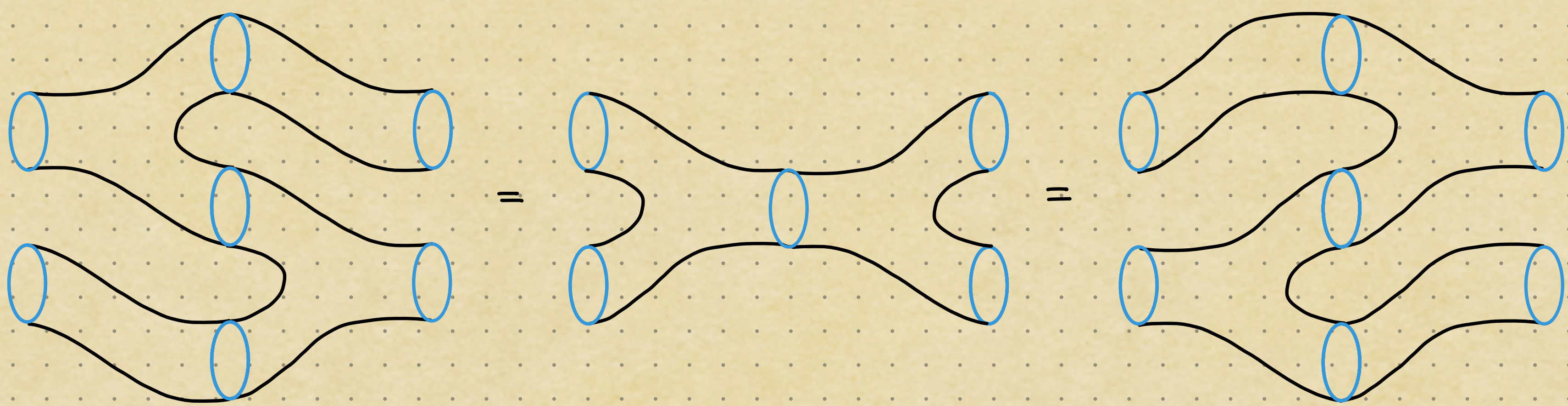
Topological Quantum Field Theories
&
The Cobordism Hypothesis
in Low Dimensions

Peter Haine



Background Needed Linear algebra: pairings, duals, tensor prod.

> Useful: Knowing what a group is and what a category is.

Plan

(1) Some motivation for TQFTs

(2) First observations & 1d TQFTs

(3) 2d TQFTs

(4) Outlook: The Cobordism hypothesis

(1) Motivation for TQFTs

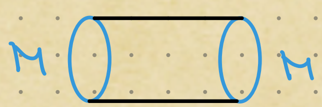
Rough Idea (physics). We should try to study objects moving through spacetime by associating

Object/ body 

$\rightsquigarrow$ Vector Space of 'States' $Z(M)$

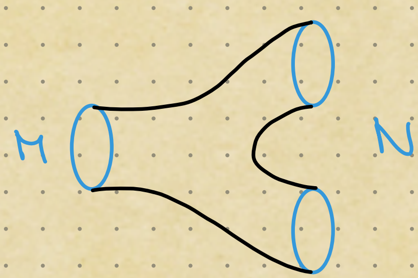
evolution of M over time

'evolution map'



