From nonabelian basechange to basechange with coefficients

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Abstract

The goal of this paper is to explain when basechange theorems for sheaves of spaces imply basechange for sheaves with coefficients in other presentable ∞-categories. We accomplish this by analyzing when the tensor product of presentable ∞-categories preserves left adjointable squares. As a sample result, we show that the Proper Basechange Theorem in topology holds with coefficients in any presentable ∞-category which is compactly generated or stable. We also prove results about the interaction between tensor products of presentable ∞-categories and various categorical constructions that are of independent interest.

Contents

0 Introduction

Let

$$
\begin{array}{ccc}
W & \xrightarrow{\bar{f}} & Y \\
{\scriptstyle \bar{g}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\
X & \xrightarrow[f]{} & Z
\end{array}
\tag{0.1}
$$

be a pullback square of locally compact Hausdorff topological spaces, and assume that the map g is proper. The classical Proper Basechange Theorem in topology [STK, Tag 09V6; SGA 4_{II}, Exposé Vbis, Théorème 4.1.1] says that for any ring R, the induced square of bounded-above[1] derived ∞-categories

$$
\begin{array}{ccc}
\mathrm{D}(W;R)_{<\infty} & \xrightarrow{\mathrm{R}\bar{f}_*} & \mathrm{D}(Y;R)_{<\infty} \\
{\scriptstyle \mathrm{R}\bar{g}_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{R}g_*} \\
\mathrm{D}(X;R)_{<\infty} & \xrightarrow[\mathrm{R}f_*]{} & \mathrm{D}(Z;R)_{<\infty}
\end{array}
\tag{0.2}
$$

is *left adjointable*. That is to say, for each object $F \in \mathrm{D}(Y;R)_{<\infty}$, the natural *exchange morphism*

$$\mathrm{L}f^*\mathrm{R}g_*(F) \to \mathrm{R}\bar{g}_*\mathrm{L}\bar{f}^*(F)$$

is an equivalence. As Lurie remarks [HTT, Remark 7.3.1.19], the classical Proper Basechange Theorem follows from the *Nonabelian Proper Basechange Theorem* [HTT, Corollary 7.3.1.18]: the induced square of ∞-categories of sheaves of *spaces*

$$
\begin{array}{ccc}
\mathrm{Sh}(W;\mathbf{Spc}) & \xrightarrow{\bar{f}_*} & \mathrm{Sh}(Y;\mathbf{Spc}) \\
{\scriptstyle \bar{g}_*}\big\downarrow & & \big\downarrow{\scriptstyle g_*} \\
\mathrm{Sh}(X;\mathbf{Spc}) & \xrightarrow[f_*]{} & \mathrm{Sh}(Z;\mathbf{Spc})
\end{array}
$$

is left adjointable.

The goal of this paper is to expand on Lurie's remark and explain when basechange results for sheaves of spaces imply basechange results for sheaves with coefficients in other presentable ∞-categories. Our inquiry is informed by the following observation: for a topological space T and ring R, the *unbounded* derived ∞-category $\mathrm{D}(T;R)$ naturally embeds as a full subcategory of the Deligne–Lurie tensor product of presentable ∞-categories

$$\mathrm{Sh}(T;\mathrm{D}(R)) \coloneqq \mathrm{Sh}(T;\mathbf{Spc}) \otimes \mathrm{D}(R)$$

[SAG, Remark 1.3.1.6, Corollary 1.3.1.8, & Corollary 2.1.2.3]. That is, $\mathrm{D}(T;R)$ embeds into the ∞-category of sheaves on T valued in the derived ∞-category of R. Moreover:

(1) The essential image of this embedding $\mathrm{D}(T;R) \hookrightarrow \mathrm{Sh}(T;\mathrm{D}(R))$ is the full subcategory spanned by the $\mathrm{D}(R)$-valued *hyper*sheaves on T. In many situations the two ∞-categories coincide, e.g., if T admits a CW structure [22] or is sufficiently finite-dimensional [HTT, Corollary 7.2.1.12, Theorem 7.2.3.6 & Remark 7.2.4.18; 14, Theorem 3.12].

[1] We use *homological* indexing. What we write as $\mathrm{D}(T;R)_{<\infty}$ is often written as $\mathrm{D}^+(T;R)$.

(2) There is a natural t-structure on the stable ∞-category $\mathrm{Sh}(T; \mathrm{D}(R))$. Moreover, the embedding $\mathrm{D}(T; R) \hookrightarrow \mathrm{Sh}(T; \mathrm{D}(R))$ is t-exact and restricts to an equivalence

$$\mathrm{D}(T; R)_{<\infty} \xrightarrow{\sim} \mathrm{Sh}(T; \mathrm{D}(R))_{<\infty}$$

on t-bounded-above objects [SAG, Corollary 2.1.2.4].

These points raise a natural question:

0.3 Question. Does the Proper Basechange Theorem hold with bounded-above dervied ∞-categories $\mathrm{D}(-; R)_{<\infty}$ replaced by the larger ∞-categories $\mathrm{Sh}(-; \mathrm{D}(R))$? If so, can this extension of the be deduced from Lurie's Nonabelian Proper Basechange Theorem by a 'formal' argument about the tensor product of presentable ∞-categories preserving adjointability?

We explain why the answer to both questions is affirmative. However, there are some important subtleties. The general setup we consider is a square of presentable ∞-categories and right adjoints

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}_*} & C \\ {\scriptstyle \bar{g}_*}\downarrow & \swarrow_{\sigma} & \downarrow{\scriptstyle g_*} \\ B & \xrightarrow[f_*]{} & D \end{array} \tag{0.4}$$

equipped with a (not necessarily invertible) natural transformation $\sigma \colon g_* \bar{f}_* \to \bar{g}_* f_*$. In this general setting, there is a natural *exchange morphism*

$$\mathrm{Ex}_\sigma \colon f^* g_* \to \bar{g}_* \bar{f}^*$$

associated to the diagram (0.4); see §1.1. Let E be another presentable ∞-category. The main subtlety is that even if the exchange morphism $f^* g_* \to \bar{g}_* \bar{f}^*$ is an equivalence, the exchange morphism associated to the tensored-up diagram

$$\begin{array}{ccc} A \otimes E & \xrightarrow{\bar{f}_* \otimes E} & C \otimes E \\ {\scriptstyle \bar{g}_* \otimes E}\downarrow & \swarrow_{\sigma \otimes E} & \downarrow{\scriptstyle g_* \otimes E} \\ B \otimes E & \xrightarrow[f_* \otimes E]{} & D \otimes E \end{array} \tag{(0.4) $\otimes E$}$$

need not be an equivalence (see Example 1.16).

However, there are many situations in which the left adjointability of (0.4) implies the left adjointability of (0.4) $\otimes E$. The following is probably the most useful result in this direction; when E is compactly generated, this applies to squares of ∞-topoi and geometric morphisms.

0.5 Theorem (Corollary 3.2 and Proposition 3.8)**.** *Consider an oriented square (0.4) of presentable ∞-categories and right adjoints. Assume that the left adjoints f^* and $\bar{f}^*$ are left exact and that square (0.4) is left adjointable. Let E be a presentable ∞-category, and assume that one of the following conditions is satisfied:*

(0.5.1) *The ∞-category E is compactly generated.*

(0.5.2) *The ∞-category E is stable and the right adjoints g_* and $\bar{g}_*$ preserve filtered colimits.*

Then the induced square (0.4) $\otimes E$ *is left adjointable.*

Throughout this paper, we also prove other adjointability results as well as results about the interaction between tensor products of presentable ∞-categories and various categorical constructions that are of independent interest.

0.6 Example (Subexample 3.15)**.** Let us return to the setting of a pullback square of locally compact Hausdorff spaces (0.1) where the morphism $g: Y \to Z$ is proper. Lurie's Nonabelian Proper Basechange Theorem and Theorem 0.5 show that if E is a presentable ∞-category which is compactly generated or stable, then the induced square of ∞-categories of E-valued sheaves

$$\begin{array}{ccc} \mathrm{Sh}(W; E) & \xrightarrow{\bar{f}_*} & \mathrm{Sh}(Y; E) \\ {\scriptstyle \bar{g}_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathrm{Sh}(X; E) & \xrightarrow[f_*]{} & \mathrm{Sh}(Z; E) \end{array} \tag{0.7}$$

is left adjointable. This generalizes the classical Proper Basechange Theorem in two important ways:

(1) Let R be an ordinary ring, and let $E = \mathrm{D}(R)$ be the unbounded derived ∞-category of R. The left adjointability of the square (0.7) generalizes the classical Proper Basechange Theorem to objects of $\mathrm{Sh}(T; \mathrm{D}(R))$ that are not bounded-above, and answers Question 0.3 in the affirmative.

