathbf{StrTop}_\infty^{\mathrm{ex}} \to \mathbf{Cat}_\infty^{\mathrm{idem}} .$$

Specifically, this functor is given by the composite

$$\mathbf{StrTop}_\infty^{\mathrm{ex}} \xrightarrow{\mathrm{Cons}} (\mathbf{Pr}^{\mathrm{R,at}})^{\mathrm{op}} \simeq \mathbf{Pr}^{\mathrm{L,at}} \xrightarrow[\sim]{(-)^{\mathrm{ex}}} \mathbf{Cat}_\infty^{\mathrm{idem}} ,$$

where the left-hand functor sends $(\mathcal{X}, P)$ to the ∞-category $\mathrm{Cons}_P(\mathcal{X})$ with functoriality given by pullback, and the right-hand functor sends an atomically generated ∞-category $\mathcal{C}$ to the ∞-category $\mathcal{C}^{\mathrm{ex}} = (\mathcal{C}^{\mathrm{at}})^{\mathrm{op}}$ given by the opposite of the subcategory of atomic objects.

3.3 Stability under coarsening. Let $(\mathcal{X}, R)$ be an exodromic stratified ∞-topos, and let $\phi \colon R \to P$ be a map of posets. In this subsection, show that $(\mathcal{X}, P)$ is also exodromic and express $\Pi_\infty(\mathcal{X}, P)$ as a localization of $\Pi_\infty(\mathcal{X}, R)$.

3.3.1 Observation. Let $(\mathcal{X}, R)$ be a stratified ∞-topos and let $\phi \colon R \to P$ be a map of posets. Since the morphism of stratified ∞-topoi $(\mathcal{X}, R) \to (\mathcal{X}, P)$ is the identity on the underlying ∞-topos $\mathcal{X}$, the pullback along $(\mathcal{X}, R) \to (\mathcal{X}, P)$ is simply the inclusion

$$\mathrm{Cons}_P(\mathcal{X}) \hookrightarrow \mathrm{Cons}_R(\mathcal{X}) .$$

3.3.2 Lemma. *Let $(\mathcal{X}, R)$ be a stratified ∞-topos and let $\phi \colon R \to P$ be a map of posets. If $(\mathcal{X}, R)$ is exodromic, then the following conditions are equivalent:*

(1) *The stratified ∞-topos $(\mathcal{X}, P)$ is exodromic.*

(2) *The full subcategory $\mathrm{Cons}_P(\mathcal{X}) \subset \mathrm{Cons}_R(\mathcal{X})$ is closed under both limits and colimits.*

Proof. Note that by Observation 3.3.1 we immediately have (1) ⇒ (2).

To show is that (2) ⇒ (1), we check the three conditions of Definition 2.2.10. First note that since $(\mathcal{X}, R)$ is exodromic, the ∞-category $\mathrm{Cons}_R(\mathcal{X})$ is atomically generated. Hence (2) and Proposition 1.1.13 imply that the full subcategory $\mathrm{Cons}_P(\mathcal{X})$ is atomically generated and the inclusion

$$\mathrm{Cons}_P(\mathcal{X}) \subset \mathrm{Cons}_R(\mathcal{X})$$

admits both a left and a right adjoint. Since $(\mathcal{X}, R)$ is exodromic, the full subcategory

$$\mathrm{Cons}_R(\mathcal{X}) \subset \mathcal{X}$$

is closed under limits and colimits; hence $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ is also closed under limits and colimits.

Write $t_* \colon \mathcal{X} \to \mathrm{Fun}(R, \mathbf{Spc})$ for the stratification, and $s_* \colon \mathcal{X} \to \mathrm{Fun}(P, \mathbf{Spc})$ for the composite $\phi_* t_*$. All that remains to be shown is that the pullback functor

$$s^* \colon \mathrm{Fun}(P, \mathbf{Spc}) \to \mathrm{Cons}_P(\mathcal{X})$$

preserves limits and colimits. For this, note that we have a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(P, \mathbf{Spc}) & \xrightarrow{\phi^*} & \mathrm{Fun}(R, \mathbf{Spc}) \\ \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle t^*} \\ \mathrm{Cons}_P(\mathcal{X}) & \hookrightarrow & \mathrm{Cons}_R(\mathcal{X}) . \end{array}$$

Here, the bottom horizontal functor is the inclusion, which is also the pullback along the refinement map $(\mathcal{X}, R) \to (\mathcal{X}, P)$. The functor ϕ^* preserves limits and colimits; by assumption both the bottom horizontal functor and t^* preserve limits and colimits. Hence s^* also preserves limits and colimits. □

To compute the exit-path ∞-category of a coarsening, we also make use of the following:

3.3.3 Lemma. *Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be functors between ∞-categories. Write $W \subset \mathrm{Mor}(\mathcal{C})$ for the collection of morphisms that GF carries to equivalences in $\mathcal{E}$. If F is a localization and G is conservative, then F induces an equivalence*

$$\mathcal{C}[W^{-1}] \simeq \mathcal{D} .$$

Proof. Since F is a localization, it suffices to show that given a morphism f in $\mathcal{C}$, the morphism $F(f)$ is an equivalence if and only if $f \in W$. To see this, note that since G is conservative, $F(f)$ is an equivalence if and only if $GF(f)$ is an equivalence. □

For the proof of stability under coarsening, recall Notations 2.1.9 and 3.2.2. We also introduce the following notation:

3.3.4 Notation. Let $(\mathcal{X}, R)$ be a stratified ∞-topos and $\phi \colon R \to P$ be a map of posets. If $(\mathcal{X}, R)$ is exodromic, write $W_P \subset \mathrm{Mor}(\Pi_\infty(\mathcal{X}, R))$ for the collection of morphisms sent to equivalences by the composite

$$\Pi_\infty(\mathcal{X}, R) \to R \to P .$$

3.3.5 Theorem (stability under coarsening)**.** *Let $(\mathcal{X}, R)$ be an exodromic stratified ∞-topos, and let $\phi \colon R \to P$ be a map of posets. Then:*

(1) *The stratified ∞-topos $(\mathcal{X}, P)$ is exodromic.*

(2) *The natural functor $\Pi_\infty(\mathcal{X}, R) \to \Pi_\infty(\mathcal{X}, P)$ induces an equivalence $\Pi_\infty(\mathcal{X}, R)[W_P^{-1}] \simeq \Pi_\infty(\mathcal{X}, P)$.*

Proof. First we prove (1). Since $(\mathcal{X}, R)$ is exodromic, Corollary 3.1.21 shows that the subcategory

$$\mathrm{Cons}_P(\mathcal{X}) \subset \mathrm{Cons}_R(\mathcal{X})$$

is closed under colimits. To prove closure under limits, let $F_\bullet \colon A \to \mathrm{Cons}_P(\mathcal{X})$ be a diagram. Write

$$F_{-\infty} := \lim_{\alpha \in A} F_\alpha$$

for the limit computed in $\mathrm{Cons}_R(\mathcal{X})$. We have to prove that for each $p \in P$, the restriction $i_p^*(F_{-\infty})$ is locally constant. Again by Corollary 3.1.17-(1), the functor

$$i_p^* \colon \mathrm{Cons}_R(\mathcal{X}) \to \mathrm{Cons}_{R_p}(\mathcal{X}_p)$$

preserves limits. Therefore,

$$i_p^*(F_{-\infty}) \simeq \lim_{\alpha \in A} i_p^*(F_\alpha) .$$

By assumption, each $i_p^*(F_\alpha)$ is a locally constant object of $\mathcal{X}_p$. Since $\mathcal{X}_p = \mathcal{X}_{R_p}$, by Corollary 3.1.17-(2), the trivially stratified ∞-topos $(\mathcal{X}_p, \{p\})$ is exodromic. Hence the subcategory

$$\mathrm{LC}(\mathcal{X}_p) \subset \mathcal{X}_p$$

is closed under limits (Recollection 1.3.8). Therefore, $i_p^*(F_{-\infty})$ is locally constant, as desired.

For item (2), note that (1) and Proposition 1.1.13 imply that the induced functor $\Pi_\infty(\mathcal{X}, R) \to \Pi_\infty(\mathcal{X}, P)$ exhibits $\Pi_\infty(\mathcal{X}, P)$ as the idempotent completion of the localization of $\Pi_\infty(\mathcal{X}, R)$ at the class of morphisms that the functor $\Pi_\infty(\mathcal{X}, R) \to \Pi_\infty(\mathcal{X}, P)$ carries to equivalences. Moreover, Corollary 3.1.17-(5) implies that the induced functor $\Pi_\infty(\mathcal{X}, P) \to P$ is conservative. Hence Lemma 3.3.3 shows that the induced functor

$$\Pi_\infty(\mathcal{X}, R) \to \Pi_\infty(\mathcal{X}, P)$$

exhibits $\Pi_\infty(\mathcal{X}, P)$ as the idempotent completion of the localization $\Pi_\infty(\mathcal{X}, R)[W_P^{-1}]$. Corollary 3.1.17-(5) shows that the natural functor $\Pi_\infty(\mathcal{X}, R) \to R$ is conservative. Thus Proposition A.2.2 shows that $\Pi_\infty(\mathcal{X}, R)[W_P^{-1}]$ is already idempotent complete, concluding the proof. □

3.3.6 Notation. Write $\mathrm{Env} : \mathbf{Cat}_\infty \to \mathbf{Spc}$ for the left adjoint to the inclusion $\mathbf{Spc} \subset \mathbf{Cat}_\infty$. For an ∞-category $\mathcal{C}$, we can compute $\mathrm{Env}(\mathcal{C})$ as the localization $\mathcal{C}[\mathcal{C}^{-1}]$ at all morphisms in $\mathcal{C}$ [14, Corollary 2.10].

3.3.7 Corollary. *Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Then there is a natural equivalence*

$$\mathrm{Env}(\Pi_\infty(\mathcal{X}, P)) \xrightarrow{\sim} \Pi_\infty(\mathcal{X}) .$$

Proof. Apply Theorem 3.3.5 to the map of posets $P \to *$. □

3.4 Checking exodromy locally. We now observe that the existence of an exit-path ∞-category can be checked by descent. This generalizes [14, Proposition 3.6-(2); 29, Proposition 3.13-(2)] to the setting of stratified ∞-topoi. We first recall two fundamental facts about ∞-topoi.

3.4.1 Recollection.

(1) The ∞-category $\mathbf{LTop}_\infty$ has all limits and colimits. Moreover, the forgetful functor $\mathbf{LTop}_\infty \to \mathbf{CAT}_\infty$ preserves limits. See [HTT, Proposition 6.3.2.3 & Corollary 6.3.4.7].

(2) A colimit in an ∞-category $\mathcal{X}$ with pullbacks is *van Kampen* if the functor

$$\mathcal{X}^{\mathrm{op}} \to \mathbf{CAT}_\infty , \; U \mapsto \mathcal{X}_{/U}$$

transforms it into a limit in $\mathbf{CAT}_\infty$. A presentable ∞-category $\mathcal{X}$ is an ∞-topos if and only if all colimits in $\mathcal{X}$ are van Kampen; see [HTT, Proposition 5.5.3.13, Theorem 6.1.3.9(3), & Proposition 6.3.2.3; 23].

3.4.2 Proposition (van Kampen)**.** *Let A be an ∞-category and let $(\mathcal{X}_\bullet, P_\bullet) : A \to \mathbf{StrTop}_\infty$ be a diagram of stratified ∞-topoi. Let $(\mathcal{X}_\infty, P_\infty)$ be a cone under $(\mathcal{X}_\bullet, P_\bullet)$. Assume that the following conditions are satisfied:*

(1) *For each $\alpha \in A$, the stratified ∞-topos $(\mathcal{X}_\alpha, P_\alpha)$ is exodromic.*

(2) *The natural pullback functors*

$$\mathcal{X}_\infty \to \lim_{\alpha \in A^{\mathrm{op}}} \mathcal{X}_\alpha \qquad \textit{and} \qquad \mathrm{Cons}_{P_\infty}(\mathcal{X}_\infty) \to \lim_{\alpha \in A^{\mathrm{op}}} \mathrm{Cons}_{P_\alpha}(\mathcal{X}_\alpha)$$

are equivalences.

Then the stratified ∞-topos $(\mathcal{X}_\infty, P_\infty)$ is exodromic and the natural functor

$$\operatorname*{colim}_{\alpha \in A} \Pi_\infty(\mathcal{X}_\alpha, P_\alpha) \to \Pi_\infty(\mathcal{X}_\infty, P_\infty)$$

is an equivalence of ∞-categories. Here the colimit is formed in $\mathbf{Cat}_\infty^{\mathrm{idem}}$.

Proof. Immediate from the definitions and the equivalence $\mathbf{Pr}^{\mathrm{L,at}} \simeq \mathbf{Cat}_\infty^{\mathrm{idem}}$ of Recollection 1.1.11. □

3.4.3 Remark (on idempotent completion)**.** Let P be a poset and write $\mathbf{Cat}_{\infty,/P}^{\mathrm{cons}} \subset \mathbf{Cat}_{\infty,/P}$ for the full subcategory spanned by those objects such that the specified functor $\mathcal{C} \to P$ is conservative. The forgetful functor

$$\mathbf{Cat}_{\infty,/P} \to \mathbf{Cat}_\infty$$

preserves colimits. The inclusion $\mathbf{Cat}_{\infty,/P}^{\mathrm{cons}} \hookrightarrow \mathbf{Cat}_{\infty,/P}$ preserves colimits (Observation A.3.5). Hence, the forgetful functor

$$\mathbf{Cat}_{\infty,/P}^{\mathrm{cons}} \to \mathbf{Cat}_\infty$$

preserves colimits. By Lemma A.1.3, every object of $\mathbf{Cat}_{\infty,/P}^{\mathrm{cons}}$ is idempotent complete. Hence in Proposition 3.4.2, if the diagram of stratifying posets is constant, then the colimit in $\mathbf{Cat}_\infty$ is already idempotent complete.

3.4.4 Corollary. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $U_\bullet : A \to \mathcal{X}$ be a diagram with $\mathrm{colim}_{\alpha \in A} U_\alpha \simeq 1_{\mathcal{X}}$. If for each $\alpha \in A$, the stratified ∞-topos $(\mathcal{X}_{/U_\alpha}, P)$ is exodromic, then the stratified ∞-topos $(\mathcal{X}, P)$ is exodromic and the natural functor*

$$\operatorname*{colim}_{\alpha \in A} \Pi_\infty(\mathcal{X}_{/U_\alpha}, P) \to \Pi_\infty(\mathcal{X}, P)$$

is an equivalence of ∞-categories.

Proof. Immediate from Proposition 3.4.2 and the fact that colimits in an ∞-topos are van Kampen (Recollection 3.4.1-(2)). □

3.5 The Künneth formula. We now prove a *Künneth formula* for the exit-path ∞-category of the product of exodromic stratified ∞-topoi. For this subsection, it may be useful to review Recollection 1.3.3 on products of ∞-topoi and tensor products of presentable ∞-categories. One key input is the Künneth formula in the unstratified setting (Proposition 1.3.19).

We start by noting that the product of stratified ∞-topoi is naturally stratified:

3.5.1 Definition (stratification of a product)**.** Let $s_* \colon \mathcal{X} \to \mathrm{Fun}(P, \mathbf{Spc})$ and $t_* \colon \mathcal{Y} \to \mathrm{Fun}(Q, \mathbf{Spc})$ be stratified ∞-topoi. We write $(\mathcal{X} \otimes \mathcal{Y}, P \times Q)$ for the stratified ∞-topos

$$s_* \otimes t_* \colon \mathcal{X} \otimes \mathcal{Y} \to \mathrm{Fun}(P, \mathbf{Spc}) \otimes \mathrm{Fun}(Q, \mathbf{Spc}) \simeq \mathrm{Fun}(P \times Q, \mathbf{Spc})\,.$$

3.5.2 Observation. In the setting of Definition 3.5.1, assume that $(\mathcal{X}, P)$ and $(\mathcal{Y}, Q)$ are exodromic stratified ∞-topoi. Then:

(1) Since s^* and t^* preserve limits and colimits,

$$s^* \otimes t^* \colon \mathrm{Fun}(P \times Q, \mathbf{Spc}) \to \mathcal{X} \otimes \mathcal{Y}$$

preserves limits and colimits.

(2) Since the inclusions $\mathrm{Cons}_P(\mathcal{X}) \hookrightarrow \mathcal{X}$ and $\mathrm{Cons}_Q(\mathcal{Y}) \hookrightarrow \mathcal{Y}$ are both left and right adjoints, the induced functor

$$\mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_Q(\mathcal{Y}) \to \mathcal{X} \otimes \mathcal{Y}$$

is fully faithful and both a left and right adjoint.

3.5.3 Lemma. *Let $(\mathcal{X}, P)$ and $(\mathcal{Y}, Q)$ be exodromic stratified ∞-topoi. The inclusion*

$$\mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_Q(\mathcal{Y}) \hookrightarrow \mathcal{X} \otimes \mathcal{Y}$$

factors through $\mathrm{Cons}_{P\times Q}(\mathcal{X} \otimes \mathcal{Y})$*.*

Proof. Let $(p, q) \in P \times Q$. Note that by the definition of $\mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_Q(\mathcal{Y})$, the composite

$$\mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_Q(\mathcal{Y}) \hookrightarrow \mathcal{X} \otimes \mathcal{Y} \xrightarrow{i_p^* \otimes i_q^*} \mathcal{X}_p \otimes \mathcal{Y}_q \tag{3.5.4}$$

factors through $\mathrm{LC}(\mathcal{X}_p) \otimes \mathrm{LC}(\mathcal{Y}_q)$. By Proposition 1.3.19, we have

$$\mathrm{LC}(\mathcal{X}_p) \otimes \mathrm{LC}(\mathcal{Y}_q) = \mathrm{LC}(\mathcal{X}_p \otimes \mathcal{Y}_q)$$

as full subcategories of $\mathcal{X}_p \otimes \mathcal{Y}_q$. Hence the functor (3.5.4) factors through $\mathrm{LC}(\mathcal{X}_p \otimes \mathcal{Y}_q)$, as desired. □

3.5.5 Proposition (Künneth formula for exodromic stratified ∞-topoi)**.** *Let $s_* \colon \mathcal{X} \to \mathrm{Fun}(P, \mathbf{Spc})$ and $t_* \colon \mathcal{Y} \to \mathrm{Fun}(Q, \mathbf{Spc})$ be exodromic stratified ∞-topoi. If P and Q are noetherian, then:*

(1) *The natural fully faithful functor*

$$\mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_Q(\mathcal{Y}) \hookrightarrow \mathrm{Cons}_{P\times Q}(\mathcal{X} \otimes \mathcal{Y})$$

is an equivalence.

(2) *The stratified ∞-topos $(\mathcal{X} \otimes \mathcal{Y}, P \times Q)$ is exodromic and the natural functor*

$$\Pi_\infty(\mathcal{X} \otimes \mathcal{Y}, P \times Q) \to \Pi_\infty(\mathcal{X}, P) \times \Pi_\infty(\mathcal{Y}, Q)$$

is an equivalence of ∞-categories.

Proof. We now proceed by noetherian induction. First, let us prove that when $Q = *$, the functor we just constructed

$$\boxtimes \colon \mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{LC}(\mathcal{Y}) \to \mathrm{Cons}_P(\mathcal{X} \otimes \mathcal{Y})$$

is an equivalence. When $P = *$, the conclusion follows from Proposition 1.3.19-(3). Otherwise, notice that Lemma 4.1.10 implies that the question is local on P. We can therefore reduce ourselves to prove that $\boxtimes$ is an equivalence for posets of the form $P_{\geq p}$. In this case, consider the following diagram:

$$\begin{array}{ccccc} \mathrm{LC}(\mathcal{X}_p) \otimes \mathrm{LC}(\mathcal{Y}) & \longleftarrow & \mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{LC}(\mathcal{Y}) & \longrightarrow & \mathrm{Cons}_{P_{>p}}(\mathcal{X}_{>p}) \otimes \mathrm{LC}(\mathcal{Y}) \\ \downarrow \boxtimes & & \downarrow \boxtimes & & \downarrow \boxtimes \\ \mathrm{LC}(\mathcal{X}_p \otimes \mathcal{Y}) & \longleftarrow & \mathrm{Cons}_P(\mathcal{X} \otimes \mathcal{Y}) & \longrightarrow & \mathrm{Cons}_{P_{>p}}(\mathcal{X}_{>p} \otimes \mathcal{Y}) . \end{array}$$

Since $\mathcal{Y}$ is monodromic, $\mathrm{LC}(\mathcal{Y})$ is compactly generated and therefore the top row is a recollement. By Lemma 3.1.9-(4), the bottom line is also a recollement. The inductive hypothesis guarantees that the outer vertical functors are equivalences. Therefore, Lemma 4.1.8-(4) implies that the same goes for the middle one. We now repeat the same argument proceeding by noetherian induction on the length of Q and for arbitrary P. Reasoning as above, we reduce ourselves to consider the following diagram:

$$\begin{array}{ccccc} \mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{LC}(\mathcal{Y}) & \longleftarrow & \mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_Q(\mathcal{Y}) & \longrightarrow & \mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_{Q_{>q}}(\mathcal{Y}_{>q}) \\ \downarrow \boxtimes & & \downarrow \boxtimes & & \downarrow \boxtimes \\ \mathrm{Cons}_{P\times\{q\}}(\mathcal{X} \otimes \mathcal{Y}_q) & \longleftarrow & \mathrm{Cons}_{P\times Q}(\mathcal{X} \otimes \mathcal{Y}) & \longrightarrow & \mathrm{Cons}_{P\times Q_{>q}}(\mathcal{X} \otimes \mathcal{Y}_{>q}) . \end{array}$$

Once again, since $(\mathcal{X}, P)$ is exodromic, $\mathrm{Cons}_P(\mathcal{X})$ is compactly generated and therefore the top row is a recollement. The same goes for the bottom row. Thus, the conclusion follows from the previous step, the inductive hypothesis and Lemma 4.1.8-(4).