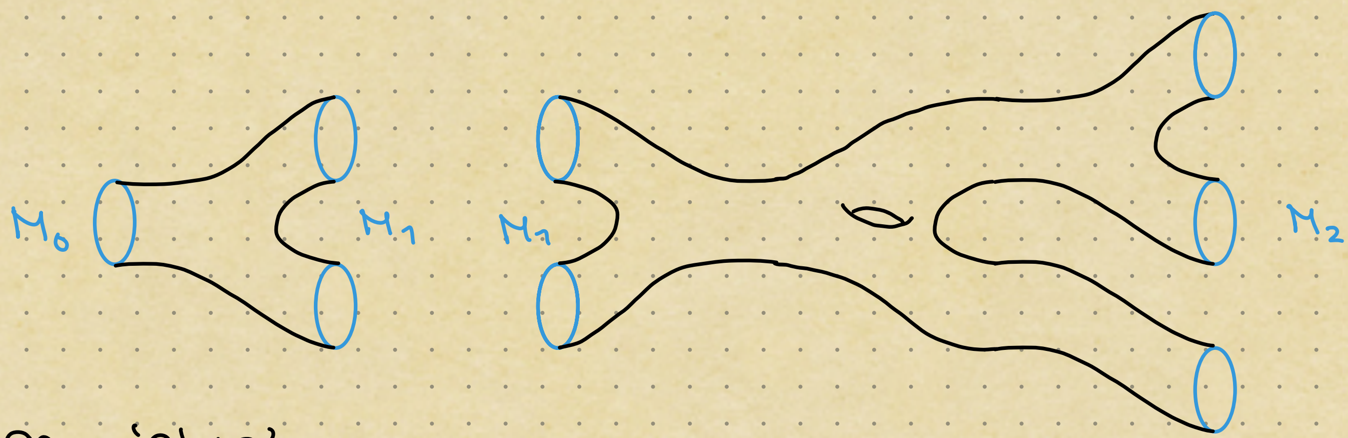
$Z(M) \longrightarrow Z(M)$

$\rightsquigarrow$

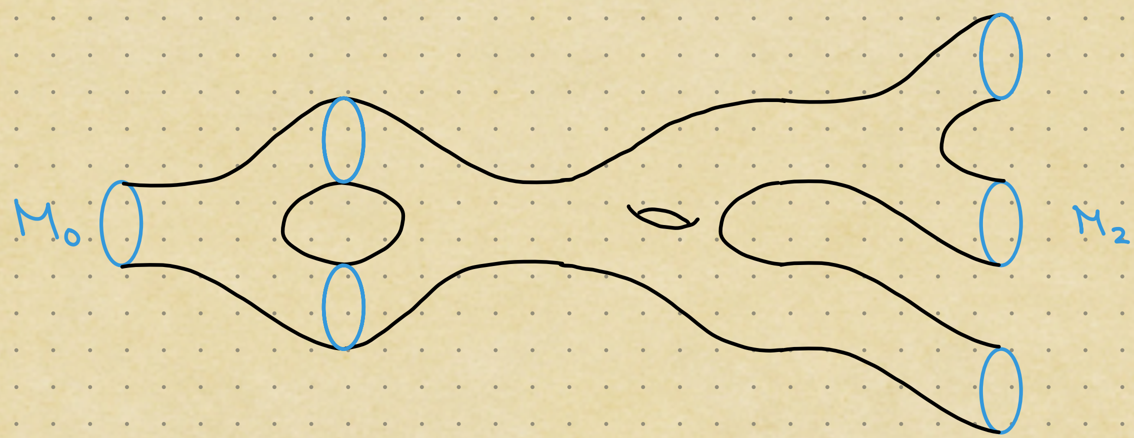


$Z(M) \longrightarrow Z(N)$

> This assignment should be subject to some relations: given two pictures



we can 'glue':



Point The map $Z(M_0) \longrightarrow Z(M_2)$ we assign this big picture should be the composite of the maps

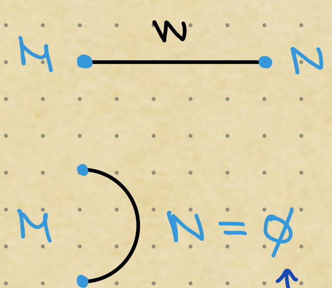
$$Z(M_0) \longrightarrow Z(M_1) \quad \text{and} \quad Z(M_1) \longrightarrow Z(M_2)$$

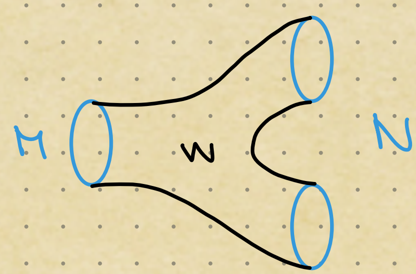
Cobordisms

Rough Definition M, N $(n-1)$-manifolds (without boundary)

A **cobordism** between M and N is an n-manifold with boundary W and an identification

$$\partial W \cong M \sqcup N$$





∅ is a manifold of every dimension

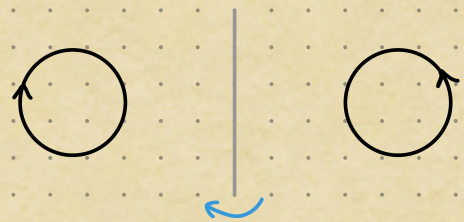
co bordant

jointly / together

bounding

Technical Point We want to be able to say which manifold is the 'start' and which is the end, so we need everything to be oriented

- Orientation of a point = assignment of + or −
- Orientation of a line = choice of 'forward' direction 
- Orientations of S^1:



reflect

- Orientation of surface in $\mathbb{R}^3$: Consistent choice of 'outward facing' normal

Simplifying Assumption All of our manifolds will be $\underbrace{\text{Compact}}_{\text{closed and bounded in } \mathbb{R}^N}$

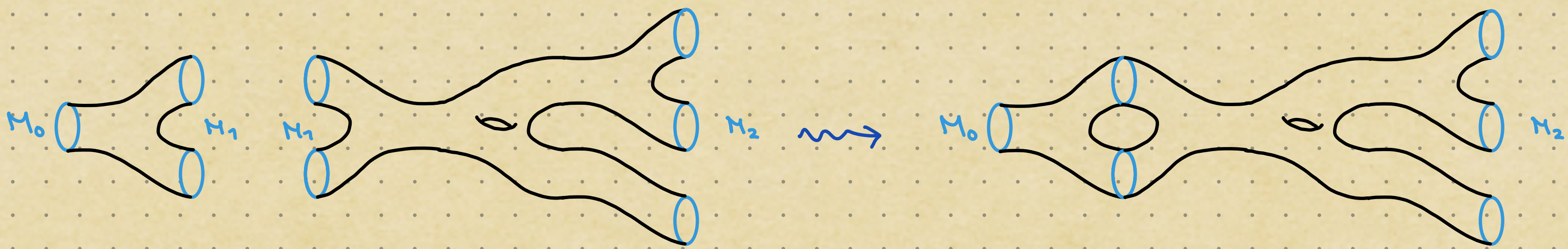
The Cobordism Category

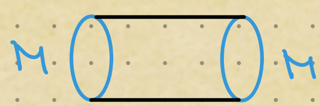
Definition Let $n \geq 1$. Write $\mathrm{Cob}(n)$ for the category with

(0) Objects: compact, oriented $(n-1)$-manifolds

(1) morphisms: diffeomorphism classes of oriented cobordisms.

