Étale homotopy theory and Exodromy

@ Frankfurt

06/16 – 06/20/2025

Goal of the lecture series. Give a "modern" introduction to étale homotopy theory

> State some applications.

Some References. Most of what we'll cover is contained in:

- Hoyois, "Higher Galois theory"
- H-Holzschuh-Wolf, "The fundamental fiber sequence in étale homotopy theory"
- H-Holzschuh-Wolf, "Nonabelian basechange theorems & étale homotopy theory"
- Barwick-Glasman-H, "Exodromy"

Lecture 1

Notation An = ∞-category of anima / spaces / ∞-groupoids

$$\mathrm{Top} \xrightarrow{\ \text{1-1}\ } \mathrm{An} \quad \text{underlying anima/homotopy type}$$

Desire of étale homotopy theory. Construct an étale homotopy type

$$\Pi_\infty^{\text{ét}} : \mathrm{Sch} \longrightarrow \mathrm{Pro}(\mathrm{An})$$

such that

$$\pi_1 \Pi_\infty^{\text{ét}}(X) \cong \pi_1^{\text{ét}}(X) \longleftarrow \text{SGA3}$$

$$H^*(\Pi_\infty^{\text{ét}}(X), R) \cong H^*_{\text{ét}}(X, R) \quad \text{discrete ring}$$

and $\Pi_\infty^{\text{ét}}(X)$ "satisfies all of the basic theorems in étale cohomology"

Plan

(1) Explain the above sentence

(2) Shape theory: construct $\Pi_\infty^{\text{ét}}$ + proof of many results

(3) Exodromy: a new/useful perspective

Background on proanima

Obs. The adjunctions $\mathrm{An} \underset{}{\overset{\tau_{\leq n}}{\rightleftarrows}} \mathrm{An}_{\leq n}$ induce adjunctions

$$\mathrm{Pro}(\mathrm{An}) \overset{\tau_{\leq n}}{\rightleftarrows} \mathrm{Pro}(\mathrm{An}_{\leq n}).$$

⚠ It is possible that a map of proanima $f: X \to Y$ induces an equivalence on all $\tau_{\leq n}$, but isn't an equivalence.

If $X \in \mathrm{An}$ regarded as a constant proanima and

$$\tau_{<\infty} K = \{ \cdots \longrightarrow \tau_{\leq n+1} K \longrightarrow \tau_{\leq n} K \longrightarrow \cdots \},$$

there is a natural map $X \longrightarrow \tau_{<\infty} X$ that's an equiv on all $\tau_{\leq n}$. However,

$$\mathrm{Map}_{\mathrm{Pro}(\mathrm{An})}(\tau_{<\infty} K, K) \simeq \underset{n \to \infty}{\mathrm{Colim}}\, \mathrm{Map}(\tau_{\leq n} K, K).$$

So every map $\tau_{<\infty} K \to K$ factors through some $\tau_{\leq n}$.
So $K \to \tau_{<\infty} K$ can't have an inverse if K isn't n-trunc. for some n.

Fact The inclusion $\mathrm{Pro}(\mathrm{An}_{<\infty}) \hookrightarrow \mathrm{Pro}(\mathrm{An})$ admits a left adjoint $\tau_{<\infty}$ that identifies

$$\mathrm{Pro}(\mathrm{An})\left[\begin{pmatrix}\text{equivs on} \\ \text{all } \tau_{\leq n}\end{pmatrix}^{-1}\right] \xrightarrow{\sim} \mathrm{Pro}(\mathrm{An}_{<\infty})$$

> We call $\tau_{<\infty}$ the **protruncation functor**

Convention We'll always protruncate and just write $\Pi_\infty^{\text{ét}}$ for $\tau_{<\infty}\Pi_\infty^{\text{ét}} = \Pi_{<\infty}^{\text{ét}}$

Def (profinite completion).

(1) An anima X is **π-finite** if $\pi_0(X)$ is finite, all homotopy groups of X are finite, and X is n-truncated for some $n \gg 0$

- $\mathrm{An}_\pi \subset \mathrm{An}$ full subcat spanned by the π-finite anima

(2) The ∞-category of **profinite anima** is $\mathrm{Pro}(\mathrm{An}_\pi)$. The inclusion $\mathrm{Pro}(\mathrm{An}_\pi) \hookrightarrow \mathrm{Pro}(\mathrm{An})$ admits a left adjoint $X \mapsto X^\wedge_\pi$ called **profinite completion**

Facts

(1) If X is a set, then $K^\wedge_\pi \simeq \beta(K)$ ← Čech-Stone compactification

- Since β doesn't preserve finite products, $(-)^\wedge_\pi$ generally doesn't either