For (2), note that by Observation 3.5.2, the pullback functor $s^* \otimes t^*$ preserves limits and colimits. Moreover, by (1), $\mathrm{Cons}_{P\times Q}(\mathcal{X} \otimes \mathcal{Y})$ is atomically generated and closed under limits and colimits in $\mathcal{X} \otimes \mathcal{Y}$. Hence, $(\mathcal{X} \otimes \mathcal{Y}, P \times Q)$ is exodromic. Finally, the equivalence

$$\mathrm{Cons}_P(\mathcal{X}) \otimes \mathrm{Cons}_Q(\mathcal{Y}) \simeq \mathrm{Cons}_{P\times Q}(\mathcal{X} \otimes \mathcal{Y})$$

shows that

$$\Pi_\infty(\mathcal{X} \otimes \mathcal{Y}, P \times Q) \simeq \Pi_\infty(\mathcal{X}, P) \times \Pi_\infty(\mathcal{Y}, Q) .$$ □

3.6 Stability properties of categorical finiteness & compactness. As explained in [32, §7], the compactness of exit-path ∞-categories can be used to prove that moduli stacks of constructible and perverse sheaves are locally geometric. Hence knowing when a stratified ∞-topos has compact exit-path ∞-category is of great utility. To complete this section, we explain why the classes of exodromic stratified ∞-topoi with finite or compact exit-path ∞-category are stable under coarsening. In §5, we use the results of this subsection to extend the representability results from [32, §7] beyond the conical situation.

Recall from [32, Definition 2.2.1] the following:

3.6.1 Definition. Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. We say that $(\mathcal{X}, P)$ is:

(1) *Categorically finite* if $\Pi_\infty(\mathcal{X}, P)$ is a finite object of $\mathbf{Cat}_\infty$. (See Recollection A.3.1.)

(2) *Categorically compact* if $\Pi_\infty(\mathcal{X}, P)$ is a compact object of $\mathbf{Cat}_\infty$.

3.6.2 Lemma. *Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos and $S \subset P$ a locally closed subposet. If $(\mathcal{X}, P)$ is categorically finite (resp., compact), then $(\mathcal{X}_S, S)$ is categorically finite (resp., compact).*

Proof. This is a special case of Proposition A.3.17. □

3.6.3 Lemma. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $U_1, \ldots, U_n \in \mathcal{X}$ be a finite set of objects such that the induced map $U_1 \sqcup \cdots \sqcup U_n \to 1_{\mathcal{X}}$ is an effective epimorphism. Assume that for all $1 \leq i_1 < \cdots < i_k \leq n$, the stratified ∞-topos $(\mathcal{X}_{/U_{i_1} \times \cdots \times U_{i_k}}, P)$ is exodromic and is categorically finite (resp., compact). Then $(\mathcal{X}, P)$ is exodromic and is categorically finite (resp., compact).*

Proof. Immediate from Corollary 3.4.4 and the fact that both finite and compact ∞-categories are closed under finite colimits in $\mathbf{Cat}_{\infty}$. □

3.6.4 Proposition. *Let $(\mathcal{X}, R)$ be an exodromic stratified ∞-topos and let $\phi \colon R \to P$ be a map of posets. If $(\mathcal{X}, R)$ is categorically finite (resp., compact), then $(\mathcal{X}, P)$ is categorically finite (resp., compact).*

Proof. The fact that $(\mathcal{X}, P)$ is exodromic follows from the stability of the class of exodromic stratified ∞-topoi under coarsening (Theorem 3.3.5-(1)). By Theorem 3.3.5-(2), there is an equivalence

$$\Pi_{\infty}(\mathcal{X}, P) \simeq \Pi_{\infty}(\mathcal{X}, R)[W_P^{-1}] .$$

Since $\Pi_{\infty}(\mathcal{X}, R)$ is a finite (resp., compact), the claim now follows from Proposition A.3.16. □

4 Exodromy with coefficients

This section concerns exodromy with coefficients in ∞-categories other than the ∞-category of spaces. In §4.1, we explain when the exodromy equivalence holds for sheaves with coefficients in more general presentable ∞-categories. In particular, exodromy with coefficients in **Spc** implies exodromy with coefficients in any compactly assembled ∞-category; see Corollary 4.1.15. Subsection 4.2 treats exodromy with coefficients in the ∞-category $\mathbf{Pr}^{\mathrm{L}}$ of presentable ∞-categories; these results are needed in forthcoming work of the second- and third-named authors [33].

4.1 Exodromy with coefficients in a presentable ∞-category.

We are also interested in when the exit-path ∞-category corepresents constructible objects with coefficients in a presentable ∞-category $\mathcal{E}$. The following slight generalization of the discussion in [32, §6.1] captures this more general situation.

4.1.1 Observation. Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos and let $\mathcal{E}$ be a presentable ∞-category. Since the ∞-category $\mathrm{Cons}_P(\mathcal{X})$ is presentable and the inclusion

$$\mathrm{Cons}_P(\mathcal{X}) \hookrightarrow \mathcal{X}$$

is both a left and a right adjoint, tensoring with $\mathcal{E}$ gives a fully faithful functor

$$\boxtimes \colon \mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} \hookrightarrow \mathrm{Sh}(\mathcal{X}; \mathcal{E})$$

that is both a left and a right adjoint.

4.1.2 Lemma. *Let $\mathcal{E}$ be a presentable ∞-category, and let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Then the functor*

$$\boxtimes \colon \mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} \hookrightarrow \mathrm{Sh}(\mathcal{X}; \mathcal{E})$$

factors through $\mathrm{Cons}_P(\mathcal{X}; \mathcal{E}) \subset \mathrm{Sh}(\mathcal{X}; \mathcal{E})$.

Proof. The functoriality of the tensor product in $\mathbf{Pr}^{\mathrm{L}}$ implies that for each $p \in P$, there is a commutative square

$$\begin{array}{ccc} \mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} & \hookrightarrow & \mathrm{Sh}(\mathcal{X}) \otimes \mathcal{E} \\ \downarrow{\scriptstyle i_p^* \otimes \mathrm{id}_{\mathcal{E}}} & & \downarrow{\scriptstyle i_p^* \otimes \mathrm{id}_{\mathcal{E}}} \\ \mathrm{LC}(\mathcal{X}_p) \otimes \mathcal{E} & \hookrightarrow & \mathrm{Sh}(\mathcal{X}_p) \otimes \mathcal{E} . \end{array}$$

Since the strata of $(\mathcal{X}, P)$ are monodromic (Corollary 3.1.17-(2)), the natural functor

$$\mathrm{LC}(\mathcal{X}_p) \otimes \mathcal{E} \to \mathrm{LC}(\mathcal{X}_p; \mathcal{E})$$

is an equivalence (Recollection 1.3.8). The claim now is immediate. □

4.1.3. In the setting of Lemma 4.1.2, we have a commutative triangle

$$\begin{array}{ccc} \mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} & \hookrightarrow & \mathrm{Cons}_P(\mathcal{X};\mathcal{E}) \\ & \searrow & \downarrow \\ & & \mathrm{Sh}(\mathcal{X};\mathcal{E}) . \end{array} \tag{4.1.4}$$

4.1.5 Definition. Let $\mathcal{E}$ be a presentable ∞-category and let $(\mathcal{X}, P)$ be a stratified ∞-topos. We say that $(\mathcal{X}, P)$ *is $\mathcal{E}$-exodromic* if the following conditions are satisfied:

(1) The stratified ∞-topos $(\mathcal{X}, P)$ is exodromic.

(2) The functor $\boxtimes \colon \mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} \hookrightarrow \mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ is an equivalence.

We collect some basic properties of $\mathcal{E}$-exodromic stratified ∞-topoi.

4.1.6 Observation. Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Since equivalences of ∞-categories are stable under retracts, the class of presentable ∞-categories $\mathcal{E}$ for which $(\mathcal{X}, P)$ is $\mathcal{E}$-exodromic is also stable under retracts.

4.1.7 Lemma. *Let $\mathcal{E}$ be a presentable ∞-category and let $(\mathcal{X}, P)$ be a $\mathcal{E}$-exodromic stratified ∞-topos. Then the equivalence*

$$\boxtimes \colon \mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} \xrightarrow{\sim} \mathrm{Cons}_P(\mathcal{X};\mathcal{E})$$

induces a canonical equivalence

$$\mathrm{Fun}(\Pi_\infty(\mathcal{X}, P), \mathcal{E}) \simeq \mathrm{Cons}_P(\mathcal{X};\mathcal{E}) .$$

Proof. Indeed, we have the following canonical equivalences:

$$\begin{aligned} \mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} &\simeq \mathrm{Fun}(\Pi_\infty(\mathcal{X}, P), \mathbf{Spc}) \otimes \mathcal{E} \\ &\simeq \mathrm{Fun}(\Pi_\infty(\mathcal{X}, P), \mathcal{E}) . && \text{[HA, Proposition 4.8.1.17]} \end{aligned}$$

The conclusion follows. □

We now prove an analogue of Corollary 3.1.17. We first need the following lemma:

4.1.8 Lemma. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be ∞-categories with finite limits and an inital object. Let*

$$\begin{array}{ccccc} \mathcal{Z}_1 & \xleftarrow{i_1^*} & \mathcal{X}_1 & \xrightarrow{j_1^*} & \mathcal{U}_1 \\ \downarrow{\scriptstyle F_{\mathcal{Z}}} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F_{\mathcal{U}}} \\ \mathcal{Z}_2 & \xleftarrow[i_2^*]{} & \mathcal{X}_2 & \xrightarrow[j_2^*]{} & \mathcal{U}_2 \end{array}$$

be a commutative diagram where each of the horizontal rows exhibits $\mathcal{X}_i$ as the recollement of $\mathcal{Z}_i$ and $\mathcal{U}_i$.

(1) *If F is essentially surjective, then $F_{\mathcal{Z}}$ and $F_{\mathcal{U}}$ are essentially surjective.*

(2) *If $F_{\mathcal{Z}}$ preserves the initial object, then the right-hand square is horizontally left adjointable. In this case, if F is fully faithful (resp., an equivalence), then the same is true of $F_{\mathcal{U}}$.*

(3) *If $F_{\mathcal{U}}$ preserves the terminal object, then the left-hand square is horizontally right adjointable. In this case, if F is fully faithful (resp., an equivalence), then the same is true of $F_{\mathcal{Z}}$.*

(4) *Assume that F is left exact. If $F_{\mathcal{Z}}$ and $F_{\mathcal{U}}$ are equivalences, then F is also an equivalence*

Proof. For (1), we prove that $F_{\mathcal{U}}$ is essentially surjective; the proof of the essential surjectivity of $F_{\mathcal{Z}}$ is identical. Since F is essentially surjective, given $u \in \mathcal{U}_2$ there exists $x \in \mathcal{X}_1$ and an equivalence $j_{2,*}(u) \simeq F(x)$. Hence the full faithfulness of $j_{2,*}$ and the commutativity of the right-hand square show that

$$u \simeq j_2^* j_{2,*}(u) \simeq j_2^*(F(x)) \simeq F_{\mathcal{U}}(j_1^*(x)) .$$

We now prove (2); item (3) follows by a dual argument. Consider the exchange transformation

$$\alpha : j_{2,!}F_{\mathcal{U}} \to F j_{1,!} .$$

Since the bottom line is a recollement, to prove that α is an equivalence it suffices to check that $j_2^*(\alpha)$ and $i_2^*(\alpha)$ are equivalences. We first deal with the former. Since the right-hand square commutes, we have $j_2^* F j_{1,!} \simeq F_{\mathcal{U}} j_1^* j_{1,!}$, so the conclusion follows from the full faithfulness of both $j_{1,!}$ and $j_{2,!}$. As for $i_2^*(\alpha)$, recall that the theory of recollements shows that both $i_2^* j_{2,!}$ and $i_1^* j_{1,!}$ are constant with value the initial object. Also, since the left-hand square commutes, we have $i_2^* F j_{1,!} \simeq F_{\mathcal{Z}} i_1^* j_{1,!}$. Since $F_{\mathcal{Z}}$ preserves the initial object, it follows that both the source and target of $i_2^*(\alpha)$ are constant with value the initial object; hence $i_2^*(\alpha)$ is an equivalence.

From the horizontal left adjointability of the right-hand square and the full faithfulness of $j_{1,!}$ and $j_{2,!}$, it immediately follows that if F is fully faithful, then $F_{\mathcal{U}}$ is also fully faithful. Finally, if F is an equivalence, then we have just seen that $F_{\mathcal{U}}$ is fully faithful and (1) shows that $F_{\mathcal{U}}$ is also essentially surjective.

We are left to prove (4). Since $F_{\mathcal{Z}}$ and $F_{\mathcal{U}}$ are equivalences, they preserve both the initial and the terminal object. Then (4) follows from the above adjointability statements and [HA, Proposition A.8.14]. □

4.1.9 Proposition. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $\mathcal{E}$ be a presentable ∞-category. Let $S \subset P$ be a locally closed subposet. If $(\mathcal{X}, P)$ is $\mathcal{E}$-exodromic and $\mathcal{E}$ is compatible with recollements (Definition 3.1.4), then $(\mathcal{X}_S, S)$ is also $\mathcal{E}$-exodromic.*

Proof. It is enough to prove that if $U \subset P$ is an open subposet with closed complement Z, then both $(\mathcal{X}_U, U)$ and $(\mathcal{X}_Z, Z)$ are $\mathcal{E}$-exodromic. First of all, we already know from Corollary 3.1.17 that these stratified ∞-topoi are exodromic. Consider now the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Cons}_U(\mathcal{X}_U) \otimes \mathcal{E} & \xleftarrow{i_U^* \otimes \mathcal{E}} & \mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} & \xrightarrow{i_Z^* \otimes \mathcal{E}} & \mathrm{Cons}_Z(\mathcal{X}_Z) \otimes \mathcal{E} \\
\downarrow{\boxtimes_U} & & \downarrow{\boxtimes} & & \downarrow{\boxtimes_Z} \\
\mathrm{Cons}_U(\mathcal{X}_U; \mathcal{E}) & \xleftarrow[i_U^*]{} & \mathrm{Cons}_P(\mathcal{X}; \mathcal{E}) & \xrightarrow[i_Z^*]{} & \mathrm{Cons}_Z(\mathcal{X}_Z; \mathcal{E}) .
\end{array}$$

Since $(\mathcal{X}, P)$ is $\mathcal{E}$-exodromic, the middle vertical functor is an equivalence. Morever, because because $(\mathcal{X}, P)$ is exodromic, the functor

$$\mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} \to \mathrm{Sh}(\mathcal{X}) \otimes \mathcal{E} \simeq \mathrm{Sh}(\mathcal{X}; \mathcal{E})$$

preserves both limits and colimits. Combining Corollary 2.2.18 and Lemma 3.1.9-(4), we see that the bottom row exhibits $\mathrm{Cons}_P(\mathcal{X}; \mathcal{E})$ as a recollement of $\mathrm{Cons}_U(\mathcal{X}_U; \mathcal{E})$ and $\mathrm{Cons}_Z(\mathcal{X}_Z; \mathcal{E})$. On the other hand, since $\mathcal{E}$ is compatible with recollements, the top row is a recollement as well. Clearly, $\boxtimes_U$ preserves the initial object. On the other hand, since $\boxtimes_Z$ is compatible with the inclusion into

$$\mathrm{Sh}(\mathcal{X}_Z) \otimes \mathcal{E} \simeq \mathrm{Sh}(\mathcal{X}_Z; \mathcal{E})$$

and since the terminal object in $\mathrm{Sh}(\mathcal{X}_Z; \mathcal{E})$ is Z-constructible thanks to Corollary 2.2.18, we conclude that $\boxtimes_Z$ preserves the terminal object as well. Thus, Lemma 4.1.8 implies that $\boxtimes_U$ and $\boxtimes_Z$ are equivalences. □

To explain why $\mathcal{E}$-exodromicity can be checked locally, we need descent for the tensor decomposition

$$\mathrm{Cons}_P(\mathcal{X}) \otimes \mathcal{E} \simeq \mathrm{Cons}_P(\mathcal{X}; \mathcal{E}) .$$

For this, we make use of the following lemma.

4.1.10 Lemma. *Let A be a small ∞-category and let $\mathcal{C}_\bullet : A \to \mathbf{CAT}_\infty$ be a diagram of ∞-categories. Assume that for each $\alpha \in A$, the ∞-category $\mathcal{C}_\alpha$ is presentable and that for each morphism $\alpha \to \beta$ in A, the transition functor $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ is both a left and a right adjoint. Then:*

(1) *The limits of $\mathcal{C}_\bullet$ when computed in $\mathbf{Pr}^{\mathrm{R}}$, $\mathbf{Pr}^{\mathrm{L}}$, or $\mathbf{CAT}_\infty$ all agree.*

(2) *For any presentable ∞-category $\mathcal{E}$, the natural morphism*

$$\lim_{\alpha\in A} \mathcal{E}\otimes\mathcal{C}_\alpha \to \mathcal{E}\otimes \lim_{\alpha\in A}\mathcal{C}_\alpha$$

in $\mathbf{Pr}^{\mathrm{L}}$ is an equivalence. (Here, both limits are computed in $\mathbf{Pr}^{\mathrm{L}}$.)

Proof. Item (1) follows from the fact that both of the forgetful functors $\mathbf{Pr}^{\mathrm{L}} \to \mathbf{CAT}_\infty$ and $\mathbf{Pr}^{\mathrm{R}} \to \mathbf{CAT}_\infty$ preserve limits [HTT, Proposition 5.5.3.13 & Theorem 5.5.3.18]. Item (2) follows from (1), the equivalence $\mathbf{Pr}^{\mathrm{R}} \simeq (\mathbf{Pr}^{\mathrm{L}})^{\mathrm{op}}$, and the fact that the functor

$$\mathcal{E}\otimes(-)\colon \mathbf{Pr}^{\mathrm{R}} \to \mathbf{Pr}^{\mathrm{R}}$$

preserves limits [HA, Remark 4.8.1.24]. □

4.1.11 Proposition. *Let $\mathcal{E}$ be a presentable ∞-category, let A be an ∞-category, and let*

$$(\mathcal{X}_\bullet, P_\bullet)\colon A \to \mathbf{StrTop}_\infty$$

be a diagram of stratified ∞-topoi. Let $(\mathcal{X}_\infty, P_\infty)$ be a cone under $(\mathcal{X}_\bullet, P_\bullet)$. Assume that the following conditions are satisfied:

(1) *For each $\alpha \in A$, the stratified ∞-topos $(\mathcal{X}_\alpha, P_\alpha)$ is $\mathcal{E}$-exodromic.*

(2) *The natural pullback functors*

$$\mathcal{X}_\infty \to \lim_{\alpha\in A^{\mathrm{op}}} \mathcal{X}_\alpha \qquad \text{and} \qquad \mathrm{Cons}_{P_\infty}(\mathcal{X}_\infty) \to \lim_{\alpha\in A^{\mathrm{op}}} \mathrm{Cons}_{P_\alpha}(\mathcal{X}_\alpha)$$

as well as

$$\mathrm{Cons}_{P_\infty}(\mathcal{X}_\infty;\mathcal{E}) \to \lim_{\alpha\in A^{\mathrm{op}}} \mathrm{Cons}_{P_\alpha}(\mathcal{X}_\alpha;\mathcal{E})$$

are equivalences.

Then the stratified ∞-topos $(\mathcal{X}_\infty, P_\infty)$ is $\mathcal{E}$-exodromic.