(2) Composition: gluing cobordisms



Note Here the cylinder 

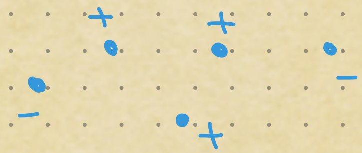
is the identity.

Operation We can take disjoint unions of cobordisms

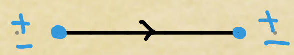
Cases of Interest

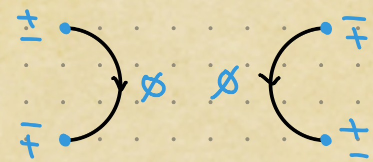
Cob(1)

Objects finite collections of signed points



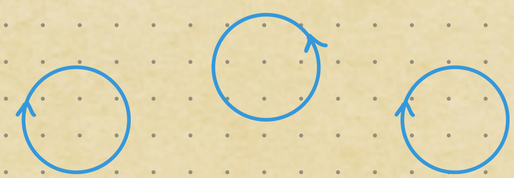
morphisms directed lines



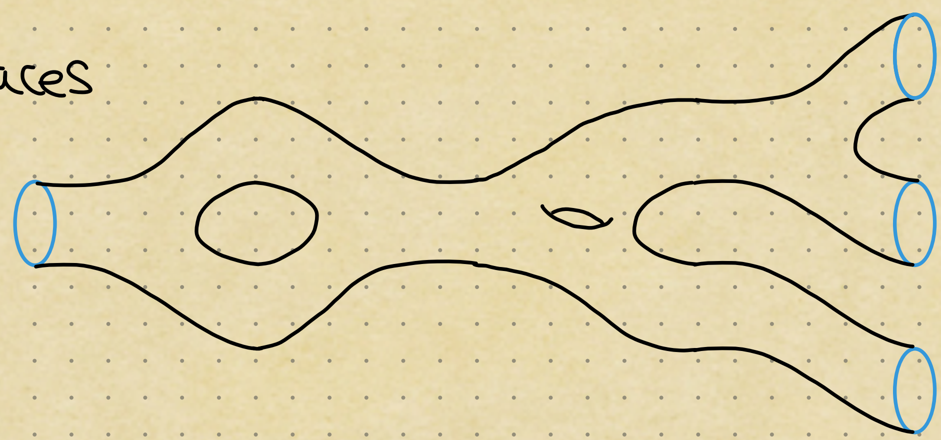


Cob(2)

Objects finite collections of oriented circles



morphisms surfaces



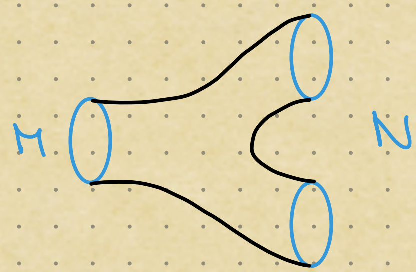
Definition (Atiyah 1988). $n \geq 1$, k field

An **n-dimensional topological quantum field theory** (TQFT) is a symmetric mononoidal functor

$$Z : \mathrm{Cob}(n) \longrightarrow \mathrm{Vect}_k .$$

Explicitly an assignment

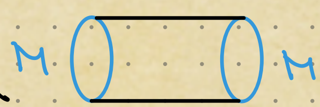
compact oriented $(n-1)$-manifold $M \longmapsto$ k-vector space $Z(M)$

cobordism 

$\longmapsto$ linear map $Z(M) \longrightarrow Z(N)$

such that

(1) Z respects gluing

(2) Z (

) is the identity

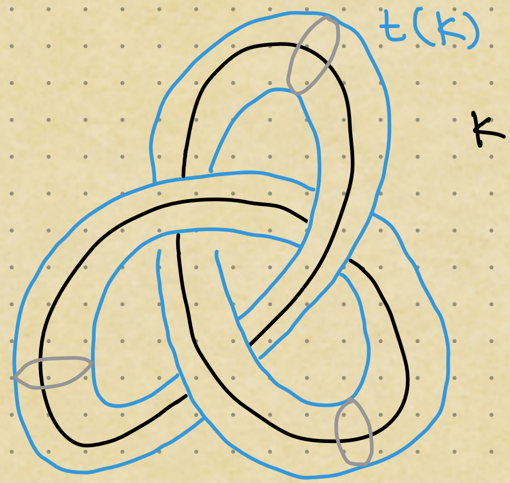
(3) $Z(\emptyset) \cong k$

(4) $Z(M \sqcup N) \cong Z(M) \otimes Z(N)$

Application: Knot Invariants

> Witten used 3d TQFTs to give invariants of knots

(1) Given a knot $K \subset S^3$, we can thicken K to a solid torus $t(K)$



(2) The 3-manifold $S^3 \setminus t(K)$ defines a cobordism from the boundary torus to the empty manifold:

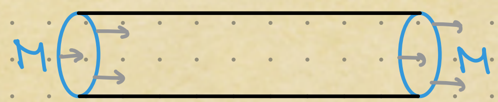
$$ev_{S^3 \setminus t(K)} : Z(T^2) \longrightarrow k.$$

(2) First Observations

Question Given an $(n-1)$-manifold M, what are some cobordisms that we always have?