(2) If X is a connected anima, then

$$\pi_1(K^\wedge_\pi) \cong \widehat{\pi_1(K)}$$

← profinite completion of groups

(3) In general, there exist discrete groups G for which $(BG)^\wedge_\pi$ has higher homotopy. In general,

$$(BG)^\wedge_\pi \longrightarrow B\hat{G}$$

is an equivalence if and only if G is **good** in the sense of Serre: for every finite G-module M, the natural map

$$H^*(\hat{G}, M) \longrightarrow H^*(G, M)$$

is an isomorphism

Basic properties of the étale homotopy type

Thm (profiniteness). If X is a noetherian geometrically unibranch scheme, then $\Pi_\infty^{ét}(X)$ is profinite

Fundamental fiber sequence. Let $X \longrightarrow S$ be a morphism of qcqs schemes where $\dim(S) = 0$ and let $\bar{s} \to S$ be a geometric point. Then there are fiber sequences

$$\Pi_\infty^{ét}(X_{\bar{s}}) \longrightarrow \Pi_\infty^{ét}(X) \longrightarrow \Pi_\infty^{ét}(S)$$

and

$$\hat{\Pi}_\infty^{ét}(X_{\bar{s}}) \longrightarrow \hat{\Pi}_\infty^{ét}(X) \longrightarrow \hat{\Pi}_\infty^{ét}(S).$$

Riemann Existence. $X/\mathbb{C}$ finite type. There is a natural map

$$|X(\mathbb{C})| \longrightarrow \Pi_\infty^{ét}(X)$$

that becomes an equivalence after profinite completion

Thm (Continuity) If $X_\bullet : I \longrightarrow \mathrm{Sch}$ is a cofiltered diagram of qcqs schemes with affine transition maps, then

$$\Pi_\infty^{\text{ét}}(\lim_{i\in I} X_i) \xrightarrow{\sim} \lim_{i\in I} \Pi_\infty^{\text{ét}}(X_i)$$

and

$$\hat{\Pi}_\infty^{\text{ét}}(\lim_{i\in I} X_i) \xrightarrow{\sim} \lim_{i\in I} \hat{\Pi}_\infty^{\text{ét}}(X_i).$$

Ex. If k is a field, then a choice of separable closure $\bar{k} > k$ gives an equivalence

$$\Pi_\infty^{\text{ét}}(\mathrm{Spec}(k)) \xrightarrow{\sim} B\mathrm{Gal}(\bar{k}/k)$$

Ex. $\Pi_\infty^{\text{ét}}(\mathrm{Spec}(\mathbb{Z})) \simeq *$ Spec($\mathbb{Z}$) has no unramified étale covers $\Rightarrow \pi_1^{\text{ét}} = 0$

Arithmetic duality $\Rightarrow H^{\geq 1}_{\text{ét}}(\mathrm{Spec}(\mathbb{Z}); \mathbb{Z}/n) = 0$

Hurewicz completes the proof

Hard!

Thm (Achinger 2017) If X is an affine connected $\mathbb{F}_p$-scheme, then $\hat{\Pi}_\infty^{\text{ét}}(X)$ is 1-truncated.

Descent

Thm (proétale hyperdescent) The functor

$$\Pi_\infty^{\text{ét}} : \mathrm{Sch} \longrightarrow \mathrm{Pro}(\mathrm{An}_{<\infty})$$

is a hypercomplete proétale cosheaf: for every proétale hypercovering $U_\bullet \to X$,

$$\operatorname*{colim}_{[n] \in \Delta^{op}} \Pi_\infty^{\text{ét}}(U_\bullet) \xrightarrow{\sim} \Pi_\infty^{\text{ét}}(X)$$

Computation If U is a [w-contractible] affine scheme, then

$$\Pi_\infty^{\text{ét}}(U) \simeq \pi_0(U) \leftarrow \text{profinite set of connected components of } U$$

U w-contr $\Rightarrow \pi_0(U)$ extr disconn

Any fflat $\underbrace{\text{weakly étale}}_{\text{flat w/ flat diagonal}}$ map $X \to U$ admits a section

Def A is <u>w-local</u> if $\mathrm{Spec}(A)_{cl} \subset \mathrm{Spec}(A)$ is closed and any conn component of $\mathrm{Spec}(A)$ has a unique closed point

Thm[BS, 1.8]. $\mathrm{Spec}(A)$ is w-contractible iff A is w-local, all local rings at closed points are strictly henselian, and $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected.