Proof. Proposition 3.4.2 implies that $(\mathcal{X}, P)$ is exodromic. Consider the following commutative square

$$\begin{array}{ccc} \mathrm{Cons}_{P_\infty}(X_\infty)\otimes\mathcal{E} & \longrightarrow & \lim_{\alpha\in A^{\mathrm{op}}} \mathrm{Cons}_{P_\alpha}(X_\alpha)\otimes\mathcal{E} \\ \downarrow & & \downarrow \\ \mathrm{Cons}_{P_\infty}(X_\infty;\mathcal{E}) & \longrightarrow & \lim_{\alpha\in A^{\mathrm{op}}} \mathrm{Cons}_{P_\alpha}(X_\alpha;\mathcal{E}) \end{array}$$

Since each $(\mathcal{X}_\alpha, P_\alpha)$ is $\mathcal{E}$-exodromic, the left vertical functor is an equivalence. Also, by assumption, the bottom horizontal functor is an equivalence. Thus it suffices to show that the top horizontal functor is an equivalence. By Lemma 4.1.10, it suffices to show that for every morphism $\alpha \to \beta$ in A^{op}, the pullback functor

$$\mathrm{Cons}_{P_\alpha}(\mathcal{X}_\alpha) \to \mathrm{Cons}_{P_\beta}(\mathcal{X}_\beta)$$

is both a left and a right adjoint. By assumption $(\mathcal{X}_\beta, P_\beta)$ and $(\mathcal{X}_\alpha, P_\alpha)$ are exodromic, so this is an immediate consequence of Theorem 3.2.3. □

4.1.12 Corollary. *Let $(\mathcal{X}, P)$ be a stratified ∞-topos and let $\mathcal{E}$ be a presentable ∞-category. Let $U_\bullet\colon A \to \mathcal{X}$ be a diagram with* $\mathrm{colim}_{\alpha\in A} U_\alpha \simeq 1_{\mathcal{X}}$. *If for each $\alpha \in A$, the stratified ∞-topos $(\mathcal{X}_{/U_\alpha}, P)$ is $\mathcal{E}$-exodromic, then the stratified ∞-topos $(\mathcal{X}, P)$ is also $\mathcal{E}$-exodromic.*

Proof. By Recollection 3.4.1 and Proposition 4.1.11, it suffices to show that the natural pullback functor

$$\mathrm{Cons}_P(\mathcal{X};\mathcal{E}) \to \lim_{\alpha\in A^{\mathrm{op}}} \mathrm{Cons}_P(\mathcal{X}_{/U_\alpha};\mathcal{E}) \tag{4.1.13}$$

is an equivalence. Notice that for every map $\alpha \to \beta$ in A^{op}, the induced pullback functor

$$\mathcal{X}_{/U_\alpha} \to \mathcal{X}_{/U_\beta}$$

is both a left and a right adjoint. Therefore, Lemma 4.1.10 implies that the pullback functor

$$\mathrm{Sh}(\mathcal{X};\mathcal{E}) \to \lim_{\alpha\in A^{\mathrm{op}}} \mathrm{Sh}(\mathcal{X}_{/U_\alpha};\mathcal{E})$$

is an equivalence. This immediately implies that (4.1.13) is fully faithful. To conclude, it is enough to observe that $F \in \mathrm{Sh}(\mathcal{X};\mathcal{E})$ is P-constructible if and only if for every $\alpha \in A$, the restriction of F to $\mathcal{X}_{/U_\alpha}$ is P-constructible. □

4.1.14 Recollection (compactly assembled ∞-categories)**.** A presentable ∞-category $\mathcal{E}$ is *compactly assembled* if $\mathcal{E}$ is a retract in $\mathbf{Pr}^{\mathrm{L}}$ of a compactly generated ∞-category [SAG, Definition 21.1.2.1 & Theorem 21.1.2.18]. If $\mathcal{E}$ is a presentable stable ∞-category, then $\mathcal{E}$ is compactly assembled if and only if $\mathcal{E}$ is dualizable in the symmetric monoidal ∞-category of presentable stable ∞-categories and left adjoints equipped with the Lurie tensor product [SAG, Proposition D.7.3.1].

4.1.15 Corollary. *Let $(\mathcal{X}, P)$ be a exodromic stratified ∞-topos and let $\mathcal{E}$ be a presentable ∞-category. Then:*

(1) *If $\mathcal{E}$ is compactly assembled, then $(\mathcal{X}, P)$ is $\mathcal{E}$-exodromic.*

(2) *If $\mathcal{E}$ is stable and P is noetherian, then $(\mathcal{X}, P)$ is $\mathcal{E}$-exodromic.*

Proof. For (1), note that by Observation 4.1.1, it suffices to prove the claim in the case that $\mathcal{E}$ is compactly generated. In this case, the proof of [30, Theorem B.9] works *verbatim*.

We now prove (2). For $p \in P$, we write $\mathcal{X}_{\geq p}$ for $\mathcal{X}_{P_{\geq p}}$. Since the sets $\{P_{\geq p}\}_{p\in P}$ form an open cover of P, by Corollary 4.1.12 it suffices to show that for every $p \in P$ the stratified ∞-topos $(\mathcal{X}_{\geq p}, P_{\geq p})$ is $\mathcal{E}$-exodromic. We prove this statement by noetherian induction. When P is a single element, the conclusion follows from Recollection 1.3.8. We are then reduced to showing that if for every $q > p$ the stratified ∞-topos $(\mathcal{X}_{\geq q}, P_{\geq q})$ is $\mathcal{E}$-exodromic, then $(X_{\geq p}, P_{\geq p})$ is also $\mathcal{E}$-exodromic. Note that

$$P_{\geq p} \smallsetminus \{p\} = P_{>p} = \bigcup_{q>p} P_{\geq q}\ .$$

Thus, Corollary 4.1.12 implies that $(\mathcal{X}_{>p}, P_{>p})$ is $\mathcal{E}$-exodromic.

Now consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{LC}(\mathcal{X}_p)\otimes\mathcal{E} & \longleftarrow & \mathrm{Cons}_P(\mathcal{X})\otimes\mathcal{E} & \longrightarrow & \mathrm{Cons}_{P_{>p}}(\mathcal{X}_{>p})\otimes\mathcal{E} \\
\downarrow{\scriptstyle\boxtimes} & & \downarrow{\scriptstyle\boxtimes} & & \downarrow{\scriptstyle\boxtimes} \\
\mathrm{LC}(\mathcal{X}_p;\mathcal{E}) & \longleftarrow & \mathrm{Cons}_P(\mathcal{X};\mathcal{E}) & \longrightarrow & \mathrm{Cons}_{P_{>p}}(\mathcal{X}_{>p};\mathcal{E})\ .
\end{array}$$

The inductive hypothesis implies that the exterior vertical functors are equivalences. Since $\mathcal{E}$ is stable, $\mathrm{Cons}_P(\mathcal{X};\mathcal{E})$ is closed under finite limits in $\mathrm{Sh}(\mathcal{X};\mathcal{E})$. Thus, Corollary 3.1.21 implies that the assumptions of Lemma 3.1.9-(4) are satisfied. It follows that the bottom line is a recollement. Since $\mathcal{E}$ is stable, it is compatible with recollements; therefore, the top line is also a recollement. Thus, Lemma 4.1.8-(4) implies that the middle functor is an equivalence as well. □

4.2 Exodromy with coefficients in $\mathbf{Pr}^{\mathrm{L}}$. Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Recall that we write $\mathbf{CAT}_\infty$ for the (very large) ∞-category of large ∞-categories. Working in a sufficiently large Grothendieck universe, $\mathbf{CAT}_\infty$ is compactly generated. Therefore, combining Lemma 4.1.7 with Corollary 4.1.15, we obtain an equivalence

$$\mathrm{Cons}_P(X;\mathbf{CAT}_\infty) \simeq \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{CAT}_\infty)\ . \tag{4.2.1}$$

In many situations it is convenient to replace $\mathbf{CAT}_\infty$ by $\mathbf{Pr}^{\mathrm{L}}$; however, since $\mathbf{Pr}^{\mathrm{L}}$ is not itself presentable, one needs some extra care.

4.2.2 Definition. Let $(\mathcal{X}, P)$ be a stratified ∞-topos. The ∞-category of $\mathbf{Pr}^{\mathrm{L}}$-valued sheaves on $\mathcal{X}$ is

$$\mathrm{Sh}(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) \coloneqq \mathrm{Fun}^{\lim}(\mathcal{X}^{\mathrm{op}},\mathbf{Pr}^{\mathrm{L}})\ .$$

4.2.3 Observation. Recall from [HTT, Proposition 5.5.3.13] that the forgetful functor $\mathbf{Pr}^{\mathrm{L}} \to \mathbf{CAT}_\infty$ preserves limits. Since $\mathrm{Sh}(\mathcal{X};\mathbf{CAT}_\infty) \coloneqq \mathcal{X} \otimes \mathbf{CAT}_\infty$, [HA, Proposition 4.8.1.17] supplies a canonical functor

$$\mathrm{Sh}(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) \to \mathrm{Sh}(\mathcal{X};\mathbf{CAT}_\infty)\,.$$

4.2.4 Definition. Let $(\mathcal{X}, P)$ be a stratified ∞-topos. The ∞-category of $\mathbf{Pr}^{\mathrm{L}}$*-valued* P*-constructible sheaves on* $\mathcal{X}$ is the fiber product

$$\mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) \coloneqq \mathrm{Sh}(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) \underset{\mathrm{Sh}(\mathcal{X};\mathbf{CAT}_\infty)}{\times} \mathrm{Cons}_P(\mathcal{X};\mathbf{CAT}_\infty)\,.$$

Although the above definition might seem ad hoc (because the restriction to strata are computed in $\mathbf{CAT}_\infty$ rather than in $\mathbf{Pr}^{\mathrm{L}}$), it is justified by the following result:

4.2.5 Proposition. *Let $(\mathcal{X}, P)$ be an exodromic stratified ∞-topos. Then the equivalence (4.2.1) induces an adjoint equivalence*

$$\Phi\colon\ \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) \leftrightarrows \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L}}) :\ \Psi\,.$$

Proof. Under the identification

$$\mathrm{Cons}_P(\mathcal{X};\mathbf{CAT}_\infty) \simeq \mathrm{Cons}_P(\mathcal{X}) \otimes \mathbf{CAT}_\infty \simeq \mathrm{Fun}^{\lim}(\mathrm{Cons}_P(\mathcal{X})^{\mathrm{op}},\mathbf{CAT}_\infty)\,,$$

the equivalence (4.2.1) is realized by the functor

$$\Phi\colon\ \mathrm{Fun}^{\lim}(\mathrm{Cons}_P(\mathcal{X})^{\mathrm{op}},\mathbf{CAT}_\infty) \to \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{CAT}_\infty)$$

given by restriction along the inclusion $\Pi_\infty(\mathcal{X},P) \hookrightarrow \mathrm{Cons}_P(\mathcal{X})^{\mathrm{op}}$. The inverse of Φ is the functor

$$\Psi\colon\ \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{CAT}_\infty) \to \mathrm{Fun}^{\lim}(\mathrm{Cons}_P(\mathcal{X})^{\mathrm{op}},\mathbf{CAT}_\infty)$$

given by right Kan extension along the same inclusion. Consider the composite

$$\mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) \longrightarrow \mathrm{Cons}_P(\mathcal{X};\mathbf{CAT}_\infty) \xrightarrow{\ \Phi\ } \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{CAT}_\infty)\,.$$

Unraveling the definitions, we see that this functor takes $F \in \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}})$ seen as a limit-preserving functor

$$F\colon\ \mathrm{Cons}_P(\mathcal{X})^{\mathrm{op}} \to \mathbf{Pr}^{\mathrm{L}}$$

to the restriction of F to $\Pi_\infty(\mathcal{X},P)$. In particular, this composite factors through $\mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L}})$. Committing a slight abuse of notation, we still denote the resulting functor as

$$\Phi\colon\ \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) \to \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L}})\,.$$

Similarly, since the forgetful functor $\mathbf{Pr}^{\mathrm{L}} \to \mathbf{CAT}_\infty$ preserves limits by [HTT, Proposition 5.5.3.13] we see that Ψ induces a well defined functor

$$\Psi\colon\ \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L}}) \to \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}})\,.$$

Since the pair (Φ,Ψ) is an adjoint equivalence and the forgetful functor $\mathbf{Pr}^{\mathrm{L}} \to \mathbf{CAT}_\infty$ is faithful and full on equivalences, we deduce that unit and counits at the level of $\mathbf{CAT}_\infty$ induce a unit and a counit transformation at the level of $\mathbf{Pr}^{\mathrm{L}}$, and therefore that they form an adjoint equivalence. ☐

4.2.6 Corollary. *Let $f_*\colon (\mathcal{X},P) \to (\mathcal{Y},Q)$ be a morphism of exodromic stratified ∞-topoi. Then the functor*

$$f^*\colon\ \mathrm{Cons}_Q(\mathcal{Y};\mathbf{CAT}_\infty) \to \mathrm{Cons}_P(\mathcal{X};\mathbf{CAT}_\infty)$$

induces a well defined functor

$$f^*\colon\ \mathrm{Cons}_Q(\mathcal{Y};\mathbf{Pr}^{\mathrm{L}}) \to \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}})$$

making the square

$$\begin{array}{ccc} \mathrm{Cons}_Q(\mathcal{Y};\mathbf{Pr}^{\mathrm{L}}) & \longrightarrow & \mathrm{Fun}(\Pi_\infty(\mathcal{Y},Q),\mathbf{Pr}^{\mathrm{L}}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle -\circ\Pi_\infty(f)} \\ \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) & \longrightarrow & \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L}}) \end{array}$$

commutative.

Proof. Recall from Theorem 3.2.3 that f_* is exodromic. Since $\mathbf{CAT}_\infty$ is compactly generated, it follows from Corollary 4.1.15 that the diagram

$$\begin{array}{ccc} \mathrm{Cons}_Q(\mathcal{Y};\mathbf{CAT}_\infty) & \longrightarrow & \mathrm{Fun}(\Pi_\infty(\mathcal{Y},Q),\mathbf{CAT}_\infty) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle -\circ\Pi_\infty(f)} \\ \mathrm{Cons}_P(\mathcal{X};\mathbf{CAT}_\infty) & \longrightarrow & \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{CAT}_\infty) \end{array}$$

commutes. Since the functor $-\circ\Pi_\infty(f)$ clearly lifts to a functor

$$-\circ\Pi_\infty(f)\colon\ \mathrm{Fun}(\Pi_\infty(\mathcal{Y},Q),\mathbf{Pr}^{\mathrm{L}}) \to \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L}})\ ,$$

it follows from Proposition 4.2.5 that the same is true of f^*. □

4.2.7 Warning. The use of constructible sheaves in Corollary 4.2.6 is fundamental. For instance, the functor

$$f^*\colon\ \mathrm{Sh}(\mathcal{Y};\mathbf{CAT}_\infty) \to \mathrm{Sh}(\mathcal{X};\mathbf{CAT}_\infty)$$

generally does not carry $\mathrm{Sh}(\mathcal{Y};\mathbf{Pr}^{\mathrm{L}})$ to $\mathrm{Sh}(\mathcal{X};\mathbf{Pr}^{\mathrm{L}})$.

4.2.8 Notation. Let $\mathbf{Pr}^{\mathrm{L},\omega} \subset \mathbf{Pr}^{\mathrm{L}}$ for the non-full subcategory with objects compactly generated presentable ∞-categories and morphisms left adjoints that preserve compact objects.

4.2.9. Recall from [7, Proposition 2.8.4] that $\mathbf{Pr}^{\mathrm{L},\omega}$ is compactly generated. In particular for an exodromic stratified ∞-topos $(\mathcal{X},P)$, Lemma 4.1.7 with Corollary 4.1.15 provide an adjoint equivalence

$$\Phi^{(\omega)}\colon\ \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L},\omega}) \leftrightarrows \mathrm{Fun}(\Pi_\infty(X,P),\mathbf{Pr}^{\mathrm{L},\omega})\colon\ \Psi^{(\omega)}\ .$$

The natural functor $\mathbf{Pr}^{\mathrm{L},\omega} \to \mathbf{Pr}^{\mathrm{L}}$ induces by composition a map

$$j\colon\ \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L},\omega}) \to \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L}})\ .$$

However, since the functor $\mathbf{Pr}^{\mathrm{L},\omega} \to \mathbf{Pr}^{\mathrm{L}}$ does *not* preserve limits, we do not get an induced functor

$$\mathrm{Sh}(\mathcal{X};\mathbf{Pr}^{\mathrm{L},\omega}) \to \mathrm{Sh}(\mathcal{X};\mathbf{Pr}^{\mathrm{L}})\ .$$

On the other hand, we have:

4.2.10 Corollary. *There exists a canonical functor*

$$\mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L},\omega}) \to \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}})$$

which makes the square

$$\begin{array}{ccc} \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L},\omega}) & \xrightarrow{\Phi^{(\omega)}_{X,P}} & \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L},\omega}) \\ \downarrow & & \downarrow{\scriptstyle j} \\ \mathrm{Cons}_P(\mathcal{X};\mathbf{Pr}^{\mathrm{L}}) & \xrightarrow[\Phi_{X,P}]{} & \mathrm{Fun}(\Pi_\infty(\mathcal{X},P),\mathbf{Pr}^{\mathrm{L}}) \end{array}$$

commute.

Proof. Thanks to Proposition 4.2.5, it is enough to define the left vertical map as $\Psi_{X,P}\circ j\circ\Phi^{(\omega)}_{X,P}$. □

5 Applications & examples

In this section, we apply the stability properties of § 3 to stratified ∞-topoi arising from topology. In § 5.1, we introduce the topological context for our results and state the stability theorem in this context (Theorem 5.1.7). Importantly, as a consequence of Theorem 3.0.1 and the exodromy theorem for conically stratified spaces [32], we deduce that for any stratified space (X,P) that *locally* admits a conical refinement, the stratified ∞-topos $(\mathrm{Sh}^{\mathrm{hyp}}(X),P)$ is exodromic (see Proposition 5.2.9). Many examples fall into this framework; see § 5.3. Of particular interest are stratified spaces coming from subanalytic geometry and real algebraic geometry. Under mild assumptions, we prove that in these geometric settings, the exit-path ∞-categories are finite (Theorems 5.3.9 and 5.3.13). In § 5.4, we use exodromy combined with these finiteness results to prove representability results for moduli stacks of constructible and perverse sheaves (see Theorems 5.4.9 and 5.4.16 and Corollary 5.4.17). This generalizes previous work of Porta–Teyssier in the conical situation [32, §7]. For use in a future paper, in § 5.5, given an exodromic stratified ∞-topos $(\mathcal{X},R)$ and map of posets $\phi\colon R\to P$, we provide a recognition criterion for when R-constructible objects are P-constructible. In § 5.6, we conclude by posing some questions about the relationship between our work and Lurie's simplicial model for exit-path ∞-categories in the setting of conically refineable stratifications.

5.1 Consequences for stratified topological spaces. To fix a topological context to apply Theorem 3.0.1, we make the following definition.

5.1.1 Definition. Let $\mathcal{E}$ be a presentable ∞-category. We say that a stratified topological space $s\colon X\to P$ is *$\mathcal{E}$-exodromic* if the stratified ∞-topos

$$s_*^{\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(X)\to \mathrm{Fun}(P,\mathbf{Spc})$$

is $\mathcal{E}$-exodromic. In this case, we write

$$\Pi_\infty(X,P) \coloneqq \Pi_\infty(\mathrm{Sh}^{\mathrm{hyp}}(X),P)\,.$$

We also have the topological version of Definition 3.6.1:

5.1.2 Definition. Let (X,P) be an exodromic stratified space. We say that (X,P) is:

(1) *Categorically finite* if $\Pi_\infty(X,P)$ is a finite object of $\mathbf{Cat}_\infty$. (See Recollection A.3.1.)

(2) *Categorically compact* if $\Pi_\infty(X,P)$ is a compact object of $\mathbf{Cat}_\infty$.

The following class of presentable ∞-categories is well-behaved from the perspective of exodromy in topology:

5.1.3 Definition. Let P be a poset. We say that a presentable ∞-category $\mathcal{E}$ is *P-admissible* if for every conically P-stratified space (X,P) the hyperrestriction functors

$$\left\{i_p^{*,\mathrm{hyp}}\colon\ \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})\to \mathrm{Sh}^{\mathrm{hyp}}(X_p;\mathcal{E})\right\}_{p\in P}$$

are jointly conservative. We say that a presentable ∞-category $\mathcal{E}$ is *admissible* if for every poset P, the ∞-category $\mathcal{E}$ is P-admissible.

5.1.4 Example [21, Lemma 5.21; 19, Lemma 2.12]**.** Let $\mathcal{E}$ be a presentable ∞-category.

(1) If $\mathcal{E}$ is compactly assembled, then $\mathcal{E}$ is admissible.

(2) If $\mathcal{E}$ is stable or an ∞-topos, then for every noetherian poset P, the ∞-category $\mathcal{E}$ is P-admissible.

5.1.5 Example [32, Theorem 5.17 & Remark 5.18]**.** Let (X,P) be a conically stratified space with locally weakly contractible strata and let $\mathcal{E}$ be a P-admissible ∞-category. Then (X,P) is $\mathcal{E}$-exodromic.

When the strata of (X,P) are locally weakly contractible, we get a particularly nice description of the objects of the exit-path ∞-category:

5.1.6 Observation (the objects of $\Pi_\infty(X,P)$). Let (X,P) be an exodromic stratified space with locally weakly contractible strata. Combining Example 1.3.9-(1) with Observation 3.1.18, we see that there is a natural identification

$$\Pi_\infty(X,P)^{\simeq} \simeq \coprod_{p \in P} \Pi_\infty(X_p)$$

between the maximal sub-∞-groupoid of $\Pi_\infty(\mathcal{X},P)$ and the coproduct of the underlying homotopy types of the strata of (X,P).

Hence each point $x \in X$ gives rise to an object $[x] \in \mathrm{Cons}_P^{\mathrm{hyp}}(X)$, and every object of $\Pi_\infty(X,P)$ is of this form. Moreover, it follows from the functoriality of the monodromy equivalence that the functor

$$\mathrm{Cons}_P^{\mathrm{hyp}}(X) \to \mathbf{Spc}$$

corepresented by $[x]$ is equivalent to the stalk functor $x^* \colon \mathrm{Cons}_P^{\mathrm{hyp}}(X) \to \mathbf{Spc}$. As a consequence, given a P-hyperconstructible hypersheaf F, every morphism $[x] \to [y]$ gives rise to a *specialization map* $x^*F \to y^*F$ on stalks.

The stability theorem for exodromic stratified ∞-topoi has the following topological consequence:

5.1.7 Theorem (stability properties of exodromic stratified spaces)**.**

(1) Stability under pulling back to locally closed subposets: *If (X,P) is an exodromic stratified space, then for each locally closed subposet $S \subset P$, the stratified space (X_S, S) is exodromic and the induced functor*

$$\Pi_\infty(X_S,S) \to \Pi_\infty(X,P)_S$$

is an equivalence. In particular, the induced functor $\Pi_\infty(X,P) \to P$ is conservative.

(2) Stability under coarsening and localization formula: *Let (X,R) be an exodromic stratified space and let $\phi \colon R \to P$ be a map of posets. Then (X,P) is exodromic and there is a natural equivalence*

$$\Pi_\infty(X,R)[W_P^{-1}] \xrightarrow{\sim} \Pi_\infty(X,P)\,.$$

(3) Functoriality: *The exodromy equivalence is functorial in all stratified maps between exodromic stratified spaces.*

(4) van Kampen: *Let (X,P) be a stratified space and let*

$$U_\bullet \colon \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathbf{Top}_{/X}$$

be an semi-simplicial étale hypercovering of X. If for each $n \geq 0$, the stratified space (U_n,P) is exodromic, then the stratified space (X,P) is exodromic. Moreover, the natural functor

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}} \Pi_\infty(U_n,P) \to \Pi_\infty(X,P)$$

is an equivalence of ∞-categories.