(2) A version of the Proper Basechange Theorem holds for sheaves of modules over any E_1-ring *spectrum* R or *animated ring* (in the terminology of [12, §5.1.4]).

0.8 Remark (unbounded derived ∞-categories)**.** There are two natural squares of right adjoints enlarging the square (0.2) appearing in the classical Proper Basechange Theorem: the square of classical unbounded derived ∞-categories

$$\begin{array}{ccc} \mathrm{D}(W; R) & \xrightarrow{\mathrm{R}\bar{f}_*} & \mathrm{D}(Y; R) \\ {\scriptstyle \mathrm{R}\bar{g}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{R}g_*} \\ \mathrm{D}(X; R) & \xrightarrow[\mathrm{R}f_*]{} & \mathrm{D}(Z; R) \end{array} \tag{0.9}$$

and the square of ∞-categories of $\mathrm{D}(R)$-valued sheaves

$$\begin{array}{ccc} \mathrm{Sh}(W; \mathrm{D}(R)) & \xrightarrow{\bar{f}_*} & \mathrm{Sh}(Y; \mathrm{D}(R)) \\ {\scriptstyle \bar{g}_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathrm{Sh}(X; \mathrm{D}(R)) & \xrightarrow[f_*]{} & \mathrm{Sh}(Z; \mathrm{D}(R)) \, . \end{array} \tag{0.10}$$

We have seen that the square (0.10) is left adjointable. Moreover, for a topological space T, the unbounded derived ∞-category $\mathrm{D}(T; R)$ is the full subcategory of $\mathrm{Sh}(T; \mathrm{D}(R))$ spanned by the hypersheaves. So the square (0.10) is really an enlargement of the square (0.9). However, the left adjointability of the square (0.10) does *not* imply the left adjointability of the square (0.9), and the square (0.9) is *not* generally left adjointable. The key point is the following: if $F \in \mathrm{D}(Y; R)$ is not t-bounded-above, then the exchange transformation

$$\mathrm{L}f^* \mathrm{R}g_*(F) \to \mathrm{R}\bar{g}_* \mathrm{L}\bar{f}^*(F) \tag{0.11}$$

associated to the square (0.9) does *not* agree with the exchange transformation

$$f^* g_*(F) \to \bar{g}_* \bar{f}^*(F)$$

associated to (0.10). The reason is that given a map of topological spaces $p\colon T \to S$, the pullback functor

$$p^* : \mathrm{Sh}(S; \mathrm{D}(R)) \to \mathrm{Sh}(T; \mathrm{D}(R))$$

does not generally carry $\mathrm{D}(S; R)$ to $\mathrm{D}(T; R)$. The inclusion $\mathrm{D}(T; R) \hookrightarrow \mathrm{Sh}(T; \mathrm{D}(R))$ admits a t-exact left adjoint $(-)^{\mathrm{hyp}} : \mathrm{Sh}(T; \mathrm{D}(R)) \to \mathrm{D}(T; R)$ called *hypercompletion*, and the left derived functor $\mathrm{L}p^* : \mathrm{D}(S; R) \to \mathrm{D}(T; R)$ is the composite

$$\mathrm{D}(S; R) \hookrightarrow \mathrm{Sh}(S; \mathrm{D}(R)) \xrightarrow{p^*} \mathrm{Sh}(T; \mathrm{D}(R)) \xrightarrow{(-)^{\mathrm{hyp}}} \mathrm{D}(T; R) .$$

This extra hypercompletion procedure is nontrivial and is what prevents the exchange transformation (0.11) from being an equivalence in general.

The t-exact inclusion $\mathrm{D}(T; R) \hookrightarrow \mathrm{Sh}(T; \mathrm{D}(R))$ restricts to an equivalence on hearts; hence the ∞-category $\mathrm{Sh}(T; \mathrm{D}(R))$ is not generally the derived ∞-category of an abelian category. Thus, if one wants a version of the Proper Basechange Theorem for unbounded complexes, one is forced to leave the world of classical derived categories and needs to work with ∞-categories. These comments are the reason Spaltenstein [34] was unable to prove a version of the Proper Basechange Theorem for arbitrary unbounded complexes. They also highlight a major advantage of working with the ∞-categories $\mathrm{Sh}(-; \mathrm{D}(R))$ over the ∞-categories $\mathrm{D}(-; R)$.

0.12 Remark. We have been aware of Theorem 0.5 for some time, and certainly results of this form are known to experts. However, we were unable to locate a source explaining the relationship between left adjointability and tensoring with a presentable ∞-category. We have written this paper because we need to use results of this form in forthcoming work; we hope that others will also find the results presented here useful.

Linear overview

In §§1 and 2, we recall the background we need about adjointability and tensor products of presentable ∞-categories. We also collect key examples of when tensoring with a presentable ∞-category does or does not preserve left adjointability. Setting up the notation and explaining the explicit descriptions of the tensor product we need takes a bit of time, but once everything is in place Theorem 0.5 is elementary. We also use these explicit descriptions to show that the tensor product preserves many properties of functors (§ 2.3) and that, in most situations that arise in nature, the tensor product preserves recollements (§2.4). In §3, we prove Theorem 0.5 and derive some consequences.

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Terminology and notations

We use the terms ∞-*category* and $(\infty,1)$-*category* interchangeably. We write **Spc** for the ∞-category of spaces. Given ∞-categories C and D with limits, we write $\mathrm{Fun}^{\lim}(C,D) \subset \mathrm{Fun}(C,D)$ for the full subcategory spanned by the limit-preserving functors $C \to D$.

In this paper, we use a small amount of the theory of $(\infty,2)$-categories; all of the $(\infty,2)$-categories we use are subcategories of the $(\infty,2)$-category $\mathbf{Cat}_\infty$ of locally small but potentialy large $(\infty,1)$-categories, functors, and natural transformations. Moreover, all functors of $(\infty,2)$-categories are subfunctors of the functor $(C,D) \mapsto \mathrm{Fun}(C,D)$. We write $\mathbf{Pr}^{\mathrm{R}} \subset \mathbf{Cat}_\infty$ for the sub-$(\infty,2)$-category of presentable $(\infty,1)$-categories, *right* adjoints, and all natural tranformations. We write $\mathbf{Pr}^{\mathrm{L}} \subset \mathbf{Cat}_\infty$ for the sub-$(\infty,2)$-category of presentable $(\infty,1)$-categories, *left* adjoints, and all natural tranformations.

1 Preliminaries on adjointability & tensor products

In this section we recall the basics of left adjointable squares and tensor products with presentable ∞-categories. Subsection 1.1 fixes our conventions on adjointability and gives some examples of adjointable squares. Subsection 1.2 recalls tensor products of presentable ∞-categories. Subsection 1.3 gives an example explaining why tensoring does not generally preserve left adjointable squares of presentable ∞-categories (Example 1.16). We also provide a class of left adjointable squares that are preserved by tensoring with any presentable ∞-category (Lemma 1.18).

1.1 Oriented squares & adjointability

We begin by fixing conventions for adjointability in an $(\infty,2)$-category.

1.1 Definition. Let $\boldsymbol{C}$ be an $(\infty,2)$-category, and A, B, C, and D objects of $\boldsymbol{C}$. We exhibit data of 1-morphisms $f_*\colon B \to D$, $g_*\colon C \to D$, $\bar{g}_*\colon A \to B$, and $\bar{f}_*\colon A \to C$, along with a 2-morphism $\sigma\colon g_*\bar{f}_* \to \bar{g}_* f_*$ by a single square

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}_*} & C \\ {\scriptstyle \bar{g}_*}\downarrow & \swarrow_{\sigma} & \downarrow{\scriptstyle g_*} \\ B & \xrightarrow[f_*]{} & D\,. \end{array} \tag{1.2}$$

We refer to such a square as an *oriented square* in $\boldsymbol{C}$.

1.3 Definition. Let $\boldsymbol{C}$ be an $(\infty,2)$-category and consider an oriented square (1.2) in $\boldsymbol{C}$.

(1.3.1) Assume that the 1-morphisms f_* and $\bar{f}_*$ admit left adjoints f^* and $\bar{f}^*$, respectively. Write $c_f\colon f^*f_* \to \mathrm{id}_B$ for the counit and $u_{\bar{f}}\colon \mathrm{id}_C \to \bar{f}_*\bar{f}^*$ for the unit. The *left exchange transformation* associated to the oriented square (1.2) is the composite 2-morphism

$$\mathrm{Ex}_\sigma\colon f^*g_* \xrightarrow{f^*g_*u_{\bar{f}}} f^*g_*\bar{f}_*\bar{f}^* \xrightarrow{f^*\sigma\bar{f}^*} f^*f_*\bar{g}_*\bar{f}^* \xrightarrow{c_f\bar{g}_*\bar{f}^*} \bar{g}_*\bar{f}^*\,.$$

We say that the square (1.2) is *(horizontally) left adjointable* if the exchange transformation $\mathrm{Ex}_\sigma\colon f^*g_* \to \bar{g}_*\bar{f}^*$ is an equivalence.