Cor The étale homotopy type

$$\Pi_\infty^{\text{ét}} : \mathrm{Sch} \longrightarrow \mathrm{Pro}(\mathrm{An}_{<\infty})$$

is the unique hypercomplete proétale cosheaf whose restriction to w-contractible affines is

$$\pi_0 \;\; \mathrm{Aff}^{wc} \longrightarrow \mathrm{Extr} \hookrightarrow \mathrm{Pro}(\mathrm{An}_{<\infty}).$$

Thm (arc-descent, H-Holzschuh-Wolf after Bhatt-Mathew)

The profinite étale homotopy type

$$\hat{\Pi}_\infty^{\text{ét}} : \mathrm{Sch}^{\mathrm{qcqs}} \longrightarrow \mathrm{Pro}(\mathrm{An}_\pi)$$

is a hypercomplete arc-cosheaf.

Cor.

(1) If

"Milnor excision"

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & X \arrow[d, "f"] \\ Z' \arrow[r, "i"] & X' \end{tikzcd} \quad (\square)$$

is a **Milnor square**, i.e., cartesian, f affine, i closed immersion, and cocartesian,

then $\widehat{\Pi}^{ét}_{\infty}$ takes $(\square)$ to a pullback

(2) Given a ring hom $A \longrightarrow B$ and a fg ideal $I \subset A$ such that $\forall n \geq 0$, we have $A/I^n \xrightarrow{\sim} B/I^nB$, the square

$$\begin{tikzcd} \mathrm{Spec}(B) \setminus V(IB) \arrow[r, hook] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\ \mathrm{Spec}(A) \setminus V(I) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd}$$

induces a pullback on profinite étale homotopy types

Künneth formulas

Def Σ set of prime numbers for p prime, $p' := \{\text{primes}\} \setminus \{p\}$

(1) A finite group G is a Σ-group if $\# G$ is in the multiplicative closure of Σ

(2) A π-finite anima X is Σ-finite if all the homotopy groups of X are Σ-groups

- $An_\Sigma \subset An_\pi$.

(3) The inclusion $Pro(An_\Sigma) \hookrightarrow Pro(An)$ admits a left adjoint $X \mapsto X^\wedge_\Sigma$ called Σ-completion

Thm. k field with absolute Galois group G_k, $X, Y/k$ qcqs

(1) If Y is proper$/k$,

$$\hat{\Pi}^{ét}_{\infty}(X \times_k Y) \xrightarrow{\sim} \hat{\Pi}^{ét}_{\infty}(X) \times_{BG_k} \hat{\Pi}^{ét}_{\infty}(Y)$$

(2) $p = \mathrm{char}(k)$. If G_k is prime-to-p, then

$$\Pi^{ét}_{\infty}(X \times_k Y)^{\wedge}_{p'} \xrightarrow{\sim} \Pi^{ét}_{\infty}(X)^{\wedge}_{p'} \times_{BG_k} \Pi^{ét}_{\infty}(Y)^{\wedge}_{p'}$$

Remark. Orgogozo (in 2003) proved **(2)** for separably closed fields.

Lecture 2

Defining the étale homotopy type: Shape theory

Idea We want a "homotopy type for ∞-topoi" that

- When applied to $Sh(T)$ for T a nice top space produces $|T|$
- When applied to the étale ∞-topos $X_{ét}$ produces $\Pi_\infty^{ét}(X)$.
- Classifies local systems

Our approach is informed by the following example.

Ex T topological space [admitting a CW structure] can be greatly generalized

Then the constant sheaf functor

$$\Gamma^* : An \longrightarrow Sh(T)$$

admits a left adjoint $\Gamma_\#$ defined as follows. First, left Kan extend

$$1\text{-}1 : Open(T) \longrightarrow An$$

to $PSh(T)$. Then restrict to $\Gamma_\#$. Thus

$$\Gamma_\#(1_{Sh(T)}) = |T|.$$

> The adjunction $\Gamma_\# \dashv \Gamma^*$ is saying that

$$\Gamma^*(K)(U) \simeq \mathrm{Map}_{\mathrm{An}}(|U|, K)$$

1-categorical analogue The constant sheaf of sets functor is given by $U \longmapsto \mathrm{Map}_{\mathrm{Set}}(\pi_0(U), K)$.

Upshot We can recover $|T|$ purely from sheaf theory (with the left adjoint $\Gamma_\#$).

⚠ The constant étale sheaf functor doesn't admit a left adjoint. But it admits a **pro-left adjoint**.

Ntn $\mathcal{X}$ ∞-topos

$\Gamma_* = \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, -) : \mathcal{X} \longrightarrow \mathrm{An}$ global sections functor

with left exact left adjoint $\Gamma^* : \mathrm{An} \longrightarrow \mathcal{X}$.