(5) Stability of finiteness/compactness: *Let (X,P) be a stratified space.*
   (a) *If (X,P) is exodromic and categorically finite (resp., compact), then for any locally closed subposet $S \subset P$, the stratified space (X_S,S) is exodromic and categorically finite (resp., compact).*
   (b) *Let $U_1, \dots, U_n$ be a* finite *open cover of X. Assume that each intersection $(U_{i_1} \cap \cdots \cap U_{i_k}, P)$ admits an refinement which is exodromic and categorically finite (resp., compact). Then (X,P) is exodromic and categorically finite (resp., compact).*

Proof. Item (1) is a special case of Corollary 3.1.17, item (2) is a special case of Theorem 3.3.5, item (3) is a special case of Theorem 3.2.3, item (4) is a special case of Corollary 3.4.4, and item (5) follows from Lemmas 3.6.2 and 3.6.3 and Proposition 3.6.4. □

Provided X is also locally weakly contractible, the classifying space of the exit-path ∞-category of (X,P) coincides with the underlying homotopy type of X:

5.1.8 Corollary. *Let (X, P) be an exodromic stratified space. If X locally weakly contractible, then the space $\mathrm{Env}(\Pi_\infty(X, P))$ is naturally equivalent to the underlying homotopy type of X.*

Proof. Note that Theorem 5.1.7-(2) shows that there is a natural equivalence

$$\mathrm{Env}(\Pi_\infty(X, P)) \xrightarrow{\sim} \Pi_\infty(X, *)$$

between the space obtained by inverting all morphisms in $\Pi_\infty(X, P)$ and the shape of the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(X)$. To conclude, recall that since X is locally weakly contractible, by Example 1.3.9-(1), the shape of $\mathrm{Sh}^{\mathrm{hyp}}(X)$ is naturally equivalent to the underlying homotopy type of X. □

We conclude this subsection with some remarks about the stability theorem.

5.1.9 Remark. Theorem 3.0.1 also applies to other topological contexts. For example, given a topological space or stack X stratified by a noetherian poset P, Ørsnes Jansen [28; 29; 30] and Clausen–Ørsnes Jansen [14] consider the stratified ∞-topos $(\mathrm{Sh}(X), P)$. Theorem 3.0.1 applies in that setting as well, giving a variant of Theorem 5.1.7 for sheaves rather than hypersheaves. In that context, many of these results were already proven by Clausen–Ørsnes Jansen and Ørsnes Jansen; see [14, Proposition 3.6; 29, Propositions 3.13 & 3.20].

5.1.10 Remark (the Künneth formula)**.** Let (X, P) and (Y, Q) be exodromic stratified spaces. The astute reader may have noticed that, unlike in Theorem 3.0.1, in Theorem 5.1.7 we have not stated that $(X\times Y, P\times Q)$ is exodromic. Neither have we stated that there is a Künneth formula

$$\Pi_\infty(X \times Y, P \times Q) \simeq \Pi_\infty(X, P) \times \Pi_\infty(Y, Q) .$$

This is because, in complete generality, we do not know if this is true.

The issue is the following: there are natural colimit-preserving functors

$$\mathrm{Sh}(X) \otimes \mathrm{Sh}(Y) \to \mathrm{Sh}(X \times Y) \quad \text{and} \quad \mathrm{Sh}^{\mathrm{hyp}}(X) \otimes \mathrm{Sh}^{\mathrm{hyp}}(Y) \to \mathrm{Sh}^{\mathrm{hyp}}(X \times Y) , \tag{5.1.11}$$

however, in general neither of these functors need be an equivalence. In particular, in the topological setting, we do not immediately deduce a Künneth formula from Proposition 3.5.5. Nonetheless, Künneth formulas still hold in many contexts. For example, if X is locally compact Hausdorff, then the left-hand functor in (5.1.11) is an equivalence [HTT, Proposition 7.3.1.11]. So if X is locally compact Hausdorff and both $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ are hypercomplete, then Theorem 3.0.1 implies the Künneth formula for the exit-path ∞-category of $(X \times Y, P \times Q)$. For another important example, in §5.2 we show that if (X, P) and (Y, Q) locally admit refinements by conical stratifications, then we have a Künneth formula. See Proposition 5.2.11.

5.2 Locally conically refineable stratifications: formal properties. Recall that if (X, P) is a conically stratified space, then for any open subset $U \subset X$, the stratified space (U, P) is also conically stratified. It is not clear if our definition of an exodromic stratified space is stable under passage to open subsets (cf. Question 3.0.3). So we introduce the following strengthening of exodromicity that applies to many examples from geometry.

5.2.1 Definition. Let $\mathcal{E}$ be a presentable ∞-category. A stratified space (X, P) is *locally $\mathcal{E}$-exodromic* if there exists a basis $\mathcal{B} \subset \mathrm{Open}(X)$ such that for each $U \in \mathcal{B}$, the stratified space (U, P) is $\mathcal{E}$-exodromic.

5.2.2 Example. Let (X, P) be a conically stratified space with locally weakly contractible strata and let $\mathcal{E}$ be a P-admissible presentable ∞-category in the sense of Definition 5.1.3. Then (X, P) is locally $\mathcal{E}$-exodromic.

In light of Theorem 5.1.7, we have the following stability properties of locally exodromic stratifications:

5.2.3 Proposition. *Let $\mathcal{E}$ be a presentable ∞-category and (X, P) a stratified space.*

(1) *If (X, P) is locally $\mathcal{E}$-exodromic, then (X, P) is $\mathcal{E}$-exodromic.*

(2) *If there exists an open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$, the stratified space (U, P) is locally $\mathcal{E}$-exodromic, then (X, P) is locally $\mathcal{E}$-exodromic.*

(3) *If (X, P) is locally $\mathcal{E}$-exodromic, then for any open subset $U \subset X$, the stratified space (U, P) is locally $\mathcal{E}$-exodromic.*

(4) *Assume that $\mathcal{E}$ is compatible with recollements. If (X, P) is locally $\mathcal{E}$-exodromic, then for any locally closed subposet $S \subset P$, the stratified space (X_S, S) is locally $\mathcal{E}$-exodromic.*

(5) *If (X, P) is locally $\mathcal{E}$-exodromic, then for any map of posets $\phi : P \to P'$, the stratified space (X, P') is locally $\mathcal{E}$-exodromic.*

Proof. Item (1) is immediate from the fact that $\mathcal{E}$-exodromicity can be checked locally (Corollary 4.1.12). Items (2) and (3) are immediate from the definitions. Item (4) follows from the definitions and the stability of $\mathcal{E}$-exodromicity under pulling back to locally closed subposets (Proposition 4.1.9). Item (5) follows from the definitions and the stability of $\mathcal{E}$-exodromicity under coarsenings (Theorem 5.1.7-(2)). □

For the examples in the rest of this subsection, it is convenient to introduce the following definition.

5.2.4 Definition. Let $s : X \to P$ be a stratified space.

(1) A *conical refinement* of (X, P) is the data of a conical stratification $t : X \to R$ of X with locally weakly contractible strata and a map of posets $\phi : R \to P$ such that $s = \phi t$. We say that (X, P) is *conically refineable* if there exists a conical refinement of (X, P).

(2) We say that (X, P) is *locally conically refineable* if there exists an open cover $\mathcal{U}$ of X such that for each $U \in \mathcal{U}$, the stratified space (U, P) is conically refineable.

First observe that locally conically refineable stratified spaces have locally weakly contractible strata (hence Observation 5.1.6 applies). In fact, even more is true; we introduce the following definition to axiomatize the categorical features of the exit-path ∞-category of a locally conically refineable stratified space.

5.2.5 Definition. We say that a stratified space (X, P) is *locally cone-like* if the following conditions are satisfied:

(1) The stratified space (X, P) is locally exodromic.

(2) The strata of X are locally weakly contractible.

(3) Every point $x \in X$ admits a fundamental system of open neighborhoods $\mathcal{U}_x$ such that for each $U \in \mathcal{U}_x$, the object $x \in \Pi_\infty(U, P)$ is initial.

5.2.6 Lemma. *Let (X, P) be a conically stratified space with locally weakly contractible strata. Then:*

(1) *The topological space X is locally weakly contractible.*

(2) *The stratified space (X, P) is locally cone-like.*

Proof. First recall that conically stratified spaces with locally weakly contractible strata are locally exodromic. We prove both items simultaneously. By [32, Proposition 2.1.18], every point $x \in X$ admits a fundamental system of open neighborhoods $\mathcal{U}_x$ such that for each $U \in \mathcal{U}_x$, the object x is initial in $\Pi_\infty(U, P)$. For any such U, [32, Corollary 6.2.7] provides a canonical equivalence

$$\Pi_\infty(U) \simeq \mathrm{Env}(\Pi_\infty(U, P)) \simeq * ,$$

where $\Pi_\infty(U)$ denotes the underlying homotopy type of U. Therefore, each U is weakly contractible, i.e., X is locally weakly contractible. □

We now analyze the stability properties of the class of locally cone-like stratified spaces. To start, we need a lemma.

5.2.7 Lemma. *Let $L : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories that exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ at a collection of morphisms. If $c \in \mathcal{C}$ is initial, then $L(c) \in \mathcal{D}$ is initial.*

Proof. Recall that for an ∞-category $\mathcal{E}$, an object $e \in \mathcal{E}$ is initial if and only if the functor $e : * \to \mathcal{E}$ that picks out e is a limit-cofinal functor. Since L is a localization, $L : \mathcal{C} \to \mathcal{D}$ is limit-cofinial [5, Proposition 5.13]. Hence the composite

$$* \xrightarrow{c} \mathcal{C} \xrightarrow{L} \mathcal{D}$$

is limit-cofinal. □

5.2.8 Lemma.

(1) *Let (X,P) be a locally cone-like stratified space. Then for each locally closed subposet $S \subset P$, the stratified space (X_S, S) is locally cone-like.*

(2) *Let (X,R) be a locally cone-like stratified space and $\phi\colon R \to P$ is a map of posets. Then the stratified space (X,P) is locally cone-like.*

(3) *If (X,P) is a stratified space and $\{U_\alpha\}_{\alpha\in A}$ is an open cover of X such that each stratified space (U_α, P) is locally cone-like, then (X,P) is locally cone-like.*

Proof. For (1), the only nontrivial condition to check is Definition 5.2.5-(3). Let $x \in X_S$ and let $\mathcal{U}_x$ be a fundamental system of open neighborhoods of x in X such that for each $U \in \mathcal{U}_x$, the object $x \in \Pi_\infty(U,P)$ is initial. Write

$$\mathcal{U}_{x,S} \coloneqq \{\, U_S \mid U \in \mathcal{U}_x \,\} .$$

Notice that $U_S = U \cap X_S$ and $\mathcal{U}_{x,S}$ is a fundamental system of open neighborhoods of x in X_S. By Theorem 5.1.7-(1), for each $U \in \mathcal{U}_x$, the natural functor

$$\Pi_\infty(U_S, S) \to \Pi_\infty(U,P)$$

is fully faithful. Since $x \in \Pi_\infty(U_S,S)$ and x is initial in the larger ∞-category $\Pi_\infty(U,P)$, we deduce that x is also initial in $\Pi_\infty(U_S,S)$.

For (2), again the only nontrivial condition to check is Definition 5.2.5-(3). Let $x \in X$ and let $\mathcal{U}_x$ be a fundamental system of open neighborhoods of x in X such that for each $U \in \mathcal{U}_x$, the object $x \in \Pi_\infty(U,R)$ is initial. Then Lemma 5.2.7 shows that $x \in \Pi_\infty(U,P)$ is also initial.

Item (3) is immediate from the definitions. □

Now we record the fundamental properties of the class of locally conically refineable stratified spaces.

5.2.9 Proposition (properties of locally conically refineable stratified spaces)**.**

(1) *Let (X,P) be a stratified space and let $\mathcal{E}$ be an admissible presentable ∞-category. If (X,P) is locally conically refineable, then (X,P) is locally $\mathcal{E}$-exodromic.*

(2) *Let (X,P) be a locally conically refineable stratified space. Then for each open subspace $U \subset X$, the stratified space (U,P) is locally conically refineable.*

(3) *Let (X,P) be a locally conically refineable stratified space. Then for each locally closed subposet $S \subset P$, the stratified space (X_S,S) is locally conically refineable.*

(4) *Let (X,R) be a locally conically refineable stratified space and $\phi\colon R \to P$ is a map of posets. Then the stratified space (X,P) is locally conically refineable.*

(5) *If (X,P) is a stratified space and $\{U_\alpha\}_{\alpha\in A}$ is an open cover of X such that each stratified space (U_α,P) is locally conically refineable, then (X,P) is locally conically refineable.*

(6) *If (X,P) is locally conically refineable, then X is locally weakly contractible. Moreover, the space*

$$\mathrm{Env}(\Pi_\infty(X,P))$$

is naturally equivalent to the underlying homotopy type of X.

(7) *If (X,P) is a locally conically refineable stratified space, then (X,P) is locally cone-like.*

Proof. Item (1) follows from Proposition 5.2.3 and the fact that conically stratified spaces with locally weakly contractible strata are $\mathcal{E}$-exodromic.

For (2), note that since the statement is local, it suffices to prove the claim when (X,P) admits a global conical refinement (X,R). Now note that since (X,R) is conically stratified, for any open subset $U \subset X$, the stratified space (U,R) is also conical.

For (3), note that since the statement is local, it suffices to prove the claim when (X,P) admits a global conical refinement (X,R). In this case, [32, Lemma 2.1.11] shows that the stratified space (X_S,R_S) is conical with locally weakly contractible strata. To conclude, note that (X_S,R_S) is a refinement of (X_S,S).

Items (4) and (5) are immediate from the definitions. For (6), note that by Lemma 5.2.6-(1), X admits an open cover by locally weakly contractible topological spaces. Hence the claim is a special case of Corollary 5.1.8. Item (7) follows from the fact that conically stratified spaces are locally cone-like (Lemma 5.2.6-(2)) and the stability properties of locally cone-like stratified spaces (Lemma 5.2.8). □

We conclude this subsection with a Künneth formula for the exit-path ∞-category of a product of locally conically refineable stratified spaces. Due the issues mentioned in Remark 5.1.10, our proof does not rely on the Künneth formula for exodromic stratified ∞-topoi (Proposition 3.5.5). Instead, we make use of the localization formula for the exit-path ∞-category of a coarsening and the following lemma.

5.2.10 Lemma. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be ∞-categories and let $W_i \subset \mathrm{Mor}(\mathcal{C}_i)$ be collections of morphisms. Then the natural functor*

$$(\mathcal{C}_1 \times \mathcal{C}_2)[(W_1 \times W_2)^{-1}] \to \mathcal{C}_1[W_1^{-1}] \times \mathcal{C}_2[W_2^{-1}]$$

is an equivalence.

Proof. This is an immediate consequence of [Ker, Tag 02LV]. □

5.2.11 Proposition (Künneth formula for locally conically refineable stratifications)**.** *Let (X,P) and (Y,Q) be locally conically refineable stratified spaces. Then:*

(1) *The product stratified space $(X \times Y, P \times Q)$ is locally conically refineable.*

(2) *The natural functor*

$$\Pi_\infty(X \times Y, P \times Q) \to \Pi_\infty(X,P) \times \Pi_\infty(Y,Q)$$

is an equivalence of ∞-categories.

(3) *The natural functor*

$$\boxtimes : \mathrm{Cons}_P^{\mathrm{hyp}}(X) \otimes \mathrm{Cons}_Q^{\mathrm{hyp}}(Y) \to \mathrm{Cons}_{P\times Q}^{\mathrm{hyp}}(X \times Y)$$

is an equivalence of ∞-categories.

Proof. Item (1) is immediate from the definitions and the fact that a product of conically stratified spaces is still conically stratified.

For (2), let

$$U_\bullet : \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathbf{Top}_{/X} \qquad \text{and} \qquad V_\bullet : \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathbf{Top}_{/Y}$$

be open semi-simplicial hypercoverings of X and Y respectively, such that for each $n \geq 0$ the stratified spaces (U_n, P) and (V_n, Q) are conically refineable. Since $\mathbf{\Delta}_{\mathrm{inj}}$ is sifted, $\mathbf{\Delta}_{\mathrm{inj}}$-indexed colimits commute with finite products in $\mathbf{Cat}_\infty$; hence Theorem 5.1.7-(4) shows that the natural functor

$$\operatorname*{colim}_{[n]\in\mathbf{\Delta}_{\mathrm{inj}}} \Pi_\infty(U_n, P) \times \Pi_\infty(V_n, Q) \to \Pi_\infty(X \times Y, P \times Q)$$

is an equivalence. We can therefore assume that (X,P) and (Y,Q) are (globally) conically refineable.

Let (X,P') and (Y,Q') be conical refinements of (X,P) and (Y,Q), respectively. Then $(X \times Y, P' \times Q')$ is conical and thus it is a conical refinement of $(X \times Y, P \times Q)$. It follows from [32, Theorem 5.4.1] and the explicit geometrical definition of the exit-path ∞-category that the natural functor

$$\Pi_\infty(X \times Y, P' \times Q') \to \Pi_\infty(X,P') \times \Pi_\infty(Y,Q')$$

is an equivalence. Unraveling the definitions, we see that $W_{P\times Q} = W_P \times W_Q$ as collection of morphisms in $\Pi_\infty(X,P') \times \Pi_\infty(Y,Q')$. The conclusion now follows from Lemma 5.2.10.

Item (3) is immediate from (2) and the fact that the functor $\mathrm{Fun}(-, \mathbf{Spc})$ carries products of ∞-categories to tensor products in $\mathbf{Pr}^{\mathrm{L}}$. □

5.3 Locally conically refineable stratifications: examples. We give some examples of locally conically refineable (hence locally exodromic) stratifications.

5.3.1 Notation (simplicial complexes)**.** Let (V,S) be an simplicial complex, and regard S as a poset ordered by inclusion. Write $\Delta^{(V,S)}$ for the geometric realization of (V,S). There is a natural stratification $\Delta^{(V,S)} \to S$ with locally contractible strata; see [HA, Definition A.6.7].

5.3.2 Example. Let (V,S) be a *locally finite* simplicial complex and let $\mathcal{E}$ be an admissible presentable ∞-category. Then the natural stratification $\Delta^{(V,S)} \to S$ is conical [HA, Proposition A.6.8]. Moreover, [HA, Theorem A.6.10] shows that

$$\Pi_\infty(\Delta^{(V,S)}, S) \simeq S \, . \tag{5.3.3}$$

By Proposition 5.2.9, we see that for any map of posets $S \to P$, the stratified space $(\Delta^{(V,S)}, P)$ is locally $\mathcal{E}$-exodromic. That is, any stratified space admitting a refinement by a locally finite triangulation is locally $\mathcal{E}$-exodromic.

5.3.4 Observation. In light of (5.3.3), given a locally finite simplicial complex (V,S), the stratified space $(\Delta^{(V,S)}, S)$ is categorically finite if and only if the set S is finite.

5.3.5 Example. The *tree stratification* of a finite simplicial complex considered by Favero–Huang [16, §4.4] is conically refineable, hence locally exodromic. Moreover, Theorem 5.1.7-(5) and Observation 5.3.4 show that the tree stratification is categorically finite.

One source of locally exodromic stratifications comes from subanalytic stratifications of real analytic spaces. Recall that subanalytic stratifications need not be conical; see Figure 1.

5.3.6 Definition. Let X be a topological space. We say that a stratification $X \to P$ is *locally finite* if for every point $x \in X$, there is an open neighborhood U of x such that U intersects only finitely many strata of (X,P).

5.3.7 Definition. A *subanalytic stratified space* is the data of a triple (M,X,P) where M is a smooth real analytic space, $X \subset M$ is a locally closed subanalytic subset, and $X \to P$ is a locally finite stratification by subanalytic subsets of M.

Subanalytic stratified spaces provide many examples of (locally) categorically finite stratified spaces:

5.3.8 Definition. Let (X,P) be a locally exodromic stratified space. We say that (X,P) is *locally categorically finite (resp., compact)* if there exists an open cover $\mathcal{U}$ such that for each $U \in \mathcal{U}$, the exodromic stratified space (U,P) is categorically finite (resp., compact).

5.3.9 Theorem. *Let (M,X,P) be a subanalytic stratified space. Then:*

(1) *The stratified space (X,P) admits a refinement by a locally finite triangulation.*

(2) *For any admissible ∞-category $\mathcal{E}$, the stratified space (X,P) is locally $\mathcal{E}$-exodromic.*

(3) *If X is compact, then (X,P) admits a refinement by a* finite *triangulation. Hence (X,P) is categorically finite.*

(4) *The stratified space (X,P) is locally categorically finite.*

(5) *If $U \Subset X$ is a relatively compact subanalytic open subset, then (U,P) is categorically finite.*

Proof. Item (1) follows from [17, §1.7] combined with [18]. Item (2) follows from (1) and Proposition 5.2.9. For (3), note that by (1), the stratified space (X,P) admits a triangulation by a locally finite simplicial complex $(\Delta^{(V,S)}, S)$. Since X is compact, the poset S is finite. The final statement in (3) follows from Theorem 5.1.7-(5) and Observation 5.3.4.