$Z : \mathrm{Cob}(n) \longrightarrow \mathrm{Vect}_k$ TQFT

'Identity' $M \times [0,1]$

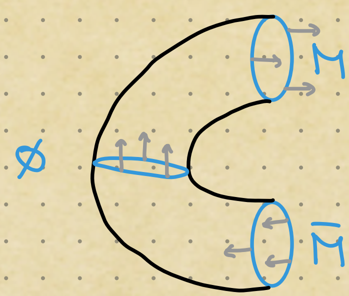


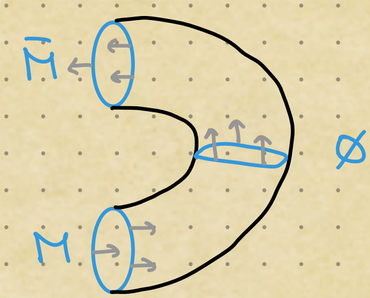
$M \longrightarrow M$

$Z(M) \xrightarrow[\text{id}]{} Z(M)$

'Macaronis'

$\bar{M}$ = M with opposite orientation

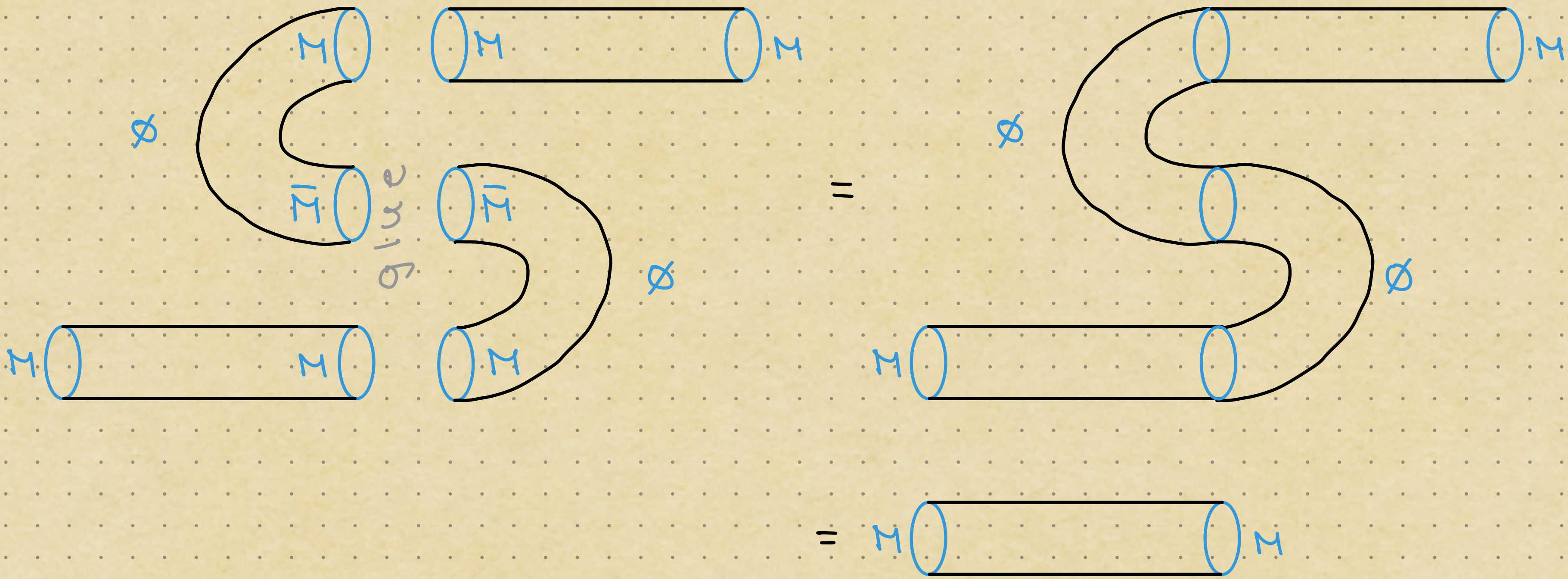




$\emptyset \longrightarrow M \sqcup \bar{M}$ $\qquad$ $\bar{M} \sqcup M \longrightarrow \emptyset$

$k \xrightarrow{\text{coev}} Z(M) \otimes Z(\bar{M})$ $\qquad$ $Z(\bar{M}) \otimes Z(M) \xrightarrow{\text{ev}} k$