(1.3.2) Assume that the 1-morphisms g_* and $\bar{g}_*$ admit right adjoints $g^!$ and $\bar{g}^!$, respectively. Write $c_{\bar{g}} \colon \bar{g}_* \bar{g}^! \to \mathrm{id}_B$ for the counit and $u_g \colon \mathrm{id}_C \to g^! g_*$ for the unit. The *right exchange transformation* associated to the oriented square (1.2) is the composite 2-morphism

$$\bar{f}_* \bar{g}^! \xrightarrow{u_g \bar{f}_* \bar{g}^!} g^! g_* \bar{f}_* \bar{g}^! \xrightarrow{g^! \sigma \bar{g}^!} g^! f_* \bar{g}_* \bar{g}^! \xrightarrow{g^! f_* c_{\bar{g}}} g^! f_* \,.$$

We say that the square (1.2) is *(vertically) right adjointable* if the exchange transformation $\bar{f}_* \bar{g}^! \to g^! f_*$ is an equivalence.

1.4 Remark. We follow Hoyois [21; 23] in calling the morphism Ex_σ the *exchange transformation*. The natural transformation Ex_σ is often referred to as a *Beck–Chevalley transformation* [7; 11, §2.2; 20, Notation 4.1.1], *basechange transformation* [5, Definition 7.1.1], or *mate transformation* [13, §1; 19; 25, §2.2]. Instead of Ex, the notations BC (for Beck–Chevalley or basechange) and β are often used [5, Definition 7.1.1; 25, §2.2; 20, Notation 4.1.1].

1.5 Remark (on notation)**.** The notation we have chosen is meant to provide an easy way to remember the specifics of left exchange transformations: the left exchange transformation goes from a composite with no bars to a composite with bars, and an oriented square is left adjointable if we can 'exchange f^* and g_*' at the cost of adding bars.

In this paper, we are mostly concerned with *left* adjointability, but right adjointability will also appear due to the following.

1.6 Observation.

(1.6.1) Since functors of $(\infty, 2)$-categories preserve adjunctions and their (co)units, functors of $(\infty, 2)$-categories preserve left/right adjointable oriented squares.

(1.6.2) If in the square (1.2) f_* and $\bar{f}_*$ admit left adjoints and g_* and $\bar{g}_*$ admit right adjoints, then (1.2) is horizontally left adjointable if and only if (1.2) is vertically right adjointable.

1.7 Notation. For the rest of this paper, we fix an oriented square of ∞-categories

(□)
$$\begin{array}{ccc} A & \xrightarrow{\bar{f}_*} & C \\ {\scriptstyle \bar{g}_*}\big\downarrow & \swArrow_{\sigma} & \big\downarrow{\scriptstyle g_*} \\ B & \xrightarrow[f_*]{} & D \,, \end{array}$$

where the functors f_* and $\bar{f}_*$ admit left adjoints f^* and $\bar{f}^*$, respectively.

1.8 Convention. Unless explicitly stated otherwise, adjointability of an oriented square of (presentable) ∞-categories (□) refers to adjointability in the $(\infty, 2)$-category $\mathbf{Cat}_\infty$.

We finish this subsection with two examples of left adjointable squares. The first is an easy-to-state version of the Smooth and Proper Basechange Theorem in algebraic geometry.

1.9 Example. Let k be an algebraically closed field and let

$$\begin{array}{ccc} W & \xrightarrow{\bar{f}} & Y \\ {\scriptstyle \bar{g}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

be a pullback square of quasiprojective k-schemes. Let ℓ be a prime number different from the characteristic of k. The Smooth and Proper Basechange Theorem in étale cohomology says that if the morphism f is smooth or the morphism g is proper, then the induced square

$$\begin{array}{ccc} \mathrm{Sh}_{\text{ét}}(W;\mathbf{Z}_\ell) & \xrightarrow{\bar{f}_*} & \mathrm{Sh}_{\text{ét}}(Y;\mathbf{Z}_\ell) \\ {\scriptstyle \bar{g}_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathrm{Sh}_{\text{ét}}(X;\mathbf{Z}_\ell) & \xrightarrow[f_*]{} & \mathrm{Sh}_{\text{ét}}(Z;\mathbf{Z}_\ell) \end{array}$$

of ∞-categories of ℓ-adic étale sheaves is left adjointable [16, Theorem 2.4.2.1].

1.10 Example. Let X and Y be spaces, and consider the canonically commutative square of presentable ∞-categories and right adjoints

(1.11)
$$\begin{array}{ccc} \mathbf{Spc} & \xrightarrow{\mathrm{Map}(X,-)} & \mathbf{Spc} \\ {\scriptstyle \mathrm{Map}(Y,-)}\downarrow & & \downarrow{\scriptstyle \mathrm{Map}(Y,-)} \\ \mathbf{Spc} & \xrightarrow[\mathrm{Map}(X,-)]{} & \mathbf{Spc}\,. \end{array}$$

The associated exchange morphism

(1.12)
$$X \times \mathrm{Map}(Y,-) \longrightarrow \mathrm{Map}(Y, X\times(-)) \simeq \mathrm{Map}(Y,X)\times \mathrm{Map}(Y,-)$$

is the product of the diagonal map $X \to \mathrm{Map}(Y,X)$ with the identity map on $\mathrm{Map}(Y,-)$. In particular, the exchange morphism (1.12) is an equivalence if and only if the diagonal $X \to \mathrm{Map}(Y,X)$ is an equivalence.

1.13. One version of the Sullivan Conjecture (proven by Carlsson [10], Lannes [26], and Miller [27; 28; 29; 30]) says that if X is a finite space and Y is a π-finite space, then the diagonal map $X \to \mathrm{Map}(Y,X)$ is an equivalence. Thus, under these hypotheses, the square (1.11) is left adjointable.

1.2 Tensor products of presentable ∞-categories

1.14 Recollection. Let S and E be presentable ∞-categories. The *tensor product* of presentable ∞-categories $S \otimes E$ along with the functor

$$\otimes\colon S \times E \to S \otimes E$$

are characterized by the following universal property: for any presentable ∞-category T, restriction along $\otimes$ defines an equivalence

$$\mathrm{Fun}^{\mathrm{colim}}(S\otimes E, T) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{colim,colim}}(S\times E, T)$$

between colimit-preserving functors $S\otimes E \to T$ and functors $S \times E \to T$ that preserve colimits separately in each variable. The tensor product of presentable ∞-categories defines a functor

$$\otimes\colon \mathbf{Pr}^{\mathrm{L}} \times \mathbf{Pr}^{\mathrm{L}} \to \mathbf{Pr}^{\mathrm{L}}$$

and can be used to equip $\mathbf{Pr}^{\mathrm{L}}$ with the structure of a symmetric monoidal $(\infty,2)$-category. See [17, Chapter 1, §6.1; 24, §4.4] for this statement for presentable stable ∞-categories; the proof is exactly the same without the stability hypothesis.

Since the $(\infty, 2)$-category $\mathbf{Pr}^{\mathrm{R}}$ of presentable ∞-categories and right adjoints is obtained from $\mathbf{Pr}^{\mathrm{L}}$ by reversing 1-morphisms and 2-morphisms, the tensor product also defines a symmetric monoidal structure on $\mathbf{Pr}^{\mathrm{R}}$. In this note, we are more interested in the tensor product on $\mathbf{Pr}^{\mathrm{R}}$. This has a very explicit description: there is a natural equivalence

$$S \otimes E \simeq \mathrm{Fun}^{\mathrm{lim}}(E^{\mathrm{op}}, S)$$

[HA, Proposition 4.8.1.17]. Moreover, there is a natural equivalence

$$(-) \otimes E \simeq \mathrm{Fun}^{\mathrm{lim}}(E^{\mathrm{op}}, -)$$

of functors of $(\infty, 2)$-categories $\mathbf{Pr}^{\mathrm{R}} \to \mathbf{Pr}^{\mathrm{R}}$. In particular, if $p_* : S \to T$ is a right adjoint functor of presentable ∞-categories, then the induced right adjoint

$$p_* \otimes E \colon S \otimes E \simeq \mathrm{Fun}^{\mathrm{lim}}(E^{\mathrm{op}}, S) \to \mathrm{Fun}^{\mathrm{lim}}(E^{\mathrm{op}}, T) \simeq T \otimes E$$

is given by post-composition with p_*. For the purposes of this work, it suffices to take

$$\mathrm{Fun}^{\mathrm{lim}}(E^{\mathrm{op}}, -) \colon \mathbf{Pr}^{\mathrm{R}} \to \mathbf{Pr}^{\mathrm{R}}$$

as the *definition* of the tensor product $(-) \otimes E$.