Now we prove (4). At the cost of shrinking M, we can assume that X is closed in M. Let $x \in X$ and let $B \subset M$ be a small ball centered at x such that $X \cap B$ intersects only finitely many strata. We claim that $(X \cap B, P)$ is categorically finite. Note that since $X \cap B$ intersects only finitely many strata, we may assume that P is finite. Extend $X \cap B \to P$ to a finite stratification $B \to P^{\triangleright}$ sending $B \smallsetminus (X \cap B)$ to the terminal object of $P^{\triangleright}$. Since P is closed in $P^{\triangleright}$, Theorem 5.1.7-(5) reduces the claim to the case where $X = B$. We thus need to show that $(B, P^{\triangleright})$ is categorically finite. Write $Q \coloneqq P^{\triangleright}$ and extend $B \to Q$ to a finite stratification $\overline{B} \to Q^{\triangleleft}$ by sending $\partial\overline{B}$ to the initial object of $Q^{\triangleleft}$. Since Q is open in $Q^{\triangleleft}$, Theorem 5.1.7-(5) reduces the claim to the case where $X = \overline{B}$. An application of (3) now shows that $(\overline{B}, Q^{\triangleleft})$ is categorically finite.

Finally, we prove (5). The closure $\overline{U}$ is again a subanalytic (see e.g., the discussion following [9, Definition 3.1]), and it is compact by assumption. In particular, it intersects only finitely many strata. As before, we can thus assume that P is finite. Extend $U \to P$ to a finite stratification $\overline{U} \to P^{\triangleleft}$ sending the boundary $\partial\overline{U} \coloneqq \overline{U} \smallsetminus U$ to the initial object of $P^{\triangleleft}$. Then P is open in $P^{\triangleleft}$, so Theorem 5.1.7-(5) reduces us to verify that $(\overline{U}, P)$ is categorically finite, and this follows directly from (3). □

5.3.10 Example. The *Bondal–Ruan stratification* of the n-torus considered by Favero–Huang [10; 16, §5.2] is subanalytic, hence locally exodromic, categorically finite, and locally categorically finite.

Stratifications of real algebraic varieties are especially well-behaved:

5.3.11 Definition. An *algebraic stratified space* is the data of a stratified space (X, P) where X is (the real points of) an algebraic variety over $\mathbf{R}$ and $X \to P$ is a finite stratification by Zariski locally closed subsets.

5.3.12 Warning. Unlike a subanalytic stratified space, an algebraic stratified space (X, P) is not presented as a subspace of a smooth algebraic variety. Note that if X is singular, such a presentation may not exist.

5.3.13 Theorem. *Let (X, P) be an algebraic stratified space. Then:*

(1) *If X is affine, (X, P) admits a categorically finite conical refinement (X, R) with R finite. Hence (X, P) is categorically finite.*

(2) *The stratified space (X, P) is locally conically refineable.*

(3) *For any admissible ∞-category $\mathcal{E}$, the stratified space (X, P) is locally $\mathcal{E}$-exodromic and locally categorically finite.*

(4) *The stratified space (X, P) is categorically finite.*

Proof. For (1), let us view X as a closed subset of $\mathbf{A}^n$. Let $\overline{X}$ be the closure of X in $\mathbf{P}^n$. Define $Q \coloneqq (P^{\triangleright})^{\triangleleft}$ and let us extend $X \to P$ as a stratification $\mathbf{P}^n \to Q$ by sending $\overline{X} \smallsetminus X$ to the initial object of Q and $\mathbf{P}^n \smallsetminus \overline{X}$ to the terminal object of Q. Then, $(\mathbf{P}^n, Q)$ is a compact subanalytic stratified space. By Theorem 5.3.9-(3), $(\mathbf{P}^n, Q)$ admits a refinement $Q' \to Q$ by a finite triangulation. Thus, $(\mathbf{P}^n, Q')$ is conically stratified with locally weakly contractible strata. Moreover, Observation 5.3.4 shows that $(\mathbf{P}^n, Q')$ is categorically finite. Since $P \subset Q$ is locally closed, (X, Q'_P) is also conically stratified with locally weakly contractible strata. Moreover, Proposition A.3.17 shows that (X, Q'_P) is categorically finite. Finally, since Q is finite, so is Q'_P.

Item (2) is an immediate consequence of (1). Item (3) follows from (1) and Proposition 5.2.9. Since X admits a finite cover by affine subsets whose iterated intersections are again affine, (4) follows from (1) and Theorem 5.1.7-(5). □

5.4 Moduli of constructible & perverse sheaves. We now use exodromy and the finiteness results of § 5.3 to study derived moduli stacks of constructible and perverse sheaves. We begin by recalling a few notions from [32, §7].

5.4.1 Recollection. Let B be an animated commutative ring (i.e., simplicial commutative ring). Write Mod_B for the ∞-category of B-modules and $\mathrm{Perf}_B \subset \mathrm{Mod}_B$ for the smallest stable full subcategory containing B and closed under retracts. The ∞-category Mod_B is compactly generated with full subcategory of compact objects Perf_B [HA, Proposition 7.2.4.2; SAG, Notation 25.2.1.1]. Also note that the shifts $B[n]$ for $n \in \mathbf{Z}$ generate Mod_B under colimits and retracts.

We are interested in the moduli of constructible sheaves with *perfect* stalks:

5.4.2 Notation. Given a stratified space (X, P) and an animated commutative ring B, we write

$$\mathrm{Cons}^{\mathrm{hyp}}_{P,\omega}(X; \mathrm{Mod}_B) \subset \mathrm{Cons}^{\mathrm{hyp}}_{P}(X; \mathrm{Mod}_B)$$

for the full subcategory spanned by the hyperconstructible hypersheaves on (X, P) whose stalks are compact objects of Mod_B.

5.4.3 Recollection. Let X be a topological space and let $L\colon \mathcal{E} \to \mathcal{D}$ be a morphism in $\mathbf{Pr}^{\mathrm{L}}$. We denote by

$$L^{\mathrm{hyp}} := (-)^{\mathrm{hyp}} \circ L \circ - \colon \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{D})$$

the induced a morphism in $\mathbf{Pr}^{\mathrm{L}}$. As recalled in [32, §2.5], the formation of L^{hyp} commutes with hypersheaf pullback. For a stratification $X \to P$, the functor L^{hyp} preserves P-hyperconstructible hypersheaves, that is, restricts to a functor

$$L^{\mathrm{hyp}} \colon \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{E}) \to \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathcal{D}) \,.$$

5.4.4 Notation. For a morphism of animated commutative rings $A \to B$, we define

$$B \otimes_A^{\mathrm{hyp}} (-) := (B \otimes_A (-))^{\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(X; \mathrm{Mod}_A) \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathrm{Mod}_B) \,.$$

5.4.5 Recollection (the derived prestack of constructible sheaves)**.** Let (X, P) be a stratified space and let A be an animated commutative ring. Following [32, §7.1], we write

$$\mathbf{Cons}_P(X) \colon \mathbf{dAff}_A^{\mathrm{op}} \to \mathbf{Spc}$$

for the derived prestack defined by sending a derived affine scheme $\mathrm{Spec}(B)$ over A to the maximal sub-∞-groupoid of $\mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathrm{Mod}_B)$ and sending a morphism of derived affine schemes $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ over A to the map on maximal sub-∞-groupoids induced by

$$C \otimes_B^{\mathrm{hyp}} (-) \colon \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathrm{Mod}_B) \to \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathrm{Mod}_C) \,.$$

5.4.6. Given a morphism of stratified spaces $f \colon (X, P) \to (Y, Q)$, pullback along f defines a map of derived prestacks

$$\mathbf{Cons}_Q(Y) \to \mathbf{Cons}_P(X) \,.$$

In the setting of exodromy, $\mathbf{Cons}_P(X)$ is a derived stack:

5.4.7 Observation. Let (X, P) be a stratified space with locally weakly contractible strata and let B be an animated commutative ring. If (X, P) is exodromic, then the exodromy equivalence

$$\mathrm{Fun}(\Pi_\infty(X, P), \mathrm{Mod}_B) \simeq \mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathrm{Mod}_B)$$

restricts to an equivalence

$$\mathrm{Fun}(\Pi_\infty(X, P), \mathrm{Perf}_B) \simeq \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathrm{Mod}_B) \,.$$

5.4.8 Lemma. *Let (X, P) be a stratified space and let A be an animated commutative ring. If (X, P) is exodromic, then the derived prestack*

$$\mathbf{Cons}_P(X) \colon \mathbf{dAff}_A^{\mathrm{op}} \to \mathbf{Spc}$$

satisfies flat hyperdescent. In particular, $\mathbf{Cons}_P(X)$ is a derived stack.

Proof. Since (X, P) is exodromic, for an animated A-algebra B, we have

$$\mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathrm{Mod}_B) \simeq \mathrm{Fun}(\Pi_\infty(X, P), \mathrm{Perf}_B) \,.$$

Hence the right-hand side preserves limits in Perf_B. The claim now follows from the fact that the assignment $B \mapsto \mathrm{Perf}_B$ satisfies flat hyperdescent [SAG, Corollary D.6.3.3 & Proposition 2.8.4.2-(10)]. □

Under compactness assumptions, the derived stack $\mathbf{Cons}_P(X)$ is even locally geometric:

5.4.9 Theorem. *Let (X, P) be an exodromic stratified space and let A be an animated commutative ring. If (X, P) is categorically compact, then:*

(1) *The derived stack $\mathbf{Cons}_P(X)$ is locally geometric and locally of finite presentation.*

(2) *Given a point $x \colon \mathrm{Spec}(B) \to \mathbf{Cons}_P(X)$ classifying a constructible sheaf $F \in \mathrm{Cons}_{P,\omega}^{\mathrm{hyp}}(X; \mathrm{Mod}_B)$, the tangent complex at x is given by*

$$x^* \mathbb{T}_{\mathbf{Cons}_P(X)} \simeq \mathrm{Hom}_{\mathrm{Cons}_P^{\mathrm{hyp}}(X; \mathrm{Mod}_B)}(F, F)[1] \,.$$

Here, the right hand side denotes the Mod_B*-enriched* Hom *of* $\mathbf{Cons}_P(X; \mathrm{Mod}_B)$*.*

Proof. Same proof as [32, Theorem 7.1.8]; in the end, the result follows combining the categorical compactness assumption with [37, Theorem 3.6 & Corollary 3.17]. □

Since coarsenings of conical stratifications are our main source of exodromic stratified spaces, it is natural to study how the moduli stacks of hyperconstructible hypersheaves behave under coarsening. To this end, we show:

5.4.10 Proposition. *Let (X, R) be a categorically compact exodromic stratified space with locally weakly contractible strata. Let $\phi\colon R \to P$ be a map of posets and let A be an animated commutative ring. Then the induced map of locally geometric derived stacks*

$$i\colon \mathbf{Cons}_P(X) \hookrightarrow \mathbf{Cons}_R(X)$$

is a representable open immersion.

Proof. From Proposition 3.6.4, we see that (X, P) is exodromic and categorically compact. Therefore, Theorem 5.4.9 implies that both $\mathbf{Cons}_P(X)$ and $\mathbf{Cons}_R(X)$ are locally geometric and locally of finite presentation. In particular, the natural map between them is automatically locally of finite presentation. To prove that i is an open immersion suffices to prove that i is étale and that the diagonal map

$$\Delta_i\colon \mathbf{Cons}_P(X) \to \mathbf{Cons}_P(X) \underset{\mathbf{Cons}_R(X)}{\times} \mathbf{Cons}_P(X)$$

is an equivalence. Theorem 3.3.5 shows that $\Pi_\infty(X, R) \to \Pi_\infty(X, P)$ exhibits $\Pi_\infty(X, P)$ as the localization of $\Pi_\infty(X, R)$ at the collection of morphism W_P. It follows that for every animated A-algebra B, the map

$$\mathbf{Cons}_P(X)(\mathrm{Spec}(B)) \to \mathbf{Cons}_R(X)(\mathrm{Spec}(B)) \tag{5.4.11}$$

is fully faithful. This immediately implies that Δ_i is an equivalence.

To prove that i is an open immersion, we are left to check that i is étale. Notice that i is automatically locally of finite presentation. Thus [**HAG-II**] implies that it suffices to show that i is formally étale, i.e., that the cotangent complex of i vanishes. We use the criterion provided in [31, Lemma 2.15]. Since (5.4.11) is fully faithful, the only thing left to check is that for every animated A-algebra B, the map

$$\mathbf{Cons}_P(X)(\mathrm{Spec}(B)) \to \mathbf{Cons}_P(\mathrm{Spec}(B_{\mathrm{red}})) \underset{\mathbf{Cons}_R(\mathrm{Spec}(B_{\mathrm{red}}))}{\times} \mathbf{Cons}_R(\mathrm{Spec}(B))$$

is surjective at the level of connected components. Therefore, let $F\colon \Pi_\infty(X, R) \to \mathrm{Perf}_B$ be a functor and assume that the induced functor

$$B_{\mathrm{red}} \otimes_B F(-)\colon \Pi_\infty(X, R) \to \mathrm{Perf}_{B_{\mathrm{red}}}$$

factors through $\Pi_\infty(X, P)$. Since $\Pi_\infty(X, R) \to \Pi_\infty(X, P)$ is a localization at W_P, this is equivalent to say that $B_{\mathrm{red}} \otimes_B F(-)$ inverts all arrows in W_P. To complete the proof, it is enough to prove that F also inverts all arrows in W_P. Therefore, let $\gamma\colon x \to y$ be a morphism in W_P and consider

$$F_\gamma \coloneqq \mathrm{fib}(F(\gamma)\colon F(x) \to F(y)) .$$

By assumption, $F(x)$ and $F(y)$ belong to Perf_B, so $F_\gamma \in \mathrm{Perf}_B$ as well. Also, we have

$$B_{\mathrm{red}} \otimes_B F_\gamma \simeq \mathrm{fib}(B_{\mathrm{red}} \otimes_B F(x) \to B_{\mathrm{red}} \otimes_B F(y)) \simeq 0 ,$$

So the conclusion follows from the cohomological Nakayama lemma [SAG, Corollary 2.7.4.4]. □

We now turn our attention to the moduli of *perverse* sheaves.

5.4.12 Notation. Let (X, P) be a stratified space, let $\mathfrak{p}\colon P \to \mathbf{Z}$ be any function, and let A be an animated commutative ring. We write

$${}^{\mathfrak{p}}\mathbf{Perv}_P(X) \subset \mathbf{Cons}_P(X)$$

for the derived subprestack of $\mathfrak{p}$-*perverse sheaves* on (X, P). See [32, §7.7] for details.

5.4.13 Recollection. Let (X, R) be a stratified space, let $\phi\colon R \to P$ be a map of posets, and let A be an animated commmutative ring. Let $\mathfrak{p}\colon P \to \mathbf{Z}$ be any function and write $\mathfrak{r}$ for the composite $\mathfrak{p}\phi\colon R \to \mathbf{Z}$. Recall from [32, Proposition 7.7.10] that if for each $p \in P$, the poset R_p is noetherian, then the square of derived prestacks

$$\begin{array}{ccc} {}^{\mathfrak{p}}\mathbf{Perv}_P(X) & \hookrightarrow & {}^{\mathfrak{r}}\mathbf{Perv}_R(X) \\ \downarrow & & \downarrow \\ \mathbf{Cons}_P(X) & \hookrightarrow & \mathbf{Cons}_R(X) \end{array} \tag{5.4.14}$$

is a pullback.

5.4.15 Recollection. Let (X, P) be a stratified space, let $\mathfrak{p}\colon P \to \mathbf{Z}$ be any function, and let A be an animated commutative ring. By [32, Proposition 7.7.8], the presheaf

$$ {}^{\mathfrak{p}}\mathbf{Perv}_P(-)\colon \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{PSh}(\mathbf{dAff}_A) $$

satisfies hyperdescent.

5.4.16 Theorem. *Let (X, R) be a conically stratified space with locally weakly contractible strata, let $\phi\colon R \to P$ be a map of posets, let $\mathfrak{p}\colon P \to \mathbf{Z}$ be any function, and let A be an animated commutative ring. Assume that for each $p \in P$, the poset R_p is noetherian. Then:*

(1) *The derived prestack ${}^{\mathfrak{p}}\mathbf{Perv}_P(X)$ satisfies flat hyperdescent. In particular, ${}^{\mathfrak{p}}\mathbf{Perv}_P(X)$ is a derived stack.*

(2) *If $\Pi_\infty(X, R)$ has finitely many equivalence classes of objects, then the morphism of derived stacks*

$$ {}^{\mathfrak{p}}\mathbf{Perv}_P(X) \hookrightarrow \mathbf{Cons}_P(X) $$

is a representable open immersion.

(3) *If (X, R) is categorically compact, then the derived stack ${}^{\mathfrak{p}}\mathbf{Perv}_P(X)$ is locally geometric and locally of finite presentation.*

Proof. Write $\mathfrak{r}$ for the composite $\mathfrak{p}\phi\colon R \to \mathbf{Z}$. For item (1), since (X, R) and (X, P) are exodromic, by Lemma 5.4.8 the prestacks $\mathbf{Cons}_P(X)$ and $\mathbf{Cons}_R(X)$ satisfy flat hyperdescent. Moreover, [32, Corollary 7.7.16] shows that ${}^{\mathfrak{r}}\mathbf{Perv}_R(X)$ satisfies flat hyperdescent. Since the square (5.4.14) from Recollection 5.4.13 is a pullback, ${}^{\mathfrak{p}}\mathbf{Perv}_P(X)$ also satisfies flat hyperdescent. Under the condition of item (2), [32, Theorem 7.7.16] shows that the morphism of derived stacks

$$ {}^{\mathfrak{r}}\mathbf{Perv}_R(X) \hookrightarrow \mathbf{Cons}_R(X) $$

is representable by an open immersion. Since (5.4.14) is a pullback, the conclusion follows.

For (3), assume that (X, R) is categorically compact. By Proposition 3.6.4, the stratified space (X, P) is also categorically compact. Hence, Theorem 5.4.9 ensures that $\mathbf{Cons}_P(X)$ and $\mathbf{Cons}_R(X)$ are locally geometric and locally of finite presentation. Moreover, [32, Theorem 7.7.16] shows that ${}^{\mathfrak{r}}\mathbf{Perv}_R(X)$ is locally geometric and locally of finite presentation. Since (5.4.14) is a pullback, the conclusion follows. □

Our work from §5.3 provides a number of examples where $\mathbf{Cons}_P(X)$ and ${}^{\mathfrak{p}}\mathbf{Perv}_P(X)$ are locally geometric and locally of finite presentation:

5.4.17 Corollary. *Let (X, P) be a stratified space, let $\mathfrak{p}\colon P \to \mathbf{Z}$ be any function, and let A be an animated commutative ring. Assume one of the following conditions:*

(1) *(X, P) admits a categorically compact conical refinement.*

(2) *(X, P) admits a refinement by a finite triangulation.*

(3) *The topological space X is compact and (X, P) admits the structure of a subanalytic stratified space in the sense of Definition 5.3.7.*

(4) *(X, P) admits the structure of an algebraic stratified space in the sense of Definition 5.3.11.*

Then the derived prestacks $\mathbf{Cons}_P(X)$ *and* ${}^{\mathfrak{p}}\mathbf{Perv}_P(X)$ *are derived stacks that are locally geometric and locally of finite presentation.*

Proof. Item (1) follows from Proposition 3.6.4 and Theorems 5.4.9 and 5.4.16. In light of Observation 5.3.4, item (2) is a special case of (1). Similarly, by Theorem 5.3.9-(3), item (3) is a special case of (2).

Let us now prove (4). Note that by Theorem 5.4.9 and Theorem 5.3.13-(4), the derived prestack $\mathbf{Cons}_P(X)$ is a derived stack that is locally geometric and locally of finite presentation. Moreover, since the properties of being a derived stack, being locally geometric, and being locally of finite presentation are stable under finite limits, Recollection 5.4.15 reduces the claim for ${}^{\mathfrak{p}}\mathbf{Perv}_P(X)$ to the case where X is affine. To conclude, note that Theorem 5.3.13-(1) shows that an affine algebraic stratified space admits a categorically compact conical refinement; the claim now follows from (1). □

5.5 A criterion for constructibility with respect to a coarsening. Let (X, R) be an exodromic stratified space with locally weakly contractible strata and let $\phi\colon R \to P$ be a map of posets. It is often useful to have a geometric recognition criterion for when an R-hyperconstructible hypersheaf is P-hyperconstructible. The goal of this subsection is to explain such a criterion: an R-hyperconstructible hypersheaf F on X is P-hyperconstructible if and only if for each morphism $\gamma\colon x \to y$ in the exit-path ∞-category $\Pi_\infty(X, R)$ that lies in a single stratum of the coarser stratification (X, P), the induced specialization map on stalks

$$y^*F \to x^*F$$

is an equivalence.[4] This criterion is an easy consequence of the exodromy equivalence and localization formula for the exit-path ∞-category of a coarsening.

5.5.1 Notation (cospecialization maps)**.** Let $\mathcal{E}$ be a presentable ∞-category and let $(\mathcal{X}, R)$ be an $\mathcal{E}$-exodromic stratified ∞-topos.

(1) Write

$$[-]\colon \Pi_\infty(\mathcal{X}, R)^{\mathrm{op}} \hookrightarrow \mathrm{Cons}_R(\mathcal{X})\,,\ x \mapsto [x]$$

for the inclusion of the subcategory of atomic objects. For each $E \in \mathcal{E}$ and $x \in \Pi_\infty(\mathcal{X}, R)$, we write $[x] \otimes E$ for the canonical object in

$$\mathrm{Cons}_R(\mathcal{X}) \otimes \mathcal{E} \simeq \mathrm{Cons}_R(\mathcal{X}; \mathcal{E})\,.$$

(2) Given a morphism $\gamma\colon x \to y$ in $\Pi_\infty(\mathcal{X}, R)$, we write

$$\mathrm{cosp}_R^\gamma \coloneqq [\gamma]\colon [y] \to [x]$$

for the corresponding morphism in $\mathrm{Cons}_R(\mathcal{X})$. We refer to cosp_R^γ as the *cospecialization map* associated to γ. Again, for general $\mathcal{E}$ and for each $E \in \mathcal{E}$, we write $\mathrm{cosp}_R^\gamma \otimes \mathrm{id}_E$ for the corresponding morphism in $\mathrm{Cons}_R(\mathcal{X}; \mathcal{E})$.