Observe The unique cofiltered limit-preserving extension of Γ^*

$$\mathrm{Pro}(\mathrm{An}) \longrightarrow \mathcal{X}$$

preserves all limits, hence has a left adjoint $\Gamma_\#$

Ntn For a scheme X, we write $X_{\acute{e}t} = \mathrm{Sh}(\acute{E}t_X, \mathrm{An})$ for the étale ∞-topos of X

Def
(1) The **shape** of an ∞-topos $\mathcal{X}$ is

$$\Pi_\infty(\mathcal{X}) := \Gamma_\#(1_{\mathcal{X}})$$

(2) For a scheme X, the **étale homotopy type** of X is

$$\Pi_\infty^{\acute{e}t}(X) := \Pi_\infty(X_{\acute{e}t}).$$

- We'll simplify things by just working with $\tau_{<\infty}\Pi_\infty(X_{\acute{e}t})$, But we'll just denote this by $\Pi_\infty^{\acute{e}t}(X)$ too.

Q. What is the meaning of the shape?

Obs. For any anima K

$$\mathrm{Map}_{\mathrm{Pro(An)}}(\Gamma_\#(1_X), K) \simeq \mathrm{Map}_X(1_X, \Gamma^*(K))$$

$$\simeq \underbrace{\Gamma_*\Gamma^*(K)}_{\text{global sections of constant sheaf w/ coefficients in } K}$$

Obs. In Pro(An)

$$\mathrm{Map}(\text{"}\lim_i\text{"} K_i, \text{"}\lim_j\text{"} L_j) \simeq \operatorname{colim}_j \mathrm{Map}(\text{"}\lim_i\text{"} K_i, L_j),$$

so a proanima is determined by mapping to all anima.

Lem. The functor

$$\begin{aligned} \mathrm{Pro}(\mathrm{An}) &\xrightarrow{\sim} \mathrm{Fun}^{\text{lex, accessible}}(\mathrm{An}, \mathrm{An})^{op} \\ K &\longmapsto \mathrm{Map}_{\mathrm{Pro}(\mathrm{An})}(K, -) \end{aligned}$$

left exact

is an equivalence of ∞-categories

> Under this identification, the left adjoints

$$\mathrm{Pro}(\mathrm{An}) \xrightarrow{\tau_{<\infty}} \mathrm{Pro}(\mathrm{An}_{<\infty}) \quad \text{and} \quad \mathrm{Pro}(\mathrm{An}) \xrightarrow{(-)^{\wedge}_{\pi}} \mathrm{Pro}(\mathrm{An}_{\pi})$$

are given by restricting to $\mathrm{An}_{<\infty}$ and An_{π}, resp.

Upshot. $\Pi_{\infty}(X) = \Gamma_* \Gamma^* \in \mathrm{Fun}^{\text{lex, acc}}(\mathrm{An}, \mathrm{An})^{op} \simeq \mathrm{Pro}(\mathrm{An})$

Cor. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi.

(1) If f^* is fully faithful, $\Pi_\infty(\mathcal{X}) \xrightarrow{\sim} \Pi_\infty(\mathcal{Y})$

(2) If f^* is fully faithful on truncated objects

$$\tau_{<\infty}\Pi_\infty(\mathcal{X}) \xrightarrow{\sim} \tau_{<\infty}\Pi_\infty(\mathcal{Y})$$

Proof

For $K \in \mathrm{An}$ or $\mathrm{An}_{<\infty}$ we have

$$\Gamma_{\mathcal{Y},*}\Gamma_{\mathcal{Y}}^*(K) \simeq \Gamma_{\mathcal{X},*} f_* f^* \Gamma_{\mathcal{X}}^*(K)$$

$$\simeq \Gamma_{\mathcal{X},*}\Gamma_{\mathcal{X}}^*(K).$$

$\square$

Ex. $\mathcal{X}^{hyp} \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathcal{X}$ inclusion of hypercomplete objects is an $\simeq$ on truncated objects. Hence

$$\tau_{<\infty}\Pi_\infty(\mathcal{X}^{hyp}) \xrightarrow{\sim} \tau_{<\infty}\Pi_\infty(\mathcal{X}).$$

Ex. For a scheme X, the pullback $\nu^*: X_{ét} \longrightarrow X_{proét}$ is fully faithful on truncated objects. Hence

$$\tau_{<\infty}\Pi_\infty(X_{proét}) \xrightarrow{\sim} \tau_{<\infty}\Pi_\infty(X_{ét})$$

Crash course on ∞-topoi

Thm. Let $\mathcal{X}$ be an ∞-category. TFAE:

(1) $\exists$ a small ∞-cat $\mathcal{C}$ and a left exact accessible localization

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, shift right, "\text{acces.}"'] \arrow[r, phantom, "\perp"] & \mathrm{PSh}(\mathcal{C}) \arrow[l, shift right, "\text{lex}"'] \end{tikzcd}$$