'S' Relation

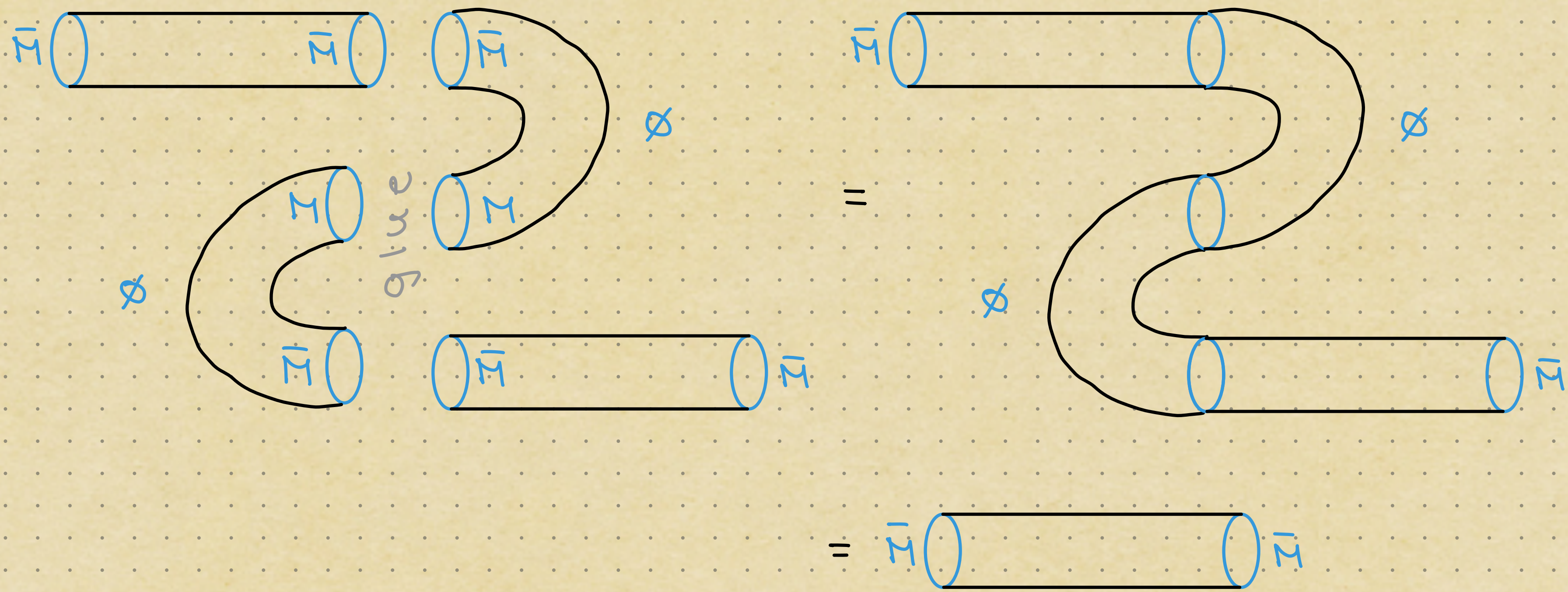


- This says that the composite

$$Z(M) \cong k \otimes_k Z(M) \xrightarrow{\text{coev} \otimes \text{id}} Z(M) \otimes_k Z(\bar{M}) \otimes_k Z(M) \xrightarrow{\text{id} \otimes \text{ev}} Z(M) \otimes_k k \cong Z(M)$$

is the identity.

'Z' Relation



- This says that the composite

$$Z(\bar{M}) \cong Z(\bar{M}) \otimes_k k \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} Z(\bar{M}) \otimes_k Z(M) \otimes_k Z(\bar{M}) \xrightarrow{\mathrm{ev} \otimes \mathrm{id}} k \otimes_k Z(\bar{M}) \cong Z(\bar{M})$$

is the identity.

Proposition Let k be a field and V a k-vector space. TFAE:

(1) V is finite dimensional

'double dual'

(2) $V \longrightarrow (V^*)^* := \mathrm{Hom}(\mathrm{Hom}(V,k),k)$ is an isomorphism

$v \longmapsto [\,\ell \longmapsto \ell(v)\,]$

(3) There exists a k-vector space $V^\vee$ and maps

$$ev : V^\vee \otimes_k V \longrightarrow k \qquad coev : k \longrightarrow V \otimes_k V^\vee$$

such that the composites

S Relation

$$V \cong k \otimes_k V \xrightarrow{coev \otimes id} V \otimes_k V^\vee \otimes_k V \xrightarrow{id \otimes ev} V \otimes_k k \cong V$$

and

Z Relation

$$V^\vee \cong V^\vee \otimes_k k \xrightarrow{id \otimes coev} V^\vee \otimes_k V \otimes_k V^\vee \xrightarrow{ev \otimes id} k \otimes_k V^\vee \cong V^\vee$$

are the identities.

Note In the setting of (3), the map

$$\begin{aligned} V^\vee &\longrightarrow \mathrm{Hom}(V, k) \\ \omega &\longmapsto [v \longmapsto ev(\omega, v)] \end{aligned}$$

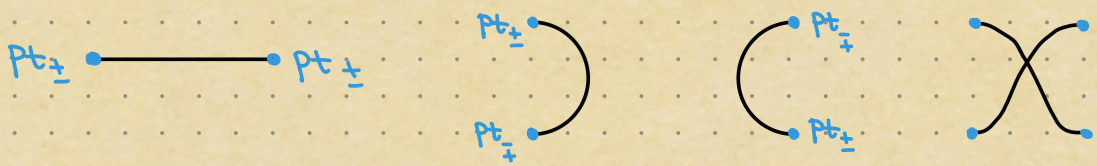
is an isomorphism.