1.15 Observation. Let $h \colon S \to S'$ and $v \colon E \to E'$ be functors between presentable ∞-categories which are both left adjoints or both right adjoints. Then the square

$$\begin{array}{ccc} S \otimes E & \xrightarrow{h \otimes E} & S' \otimes E \\ {\scriptstyle S \otimes v}\downarrow & & \downarrow{\scriptstyle S' \otimes v} \\ S \otimes E' & \xrightarrow[h \otimes E']{} & S' \otimes E' \end{array}$$

canonically commutes: both composites are identified with $h \otimes v$.

1.3 Interaction between tensor products and adjointability

Now we give an example showing that the functor $(-) \otimes E \colon \mathbf{Pr}^{\mathrm{R}} \to \mathbf{Pr}^{\mathrm{R}}$ need not preserve left adjointability of oriented squares (so that Theorem 0.5 is not completely trivial). The problem here is that we are interested in adjointability in $\mathbf{Cat}_\infty$, rather the much stronger notions of adjointability in $\mathbf{Pr}^{\mathrm{R}}$ or $\mathbf{Pr}^{\mathrm{L}}$. Said differently, the composite functors $f^* g_*$ and $\bar{g}_* \bar{f}^*$ involved in the exchange transformation are not generally right or left adjoints. Hence the condition that the exchange morphism $f^* g_* \to \bar{g}_* \bar{f}^*$ be an equivalence is not expressible internally to $\mathbf{Pr}^{\mathrm{R}}$ or $\mathbf{Pr}^{\mathrm{L}}$, thus need not be preserved by the functor of $(\infty, 2)$-categories $(-) \otimes E$.

The following example is a variant of Example 1.10; we learned of it from Lurie.

1.16 Example. Let p be a prime number and let $Y = \mathrm{BC}_p$ be the classifying space of the cyclic group C_p of order p. Let X be a connected finite space with $\pi_1(X) \cong \mathrm{C}_p$. (For example, take $p = 2$ and $X = \mathbf{RP}^2$.) As a special case of (1.13), the square of presentable ∞-categories

(1.17)
$$\begin{array}{ccc} \mathbf{Spc} & \xrightarrow{\mathrm{Map}(X,-)} & \mathbf{Spc} \\ {\scriptstyle \mathrm{Map}(\mathrm{BC}_p,-)}\downarrow & & \downarrow{\scriptstyle \mathrm{Map}(\mathrm{BC}_p,-)} \\ \mathbf{Spc} & \xrightarrow[\mathrm{Map}(X,-)]{} & \mathbf{Spc} \end{array}$$

is left adjointable. We claim that the induced square of presentable ∞-categories

$$\begin{array}{ccc} \mathbf{Spc}_{\leq 1} & \xrightarrow{\mathrm{Map}(\tau_{\leq 1} X,-)} & \mathbf{Spc}_{\leq 1} \\ {\scriptstyle \mathrm{Map}(\mathrm{BC}_p,-)}\downarrow & & \downarrow{\scriptstyle \mathrm{Map}(\mathrm{BC}_p,-)} \\ \mathbf{Spc}_{\leq 1} & \xrightarrow[\mathrm{Map}(\tau_{\leq 1} X,-)]{} & \mathbf{Spc}_{\leq 1} \end{array} \tag*{(1.17) $\otimes \mathbf{Spc}_{\leq 1}$}$$

is *not* left adjointable. To see this note that the square (1.17) $\otimes\,\mathbf{Spc}_{\leq 1}$ is left adjointable if and only if the diagonal morphism

$$\delta\colon \mathrm{BC}_p \simeq \tau_{\leq 1} X \longrightarrow \mathrm{Map}(\mathrm{BC}_p, \tau_{\leq 1} X) \simeq \mathrm{Map}(\mathrm{BC}_p, \mathrm{BC}_p)$$

is an equivalence. However, the morphism δ is not an equivalence: $\pi_0 \mathrm{BC}_p \simeq *$ and

$$\pi_0 \mathrm{Map}(\mathrm{BC}_p, \mathrm{BC}_p) \cong \mathrm{Hom}_{\mathbf{Ab}}(\mathrm{C}_p, \mathrm{C}_p) \cong \mathrm{C}_p \ .$$

When the functors g_* and $\bar{g}_*$ are left adjoints, the requirement that the exchange transformation $f^* g_* \to \bar{g}_* \bar{f}^*$ be an equivalence *is* expressible internally to $\mathbf{Pr}^{\mathrm{L}}$, hence is preserved by tensoring with any presentable ∞-category:

1.18 Lemma. *Let E be a presentable ∞-category. Assume that:*

(1.18.1) ($\square$) *is an oriented square in* $\mathbf{Pr}^{\mathrm{R}}$.

(1.18.2) *The right adjoints $g_*\colon C \to D$ and $\bar{g}_*\colon A \to B$ admit right adjoints $g^!$ and $\bar{g}^!$, respectively.*

(1.18.3) *The oriented square* ($\square$) *is left adjointable.*

Then the functors $g_* \otimes E$ and $\bar{g}_* \otimes E$ are left adjoints and the oriented square ($\square$) $\otimes E$ is left adjointable.

Proof. Assumption (1.18.2) implies and $g_* \otimes E$ and $\bar{g}_* \otimes E$ are left adjoint to $g^! \otimes E$ and $\bar{g}^! \otimes E$, respectively. Also note that by assumption g_* and $\bar{g}_*$ admit right adjoints in the $(\infty, 2)$-category $\mathbf{Pr}^{\mathrm{R}}$. In light of Observation 1.6, assumption (1.18.3) says implies that the square ($\square$) is vertically right adjointable in the $(\infty, 2)$-category $\mathbf{Pr}^{\mathrm{R}}$. Since functors of $(\infty, 2)$-categories preserve right ajointability, the square ($\square$)$\otimes E$ is vertically right adjointable in the $(\infty, 2)$-category $\mathbf{Pr}^{\mathrm{R}}$. Hence ($\square$)$\otimes E$ is vertically right adjointable in the $(\infty, 2)$-category $\mathbf{Cat}_\infty$. Again applying Observation 1.6, we conclude that the square is horizontally left adjointable in the $(\infty, 2)$-category $\mathbf{Cat}_\infty$. $\square$

2 Compactly generated ∞-categories

In this section we recall a few facts about compactly generated ∞-categories (§2.1) and give an explicit description of the tensor product with a compactly generated ∞-category (§ 2.2). We then give two useful applications of this description:

(1) In § 2.3, we show that many properties of a left adjoint between presentable ∞-categories are preserved by tensoring with a compactly generated ∞-category.

(2) In § 2.4, we show that recollements of presentable ∞-categories are preserved by tensoring with a compactly generated or stable presentable ∞-category.

2.1 Notations & definitions

2.1 Notation. Let E be an ∞-category with filtered colimits. We write $E^{c} \subset E$ for the full subcategory spanned by the compact objects.

Recall that if E is compactly generated, then $E^{c} \subset E$ is closed under finite colimits and retracts. Moreover, E is the Ind-*completion* of E^{c}. That is, E is obtained from E^{c} by freely adjoining filtered colimits.

2.2 Recollection (projectively generated ∞-categories)**.** Let E be an ∞-category with sifted colimits, and let $X \in E$. We say that X is *projective* if $\mathrm{Map}_E(X,-)\colon E \to \mathbf{Spc}$ preserves geometric realizations of simplicial objects. We say that X is *compact projective* if X is compact and projective, i.e., $\mathrm{Map}_E(X,-)\colon E \to \mathbf{Spc}$ preserves sifted colimits. We write $E^{\mathrm{cpr}} \subset E$ for the full subcategory spanned by the compact projective objects.

We say that E is *projectively generated* if there is a small collection of *compact* projective objects of E that generate E under small colimits [HTT, Definition 5.5.8.23]. In this case, E^{cpr} is closed under finite coproducts and retracts in E. Moreover, E is the *nonabelian derived category* of E^{cpr}. That is, E is obtained from E^{cpr} by freely adjoining sifted colimits. See [HTT, Propositions 5.5.8.15 & 5.5.8.25].

2.3 Example. The following ∞-categories are projectively generated: the ∞-category of spaces, the ∞-category of connective modules over a connective E_1-ring spectrum [HA, Proposition 7.1.4.15], the ∞-category of animated (aka simplicial commutative) rings, and (up to set-theoretic issues) the ∞-category of condensed/pyknotic spaces [5, §13.3; 6; 33].

2.2 Tensor products with compactly generated ∞-categories

Now we provide alternative models for tensor products with compactly generated ∞-categories. The key point is that these alternative models give us access to an explicit description of the action of the Lurie tensor product on a left adjoint functor. These observations are known to experts (see [2, §2.3.1; 31, §B.1]); we have included the material here because we were unable to locate a reference saying everything we need.

We begin with some terminology.