(2) Giraud's axioms: $\mathcal{X}$ is presentable and

(a) Coproducts are disjoint: $\forall\, U, V \in \mathcal{X}$, the square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \sqcup V \end{tikzcd}$$

is a pullback

(b) Colimits are universal: for every diagram $U_\bullet : A \to \mathcal{X}$, objects $V, W \in \mathcal{X}$ and morphisms

$$\operatorname*{colim}_{\alpha \in A} U_\alpha \longrightarrow W \longleftarrow V$$

we have

$$\operatorname*{colim}_{\alpha \in A} \left(U_\alpha \times_W V\right) \xrightarrow{\sim} \left(\operatorname*{colim}_{\alpha \in A} U_\alpha\right) \times_W V.$$

(c) **Groupoid objects are effective**: if $U_\bullet : \Delta^{op} \to \mathcal{X}$ is a simplicial object so that for every partition $[n] = S \cup T$ with $S \cap T = \{i\}$ a single point, then

$$\begin{tikzcd} U_n \simeq U_\bullet([n]) \arrow[r] \arrow[d] & U_\bullet(T) \arrow[d] \\ U_\bullet(S) \arrow[r] & U_\bullet(\{i\}) \simeq U_0 \end{tikzcd}$$

is a pullback,

then $U_\bullet \simeq \underbrace{\text{Čech nerve of } U_0 \longrightarrow |U_\bullet|}$

$f : X \to Y$ Čech nerve is $\cdots \; X \times_Y X \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} X$

$\operatorname{colim}_{\alpha \in A} U_\alpha \qquad \alpha \in A^{op} \quad U_\alpha$

(3) $\mathcal{X}$ is presentable and colimits in $\mathcal{X}$ are **van Kampen**: the functor

$$\begin{tikzcd}
\mathcal{X}^{op} \arrow[r] & \mathrm{Cat}_\infty^{large} \\
U \arrow[r, mapsto] \arrow[d] & \mathcal{X}_{/U} \\
V \arrow[r, mapsto] & \mathcal{X}_{/V} \arrow[u, "U \times_V (-)"']
\end{tikzcd}$$

preserves limits, i.e., $\mathcal{X}_{/\operatorname{colim}_{\alpha \in A} U_\alpha} \xrightarrow{\sim} \lim_{\alpha \in A^{op}} \mathcal{X}_{/U_\alpha}$

⚠ Note that we're not saying that there is an ∞-site $(\mathcal{C}, \tau)$ and an equivalence $\mathcal{X} \simeq Sh_\tau(\mathcal{C})$.
This is expected to be false!

Cor (of (3)). If $\mathcal{X}$ is an ∞-topos, then for any $U \in \mathcal{X}$, the ∞-cat $\mathcal{X}_{/U}$ is an ∞-topos.

Ex. X scheme, $U \in \acute{E}t_X$, $h(U) \in X_{\acute{e}t}$ sheaf represented by U.
Then
$$(X_{\acute{e}t})_{/h(U)} \simeq U_{\acute{e}t}.$$

Def. Given ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, a **geometric morphism** $f_*: \mathcal{X} \longrightarrow \mathcal{Y}$ is a right adjoint whose left adjoint $f^*: \mathcal{Y} \longrightarrow \mathcal{X}$ is left exact (= preserves finite limits).

- RTop_∞ := ∞-topoi and geometric morphisms
- LTop_∞ := ∞-topoi and left exact left adjoints

$$\mathrm{LTop}_\infty \simeq (\mathrm{RTop}_\infty)^{op}$$

Thm [HTT, Prop. 6.3.2.3 & Cor. 6.3.4.7]. LTop_∞ has all limits and colimits. Moreover, the forgetful functor

$$\mathrm{LTop}_\infty \longrightarrow \mathrm{Cat}_\infty^{\mathrm{large}}$$

preserves limits.

Obs. The functor

$$\begin{aligned} \mathrm{An} &\longrightarrow \mathrm{RTop}_\infty \\ K &\longmapsto \mathrm{Fun}(K, \mathrm{An}) \simeq \mathrm{An}_{/K} \end{aligned}$$ w/ right Kan extension functoriality

is left exact and fully faithful. Hence extends to a right adjoint

$$\lambda : \mathrm{Pro}(\mathrm{An}) \longrightarrow \mathrm{RTop}_\infty$$

Thm/Def (Lurie). The Shape $\Pi_\infty : \mathrm{RTop}_\infty \longrightarrow \mathrm{Pro}(\mathrm{An})$ is the left adjoint to λ.

Corollary. For any ∞-topos X and diagram $U_\bullet : I \longrightarrow X$,

$$\operatorname*{colim}_{i \in I} \Pi_\infty(X_{/U_i}) \xrightarrow{\sim} \Pi_\infty(X_{/\operatorname*{colim}_i U_i})$$

Cor. $\Pi_\infty^{\text{ét}} : \mathrm{Sch} \to \mathrm{Pro}(\mathrm{An}_{<\infty})$ is a hypercomplete proétale cosheaf.