Terminology Condition (3) is called dualizability.

Corollary $Z : \mathrm{Cob}(n) \longrightarrow \mathrm{Vect}_k$ n-dim. TQFT
For every $(n-1)$-manifold M, the vector space $Z(M)$ is finite-dim. with dual $Z(\bar{M})$.

1d TQFTs

> There aren't so many cobordisms between 0-manifolds. They're all generated by



Theorem Evaluation on the positively oriented point pt_+ defines an equivalence of categories

$$\left\{ \begin{array}{c} \text{1d TQFTs} \\ \text{over } k \end{array} \right\} = \mathrm{Fun}^{\otimes}(\mathrm{Cob}(1), \mathrm{Vect}_k) \xrightarrow{\sim} \left\{ \begin{array}{c} \text{finite dim.} \\ k\text{-vector spaces} \end{array} \right\}.$$

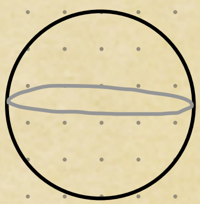
$$Z \longmapsto Z(pt_+).$$

(3) 2d TQFTs

Goal Explain the classification of 2d TQFTs

Classification of surfaces Compact orientable surfaces are classified up to diffeomorphism by their

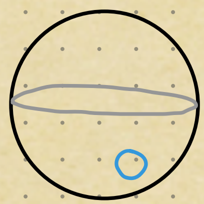
genus (# of 'holes') and number of boundary components



Sphere

genus = 0

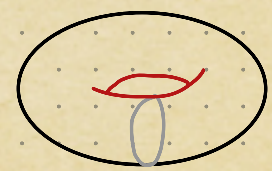
∂ components = 0



Sphere with disk removed

genus = 0

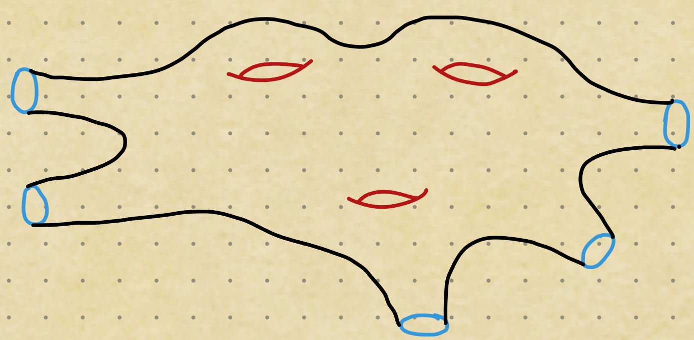
∂ components = 1



Torus

genus = 1

∂ components = 0

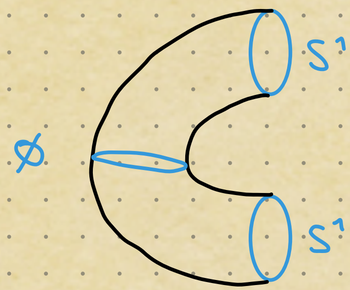


genus = 3 ∂ components = 5

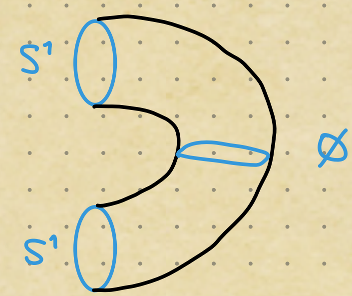
First Cobordisms to observe

Macaronis

$$Z(S^1) \xrightarrow[\text{id}]{} Z(S^1)$$

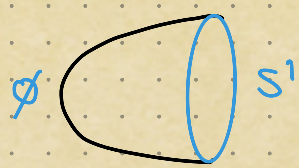


$$k \xrightarrow{\text{coev}} Z(S^1)^{\otimes 2}$$

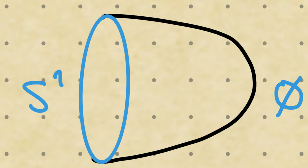


$$Z(S^1)^{\otimes 2} \xrightarrow{\text{ev}} k$$

Caps

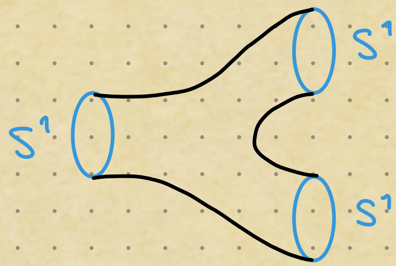


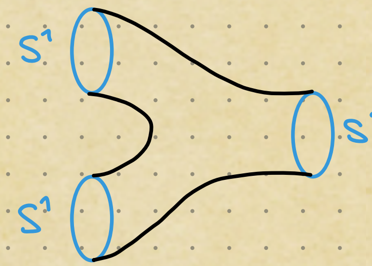
$$k \xrightarrow{\text{unit}} Z(S^1)$$



$$Z(S^1) \xrightarrow{\text{counit}} k$$

Pants





$Z(S^1) \xrightarrow{\text{comult}} Z(S^1)^{\otimes 2}$

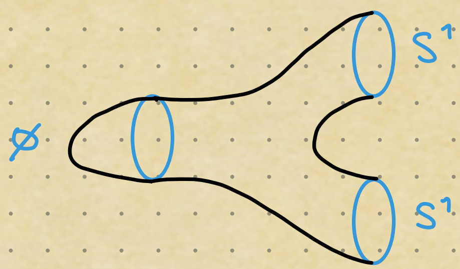
$Z(S^1)^{\otimes 2} \xrightarrow{\text{mult}} Z(S^1)$