5.5.2 Observation (specialization maps)**.** Let (X, R) be an exodromic stratified space with locally weakly contractible strata. In light of Observation 5.1.6, given a R-hyperconstructible hypersheaf F and a morphism $\gamma\colon x \to y$ in $\Pi_\infty(X, R)$, applying $\mathrm{Map}(-, F)$ to the cospecialization map

$$\mathrm{cosp}_R^\gamma\colon [y] \to [x]$$

yields a *specialization map* $x^*F \to y^*F$ on stalks.

5.5.3 Recollection. Let $\mathcal{D}_0$ be a small ∞-category and let $W \subset \mathrm{Mor}(\mathcal{D}_0)$ be a class of morphisms. Write $L\colon \mathcal{D}_0 \to \mathcal{D}_0[W^{-1}]$ for the localization functor. Then, by the definition of localization, the induced pullback functor

$$L^*\colon \mathrm{PSh}(\mathcal{D}_0[W^{-1}]) \to \mathrm{PSh}(\mathcal{D}_0)$$

is fully faithful with image those $F\colon \mathcal{D}_0^{\mathrm{op}} \to \mathbf{Spc}$ that carry morphisms in W to equivalences.

5.5.4 Proposition. *Let* $\mathcal{D}_0$ *be a small* ∞*-category,* $W \subset \mathrm{Mor}(\mathcal{D}_0)$ *a class of morphisms, and* $\mathcal{E}$ *a presentable* ∞*-category. Write* $L\colon \mathcal{D}_0 \to \mathcal{D}_0[W^{-1}]$ *for the localization functor. Then:*

[4] We do not make use of this result in the present paper, but need it in future work.

(1) *Let $F \in \mathrm{PSh}(\mathcal{D}_0; \mathcal{E})$ and let f be a morphism in $\mathcal{D}_0$. Then the full subcategory of $\mathcal{E}$ spanned by those objects $E \in \mathcal{E}$ such that F is $よ(f) \otimes \mathrm{id}_E$-local is closed under colimits and retracts.*

(2) *An object $F \in \mathrm{PSh}(\mathcal{D}_0; \mathcal{E})$ is in the image of the fully faithful pullback functor*

$$L^* \colon \mathrm{PSh}(\mathcal{D}_0[W^{-1}]; \mathcal{E}) \hookrightarrow \mathrm{PSh}(\mathcal{D}_0; \mathcal{E})$$

if and only if for each $w \in W$ and $E \in \mathcal{E}$, the object F is $よ(w) \otimes \mathrm{id}_E$-local.

Proof. Immediate from Recollection 5.5.3 and the definitions. □

5.5.5 Corollary. *Let $\mathcal{E}$ be a presentable ∞-category, let $(\mathcal{X}, R)$ be an $\mathcal{E}$-exodromic stratified ∞-topos, let $\phi \colon R \to P$ be a map of posets, let $F \in \mathrm{Cons}_R(\mathcal{X}; \mathcal{E})$, and let $\gamma \colon x \to y$ be a morphism in $\Pi_\infty(\mathcal{X}, R)$. Then:*

(1) *The full subcategory of $\mathcal{E}$ spanned by those objects $E \in \mathcal{E}$ such that F is $(\mathrm{cosp}_R^\gamma \otimes \mathrm{id}_E)$-local is closed under colimits and retracts.*

(2) *The R-constructible object F is P-constructible if and only if for each $\gamma \in W_P$ and $E \in \mathcal{E}$, the object F is $\mathrm{cosp}_R^\gamma \otimes \mathrm{id}_E$-local.*

Proof. In light of the exodromy equivalence and the localization formula for the exit-path ∞-category of a coarsening (Theorem 3.3.5), this result is a special case of Proposition 5.5.4. □

5.6 Relationship to Lurie's simplicial model for exit-paths. We conclude with some remarks and questions regarding the relationship between the exit-path ∞-category in the conically refineable setting and Lurie's simplicial model for exit-paths $\mathrm{Sing}(X, R)$. See [HA, Definition A.6.2; 32, §2] for background on the simplicial model.

5.6.1 Recollection. Let (X, R) be a conically stratified space with locally weakly contractible strata. Then Lurie's exit-path simplicial set $\mathrm{Sing}(X, R)$ is an ∞-category [HA, Theorem A.6.4]. Moreover, (X, R) is exodromic in the sense of Definition 5.1.1 and [32, Theorem 5.4.1] implies that there is an equivalence of ∞-categories

$$\Pi_\infty(X, R) \simeq \mathrm{Sing}(X, R) .$$

That is, [32, Theorem 5.4.1] provides an *explicit* simplicial model for the exit-path ∞-category.

5.6.2 Observation. Let (X, R) be a conically stratified space with locally weakly contractible strata and let $\phi \colon R \to P$ be a map of posets. In general, the exit-path simplicial set $\mathrm{Sing}(X, P)$ need not be an ∞-category. Write $\widetilde{\mathrm{Sing}}(X, P)$ for the fibrant replacement of $\mathrm{Sing}(X, P)$ in the Joyal model structure on simplicial sets over (the nerve of) P. By construction, the composite

$$\Pi_\infty(X, R) \simeq \mathrm{Sing}(X, R) \longrightarrow \mathrm{Sing}(X, P) \longrightarrow \widetilde{\mathrm{Sing}}(X, P)$$

carries all morphisms in W_P to equivalences. By Theorem 5.1.7 and the universal property of the localization, this induces a functor

$$\Pi_\infty(X, P) \simeq \Pi_\infty(X, R)[W_P^{-1}] \longrightarrow \widetilde{\mathrm{Sing}}(X, P) .$$

Moreover, [20, Lemma 2.5.2] and Theorem 5.1.7-(1) imply that for each $p \in P$, the induced map on strata

$$\Pi_\infty(X, P) \times_P \{p\} \to \widetilde{\mathrm{Sing}}(X, P) \times_P \{p\}$$

is an equivalence of ∞-groupoids.

5.6.3. Note that *if* the functor $\Pi_\infty(X, P) \to \widetilde{\mathrm{Sing}}(X, P)$ is an equivalence of ∞-categories, then Proposition 5.2.9 implies that there is an equivalence of ∞-categories

$$\mathrm{Cons}_P^{\mathrm{hyp}}(X) \simeq \mathrm{Fun}(\mathrm{Sing}(X, P), \mathbf{Spc}) .$$

That is, even though Lurie's exit-path simplicial set $\mathrm{Sing}(X, P)$ may not be an ∞-category, $\mathrm{Sing}(X, P)$ still corepresents hyperconstructible hypersheaves.

5.6.4 Question. In the setting of Observation 5.6.2, is the functor

$$\Pi_\infty(X,P) \to \widetilde{\mathrm{Sing}}(X,P)$$

an equivalence of ∞-categories? If not, what are some mild conditions on the stratified space (X,P) that guarantee that this functor is an equivalence?

Appendix A Inverting arrows over a poset

Let P be a poset. In Theorem 3.0.1, we are interested in the following situation: we have an ∞-category $\mathcal{C}$ and functor $F\colon \mathcal{C} \to P$, and we want to form the localization of $\mathcal{C}$ at the set W_P of morphisms that F carries to identities in P. There are two goals of this appendix. First, we show that for each $p \in P$, the fiber of $\mathcal{C}[W_P^{-1}]$ over p coincides with the classifying space of the fiber $\mathcal{C} \times_P \{p\}$; see Proposition A.2.2. From this we deduce that the natural functor $\mathcal{C}[W_P^{-1}] \to P$ is conservative and that $\mathcal{C}[W_P^{-1}]$ is idempotent complete. Second, we show that if $\mathcal{C}$ is finite (resp., compact), then the localization $\mathcal{C}[W_P^{-1}]$ is also finite (resp., compact). See Proposition A.3.16.

In §A.1, we review some basic facts about ∞-categories with a conservative functor to a poset. Subsection A.2 proves structural results about the localization $\mathcal{C}[W_P^{-1}]$. In §A.3, we explain various characterizations of finiteness and compactness in the ∞-category of ∞-categories with a conservative functor to the poset P. We use these characterizations to prove stability properties of finite and compact ∞-categories with over P.

A.1 Layered ∞-categories. We start by collecting background material about the types of ∞-categories that arise as exit-path ∞-categories of stratified spaces.

A.1.1 Recollection. Let $F\colon \mathcal{C} \to P$ be a functor from an ∞-category to a poset. The following are equivalent:

(1) The functor $F\colon \mathcal{C} \to P$ is conservative.

(2) For each $p \in P$, the fiber $\mathcal{C} \times_P \{p\}$ is an ∞-groupoid.

A.1.2 Recollection. Let $\mathcal{C}$ be an ∞-category. The following are equivalent:

(1) There exists a poset P and a conservative functor $\mathcal{C} \to P$.

(2) For each $x \in \mathcal{C}$, every endomorphism $x \to x$ is an equivalence.

If these equivalent conditions are satisfied, we say that $\mathcal{C}$ is a *layered ∞-category*. By the stratified homotopy hypothesis, an ∞-category $\mathcal{C}$ is layered if and only if $\mathcal{C}$ is equivalent to the exit-path ∞-category of a stratified space; see [20, Theorem 0.1.1] for a precise formulation of this result.

An important fact is that layered ∞-categories are idempotent complete. For this, recall Notation 3.3.6.

A.1.3 Lemma. *Let $\mathcal{C}$ be layered ∞-category. Then:*

(1) *If $e\colon x \to x$ is a morphism in $\mathcal{C}$ such that there exists an equivalence $e^2 \simeq e$, then $e \simeq \mathrm{id}_x$.*

(2) *The ∞-category $\mathcal{C}$ is idempotent complete.*

Proof. For (1), note that since $\mathcal{C}$ is layered, the morphism e is an equivalence. Since $e^2 \simeq e$, the fact that e is invertible implies that $e \simeq \mathrm{id}_x$. For (2), observe that since $\mathcal{C}$ is layered, every idempotent $e\colon \mathbf{Idem} \to \mathcal{C}$ factors through the maximal sub-∞-groupoid $\mathcal{C}^{\simeq}$ of $\mathcal{C}$. Hence e descends to a functor $\mathrm{Env}(\mathbf{Idem}) \to \mathcal{C}^{\simeq}$. Since $\mathrm{Env}(\mathbf{Idem})$ is contractible [HTT, Lemma 4.4.5.10], we conclude that e splits. □

A.2 Strata of localizations. The purpose of this subsection is to prove a fundamental proposition about the types of localizations that appear in Theorem 3.0.1-(3). To state it, we need to fix some notation.

A.2.1 Notation. Let $F\colon \mathcal{C} \to P$ be a functor from an ∞-category to a poset.

(1) Given a subposet $S \subset P$, we write $F_S\colon \mathcal{C}_S \to S$ for the basechange of $F\colon \mathcal{C} \to P$ to S.

(2) We write $W_P \subset \mathrm{Mor}(\mathcal{C})$ for the set of morphisms in $\mathcal{C}$ that F sends to equivalences (i.e., identities) in P.

By construction, functor F uniquely extends to a functor $\mathcal{C}[W_P^{-1}] \to P$.

A.2.2 Proposition. *Let $F: \mathcal{C} \to P$ be a functor between ∞-categories where P is a poset. Then:*

(1) *For each locally closed subposet $S \subset P$, the induced functor $\mathcal{C}_S[W_S^{-1}] \to \mathcal{C}[W_P^{-1}]_S$ is an equivalence.*

(2) *The induced functor $\mathcal{C}[W_P^{-1}] \to P$ is conservative. In particular, the ∞-category $\mathcal{C}[W_P^{-1}]$ is idempotent complete.*

Since localizations do not generally commute with pullbacks, Proposition A.2.2 is not completely formal. To prove Proposition A.2.2, we recall the following description of localizations.

A.2.3 Recollection (localizations as pushouts)**.** Let $\mathcal{C}$ be an ∞-category and let $W \subset \operatorname{Mor}(\mathcal{C})$ be a class of morphisms. The localization $\mathcal{C}[W^{-1}]$ can be defined as the pushout

$$\begin{array}{ccc} \coprod_{w \in W} [1] & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \coprod_{w \in W} * & \longrightarrow & \mathcal{C}[W^{-1}] . \end{array}$$

Here, the top horizontal functor is the induced by the functors $[1] \to \mathcal{C}$ that pick out each morphism $w \in W$.

Hence Proposition A.2.2 amounts to commuting the pullback $S \times_P (-)$ past the pushout defining the localization $\mathcal{C}[W_P^{-1}]$. To explain why we can do this, we recall some categorical notions.

A.2.4 Recollection. A functor $F: \mathcal{C} \to \mathcal{D}$ is an *exponentiable fibration* if the right adjoint pullback functor

$$\mathcal{C} \times_{\mathcal{D}} (-): \mathbf{Cat}_{\infty,/\mathcal{D}} \to \mathbf{Cat}_{\infty,/\mathcal{C}}$$

is also a left adjoint. Note that the class of exponentiable fibrations is closed under basechange.

A.2.5 Example [5, Lemma 2.15]**.** Cartesian and cocartesian fibrations are exponentiable fibrations. In particular, right and left fibrations are exponentiable fibrations.

Recall that for any ∞-category $\mathcal{C}$, the unique functor $\mathcal{C} \to *$ is both a cartesian and a cocartesian fibration. In this case, the right adjoint to $\mathcal{C} \times (-): \mathbf{Cat}_\infty \to \mathbf{Cat}_{\infty,/\mathcal{C}}$ is given by sending $\mathcal{B} \to \mathcal{C}$ to the ∞-category of sections $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{B})$.

A.2.6 Lemma. *Let P be a poset.*

(1) *If $U \subset P$ is an open subposet, then the inclusion $U \hookrightarrow P$ is a left fibration.*

(2) *If $Z \subset P$ is a closed subposet, then the inclusion $Z \hookrightarrow P$ is a right fibration.*

(3) *If $S \subset P$ is a locally closed subposet, then the inclusion $S \hookrightarrow P$ is an exponentiable fibration.*

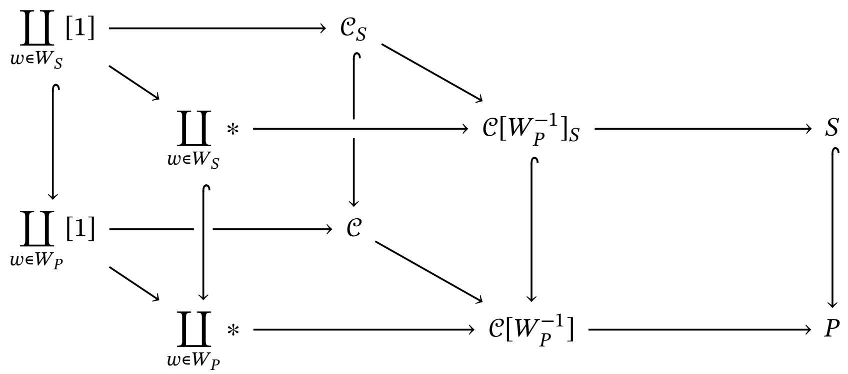
Proof. For (1), first observe that the inclusion $\{1\} \hookrightarrow \{0 < 1\}$ is a left fibration. Let $\chi_U: P \to \{0 < 1\}$ be the map sending U to 1 and $P \smallsetminus U$ to 0. Then we have a pullback square

$$\begin{array}{ccc} U & \hookrightarrow & P \\ \downarrow & & \downarrow \chi_U \\ \{1\} & \hookrightarrow & \{0 < 1\} . \end{array}$$

The claim now follows from the fact that the class of left fibrations is closed under basechange.

Item (2) follows from (1) by passing to opposite posets. Item (3) follows from (1), (2), Example A.2.5, and the fact that exponentiable fibrations are closed under composition. □

Proof of Proposition A.2.2. For (1), consider the commutative diagram



Notice that by Recollection A.2.3, the bottom face is a pushout. Moreover, all of the vertical faces are pullbacks. Since the inclusion $S \hookrightarrow P$ is an exponentiable fibration (Lemma A.2.6), the top face is also a pushout; again applying Recollection A.2.3 completes the proof.

For (2), note that by Recollection A.1.1, to show that $\mathcal{C}[W_P^{-1}] \to P$ is conservative, we need to show that each fiber $\mathcal{C}[W_P^{-1}]_p$ is an ∞-groupoid. To see this, note that for each $p \in P$, part (1) provides an identification

$$\mathcal{C}[W_P^{-1}]_p \simeq \mathcal{C}_p[W_p^{-1}] .$$

To complete the proof, observe that W_p is the set of *all* morphisms in $\mathcal{C}_p$. □

A.3 Compactness. The goal of this subsection is to characterize the compact objects of $\mathbf{Cat}_{\infty,/P}$ as well as the compact objects of the full subcategory spanned by the conservative functors $\mathcal{C} \to P$ (Lemma A.3.10 and Corollary A.3.11). We then use this to explain why the assignment $\mathcal{C} \mapsto \mathcal{C}[W_P^{-1}]$ and pulling back to a locally closed subposet $S \subset P$ both preserve compactness; see Propositions A.3.16 and A.3.17. We begin by introducing some notation.

A.3.1 Recollection (finite & compact ∞-categories)**.** Write $\mathbf{Cat}_\infty^{\mathrm{fin}} \subset \mathbf{Cat}_\infty$ for the smallest full subcategory closed under pushouts and containing the ∞-categories $\varnothing$, $*$, and $[1]$. An ∞-category $\mathcal{C}$ is *finite* if $\mathcal{C} \in \mathbf{Cat}_\infty^{\mathrm{fin}}$. In particular, $\mathbf{Cat}_\infty^{\mathrm{fin}}$ is closed under finite colimits in $\mathbf{Cat}_\infty$. Equivalently, an ∞-category $\mathcal{C}$ is finite if and only if $\mathcal{C}$ is categorically equivalent to a simplicial set with only finitely many nondegenerate simplicies [40, Corollary 2.3].

Importantly, the full subcategory $\mathbf{Cat}_\infty^{\omega} \subset \mathbf{Cat}_\infty$ of compact ∞-categories is the smallest full subcategory containing $\mathbf{Cat}_\infty^{\mathrm{fin}}$ and closed under retracts.

A.3.2 Notation. Let P be a poset and write $\mathbf{Cat}_{\infty,/P}^{\mathrm{cons}} \subset \mathbf{Cat}_{\infty,/P}$ for the full subcategory spanned by those objects such that the specified functor $\mathcal{C} \to P$ is conservative.

We now establish some pleasant features of the inclusion $\mathbf{Cat}_{\infty,/P}^{\mathrm{cons}} \subset \mathbf{Cat}_{\infty,/P}$. See [8, §2.2] for a related discussion.

A.3.3 Observation. Let P be a poset. Then Proposition A.2.2 implies that the functor

$$\mathbf{Cat}_{\infty,/P} \to \mathbf{Cat}_{\infty,/P}^{\mathrm{cons}}$$

given by the assignment $\mathcal{C} \mapsto \mathcal{C}[W_P^{-1}]$ is left adjoint to the inclusion.

We introduce a more convenient notation for this left adjoint.

A.3.4 Notation. Given a poset P, write $\mathrm{Env}_P : \mathbf{Cat}_{\infty,/P} \to \mathbf{Cat}_{\infty,/P}^{\mathrm{cons}}$ for the left adjoint to the inclusion.

A.3.5 Observation. The inclusion $\mathbf{Cat}_{\infty,/P}^{\mathrm{cons}} \subset \mathbf{Cat}_{\infty,/P}$ also admits a right adjoint

$$\iota_P : \mathbf{Cat}_{\infty,/P} \to \mathbf{Cat}_{\infty,/P}^{\mathrm{cons}}$$

defined as follows. Given a functor $F\colon \mathcal{C} \to P$, let $\iota_P(\mathcal{C}) \subset \mathcal{C}$ be the largest subcategory containing all objects such that the composite

$$\iota_P(\mathcal{C}) \longrightarrow \mathcal{C} \xrightarrow{F} P$$

is conservative. Equivalently, $\iota_P(\mathcal{C}) \subset \mathcal{C}$ is the subcategory containing all objects such that a morphism $f\colon x \to y$ in $\mathcal{C}$ lies in $\iota_P(\mathcal{C})$ if and only if one of the following disjoint conditions is satisfied:

(1) The morphism f is an equivalence in $\mathcal{C}$.

(2) The elements $F(x)$ and $F(y)$ of the poset P are not equal.

A.3.6 Observation. By definition, that the inclusion $\iota_P(\mathcal{C}) \to \mathcal{C}$ restricts to an equivalence on maximal sub-∞-groupoids.

In order to understand when $\mathrm{Env}_P(\mathcal{C})$ is compact, we make use of the following general fact:

A.3.7 Recollection [HTT, Proposition 5.5.7.2]**.** Let $f^*\colon \mathcal{D} \rightleftarrows \mathcal{C} : f_*$ be an adjunction between ∞-categories that admit filtered colimits. If f_* preserves filtered colimits, then f^* preserves compact objects. As a consequence, if f^* admits a further left adjoint $f_\sharp$, then $f_\sharp$ preserves compact objects.

A.3.8 Recollection. The right adjoint $(-)^{\simeq}\colon \mathbf{Cat}_\infty \to \mathbf{Spc}$ to the inclusion preserves filtered colimits.