Lecture 3:

Monodromy, Künneth formulas,

exodromy, & applications

Locally constant objects & monodromy

Def. $\mathcal{X}$ an ∞-topos. An object $L \in \mathcal{X}$ is **locally constant** if there exists an effective epimorphism $\coprod_{\alpha \in A} U_\alpha \twoheadrightarrow 1_{\mathcal{X}}$

such that for each $\alpha \in A$, there exist anima $(K_\alpha)_{\alpha \in A}$ and equivalences

$$L \times U_\alpha \simeq \Gamma^*_{\mathcal{X}_{/U_\alpha}}(K_\alpha) \quad \text{in } \mathcal{X}_{/U_\alpha}.$$

$$\simeq \Gamma^*_{\mathcal{X}}(K_\alpha) \times U_\alpha$$

- We say that L is **lisse** if, in addition, A can be taken to be finite and each K_α can be taken to be π-finite.

- We say that L is **Σ-torsion lisse** if, in addition, A can be taken to be finite and each K_α can be taken to be Σ-finite.

- $\mathcal{X}^{\text{lisse}} \subset \mathcal{X}$ full subcategory spanned by the lisse objects.

Ntn. Given an ∞-category D, write

$$\mathrm{Fun}^{\mathrm{cts}}(-, D) : \mathrm{Pro}(\mathrm{Cat}_\infty)^{op} \longrightarrow \mathrm{Cat}_\infty$$

for the unique colimit-preserving extension of $\mathrm{Fun}(-, D) : \mathrm{Cat}_\infty^{op} \longrightarrow \mathrm{Cat}_\infty$:

$$\mathrm{Fun}^{\mathrm{cts}}(\text{``}\lim_{i \in I}\text{''} C_i, D) = \operatorname*{colim}_{i \in I^{op}} \mathrm{Fun}(C_i, D)$$

Monodromy Thm. X ∞-topos.
There is a natural equivalence

$$X^{\mathrm{lisse}} \simeq \mathrm{Fun}^{\mathrm{cts}}(\Pi_\infty(X)^{\wedge}_{\pi}, \mathrm{An}_\pi).$$

Slogan "The profinite shape classifies lisse local systems"

- In fact

Thm (Lurie). $\lambda : \mathrm{Pro}(\mathrm{An}_\pi) \longrightarrow \mathrm{RTop}_\infty$ is fully faithful with explicit image.

Basechange + Künneth

Nonabelian smooth/proper basechange. Let

$$\begin{tikzcd} W \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a pullback square of qcqs schemes, Σ = primes invertible on Z.

(1) If g is proper, then for every constructible étale sheaf of anima $F \in Y_{ét}$,

$$f^* g_*(F) \xrightarrow{\sim} \bar{g}_* \bar{f}^*(F)$$

(2) If f is prosmooth, then for every Σ-torsion constructible étale sheaf $F \in Y_{ét}$,

$$f^* g_*(F) \xrightarrow{\sim} \bar{g}_* \bar{f}^*(F).$$

General Prop. k sep closed field. Consider a commutative square
Σ any set of primes

$$\begin{tikzcd} W \arrow[r, "\bar{f}"] \arrow[d, "\bar{g}"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & \mathrm{Spec}(k). \end{tikzcd}$$

If for every constant Σ-torsion étale sheaf $\mathcal{F}$ on Y, we have

$$f^* g_*(\mathcal{F}) \xrightarrow{\sim} \bar{g}_* \bar{f}^*(\mathcal{F}),$$

then

$$\Pi_\infty^{\text{ét}}(W)^\wedge_\Sigma \longrightarrow \Pi_\infty^{\text{ét}}(X)^\wedge_\Sigma \times \Pi_\infty^{\text{ét}}(Y)^\wedge_\Sigma.$$

Upshot. Applying this + basechange gives the Künneth formulas of the first lecture (modulo removing smoothness, which we can do with alterations + arc-descent).

Obs. Composition defines a monoidal structure $\circ$ on $\mathrm{Fun}(\mathrm{An}, \mathrm{An})^{op}$. This restricts to

$$\mathrm{Pro}(\mathrm{An}) \simeq \mathrm{Fun}^{lex, acc}(\mathrm{An}, \mathrm{An})^{op}.$$

$\mathrm{id}_{\mathrm{An}}$ is the unit for both $\circ$ and $\times$:

$$\rightsquigarrow \quad K \circ L \longrightarrow K \times L.$$

Lem. Σ any set of primes, X, Y qcqs schemes. Then

$$\Pi_\infty^{\text{ét}}(X) \circ \Pi_\infty^{\text{ét}}(Y) \longrightarrow \Pi_\infty^{\text{ét}}(X) \times \Pi_\infty^{\text{ét}}(Y)$$

becomes an equivalence after Σ-completion.