gives $Z(S^1)$ a product

$(a,b) \longmapsto a \cdot b$

First Relations

Relation 1



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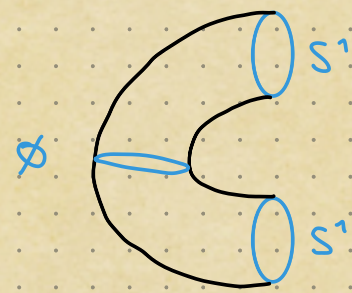
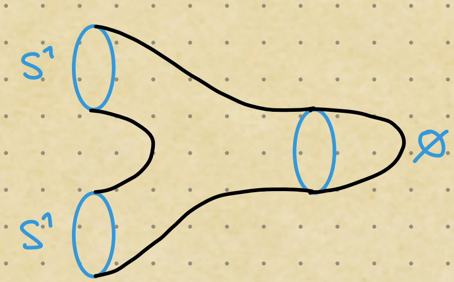


$k \xrightarrow{\text{unit}} Z(S^1) \xrightarrow{\text{comult}} Z(S^1)^{\otimes 2}$

$k \xrightarrow{\text{coev}} Z(S^1)^{\otimes 2}$

Relation 2



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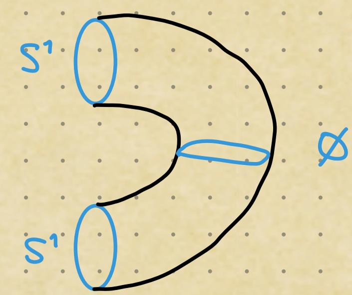
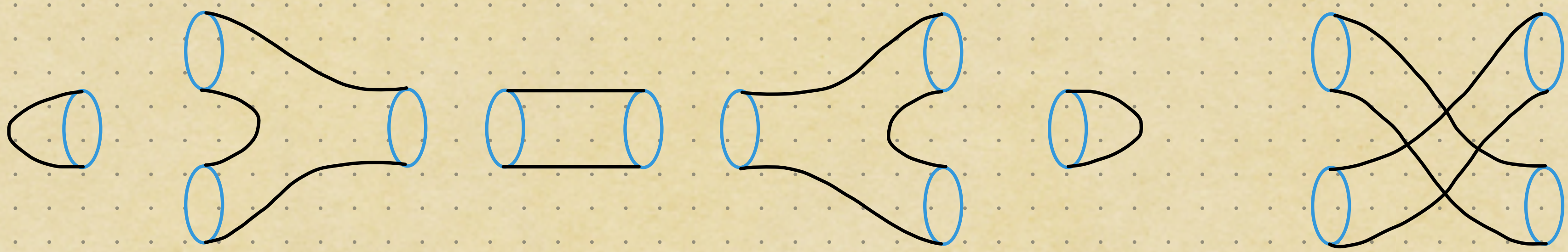


$Z(S^1)^{\otimes 2} \xrightarrow{\text{mult}} Z(S^1) \xrightarrow{\text{counit}} k$

$Z(S^1)^{\otimes 2} \xrightarrow{\text{ev}} k$

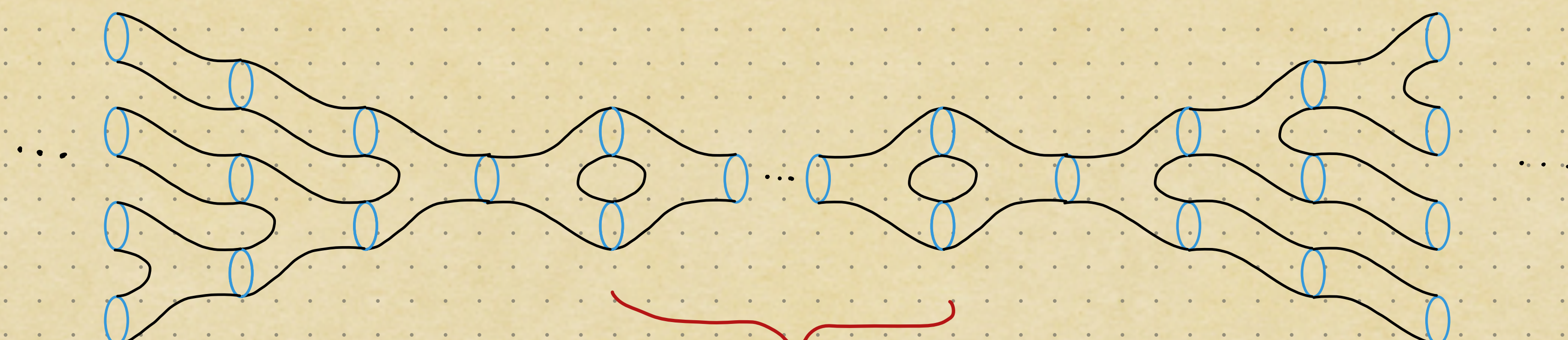
Proposition Cob(2) is generated under composition and disjoint union by the following six cobordisms