A.3.9 Lemma. *Let P be a poset. Then:*

(1) *The functor $\iota_P\colon \mathbf{Cat}_{\infty,/P} \to \mathbf{Cat}^{\mathrm{cons}}_{\infty,/P}$ preserves filtered colimits.*

(2) *The inclusion $\mathbf{Cat}^{\mathrm{cons}}_{\infty,/P} \hookrightarrow \mathbf{Cat}_{\infty,/P}$ preserves compact objects.*

(3) *The functor $\mathrm{Env}_P\colon \mathbf{Cat}_{\infty,/P} \to \mathbf{Cat}^{\mathrm{cons}}_{\infty,/P}$ preserves compact objects.*

Proof. To prove (1), let $\mathcal{C}_\bullet\colon A \to \mathbf{Cat}_{\infty,/P}$ be a filtered diagram with colimit $\mathcal{C}_\infty$. Write $F_\infty\colon \mathcal{C}_\infty \to P$ for the structure functor, and for each $\alpha \in A$, write $\lambda_\alpha\colon \mathcal{C}_\alpha \to \mathcal{C}_\infty$ for the leg of the colimit cone. By the explicit description of filtered coimits in $\mathbf{Cat}_\infty$, to show that the natural functor

$$\operatorname*{colim}_{\alpha\in A} \iota_P(\mathcal{C}_\alpha) \to \iota_P(\mathcal{C}_\infty)$$

is an equivalence, it suffices to show that if $f\colon x \to y$ is a morphism in $\mathcal{C}_\infty$ and f is an equivalence or $F_\infty(x) \neq F_\infty(y)$, then f is in the image of one of the canonical functors

$$\iota_P(\mathcal{C}_\alpha) \longrightarrow \mathcal{C}_\alpha \xrightarrow{\lambda_\alpha} \mathcal{C}_\infty .$$

The case where f is an equivalence follows from the fact that the functor $(-)^{\simeq}\colon \mathbf{Cat}_\infty \to \mathbf{Spc}$ preserves filtered colimits and each inclusion $\iota_P(\mathcal{C}) \to \mathcal{C}$ restricts to an equivalence on maximal sub-∞-groupoids (Observation A.3.6).

In the case where $F_\infty(x) \neq F_\infty(y)$, notice that by the explicit description of filtered coimits in $\mathbf{Cat}_\infty$, there exists an index $\alpha \in A$ and morphism $f'\colon x' \to y'$ in $\mathcal{C}_\alpha$ such that $f \simeq \lambda_\alpha(f')$; to complete the proof of (1), it suffices to show that f' is in the subcategory $\iota_P(\mathcal{C}_\alpha)$. Since $\lambda_\alpha(x') \simeq x$ and $\lambda_\alpha(y') \simeq y$ and we have $F_\infty(x) \neq F_\infty(y)$, we deduce that the composite $F_\infty\lambda_\alpha\colon \mathcal{C}_\alpha \to P$ carries x' and y' to distinct elements of P. Hence the morphism f' is in the subcategory $\iota_P(\mathcal{C}_\alpha)$, as desired.

To finish the proof, observe that Recollection A.3.7 shows that (1) implies (2) and (3). □

Using Lemma A.3.9, we can now give a characterization of the compact objects of $\mathbf{Cat}^{\mathrm{cons}}_{\infty,/P}$.

A.3.10 Lemma. *Let $\mathcal{D}$ be an ∞-category. An object $F\colon \mathcal{C} \to \mathcal{D}$ of $\mathbf{Cat}_{\infty,/\mathcal{D}}$ is compact if and only if the ∞-category $\mathcal{C}$ is compact in $\mathbf{Cat}_\infty$.*

Proof. Since the unique functor $\mathcal{D} \to *$ is an exponentiable fibration (Example A.2.5), Recollection A.3.7 shows that the forgetful functor $\mathbf{Cat}_{\infty,/\mathcal{D}} \to \mathbf{Cat}_\infty$ preserves compact objects. Hence all that remains to be proven is that if $\mathcal{C} \in \mathbf{Cat}_\infty$ is compact, then $F \colon \mathcal{C} \to \mathcal{D}$ is compact in $\mathbf{Cat}_{\infty,/\mathcal{D}}$. For this, consider a filtered diagram $\mathcal{D}_\bullet \colon A \to \mathbf{Cat}_{\infty,/\mathcal{D}}$. Note that we have a pullback square

$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{Cat}_{\infty,/\mathcal{D}}}(\mathcal{C}, \operatorname{colim}_{\alpha\in A} \mathcal{D}_\alpha) & \longrightarrow & \mathrm{Map}_{\mathbf{Cat}_\infty}(\mathcal{C}, \operatorname{colim}_{\alpha\in A} \mathcal{D}_\alpha) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \{F\} & \longrightarrow & \mathrm{Map}_{\mathbf{Cat}_\infty}(\mathcal{C}, \mathcal{D})\,. \end{array}$$

Since $\mathcal{C}$ is compact in $\mathbf{Cat}_\infty$, the natural map

$$\operatorname*{colim}_{\alpha\in A} \mathrm{Map}_{\mathbf{Cat}_\infty}(\mathcal{C}, \mathcal{D}_\alpha) \to \mathrm{Map}_{\mathbf{Cat}_\infty}(\mathcal{C}, \operatorname{colim}_{\alpha\in A} \mathcal{D}_\alpha)$$

is an equivalence. The fact that colimits are universal in **Spc** completes the proof. □

A.3.11 Corollary. *Let P be a poset and let $F \colon \mathcal{C} \to P$ be a conservative functor from an ∞-category. Then the following are equivalent:*

(1) *The object $F \colon \mathcal{C} \to P$ of $\mathbf{Cat}^{\mathrm{cons}}_{\infty,/P}$ is compact.*

(2) *The object $F \colon \mathcal{C} \to P$ of $\mathbf{Cat}_{\infty,/P}$ is compact.*

(3) *The ∞-category $\mathcal{C}$ is a compact object of $\mathbf{Cat}_\infty$.*

Proof. The fact that both the inclusion $\mathbf{Cat}^{\mathrm{cons}}_{\infty,/P} \hookrightarrow \mathbf{Cat}_{\infty,/P}$ and its left adjoint Env_P preserve compact objects (Lemma A.3.9) shows that (1) ⇔ (2). Lemma A.3.10 shows that (2) ⇔ (3). □

A.3.12 Remark. Corollary A.3.11 was mentioned in [40, Remark 2.14].

Finiteness is also a well-behaved notion in $\mathbf{Cat}_{\infty,/P}$:

A.3.13 Definition. Given an ∞-category $\mathcal{D}$, we say that an object $F \colon \mathcal{C} \to \mathcal{D}$ of $\mathbf{Cat}_{\infty,/\mathcal{D}}$ is *finite* if the ∞-category $\mathcal{C}$ is finite.

Given a poset P, we say that an object $F \colon \mathcal{C} \to P$ of $\mathbf{Cat}^{\mathrm{cons}}_{\infty,/P}$ is finite if the ∞-category $\mathcal{C}$ is finite.

A.3.14 Notation. For the sake of convenience, let us write $[-1] \coloneqq \varnothing$ for the empty poset.

A.3.15 Observation. Let $\mathcal{D}$ be an ∞-category. Then the full subcategory

$$\mathbf{Cat}^{\mathrm{fin}}_{\infty,/\mathcal{D}} \subset \mathbf{Cat}_{\infty,/\mathcal{D}}$$

spanned by the finite objects is the smallest subcategory closed under pushouts and containing all objects of the form $\sigma \colon [n] \to \mathcal{D}$ where $-1 \le n \le 1$. Similarly,

$$\mathbf{Cat}^{\omega}_{\infty,/\mathcal{D}} \subset \mathbf{Cat}_{\infty,/\mathcal{D}}$$

is the smallest full subcategory containing $\mathbf{Cat}^{\mathrm{fin}}_{\infty,/\mathcal{D}}$ and closed under retracts.

We conclude by recording some important operations that preserve finiteness and compactness.

A.3.16 Proposition. *Let $F \colon \mathcal{C} \to P$ be a functor from an ∞-category to a poset. If $\mathcal{C}$ is a finite (resp., compact) object of $\mathbf{Cat}_\infty$, then the ∞-category*

$$\mathrm{Env}_P(\mathcal{C}) = \mathcal{C}[W_P^{-1}]$$

is a finite (resp., compact) object of $\mathbf{Cat}_\infty$.

Proof. In light of Observation A.3.15, it suffices to show that Env_P preserves finite objects. Moreover, to prove this, it suffices to show that for $-1 \le n \le 1$ and each map of posets $\sigma \colon [n] \to P$, the localization $\mathrm{End}_P([n])$ is finite. If $n = -1$ or $n = 0$, then $\mathrm{Env}_P([n]) = [n]$, so the claim is clear.

If $n = 1$, then there are two cases. First, if the map $\sigma \colon [1] \to P$ is constant, then the class W_P consists of all morphisms in P, hence $\mathrm{Env}_P([1]) \simeq *$ is finite. Second, if the map $\sigma \colon [1] \to P$ is *not* constant, then the class W_P consists of only the identity morphisms in P, hence $\mathrm{Env}_P([1]) \simeq [1]$ is finite. □

A.3.17 Proposition. *Let P be a poset and let $S \subset P$ be a locally closed subposet. Then the basechange functor*

$$S \times_P (-) \colon \mathbf{Cat}_{\infty,/P} \to \mathbf{Cat}_{\infty,/S}$$

preserves finite and compact objects.

Proof. Since the inclusion $S \hookrightarrow P$ is an exponentiable fibration (Lemma A.2.6), the functor $S \times_P (-)$ preserves colimits. Hence by Observation A.3.15, it suffices to prove that $S \times_P (-)$ preserves finite objects. Moreover, to prove this, it suffices to show that for $-1 \leq n \leq 1$ and each map of posets $\sigma \colon [n] \to P$, the basechange $S \times_P [n]$ is finite. To conclude, observe that since $S \subset P$ is locally closed, $S \times_P [n] \subset [n]$ is also locally closed; hence, there exists $-1 \leq m \leq n$ such that $S \times_P [n] \cong [m]$. □

The following application of Proposition A.3.17 is not needed in the present paper, but is quite useful:

A.3.18 Lemma. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Then the join $\mathcal{C} \star \mathcal{D}$ is finite (resp., compact) if and only if both $\mathcal{C}$ and $\mathcal{D}$ are finite (resp., compact).*

Proof. By definition, the join $\mathcal{C} \star \mathcal{D}$ is the colimit in $\mathbf{Cat}_\infty$ of the diagram

$$\begin{array}{ccccc} & \mathcal{C} \times \mathcal{D} \times \{0\} & & \mathcal{C} \times \mathcal{D} \times \{1\} & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ \mathcal{C} & & \mathcal{C} \times \mathcal{D} \times [1] & & \mathcal{D}\,, \end{array}$$

where the outermost functors are the projections. Furthermore, the unique functors $\mathcal{C} \to \{0\}$ and $\mathcal{D} \to \{1\}$ induce a functor

$$\mathcal{C} \star \mathcal{D} \longrightarrow \{0\} \star \{1\} \cong [1]$$

with fibers $(\mathcal{C} \star \mathcal{D})_0 \simeq \mathcal{C}$ and $(\mathcal{C} \star \mathcal{D})_1 \simeq \mathcal{D}$. In particular, the forward implication follows from the fact that finite (resp., compact) ∞-categories are stable under finite products and finite colimits. The reverse implcation follows from Proposition A.3.17 applied to the induced functor $\mathcal{C} \star \mathcal{D} \to [1]$. □

Of particular interest are cones:

A.3.19 Corollary. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is finite (resp., compact) if and only if the cone $\mathcal{C}^{\triangleleft}$ is finite (resp., compact).*

APPENDIX B COMPLEMENTS ON ∞-TOPOI

The purpose of this appendix is to prove some fundamental results about ∞-topoi that are used in the main body of the paper. In § B.1, we recall the basics of étale geometric morphisms as well as open and closed immersions of ∞-topoi. In § B.2, we explain how hypercompletion interacts with étale geometric morphisms. In § B.3, we prove that the hypercompletion of a recollement of ∞-topoi is still a recollement (Proposition B.3.5). We then use this to explain how hypercompletion interacts with locally closed immersions of ∞-topoi (Corollary B.3.7 and Lemma 2.4.2).

B.1 Open and closed subtopoi. In this subsection, we recall the notions of open and closed immersions of ∞-topoi and how they give rise to recollements. In order to discuss open immersions, we start with the more general notion of a *étale* geometric morphisms. For more background on étale geometric morphisms, the reader should consult [HTT, §6.3.5].

B.1.1 Recollection (étale geometric morphisms)**.** Let $\mathcal{X}$ be an ∞-topos and $U \in \mathcal{X}$. Then the overcategory $\mathcal{X}_{/U}$ is an ∞-topos. Moreover, the forgetful functor $p_\sharp \colon \mathcal{X}_{/U} \to \mathcal{X}$ admits a right adjoint $p^* \colon \mathcal{X} \to \mathcal{X}_{/U}$ given by the assignment $X \mapsto X \times U$. Since colimits are universal in $\mathcal{X}$, the functor p^* admits a further right adjoint $p_* \colon \mathcal{X}_{/U} \to \mathcal{X}$. See [HTT, Proposition 6.3.5.1]. We always regard the ∞-topos $\mathcal{X}_{/U}$ as an ∞-topos over $\mathcal{X}$ via the natural geometric morphism $p_* \colon \mathcal{X}_{/U} \to \mathcal{X}$.

Let $e_* \colon \mathcal{W} \to \mathcal{X}$ be a geometric morphism of ∞-topoi. Then the following conditions are equivalent:

(1) There exists an object $U \in \mathcal{X}$ and an equivalence $\mathcal{W} \xrightarrow{\sim} \mathcal{X}_{/U}$ of ∞-topoi over $\mathcal{X}$.

(2) The functor e^* admits a left adjoint $e_\sharp \colon \mathcal{W} \to \mathcal{X}$ and the induced functor

$$e_\sharp \colon \mathcal{W} \to \mathcal{X}_{/e_\sharp(1_{\mathcal{W}})}$$

is an equivalence of ∞-categories.

(3) The functor e^* admits a conservative left adjoint $e_\sharp \colon \mathcal{W} \to \mathcal{X}$ and for all maps $X \to Z$ in $\mathcal{X}$, objects $Y \in \mathcal{W}$, and maps $f_\sharp(Y) \to Z$, the natural map

$$e_\sharp\left(e^*(X) \underset{e^*(Z)}{\times} Y\right) \to X \times_Z e_\sharp(Y)$$

is an equivalence.

See [HTT, Proposition 6.3.5.11]. We call a geometric morphism satisfying these equivalent conditions an *étale* geometric morphism.

B.1.2 Recollection (open immersions)**.** Let $j_* \colon \mathcal{U} \to \mathcal{X}$ be a geometric morphism of ∞-topoi. Then the following conditions are equivalent:

(1) There exists a (-1)-truncated object $U \in \mathcal{X}$ and an equivalence $\mathcal{U} \simeq \mathcal{X}_{/U}$ of ∞-topoi over $\mathcal{X}$.

(2) The geometric morphism $j_* \colon \mathcal{U} \to \mathcal{X}$ is étale and $j_\sharp(1_{\mathcal{U}}) \in \mathcal{X}$ is (-1)-truncated.

(3) The geometric morphism $j_* \colon \mathcal{U} \to \mathcal{X}$ is étale and the functor j_* is fully faithful.

We call a geometric morphism satisfying these equivalent conditions an *open immersion* of ∞-topoi. Also notice that in this situation, $j_\sharp$ is fully faithful. For open immersions of ∞-topoi, we write $j_! \coloneqq j_\sharp$.

B.1.3 Recollection (closed immersions)**.** Let $\mathcal{X}$ be an ∞-topos and let $U \in \mathcal{X}$ be a (-1)-truncated object. We write

$$\mathcal{X}_{\smallsetminus U} \subset \mathcal{X}$$

for the full subcategory spanned by those objects F such that the projection $\mathrm{pr}_2 \colon F \times U \to U$ is an equivalence. The inclusion $\mathcal{X}_{\smallsetminus U} \subset \mathcal{X}$ is accessible and admits a left exact left adjoint [HTT, Proposition 7.3.2.3]. In particular, $\mathcal{X}_{\smallsetminus U}$ is an ∞-topos and the inclusion $\mathcal{X}_{\smallsetminus U} \hookrightarrow \mathcal{X}$ is a geometric morphism. We call the ∞-topos $\mathcal{X}_{\smallsetminus U}$ the *closed complement* of the open subtopos $\mathcal{X}_{/U}$.

We say that a geometric morphism of ∞-topoi $i_* \colon \mathcal{Z} \to \mathcal{X}$ is a *closed immersion* if there exists a (-1)-truncated object $U \in \mathcal{X}$ such that i_* factors through $\mathcal{X}_{\smallsetminus U}$ and restricts to an equivalence $i_* \colon \mathcal{Z} \simeq \mathcal{X}_{\smallsetminus U}$.

B.1.4 Definition. Let $f_* \colon \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. We say that f_* is a *locally closed immersion* if there exists a factorization $f_* \simeq j_* i_*$ where i_* is a closed immersion and j_* is an open immersion.

B.1.5 Recollection. Let X be a topological space and let $j \colon U \hookrightarrow X$ be an open subspace with closed complement $i \colon Z \hookrightarrow X$. Also write $U \in \mathrm{Sh}(X)$ for the sheaf represented by the open subset $U \subset X$. Then:

(1) The geometric morphism $j_* \colon \mathrm{Sh}(U) \hookrightarrow \mathrm{Sh}(X)$ is an open immersion that identifies $\mathrm{Sh}(U)$ with $\mathrm{Sh}(X)_{/U}$.

(2) The geometric morphism $i_* \colon \mathrm{Sh}(Z) \hookrightarrow \mathrm{Sh}(X)$ is a closed immersion that identifies $\mathrm{Sh}(Z)$ with $\mathrm{Sh}(X)_{\smallsetminus U}$.

See [HTT, Corollary 7.3.2.10].

As a consequence, locally closed immersions of topological spaces induce locally closed immersions of ∞-topoi of sheaves.

The key feature of open and closed immersions is that they give rise to recollements:

B.1.6 Recollection (open-closed recollement)**.** Let $\mathcal{X}$ be an ∞-topos and $U \in \mathcal{X}$ a (-1)-truncated object. Write $i_* \colon \mathcal{X}_{\smallsetminus U} \hookrightarrow \mathcal{X}$ and $j_* \colon \mathcal{X}_{/U} \to \mathcal{X}$ for the complementary closed and open geometric morphisms. Then the functors

$$i^* \colon \mathcal{X} \to \mathcal{X}_{\smallsetminus U} \qquad \text{and} \qquad j^* \colon \mathcal{X} \to \mathcal{X}_{/U}$$

exhibit $\mathcal{X}$ as the recollement of $\mathcal{X}_{\smallsetminus U}$ and $\mathcal{X}_{/U}$.

In light of Recollections B.1.5 and B.1.6, we see:

B.1.7 Example. Let X be a topological space and let $i \colon Z \hookrightarrow X$ be a closed subspace with open complement $j \colon U \hookrightarrow X$. Then the functors

$$i^* \colon \mathrm{Sh}(X) \to \mathrm{Sh}(Z) \qquad \text{and} \qquad j^* \colon \mathrm{Sh}(X) \to \mathrm{Sh}(U)$$

exhibit $\mathrm{Sh}(X)$ as the recollement of $\mathrm{Sh}(Z)$ and $\mathrm{Sh}(U)$.

Étale geometric morphisms and closed immersions also behave well under basechange.

B.1.8 Proposition. *Let $f_* \colon \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi and let $V \in \mathcal{Y}$. Then:*

(1) *The induced square*

$$\begin{array}{ccc} \mathcal{X}_{/f^*(V)} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f_* \\ \mathcal{Y}_{/V} & \longrightarrow & \mathcal{Y} \end{array}$$

is a pullback square in $\mathbf{RTop}_\infty$.

(2) *If V is (-1)-truncated, then the induced square*

$$\begin{array}{ccc} \mathcal{X}_{\smallsetminus f^*(V)} & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f_* \\ \mathcal{Y}_{\smallsetminus V} & \hookrightarrow & \mathcal{Y} \end{array}$$

is a pullback square in $\mathbf{RTop}_\infty$.

Proof. For (1), see [HTT, Remark 6.3.5.8]. For (2), see [HTT, Proposition 7.3.2.12]. ☐

B.1.9. As a consequence of Proposition B.1.8 the properties being étale, an open immersion, a closed immersion, or a locally closed immersion are all stable under basechange in $\mathbf{RTop}_\infty$.

In general, the functor sending a topological space X to the ∞-topos $\mathrm{Sh}(X)$ does not preserve pullbacks. However, Proposition B.1.8 implies that the assignment $X \mapsto \mathrm{Sh}(X)$ *does* preserve pullbacks along locally closed immersions:

B.1.10 Corollary. *Let*

$$\begin{array}{ccc} S & \overset{\bar{\imath}}{\hookrightarrow} & X \\ \downarrow & \lrcorner & \downarrow f \\ T & \underset{i}{\hookrightarrow} & Y \end{array}$$

be a pullback square of topological spaces where i is a locally closed immersion. Then the induced square of ∞-topoi

$$\begin{array}{ccc} \mathrm{Sh}(S) & \overset{\bar{\imath}_*}{\hookrightarrow} & \mathrm{Sh}(X) \\ \downarrow & & \downarrow f_* \\ \mathrm{Sh}(T) & \underset{i_*}{\hookrightarrow} & \mathrm{Sh}(Y) \end{array}$$

is a pullback square in $\mathbf{RTop}_\infty$.