Thm (H.) In the above setting,

$$\left(\Pi_\infty^{\text{ét}}(X) \times \Pi_\infty^{\text{ét}}(Y)\right)^\wedge_\Sigma \xrightarrow{\sim} \Pi_\infty^{\text{ét}}(X)^\wedge_\Sigma \times \Pi_\infty^{\text{ét}}(Y)^\wedge_\Sigma.$$

Key. $\Pi_\infty^{\text{ét}}(X)$, $\Pi_\infty^{\text{ét}}(Y)$ can be written as Δ^{op}-indexed colimits in $\mathrm{Pro}(\mathrm{An}_{<\infty})$

Proof of general prop.

For $K \in \mathrm{An}_\Sigma$, we have

$$
\begin{aligned}
\Gamma_{W,*}\Gamma_W^*(K) &\simeq f_*\bar{g}_*\bar{f}^*g^*(K) \\
&\xleftarrow{\simeq} f_*f^*g_*g^*(K) \\
&= ((f_*f^*)\circ(g_*g^*))(K) \\
&= (\Pi_\infty^{\text{ét}}(X)\circ\Pi_\infty^{\text{ét}}(Y))^\wedge_\Sigma(K) \\
&\simeq (\Pi_\infty^{\text{ét}}(X)^\wedge_\Sigma \times \Pi_\infty^{\text{ét}}(Y)^\wedge_\Sigma)(K).
\end{aligned}
$$

$\square$

Obs. If one knows that:

(1) $\Pi_\infty^{\text{ét}}$ is finitary

(2) $\hat{\Pi}_\infty^{\text{ét}}$ is a hypercomplete h-cosheaf

(3) Δ^{op}-indexed colimits in $\mathrm{Pro}(\mathrm{An}_\pi)$ are stable under pullback,

then by

- Reducing to X/k finite type
- Using alterations to find an h-hypercover of X by smooth schemes

one can remove the smoothness assumptions on X

Q What about relative Künneth formulas from Lecture 1?

- These will follow once we have the fundamental fiber sequence

A quick overview of exodromy

most simple case of "exit-path" categories in topology

Nth. $B: Cat_\infty \longrightarrow An$ left adjoint to the inclusion

Two ideas. If T is a top space admitting a triangulation with poset of simplices P, then

(1) $\left\{ \begin{array}{l} \text{sheaves on } T \text{ constructible} \\ \text{wrt the triangulation} \end{array} \right\} \xrightarrow{\sim} Fun(P, An)$

(2) $|T| \simeq BP$

Idea. We should have an algebraic geometry version of this

$\rightsquigarrow$ but we'll deal with all stratifications at once

Nth. Cat_∞^π = ∞-categories with finitely many objects up to $\simeq$ and finite mapping anima

$Cat_\infty^\pi \cap Cat_1$ = finite 1-categories

Exodromy. X qcqs scheme. In work with Barwick–Glasman, we defined a profinite 1-category $\mathrm{Gal}(X) \in \mathrm{Pro}(\mathrm{Cat}_1^{\pi})$ such that:

(1) $\lim : \mathrm{Pro}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{Cat}_\infty$

$$\mathrm{Gal}(X) \longmapsto \mathrm{Pt}(X_{\acute{e}t}) := \mathrm{Fun}^{\mathrm{lex\ leftadj}}(X_{\acute{e}t}, \mathrm{An})$$

Computed by Grothendieck school

(2) $\mathrm{Pro}(\mathrm{Cat}_\infty) \xrightarrow{\mathrm{Pro}(B)} \mathrm{Pro}(\mathrm{An}) \xrightarrow{\tau_{<\infty}} \mathrm{Pro}(\mathrm{An}_{<\infty})$

$$\mathrm{Gal}(X) \longmapsto \Pi_\infty^{\acute{e}t}$$

(3) There's a natural equivalence

$$X_{\acute{e}t}^{\mathrm{cons}} \simeq \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \mathrm{An}_{\pi})$$

constructible étale sheaves of anima

Ex. If X is a 0-dim qcqs scheme, then $\mathrm{Gal}(X)$ is a 1-truncated profinite anima. Hence $\Pi_\infty^{\acute{e}t}(X) \simeq \mathrm{Gal}(X)$.