2.4 Definition. Let $\mathcal{K}$ be a collection of ∞-categories.

(2.4.1) We say that an ∞-category I *admits $\mathcal{K}$-shaped limits* if for each $K \in \mathcal{K}$, the ∞-category I admits limits of K-shaped diagrams.

(2.4.2) Given ∞-categories I and S that admit $\mathcal{K}$-shaped limits, we say that a functor $F\colon I \to S$ *preserves limits of $\mathcal{K}$-shaped diagrams* if for each $K \in \mathcal{K}$, the functor F preserves limits of K-shaped diagrams.

(2.4.3) Given ∞-categories I and S that admit $\mathcal{K}$-shaped limits, we write $\mathrm{Fun}^{\mathcal{K}\text{-lim}}(I,S) \subset \mathrm{Fun}(I,S)$ for the full subcategory spanned by those functors that preserve limits of $\mathcal{K}$-shaped diagrams.

(2.4.4) We write $\mathbf{Cat}_\infty^{\mathcal{K}\text{-lim}} \subset \mathbf{Cat}_\infty$ for the (non-full) sub-$(\infty,2)$-category of ∞-categories admitting $\mathcal{K}$-shaped limits, functors preserving $\mathcal{K}$-shaped limits, and all natural transformation.

(2.4.5) If $\mathcal{K}$ is the collection of *finite ∞-categories*, we write $\mathrm{Fun}^{\mathrm{lex}} \coloneqq \mathrm{Fun}^{\mathcal{K}\text{-lim}}$ and $\mathbf{Cat}_\infty^{\mathrm{lex}} \coloneqq \mathbf{Cat}_\infty^{\mathcal{K}\text{-lim}}$.

(2.4.6) If $\mathcal{K}$ is the collection of *finite sets*, we write $\mathrm{Fun}^{\times} \coloneqq \mathrm{Fun}^{\mathcal{K}\text{-lim}}$ and $\mathbf{Cat}_\infty^{\mathrm{fp}} \coloneqq \mathbf{Cat}_\infty^{\mathcal{K}\text{-lim}}$.

2.5 Observation. Let S and E be presentable ∞-categories.

(2.5.1) If E is compactly generated, then restriction along the inclusion $E^{c,op} \hookrightarrow E^{op}$ defines an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{lim}}(E^{op}, S) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(E^{c,op}, S) .$$

Hence the tensor product $(-)\otimes E$ fits into a commutative square of functors of $(\infty, 2)$-categories

$$\begin{array}{ccc} \mathbf{Pr}^{\mathrm{R}} & \xrightarrow{(-)\otimes E} & \mathbf{Pr}^{\mathrm{R}} \\ \downarrow & & \downarrow \\ \mathbf{Cat}_{\infty}^{\mathrm{lex}} & \xrightarrow[\mathrm{Fun}^{\mathrm{lex}}(E^{c,op},-)]{} & \mathbf{Cat}_{\infty}^{\mathrm{lex}} . \end{array}$$

Here the vertical functors are inclusions of non-full subcategories.

(2.5.2) If E is projectively generated, then restriction along the inclusion $E^{cpr,op} \hookrightarrow E^{op}$ defines an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{lim}}(E^{op}, S) \xrightarrow{\sim} \mathrm{Fun}^{\times}(E^{cpr,op}, S) .$$

Hence the tensor product $(-)\otimes E$ fits into a commutative square of functors of $(\infty, 2)$-categories

$$\begin{array}{ccc} \mathbf{Pr}^{\mathrm{R}} & \xrightarrow{(-)\otimes E} & \mathbf{Pr}^{\mathrm{R}} \\ \downarrow & & \downarrow \\ \mathbf{Cat}_{\infty}^{\mathrm{fp}} & \xrightarrow[\mathrm{Fun}^{\times}(E^{cpr,op},-)]{} & \mathbf{Cat}_{\infty}^{\mathrm{fp}} . \end{array}$$

Here the vertical functors are inclusions of non-full subcategories.

2.6 Observation. Let E be a compactly generated ∞-category and $p^* : T \to S$ be a *left exact* left adjoint between presentable ∞-categories with right adjoint p_*. Note that we have a commutative diagram of ∞-categories

$$\begin{array}{ccccc} S \otimes E & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{lim}}(E^{op}, S) & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{lex}}(E^{c,op}, S) \\ {\scriptstyle p_*\otimes E}\downarrow & & {\scriptstyle p_*\circ-}\downarrow & & \downarrow{\scriptstyle p_*\circ-} \\ T \otimes E & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{lim}}(E^{op}, T) & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{lex}}(E^{c,op}, T) . \end{array}$$

Moreover, since p^* is left exact, the functor

$$p^* \circ - \colon \mathrm{Fun}^{\mathrm{lex}}(E^{c,op}, T) \to \mathrm{Fun}^{\mathrm{lex}}(E^{c,op}, S)$$

given by post-composition with p^* is left adjoint to the functor given by post-composition with p_*. Hence we have a commutative square of ∞-categories

$$\begin{array}{ccc} T \otimes E & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{lex}}(E^{c,op}, T) \\ {\scriptstyle p^*\otimes E}\downarrow & & \downarrow{\scriptstyle p^*\circ-} \\ S \otimes E & \xrightarrow{\sim} & \mathrm{Fun}^{\mathrm{lex}}(E^{c,op}, S) . \end{array}$$

2.7 Variant. Let E be a projectively generated ∞-category and $p^*: T \to S$ be a left adjoint functor between presentable ∞-categories that preserves finite products. Then we have a commutative square of ∞-categories

$$\begin{array}{ccc} T \otimes E & \xrightarrow{\sim} & \mathrm{Fun}^{\times}(E^{\mathrm{cpr,op}}, T) \\ {\scriptstyle p^* \otimes E} \downarrow & & \downarrow {\scriptstyle p^* \circ -} \\ S \otimes E & \xrightarrow{\sim} & \mathrm{Fun}^{\times}(E^{\mathrm{cpr,op}}, S) . \end{array}$$

Observations 2.5 and 2.6 and Variant 2.7 highlight that tensoring with a compactly or projectively generated ∞-category has (unexpected) additional functoriality.

2.3 Application: properties of left adjoints

We now give two applications of Observation 2.6 and Variant 2.7. For the first, our motivation is the following question: given a conservative family of points of an ∞-topos, does the family remain conservative after tensoring with a presentable ∞-category? This is true as long as we tensor with a compactly generated ∞-category:

2.8 Lemma. *Let $\{p_i^* : T \to S_i\}_{i \in I}$ be a jointly conservative family of left adjoint functors between presentable ∞-categories, and let E be a presentable ∞-category. Assume that one of the following conditions holds:*

(2.8.1) *The ∞-category E is compactly generated and the functors $\{p_i^*\}_{i \in I}$ are left exact.*

(2.8.2) *The ∞-category E is projectively generated and the functors $\{p_i^*\}_{i \in I}$ preserve finite products.*

Then the family of left adjoints $\{p_i^* \otimes E : T \otimes E \to S_i \otimes E\}_{i \in I}$ is jointly conservative.

Proof. By Observation 2.6, in situation (2.8.1) it suffices to show that the collection of functors

$$\left\{ p_i^* \circ - : \mathrm{Fun}^{\mathrm{lex}}(E^{\mathrm{c,op}}, T) \to \mathrm{Fun}^{\mathrm{lex}}(E^{\mathrm{c,op}}, S_i) \right\}_{i \in I}$$

is jointly conservative. Similarly, by Variant 2.7, in situation (2.8.2) it suffices to show that the collection of functors

$$\{ p_i^* \circ - : \mathrm{Fun}^{\times}(E^{\mathrm{cpr,op}}, T) \to \mathrm{Fun}^{\times}(E^{\mathrm{cpr,op}}, S_i) \}_{i \in I}$$

is jointly conservative. These assertions are immediate from the assumption that the functors $\{p_i^*\}_{i \in I}$ are jointly conservative. □

The following example shows the necessity of the hypothesis that the left adjoints are left exact:

2.9 Example. Let X be a space. By the straightening/unstraightening theorem, the functor

$$\mathrm{colim}_X : \mathrm{Fun}(X, \mathbf{Spc}) \to \mathbf{Spc}$$

is conservative. However, after tensoring with the ∞-category $\mathbf{Spt}$ of spectra, the resulting functor

$$\mathrm{colim}_X : \mathrm{Fun}(X, \mathbf{Spt}) \to \mathbf{Spt}$$

need not be conservative. For example, let $X = \mathrm{BC}_2$ and $F : \mathrm{BC}_2 \to \mathbf{Spt}$ be the Eilenberg–MacLane spectrum $\mathbf{Q}$ with C_2-action given by the sign representation. The homotopy fixed point spectrum $\mathbf{Q}^{\mathrm{hC}_2} = \mathrm{colim}_{\mathrm{BC}_2} F$ is trivial.