Proof. Note that by factoring i as a closed immersion followed by an open immersion, it suffices to treat the cases of closed immersions and open immersions separately. Since $S = f^{-1}(T)$, the claim is immediate from Recollection B.1.5 and Proposition B.1.8. ☐

B.2 Hypercompleteness & étale geometric morphisms. The purpose of this subsection is to prove the following characterization the hypercomplete objects of the slice ∞-topos over a hypercomplete object.

B.2.1 Proposition. *Let $\mathcal{X}$ be an ∞-topos and let $U \in \mathcal{X}$ be a hypercomplete object. Write $e_* : \mathcal{X}_{/U} \to \mathcal{X}$ for the natural geometric morphism. For an object $[p : X \to U] \in \mathcal{X}_{/U}$, the following are equivalent:*

(1) *The object $p : X \to U$ is a hypercomplete object of $\mathcal{X}_{/U}$.*

(2) *The object X is a hypercomplete object of $\mathcal{X}$.*

In particular, there is a natural identification

$$(\mathcal{X}^{\mathrm{hyp}})_{/U} = (\mathcal{X}_{/U})^{\mathrm{hyp}}$$

as full subcategories of $\mathcal{X}_{/U}$.

B.2.2 Corollary. *Let $\mathcal{X}$ be an ∞-topos and let $U \in \mathcal{X}$. If $\mathcal{X}$ is hypercomplete, then the ∞-topos $\mathcal{X}_{/U}$ is hypercomplete.*

To prove Proposition B.2.1, we need a few technical lemmas. The first is a slight refinement of the statement of [HA, Lemma A.2.6]:

B.2.3 Lemma. *Let $e_* : \mathcal{W} \to \mathcal{X}$ be a geometric morphism of ∞-topoi. Assume that e^* admits a left adjoint $e_\sharp : \mathcal{W} \to \mathcal{X}$. Then:*

(1) *For each $-2 \leq n \leq \infty$, the functor $e_\sharp$ preserves n-connected maps.*

(2) *The functor $e^* : \mathcal{X} \to \mathcal{W}$ preserves hypercomplete objects.*

B.2.4 Lemma. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories.*

(1) *Let $\mathcal{I}$ be an ∞-category. Assume that $\mathcal{C}$ and $\mathcal{D}$ admit $\mathcal{I}$-shaped colimits and that F preserves $\mathcal{I}$-shaped colimits. If F is conservative, then F reflects $\mathcal{I}$-shaped colimits.*

(2) *Assume that $\mathcal{C}$ and $\mathcal{D}$ admit pullbacks and geometric realizations of simplicial objects and that F preserves pullbacks and geometric realizations. If F is conservative, then F reflects effective epimorphisms.*

Proof. For (1), let $X_\bullet : \mathcal{I}^{\triangleright} \to \mathcal{C}$ be a diagram, and assume that the composite diagram $F \circ X_\bullet : \mathcal{I}^{\triangleright} \to \mathcal{D}$ is a colimit diagram. Write X_∞ for the value of the cone point and let $\lambda : \operatorname{colim}_{i \in \mathcal{I}} X_i \to X_\infty$ denote the natural map. Then $F(\lambda)$ factors as a composite of natural maps

$$F\left(\operatorname*{colim}_{i \in \mathcal{I}} X_i\right) \longrightarrow \operatorname*{colim}_{i \in \mathcal{I}} F(X_i) \longrightarrow F(X_\infty)\,.$$

Since F preserves colimits, the left-hand map is an equivalence; since $F \circ X_\bullet$ is a colimit diagram, the right-hand map is also an equivalence. Since F is conservative, we deduce that λ is an equivalence, i.e., that $X_\bullet$ is a colimit diagram, as desired.

Item (2) is immediate from the definition of an effective epimorphism combined with item (1) and its dual. □

B.2.5 Lemma. *Let $\mathcal{X}$ be an ∞-topos and let $\{f_\alpha^* : \mathcal{X} \to \mathcal{X}_\alpha\}_{\alpha \in A}$ be a jointly conservative family of functors between ∞-topoi that each preserve pullbacks and geometric realizations of simplicial objects. Let $-2 \leq n \leq \infty$ and let $\phi : U \to V$ be a morphism in $\mathcal{X}$. Then the following are equivalent:*

(1) *The morphism ϕ is n-connected.*

(2) *For each $\alpha \in A$, the morphism $f_\alpha^*(\phi)$ is n-connected.*

Proof. Since functors that preserve pullbacks and geometric realizations of simplicial objects preserve n-connectedness, (1) ⇒ (2). For the implication (2) ⇒ (1), first note a morphism ϕ is ∞-connected map if and only if for each $n < \infty$, the morphism ϕ is n-connected. So it suffices to treat the case of finite n. Write $\mathcal{Y}$ for the product of ∞-categories $\prod_{\alpha \in A} \mathcal{X}_\alpha$ and $f^* : \mathcal{X} \to \mathcal{Y}$ for the functor induced by the functors $f_\alpha^* : \mathcal{X} \to \mathcal{X}_\alpha$ by the universal property of the product. Note that $\mathcal{Y}$ is an ∞-topos and since limits and colimits in $\mathcal{Y}$ are computed levelwise, f^* also preserves pullbacks and effective epimorphisms. Moreover, the statement (2) is equivalent to the statement:

(3) The morphism $f^*(\phi)$ is n-connected.

So we instead prove that (3) ⇒ (1).

We prove the claim by induction on n. The case $n = -2$ is clear; every morphism is (-2)-connected. For the case $n = -1$, recall that a morphism ϕ is (-1)-connected if and only if ϕ is an effective epimorphism. The claim now follows from Lemma B.2.4-(2).

For the inductive step, assume that $n \geq 0$, and that we know that for all $k \leq n$, the functor $f^* : \mathcal{X} \to \mathcal{Y}$ reflects k-connectedness. Let $\phi : U \to V$ be a morphism of $\mathcal{X}$ such that $f^*(\phi)$ is n-connected. That is $f^*(\phi)$ is an effective epimorphism and the diagonal

$$\Delta_{f^*(\phi)} : f^*(U) \to f^*(U) \underset{f^*(V)}{\times} f^*(U)$$

is $(n-1)$-connected. By the base case, ϕ is an effective epimorphism. Moreover, since f^* preserves pullbacks,

$$\Delta_{f^*(\phi)} \simeq f^*(\Delta_\phi) .$$

The inductive hypothesis then show that Δ_ϕ is $(n-1)$-connected. Thus ϕ is n-connected, as desired. □

B.2.6 Corollary. *Let $e_* : \mathcal{W} \to \mathcal{X}$ be an étale geometric morphism of ∞-topoi and let ϕ be a morphism in $\mathcal{W}$. Then for each $-2 \leq n \leq \infty$, the morphism ϕ in $\mathcal{W}$ is n-connected if and only if $e_\sharp(\phi)$ is n-connected.*

Proof. Since the forgetful functor $e_\sharp : \mathcal{W} \to \mathcal{X}$ is a conservative left adjoint that preserves pullbacks, this is a special case of Lemma B.2.5. □

Now we are ready to prove Proposition B.2.1.

Proof of Proposition B.2.1. We start by proving that (1) ⇒ (2). Let $\phi : V \to V'$ be an ∞-connected map in $\mathcal{X}$. We need to show that $\mathrm{Map}_{\mathcal{X}}(-, X)$ inverts ϕ. Consider the commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{X}}(V', X) & \xrightarrow{p\circ -} & \mathrm{Map}_{\mathcal{X}}(V', U) \\ \downarrow{\scriptstyle -\circ\phi} & & \downarrow{\scriptstyle -\circ\phi} \\ \mathrm{Map}_{\mathcal{X}}(V, X) & \xrightarrow[p\circ -]{} & \mathrm{Map}_{\mathcal{X}}(V, U) . \end{array}$$

Since ϕ is ∞-connected and U is hypercomplete, the right-hand vertical map is an equivalence. Thus to show that the left-hand vertical map is an equivalence, it suffices to show that for each map $q : V' \to U$, the induced map on horizontal fibers is an equivalence.

For this, regard V and V' as objects of $\mathcal{X}_{/U}$ via the structure maps $q\phi$ and q, respectively; then ϕ defines a map

$$[q\phi : V \to U] \to [q : V' \to U]$$

in $\mathcal{X}_{/U}$. By the definition of the mapping spaces in an overcategory, we have a commutative square

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{X}_{/U}}(V', X) & \xrightarrow{\sim} & \{q\} \underset{\mathrm{Map}_{\mathcal{X}}(V', U)}{\times} \mathrm{Map}_{\mathcal{X}}(V', X) \\ \downarrow{\scriptstyle -\circ\phi} & & \downarrow \\ \mathrm{Map}_{\mathcal{X}_{/U}}(V, X) & \xrightarrow{\sim} & \{q\phi\} \underset{\mathrm{Map}_{\mathcal{X}}(V, U)}{\times} \mathrm{Map}_{\mathcal{X}}(V, X) , \end{array} \tag{B.2.7}$$

where the horizontal maps are equivalences and the vertical maps are given by precomposition with ϕ. Since ϕ is an ∞-connected map in $\mathcal{X}$, by Corollary B.2.6, ϕ is also an ∞-connected map when regarded as a map $V \to V'$ in $\mathcal{X}_{/U}$. Since X is a hypercomplete object of $\mathcal{X}_{/U}$, we deduce that the left-hand vertical map in (B.2.7) is an equivalence. Thus the right-hand vertical map is also an equivalence, as desired.

Now we prove that (2) ⇒ (1). Assume that X is hypercomplete when regarded as an object of $\mathcal{X}$. Let $\phi : V \to V'$ be an ∞-connected map in $\mathcal{X}_{/U}$, and write $q : V \to V'$ for the structure map. We need to show that the functor $\mathrm{Map}_{\mathcal{X}_{/U}}(-, X)$ inverts ϕ. Again consider the square (B.2.7). Since ϕ is an ∞-connected map of $\mathcal{X}_{/U}$, Corollary B.2.6 shows that ϕ is also ∞-connected when regarded as a map of $\mathcal{X}$. Since U and X are hypercomplete when regarded as objects of $\mathcal{X}$, the right-hand vertical map in (B.2.7) is an equivalence; hence the left-hand vertical map is also an equivalence, as desired. □

B.3 The hypercompletion of a recollement. This subsection has two goals. The first is to show that the hypercompletion of a recollement of ∞-topoi is still a recollement (Proposition B.3.5). The second is to show that hypercompletion preserves pullbacks along locally closed immersions of ∞-topoi (Proposition B.3.8).

We begin by using Proposition B.2.1 to describe the hypercomplete objects of a locally closed subtopos. To do this, we first observe that the pushforward along a closed immersion preserves ∞-connectedness and detects hypercompleteness.

B.3.1 Lemma. *Let $i_* \colon \mathcal{Z} \to \mathcal{X}$ be a closed immersion of ∞-topoi and ϕ a map in $\mathcal{Z}$. For each $-2 \leq n \leq \infty$, the following are equivalent:*

(1) *The map ϕ is an n-connected map of $\mathcal{Z}$.*

(2) *The map $i_*(\phi)$ is an n-connected map of $\mathcal{X}$.*

Proof. First we show that (2) ⇒ (1). Let $j_* \colon \mathcal{U} \hookrightarrow \mathcal{X}$ denote the open complement of $\mathcal{Z}$. Since i^* and j^* are jointly conservative, by Lemma B.2.5 we need to show that if ϕ is n-connected, then $i^*i_*(\phi)$ and $j^*i_*(\phi)$ are n-connected. Since i_* is fully faithful, $i^*i_*(\phi) \simeq \phi$. Thus our assumption on ϕ says that $i^*i_*(\phi)$ is n-connected. Also, j^*i_* is constant with value the terminal object, hence $j^*i_*(\phi)$ is an equivalence.

To see that (2) ⇒ (1), note that since $\phi \simeq i^*i_*(\phi)$, the claim immediately follows from the fact that i^* preserves n-connected maps. □

B.3.2 Lemma. *Let $i_* \colon \mathcal{S} \to \mathcal{X}$ be a fully faithful geometric morphism of ∞-topoi. If i_* preserves ∞-connected maps, then an object $F \in \mathcal{S}$ is hypercomplete if and only if $i_*(F) \in \mathcal{X}$ is hypercomplete.*

Proof. Since pushforwards preserve hypercompleteness, it suffices to show that if $i_*(F)$ is hypercomplete, then F is hypercomplete. Let $\phi \colon V \to V'$ be an ∞-connected map of $\mathcal{S}$. By assumption, the morphism $i_*(\phi)$ is also ∞-connected. Since $i_*(F)$ is hypercomplete, we deduce that the induced map

$$-\circ i_*(\phi) \colon \operatorname{Map}_{\mathcal{X}}(i_*(V'), i_*(F)) \to \operatorname{Map}_{\mathcal{X}}(i_*(V), i_*(F))$$

is an equivalence. Since i_* is fully faithful, the map

$$-\circ\phi \colon \operatorname{Map}_{\mathcal{S}}(V', F) \to \operatorname{Map}_{\mathcal{S}}(V, F)$$

is also an equivalence. □

B.3.3 Proposition. *Let $i_* \colon \mathcal{S} \hookrightarrow \mathcal{X}$ be a locally closed immersion of ∞-topoi. Then an object $F \in \mathcal{S}$ is hypercomplete if and only if $i_*(F) \in \mathcal{X}$ is hypercomplete.*

Proof. Since pushforwards preserve hypercompleteness, it suffices to show that if $i_*(F)$ is hypercomplete, then F is hypercomplete. By writing i_* as the composite of a closed immersion followed by an open immersion, we are reduced to treating the cases where i_* is a closed or an open immersion.

If i_* is an open immersion, note that by Lemma B.2.3, the functor i^* preserves hypercompletenss. Since i_* is fully faithful and $i_*(F)$ is hypercomplete, we deduce that $i^*i_*(F) \simeq F$ is hypercomplete.

If i_* is a closed immersion, then Lemma B.3.1 shows that i_* preserves ∞-connected maps. The claim now follows from Lemma B.3.2. □

B.3.4 Corollary. *Let $i_* \colon \mathcal{S} \hookrightarrow \mathcal{X}$ be a locally closed immersion of ∞-topoi. If $\mathcal{X}$ is hypercomplete, then $\mathcal{S}$ is hypercomplete.*

We are now ready to show that the hypercompletion of a recollement remains a recollement:

B.3.5 Proposition. *Let $\mathcal{X}$ be an ∞-topos and let $U \in \mathcal{X}$ be a (-1)-truncated object. Write $i_* \colon \mathcal{X}_{\smallsetminus U} \hookrightarrow \mathcal{X}$ and $j_* \colon \mathcal{X}_{/U} \hookrightarrow \mathcal{X}$ for the natural geometric morphisms. Then:*

(1) *There are natural identifications*

$$(\mathcal{X}_{/U})^{\mathrm{hyp}} = (\mathcal{X}^{\mathrm{hyp}})_{/U} \qquad \text{and} \qquad (\mathcal{X}_{\smallsetminus U})^{\mathrm{hyp}} = (\mathcal{X}^{\mathrm{hyp}})_{\smallsetminus U}$$

as full subcategories of $\mathcal{X}$.

(2) *The functors*

$$i^{*,\mathrm{hyp}} : \mathcal{X}^{\mathrm{hyp}} \to (\mathcal{X}_{\smallsetminus U})^{\mathrm{hyp}} \qquad \textit{and} \qquad j^{*,\mathrm{hyp}} : \mathcal{X}^{\mathrm{hyp}} \to (\mathcal{X}_{/U})^{\mathrm{hyp}}$$

exhibit $\mathcal{X}^{\mathrm{hyp}}$ *as the recollement of* $(\mathcal{X}_{\smallsetminus U})^{\mathrm{hyp}}$ *and* $(\mathcal{X}_{/U})^{\mathrm{hyp}}$.

Proof. For (1), note that the left-hand identification is a special case of Proposition B.2.1. For the right-hand identification, note that Corollary B.3.4 implies that

$$(\mathcal{X}_{\smallsetminus U})^{\mathrm{hyp}} = \mathcal{X}^{\mathrm{hyp}} \cap \mathcal{X}_{\smallsetminus U}$$

as full subcategories of $\mathcal{X}$. Since U is hypercomplete and $\mathcal{X}^{\mathrm{hyp}} \subset \mathcal{X}$ is closed under finite products, unpacking definitions we see that

$$\mathcal{X}^{\mathrm{hyp}} \cap \mathcal{X}_{\smallsetminus U} = (\mathcal{X}^{\mathrm{hyp}})_{\smallsetminus U} \,.$$

Finally, (2) is an immediate consequence of (1) and the open-closed recollement associated to a (-1)-truncated object. ☐

B.3.6 Example. Let X be a topological space and let $i : Z \hookrightarrow X$ be a closed subspace with open complement $j : U \hookrightarrow X$. From Example B.1.7 and Proposition B.3.5, we deduce that the functors

$$i^{*,\mathrm{hyp}} : \mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathrm{Sh}^{\mathrm{hyp}}(Z) \qquad \text{and} \qquad j^{*,\mathrm{hyp}} : \mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathrm{Sh}^{\mathrm{hyp}}(U)$$

exhibit $\mathrm{Sh}^{\mathrm{hyp}}(X)$ as the recollement of $\mathrm{Sh}^{\mathrm{hyp}}(Z)$ and $\mathrm{Sh}^{\mathrm{hyp}}(U)$.

In the remainder of this subsection, we use Proposition B.3.5 to prove some compatibilities between hypercompletion and pulling back along locally closed immersions. Note that since the inclusion of hypercomplete ∞-topoi into all ∞-topoi does not preserve limits, these results do not immediately follow from formal considerations.

B.3.7 Corollary. *Let* $i_* : \mathcal{S} \hookrightarrow \mathcal{X}$ *be a locally closed immersion of ∞-topoi. Then the natural square*

$$\begin{array}{ccc} \mathcal{S}^{\mathrm{hyp}} & \xhookrightarrow{i_*^{\mathrm{hyp}}} & \mathcal{X}^{\mathrm{hyp}} \\ \downarrow & & \downarrow \\ \mathcal{S} & \xhookrightarrow[i_*]{} & \mathcal{X} \end{array}$$

is a pullback square in $\mathbf{RTop}_\infty$.

Proof. By factoring i_* as the composite of a closed immersion followed by an open immersion, it suffices to treat the cases of closed and open immersions separately. These cases follow from Proposition B.3.5-(1) and the explicit description of the pullbacks along open and closed immersions of ∞-topoi (Proposition B.1.8). ☐

B.3.8 Proposition. *Let*

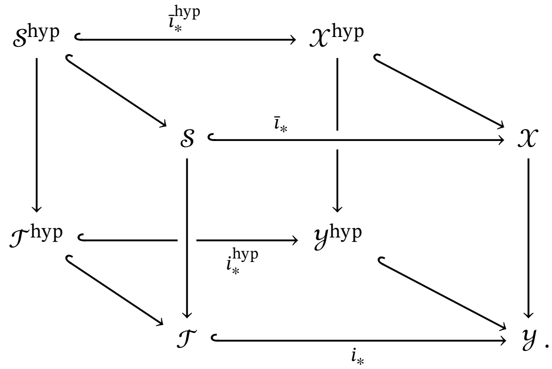
$$\begin{array}{ccc} \mathcal{S} & \xhookrightarrow{\bar{i}_*} & \mathcal{X} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{T} & \xhookrightarrow[i_*]{} & \mathcal{Y} \end{array}$$

be a pullback square of ∞-topoi where i_* *is a locally closed immersion. Then the induced square*

$$\begin{array}{ccc} \mathcal{S}^{\mathrm{hyp}} & \xhookrightarrow{\bar{i}_*^{\mathrm{hyp}}} & \mathcal{X}^{\mathrm{hyp}} \\ \downarrow & & \downarrow \\ \mathcal{T}^{\mathrm{hyp}} & \xhookrightarrow[i_*^{\mathrm{hyp}}]{} & \mathcal{Y}^{\mathrm{hyp}} \end{array}$$

is also a pullback square in $\mathbf{RTop}_\infty$.

Proof. Consider the commutative cube of ∞-topoi



By assumption, the front vertical face is a pullback square. Since i_* and $\bar{\imath}_*$ are locally closed immersions, Corollary B.3.7 shows that the top and bottom horizontal faces are pullback squares. By the gluing lemma for pullbacks, the back vertical face is also a pullback square. ☐

In general, the functor sending a topological space X to the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(X)$ does not preserve pullbacks. However, the assignment $X \mapsto \mathrm{Sh}^{\mathrm{hyp}}(X)$ *does* preserve pullbacks along locally closed immersions:

B.3.9 Corollary. *Let*

$$\begin{array}{ccc} S & \xhookrightarrow{\bar{\imath}} & X \\ \downarrow & \lrcorner & \downarrow f \\ T & \xhookrightarrow[i]{} & Y \end{array}$$

be a pullback square of topological spaces where i is a locally closed immersion. Then the induced square of ∞*-topoi*

$$\begin{array}{ccc} \mathrm{Sh}^{\mathrm{hyp}}(S) & \xhookrightarrow{\bar{\imath}_*^{\mathrm{hyp}}} & \mathrm{Sh}^{\mathrm{hyp}}(X) \\ \downarrow & & \downarrow f_*^{\mathrm{hyp}} \\ \mathrm{Sh}^{\mathrm{hyp}}(T) & \xhookrightarrow[i_*^{\mathrm{hyp}}]{} & \mathrm{Sh}^{\mathrm{hyp}}(Y) \end{array}$$

is a pullback square in $\mathbf{RTop}_\infty$.

Proof. By Corollary B.1.10, the claim is true *before* hypercompletion. Proposition B.3.8 shows that the claim remains true after hypercompletion. ☐

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