Prop (fundamental fiber sequence). $f: X \to S$ morphism of qcqs schemes with $\dim(S) = 0$, $\bar{s} \to S$ geom point. the naturally null sequence

$$\Pi_\infty^{\text{ét}}(X_{\bar{s}}) \longrightarrow \Pi_\infty^{\text{ét}}(X) \longrightarrow \Pi_\infty^{\text{ét}}(S) \simeq \mathrm{Gal}(S)$$

is a fiber sequence

Proof Sketch

Easy (basically SGA4). The square of profinite categories

$$\begin{tikzcd}
\mathrm{Gal}(X_{\bar{s}}) \arrow[r] \arrow[d] & \mathrm{Gal}(X) \arrow[d] \\
* \simeq \mathrm{Gal}(\bar{s}) \arrow[r] & \mathrm{Gal}(S)
\end{tikzcd}$$

is a pullback

> Also know that $\mathrm{Gal}(S)$ is a profinite anima.

Def A localization $D \underset{\hookleftarrow}{\overset{L}{\underset{\perp}{\rightleftarrows}}} C$ is locally cartesian if given

$$\underbrace{x \longrightarrow z}_{\in\, C} \longleftarrow y \quad \text{we have} \quad L(x \times_z y) \xrightarrow{\sim} x \times_z L(y).$$

Easy fact $B: \mathrm{Cat}_\infty \longrightarrow \mathrm{An}$ is locally cartesian

$\Rightarrow \mathrm{Pro}(B): \mathrm{Pro}(\mathrm{Cat}_\infty) \longrightarrow \mathrm{Pro}(\mathrm{An})$

is locally cartesian.

> To conclude, note that $\mathrm{Gal}(\bar{s}) \simeq *$ and $\mathrm{Gal}(s)$ are profinite anima. $\square$

Useful Exactly the same proof works for $\Pi_\infty^{\mathrm{cond}}$.

Q. What about after profinite completion?

Prop (H.–Holzschuh–Wolf, after Lurie). For any set of primes Σ, the localization

$$(-)^\wedge_\Sigma : \mathrm{Pro}(\mathrm{An}) \longrightarrow \mathrm{Pro}(\mathrm{An}_\Sigma) \quad \text{is locally cartesian.}$$

Upshot. Since $\mathrm{Gal}(S)$ is a profinite anima, we still get the fiber sequence after profinite completion.

Applications of étale homotopy theory

Application: Anabelian geometry

Thm (A. Schmidt – Stix). $k \supset \mathbb{Q}$ finitely generated extension, X, Y smooth k-varieties st $\exists$ embeddings $X, Y \xhookrightarrow[\text{loc closed}]{} \prod$ hyperbolic curves

The the natural map

$$\mathrm{Isom}_k(X,Y) \longrightarrow \pi_0 \mathrm{Equiv}_{BG_k}(\Pi_\infty^{\text{ét}}(X), \Pi_\infty^{\text{ét}}(Y))$$

is a split injection with a natural retraction

Cor. If $\Pi_\infty^{\text{ét}}(X) \simeq \Pi_\infty^{\text{ét}}(Y)$ over BG_k, then $X \cong Y$ as k-schemes.

Thm (Holschbach – J. Schmidt – Stix) k alg closed field $\mathrm{char}(k) > 0$, X/k smooth k-variety

Then $\Pi_\infty^{\text{ét}}(X) \simeq *$ if and only if $X \cong \mathrm{Spec}(k)$.

See also. Holzschuh's thesis for more applications

Application:
The Artin-Tate Pairing

Setup. $X/\mathbb{F}_q$ smooth geometrically connected surface dim=2d

$\ell \nmid q$ prime $\qquad \mathrm{Br}_{nd}(X) := \dfrac{\mathrm{Br}(X)}{\text{divisible elts}}$

> M. Artin & Tate defined a pairing

$$\mathrm{Br}_{nd}(X)[\ell^\infty] \times \mathrm{Br}_{nd}(X)[\ell^\infty] \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Really defined on $H^{2d}_{\acute{e}t}(X; \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{nd} \xrightarrow{\sim} H^{2d+1}_{\acute{e}t}(X; \mathbb{Z}_\ell(d))_{tors}$

Conj (Tate, 1966) The Artin-Tate pairing is alternating

History

1967 & 86 Manin gave counterexamples

1996 Urabe found mistakes is Manin's work

Thm (Feng, 2017) Tate's conjecture is true

- uses analogies from arithmetic topology

Idea. Because of the fiber sequence

$$\Pi_\infty^{\text{ét}}(X_{\overline{\mathbb{F}}_q}) \longrightarrow \Pi_\infty^{\text{ét}}(X) \longrightarrow B\hat{\mathbb{Z}} \simeq (S^1)^\wedge_\pi$$

"$\mathbb{R}$-dim" "dim 4" "dim 5" "dim 1"

$\Pi_\infty^{\text{ét}}(X)$ behaves like a real 5-manifold

Artin-Tate pairing $\longleftrightarrow$ linking form on an orientable $(4d+1)$-manifold

- Feng used the étale homotopy type to construct Steenrod operations on étale cohomology + analogies from topology to prove the conjecture