Here is a related result. If p_* is a fully faithful right adjoint, then Recollection 1.14 immediately implies that for any presentable ∞-category E, the functor $p_* \otimes E$ is fully faithful. This is not generally true for left adjoints, but it is true in the setting of Observation 2.6 and Variant 2.7:

2.10 Lemma. *Let $p^* : T \hookrightarrow S$ be a fully faithful left adjoint functor between presentable ∞-categories, and let E be a presentable ∞-category. Assume that one of the following conditions holds:*

(2.10.1) *The ∞-category E is compactly generated and p^* is left exact.*

(2.10.2) *The ∞-category E is projectively generated and p^* preserves finite products.*

Then the left adjoint $p^* \otimes E : T \otimes E \to S \otimes E$ fully faithful.

2.4 Application: recollements

Let X be an ∞-category with finite limits. Recall that fully faithful functors

$$i_* : Z \hookrightarrow X \qquad \text{and} \qquad j_* : U \hookrightarrow X$$

are said to exhibit X as the *recollement* of Z and U if:[2]

(1) The functors i_* and j_* admit left exact left adjoints i^* and j^*, respectively.

(2) The functor $j^* i_* : Z \to U$ is constant at the terminal object of U.

(3) The functors $i^* : X \to Z$ and $j^* : X \to U$ are jointly conservative.

See [HA, §A.8; 4]. Primarily due to the requirement that i^* and j^* are jointly conservative, given a recollement of presentable ∞-categories, it is not obvious if it remains a recollement after tensoring with another presentable ∞-category. We finish this section by showing that tensoring with a compactly generated or stable ∞-category preserves recollements (Corollary 2.12 and Proposition 2.20).

The following is immediate from the definitions.

2.11 Proposition. *Let $\mathcal{K}$ be a collection of ∞-categories, let I be a small ∞-category with $\mathcal{K}$-shaped limits, and let $i_* : Z \hookrightarrow X$ and $j_* : U \hookrightarrow X$ be fully faithful right adjoints between ∞-categories that admit $\mathcal{K}$-shaped limits. Assume that i_* and j_* exhibit X as the recollement of Z and U and that the left adjoints i^* and j^* preserve $\mathcal{K}$-shaped limits. Then the functors*

$$i_* \circ - : \mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, Z) \hookrightarrow \mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, X) \qquad \textit{and} \qquad j_* \circ - : \mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, U) \hookrightarrow \mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, X)$$

exhibit $\mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, X)$ as the recollement of $\mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, Z)$ and $\mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, U)$.

The following consequence was previously recorded by Aizenbud and Carmeli [1, Lemma 3.0.10].

2.12 Corollary. *Let E be a presentable ∞-category, and let $i_* : Z \hookrightarrow X$ and $j_* : U \hookrightarrow X$ be fully faithful right adjoints of presentable ∞-categories that exhibit X as the recollement of Z and U. If E is compactly generated, then $i_* \otimes E$ and $j_* \otimes E$ exhibit $X \otimes E$ as the recollement of $Z \otimes E$ and $U \otimes E$.*

Proof. Combine Observations 2.5 and 2.6 with Proposition 2.11 in the case that $\mathcal{K}$ is the collection of finite ∞-categories and $I = E^{\mathrm{c,op}}$. □

[2] Here we use the convention for the open and closed pieces of a recollement from the theory of constructible sheaves.

Now we use Corollary 2.12 and properties of recollements of stable ∞-categories to show that tensoring with a presentable stable ∞-category preserves recollements.

2.13 Notation. We write **Spt** for the ∞-category of spectra. Recall that **Spt** is compactly generated, and for any presentable stable ∞-category E, there is a natural equivalence $\Omega^\infty_E : \mathbf{Spt} \otimes E \simeq E$ [HA, Proposition 1.4.2.21 & Example 4.8.1.23].

2.14 Observation. Let E be a presentable stable ∞-category and $p_* : S \to T$ a right adjoint between presentable ∞-categories. Since E is stable, $p_* \otimes \mathbf{Spt} \otimes E \simeq p_* \otimes E$. Thus, if $p_* \otimes \mathbf{Spt}$ admits a right adjoint, then $p_* \otimes E$ admits a right adjoint. Similarly, if $p^* \otimes \mathbf{Spt}$ admits a left adjoint, then $p^* \otimes E$ admits a left adjoint.

2.15 Recollection. Let $i_* : Z \hookrightarrow X$ and $j_* : U \hookrightarrow X$ be functors that exhibit X as the *recollement* of Z and U. If the ∞-category Z has an initial object, then j^* admits a fully faithful left adjoint $j_! : U \hookrightarrow X$ [HA, Corollary A.8.13]. If, moreover, X has a zero object, then i_* admits a right adjoint $i^! : X \to Z$ defined by taking the fiber

$$i^! \coloneqq \mathrm{fib}(\mathrm{id}_X \to j_* j^*)$$

of the unit $\mathrm{id}_X \to j_* j^*$ [HA, Remark A.8.5].

If X is stable, then Z and U are also stable. Moreover, there is a canonical fiber sequence

$$j_! j^* \longrightarrow \mathrm{id}_X \longrightarrow i_* i^* , \tag{2.16}$$

where the first morphism is the counit and the second is the unit [HA, Proposition A.8.17; 32, 1.17].

Note that given adjoints of stable ∞-categories $j_! \dashv j^*$ and $i^* \dashv i_*$, the existence of a fiber sequence (2.16) implies that i^* and j^* are jointly conservative. To show that tensoring with a presentable stable ∞-category E preserves recollements, we prove that such a fiber sequence always exists by embedding E in a compactly generated stable ∞-category. We first check that the relevant adjoints exist.

2.17 Notation. Let T be a presentable ∞-category and let $i_* : Z \hookrightarrow X$ and $j_* : U \hookrightarrow X$ be fully faithful right adjoints of presentable ∞-categories that exhibit X as the recollement of Z and U. We write $i^*_T \coloneqq i^* \otimes T$ and $j^*_T \coloneqq j^* \otimes T$, and write $i^T_* \coloneqq i_* \otimes T$ and $j^T_* \coloneqq j_* \otimes T$. If i^T_* admits a right adjoint, we denote this adjoint by $i^!_T$; if j^*_T admits a left adjoint, we denote this adjoint by $j^T_!$.

2.18 Lemma. *Let E be a presentable ∞-category, and let $i_* : Z \hookrightarrow X$ and $j_* : U \hookrightarrow X$ be fully faithful right adjoints of presentable ∞-categories that exhibit X as the recollement of Z and U. If E is stable, then:*

(2.18.1) *The functor i^E_* admits a right adjoint $i^!_E$.*

(2.18.2) *The functor j^*_E admits a left adjoint $j^E_!$.*

(2.18.3) *The composite $j^*_E i^E_* : Z \otimes E \to U \otimes E$ is constant at the terminal object of $U \otimes E$.*

Proof. By Corollary 2.12, $i^{\mathbf{Spt}}_*$ and $j^{\mathbf{Spt}}_*$ exhibit $X \otimes \mathbf{Spt}$ as the recollement of $Z \otimes \mathbf{Spt}$ and $U \otimes \mathbf{Spt}$. Thus (2.18.1) and (2.18.2) follow from Observation 2.14. For (2.18.3), note that since E is stable, we have a commuative diagram of right adjoints

$$
\begin{array}{ccccc}
\mathrm{Fun}^{\lim}(E^{\mathrm{op}}, Z \otimes \mathbf{Spt}) & \xhookrightarrow{i_*^{\mathrm{Spt}} \circ -} & \mathrm{Fun}^{\lim}(E^{\mathrm{op}}, X \otimes \mathbf{Spt}) & \xrightarrow{j_{\mathrm{Spt}}^* \circ -} & \mathrm{Fun}^{\lim}(E^{\mathrm{op}}, U \otimes \mathbf{Spt}) \\
\uparrow \wr & & \uparrow \wr & & \uparrow \wr \\
Z \otimes \mathbf{Spt} \otimes E & \xhookrightarrow{i_*^{\mathrm{Spt}} \otimes E} & X \otimes \mathbf{Spt} \otimes E & \xrightarrow{j_{\mathrm{Spt}}^* \otimes E} & U \otimes \mathbf{Spt} \otimes E \\
\wr \downarrow \Omega^\infty_{Z \otimes E} & & \wr \downarrow \Omega^\infty_{X \otimes E} & & \Omega^\infty_{U \otimes E} \downarrow \wr \\
Z \otimes E & \xhookrightarrow[i_*^E]{} & X \otimes E & \xrightarrow[j_E^*]{} & U \otimes E\,.
\end{array}
$$

By Corollary 2.12 the composite $j_{\mathrm{Spt}}^* i_*^{\mathrm{Spt}}$ is constant at the terminal object of $U \otimes \mathbf{Spt}$, completing the proof. ☐

2.19 Recollection [HA, Proposition 1.4.4.9]. An ∞-category E is presentable and stable if and only if there exists a small ∞-category E_0 such that E is equivalent to an accessible exact localization of $\mathrm{Fun}(E_0, \mathbf{Spt})$.

2.20 Proposition. *Let E be a presentable ∞-category, and let $i_* \colon Z \hookrightarrow X$ and $j_* \colon U \hookrightarrow X$ be fully faithful right adjoints of presentable ∞-categories that exhibit X as the recollement of Z and U. If E is stable, then $i_* \otimes E$ and $j_* \otimes E$ exhibit $X \otimes E$ as the recollement of $Z \otimes E$ and $U \otimes E$.*

Proof. Since $X \otimes E$, $Z \otimes E$, and $U \otimes E$ are stable, the left adjoints i_E^* and j_E^* are exact. In light of Lemma 2.18, the remaining point to check is that the functors i_E^* and j_E^* are jointly conservative. To do this, use Recollection 2.19 to choose a compactly generated stable ∞-category E' and fully faithful right adjoint $E \hookrightarrow E'$ with exact left adjoint $L \colon E' \to E$. For a presentable ∞-category T, write $L_T \coloneqq T \otimes L$.

By Corollary 2.12, $i_*^{E'}$ and $j_*^{E'}$ exhibit $X \otimes E'$ as the recollement of $Z \otimes E'$ and $U \otimes E'$. Since E' is stable, there is a fiber sequence

$$
j_!^{E'} j_{E'}^* \longrightarrow \mathrm{id}_{X \otimes E'} \longrightarrow i_*^{E'} i_{E'}^* \tag{2.21}
$$

of left adjoint functors. Applying Observation 1.15 and Lemma 2.18, we see that

$$
L_X j_!^{E'} j_{E'}^* \simeq j_!^E L_U j_{E'}^* \simeq j_!^E j_E^* L_X
$$

and

$$
L_X i_*^{E'} i_{E'}^* \simeq i_*^E L_Z i_{E'}^* \simeq i_*^E i_E^* L_X\,.
$$

Thus the fiber sequence (2.21) localizes to a fiber sequence of left adjoints

$$
j_!^E j_E^* \longrightarrow \mathrm{id}_{X \otimes E} \longrightarrow i_*^E i_E^*\,. \tag{2.22}
$$

To see that i_E^* and j_E^* are jointly conservative, note that if $F \in X \otimes E$ and $j_E^*(F) = 0$ and $i_E^*(F) = 0$, then the fiber sequence (2.22) shows that $F = 0$. ☐

2.23 Corollary. *Let E be a presentable ∞-category, and let $i_* \colon Z \hookrightarrow X$ and $j_* \colon U \hookrightarrow X$ be fully faithful right adjoints of presentable ∞-categories that exhibit X as the recollement of Z and U. If X is stable, then $i_* \otimes E$ and $j_* \otimes E$ exhibit $X \otimes E$ as the recollement of $Z \otimes E$ and $U \otimes E$.*

Proof. Since X is stable, both Z and U are stable (Recollection 2.15) and we have a commutative diagram

$$\begin{array}{ccccc} Z \otimes \mathbf{Spt} & \xrightarrow{i_* \otimes \mathrm{Sp}} & X \otimes \mathbf{Spt} & \xleftarrow{j_* \otimes \mathrm{Sp}} & U \otimes \mathbf{Spt} \\ \downarrow{\scriptstyle \Omega_Z^\infty} \wr & & \downarrow{\scriptstyle \Omega_X^\infty} \wr & & \wr \downarrow{\scriptstyle \Omega_U^\infty} \\ Z & \xrightarrow[i_*]{} & X & \xleftarrow[j_*]{} & U . \end{array}$$

Hence the claim is equivalent to showing that $i_* \otimes (\mathbf{Spt} \otimes E)$ and $j_* \otimes (\mathbf{Spt} \otimes E)$ exhibit $X \otimes (\mathbf{Spt} \otimes E)$ as the recollement of $Z \otimes (\mathbf{Spt} \otimes E)$ and $U \otimes (\mathbf{Spt} \otimes E)$. Since $\mathbf{Spt} \otimes E$ is stable, Proposition 2.20 completes the proof. ☐

2.24 Remark. If $i_* : Z \hookrightarrow X$ and $j_* : U \hookrightarrow X$ form a recollement of presentable ∞-categories, and X is an ∞-topos, then Z and U are ∞-topoi [HA, Proposition A.8.15]. Moreover, if E is another ∞-topos, then $i_* \otimes E$ and $j_* \otimes E$ exhibit $X \otimes E$ as the recollement of $Z \otimes E$ and $U \otimes E$ [HTT, Remark 6.3.5.8 & Proposition 7.3.2.12; HA, Example 4.8.1.19 & Proposition A.8.15]. In light of this and Corollaries 2.12 and 2.23 and Proposition 2.20, in basically all situations one naturally runs into, the tensor product preserves recollements.

3 Adjointability results

In this section, we use the explicit descriptions of the tensor product with a compactly generated ∞-category from §2.2 to explain which operations on an oriented square (☐) of presentable ∞-categories preserve left adjointability. In particular, we prove Theorem 0.5.

In §3.1, we make a general observation (Proposition 3.1) that immediately takes care of the compactly generated case of Theorem 0.5. Proposition 3.1 also has some other useful consequences; see Example 3.5 and Corollary 3.6. In §3.2, we take care of the stable case of Theorem 0.5. In §3.3, we explain consequences of Theorem 0.5 and Lurie's Nonabelian Proper Basechange Theorem.

3.1 Adjointability & ∞-categories of functors

To prove the compactly generated case of Theorem 0.5, we appeal to the improved functoriality of the tensor product explained in Observation 2.5.

3.1 Proposition. *Let $\mathcal{K}$ be a collection of ∞-categories and let I be a small ∞-category. Assume that:*

(3.1.1) *The ∞-category I admits $\mathcal{K}$-shaped limits and (☐) is an oriented square in $\mathbf{Cat}_\infty^{\mathcal{K}\text{-lim}}$.*

(3.1.2) *The left adjoints $f^* : D \to B$ and $\bar{f}^* : C \to A$ preserve $\mathcal{K}$-shaped limits.*

(3.1.3) *The oriented square (☐) is left adjointable.*

Then the induced oriented square

$$\mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, (☐)) \qquad \begin{array}{ccc} \mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, A) & \xrightarrow{\bar{f}_* \circ -} & \mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, C) \\ {\scriptstyle \bar{g}_* \circ -}\downarrow & \swarrow_{\sigma \circ -} & \downarrow{\scriptstyle g_* \circ -} \\ \mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, B) & \xrightarrow[f_* \circ -]{} & \mathrm{Fun}^{\mathcal{K}\text{-lim}}(I, D) \end{array}$$

is left adjointable.

Proof. By the assumptions, the functors f_* and $\bar{f}_*$ admit left adjoints in the $(\infty, 2)$-category $\mathbf{Cat}_\infty^{\mathcal{K}\text{-lim}}$. The fact that functors of $(\infty, 2)$-categories preserves left adjointable squares completes the proof. $\square$

3.2 Corollary. *Let E be a compactly generated ∞-category. Assume that:*

(3.2.1) *$(\square)$ is an oriented square in $\mathbf{Pr}^{\mathrm{R}}$.*

(3.2.2) *The left adjoints $f^* : D \to B$ and $\bar{f}^* : C \to A$ are left exact.*

(3.2.3) *The oriented square $(\square)$ is left adjointable.*

Then the oriented square $(\square) \otimes E$ is left adjointable.

Proof. Combine Observations 2.5 and 2.6 with Proposition 3.1 in the case that $\mathcal{K}$ is the collection of finite ∞-categories and $I = E^{\mathrm{c},\mathrm{op}}$. $\square$

3.3 Warning. Note that the ∞-category $\mathbf{Spc}_{\leq 1}$ of 1-truncated spaces is compactly generated. Hence Example 1.16 shows that the assumption (3.2.2) cannot be removed.

3.4 Corollary. *Let I be a small ∞-category with finite products. Assume that:*

(3.4.1) *$(\square)$ is an oriented square in $\mathbf{Cat}_\infty^{\mathrm{fp}}$.*

(3.4.2) *The left adjoints $f^* : D \to B$ and $\bar{f}^* : C \to A$ preserve finite products.*

(3.4.3) *The oriented square $(\square)$ is left adjointable.*

Then the oriented square $\mathrm{Fun}^{\times}(I, (\square))$ is left adjointable.

Proof. Combine Observation 2.5 and Variant 2.7 with Proposition 3.1 in the case that $\mathcal{K}$ is the collection of finite sets. $\square$

Corollary 3.4 has some nice consequences:

3.5 Example (algebras over Lawvere theories)**.** Let L be a Lawvere theory in the ∞-categorical sense [8; 9; 15, Chapter 3]. In the setting of Corollary 3.4, the induced square of ∞-categories of L-algebras

$$\begin{array}{ccc} \mathrm{Alg}_L(A) & \longrightarrow & \mathrm{Alg}_L(C) \\ \big\downarrow & \swArrow & \big\downarrow \\ \mathrm{Alg}_L(B) & \longrightarrow & \mathrm{Alg}_L(D)\,, \end{array}$$

is left adjointable. In particular, letting $L = \mathrm{Span}(\mathbf{Set}^{\mathrm{fin}})$ be the $(2,1)$-category of spans of finite sets [3, §3], we see that the formation of commutative monoid objects preserves left adjointability of oriented squares in which all functors preserve finite products.

Another special case of Corollary 3.4 is given by tensoring with a projectively generated ∞-category (here take $I = E^{\mathrm{cpr},\mathrm{op}}$):

3.6 Corollary. *Let E be a projectively generated ∞-category. Assume that:*

(3.6.1) *$(\square)$ is an oriented square in $\mathbf{Pr}^{\mathrm{R}}$.*

(3.6.2) *The left adjoints $f^* : D \to B$ and $\bar{f}^* : C \to A$ preserve finite products.*

(3.6.3) *The oriented square $(\square)$ is left adjointable.*

Then the oriented square $(\square) \otimes E$ is left adjointable.

3.2 Adjointability & preservation of filtered colimits

In many ∞-categories that arise in algebra and sheaf theory, filtered colimits commute with finite limits. For example, filtered colimits commute with finite limits in: compactly generated ∞-categories, Grothendieck abelian categories [STK, Tag 079A; HA, Definition 1.3.5.1; 18], Grothendieck prestable ∞-categories [SAG, Definition C.1.4.2], stable ∞-categories, and n-topoi for each $0 \leq n \leq \infty$. The goal of this subsection is to show that in these situations, if the vertical right adjoints g_* and $\bar{g}_*$ preserve filtered colimits and ($\square$) becomes left adjointable after tensoring with the ∞-category of spectra, then ($\square$) becomes left adjointable after tensoring with *any* stable presentable ∞-category. Combined with Corollary 3.2, this allows us to generalize the Proper Basechange Theorem in topology to sheaves with values in presentable ∞-categories which are compactly generated or stable (Subexample 3.15).

In order to state this result, we recall some terminology.

3.7 Recollection. Let S be an ∞-category with finite limits and filtered colimits. We say that *filtered colimits in S are left exact* if for each small filtered ∞-category I, the functor

$$\operatorname{colim}_I \colon \operatorname{Fun}(I, S) \to S$$

is left exact.

3.8 Proposition. *Let E be a stable presentable ∞-category. Assume that:*

(3.8.1) *The ∞-categories A, B, C, and D in the oriented square ($\square$) are presentable and filtered colimits are left exact in each ∞-category. Moreover, all functors in ($\square$) are right adjoints.*

(3.8.2) *The right adjoints g_* and $\bar{g}_*$ preserve filtered colimits.*

(3.8.3) *The oriented square ($\square$) $\otimes$ **Spt** is left adjointable.*

Then the functors $g_* \otimes E$ and $\bar{g}_* \otimes E$ are left adjoints and the oriented square ($\square$) $\otimes E$ is left adjointable.

3.9 Remark. Since the ∞-category **Spt** is compactly generated, Corollary 3.2 shows that if the oriented square ($\square$) is left adjointable and the left adjoints f^* and $\bar{f}^*$ are left exact, then (3.8.3) is satisfied. In particular, hypotheses (3.8.1) and (3.8.3) are satisfied for left adjointable squares of ∞-topoi and geometric morphisms.

To prove Proposition 3.8, we begin with a few basic lemmas. The key point is that the assumption that g_* and $\bar{g}_*$ preserve filtered colimits implies that $g_* \otimes$ **Spt** and $\bar{g}_* \otimes$ **Spt** are left adjoints (Corollary 3.12). Thus we are in the situation to apply Lemma 1.18.

The following is immediate from the definitions.

3.10 Lemma. *Let I be an ∞-category with finite limits and let S be an ∞-category with finite limits and filtered colimits. If filtered colimits in S are left exact, then $\operatorname{Fun}^{\mathrm{lex}}(I, S) \subset \operatorname{Fun}(I, S)$ is closed under filtered colimits.*

3.11 Corollary. *Let $p_* \colon S \to T$ be a right adjoint between presentable ∞-categories in which filtered colimits are left exact. Let E be a compactly generated ∞-category. If p_* preserves filtered colimits, then $p_* \otimes E$ preserves filtered colimits.*

Proof. Consider the commutative diagram of ∞-categories

$$\begin{array}{ccccc}
S \otimes E & \xrightarrow{\sim} & \operatorname{Fun}^{\mathrm{lex}}(E^{\mathrm{op}}, S) & \hookrightarrow & \operatorname{Fun}(E^{\mathrm{c,op}}, S) \\
{\scriptstyle p_* \otimes E}\downarrow & & {\scriptstyle p_* \circ -}\downarrow & & \downarrow{\scriptstyle p_* \circ -} \\
T \otimes E & \xrightarrow{\sim} & \operatorname{Fun}^{\mathrm{lex}}(E^{\mathrm{op}}, T) & \hookrightarrow & \operatorname{Fun}(E^{\mathrm{c,op}}, T) .
\end{array}$$

Since p_* preserves filtered colimits, the rightmost vertical functor preserves filtered colimits. The claim now follows from Lemma 3.10. □

3.12 Corollary. *Let $p_* : S \to T$ be a right adjoint between presentable ∞-categories in which filtered colimits are left exact. Let E be a stable presentable ∞-category. If p_* preserves filtered colimits, then the right adjoint functor $p_* \otimes E$ is also a left adjoint.*

Proof. By Observation 2.14, it suffices to show that $p_* \otimes \mathbf{Spt}$ is a left adjoint. Since $S \otimes \mathbf{Spt}$ and $T \otimes \mathbf{Spt}$ are stable and $p_* \otimes \mathbf{Spt}$ is exact, by the Adjoint Functor Theorem, it suffices to show that $p_* \otimes \mathbf{Spt}$ preserves filtered colimits. Corollary 3.11 completes the proof. □

Proof of Proposition 3.8. Corollary 3.12 shows that $g_* \otimes \mathbf{Spt}$ and $\bar{g}_* \otimes \mathbf{Spt}$ are left adjoints. Since E is stable, $(\square) \otimes \mathbf{Spt} \otimes E \simeq (\square) \otimes E$; applying Lemma 1.18 to the oriented square $(\square) \otimes \mathbf{Spt}$ completes the proof. □

3.3 Consequences of the Nonabelian Proper Basechange Theorem

We finish by explaining how Corollary 3.2 and Proposition 3.8 answer Question 0.3.

3.13 Example. Let

$$\begin{array}{ccc} \boldsymbol{W} & \xrightarrow{\bar{f}_*} & \boldsymbol{Y} \\ {\scriptstyle \bar{g}_*}\downarrow & \lrcorner & \downarrow{\scriptstyle g_*} \\ \boldsymbol{X} & \xrightarrow[f_*]{} & \boldsymbol{Z} \end{array} \tag{3.14}$$

be a pullback square in the ∞-category of ∞-topoi and (right adjoints in) geometric morphisms. Assume that the geometric morphism g_* is *proper* in the sense of [HTT, Definition 7.3.1.4]. Then the square (3.14) is left adjointable, the geometric morphism $\bar{g}_*$ is also proper, and the functors g_* and $\bar{g}_*$ preserve filtered colimits [HTT, Remark 7.3.1.5]. Applying Corollary 3.2 and Proposition 3.8, we see that if E is a presentable ∞-category which is stable or compactly generated, then the square (3.14) $\otimes E$ is left adjointable.

3.15 Subexample. Let

$$\begin{array}{ccc} W & \xrightarrow{\bar{f}} & Y \\ {\scriptstyle \bar{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

be a pullback square of locally compact Hausdorff topological spaces, and assume that the map g is proper. By [HTT, Theorem 7.3.1.16], the geometric morphism $g_* : \mathrm{Sh}(Y; \mathbf{Spc}) \to \mathrm{Sh}(Z; \mathbf{Spc})$ is proper. As a special case of Example 3.13, we see that if E is a presentable ∞-category which is stable or compactly generated, then the induced square of ∞-categories of E-valued sheaves

$$\begin{array}{ccc} \mathrm{Sh}(W; E) & \xrightarrow{\bar{f}_*} & \mathrm{Sh}(Y; E) \\ {\scriptstyle \bar{g}_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathrm{Sh}(X; E) & \xrightarrow[f_*]{} & \mathrm{Sh}(Z; E) \end{array}$$

is left adjointable.

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