This 'swap' cobordism says $X \sqcup Y \cong Y \sqcup X$

Proof Idea

We can always put a surface in 'normal form':



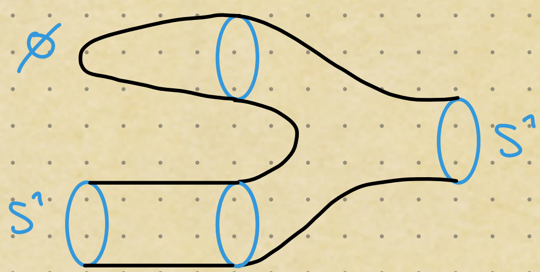
genus many times

$\square$

Unit / Counit Relations

Unit Relation

' $1 \cdot x = x = x \cdot 1$ '

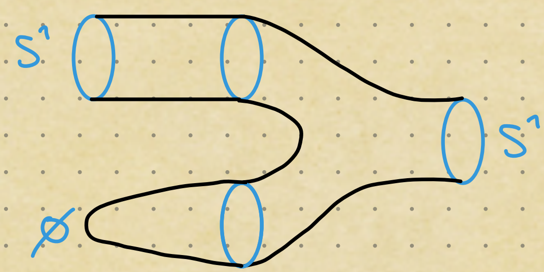


$=$

$$k \otimes_k Z(S^1) \xrightarrow[\text{unit} \otimes \text{id}]{} Z(S^1)^{\otimes 2} \xrightarrow[\text{mult}]{} Z(S^1)$$

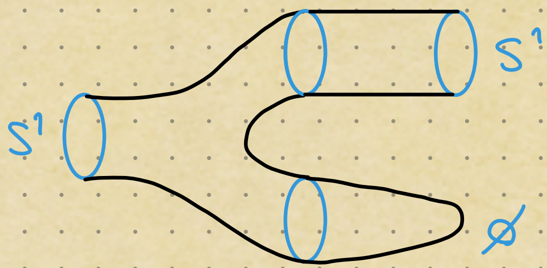
$$Z(S^1) \xrightarrow[\text{id}]{} Z(S^1)$$

$=$



$$Z(S^1) \otimes_k k \xrightarrow[\text{id} \otimes \text{unit}]{} Z(S^1)^{\otimes 2} \xrightarrow[\text{mult}]{} Z(S^1)$$

Counit Relation

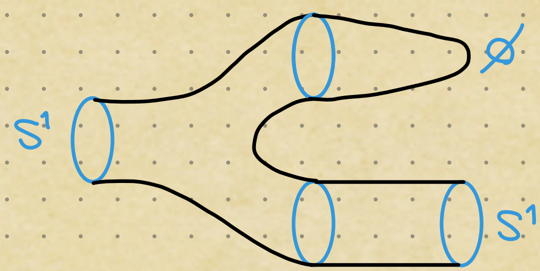


$=$

$$Z(S^1) \xrightarrow[\text{comult}]{} Z(S^1)^{\otimes 2} \xrightarrow[\substack{\text{id} \\ \otimes \\ \text{counit}}]{} Z(S^1) \underset{K}{\otimes K}$$

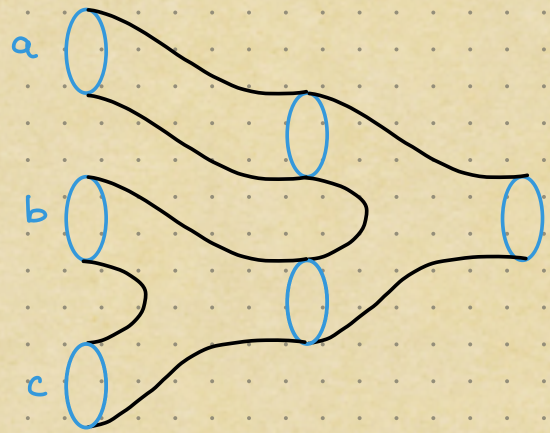
$$Z(S^1) \xrightarrow[\text{id}]{} Z(S^1)$$

$=$



$$Z(S^1) \xrightarrow[\text{comult}]{} Z(S^1)^{\otimes 2} \xrightarrow[\substack{\text{counit} \\ \otimes \\ \text{id}}]{} \underset{K}{K \otimes} Z(S^1)$$

Associativity Relation



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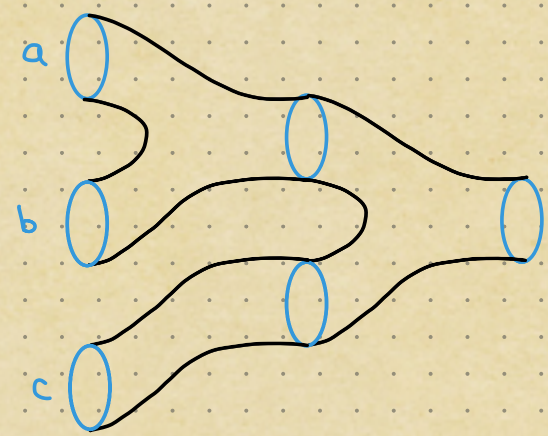
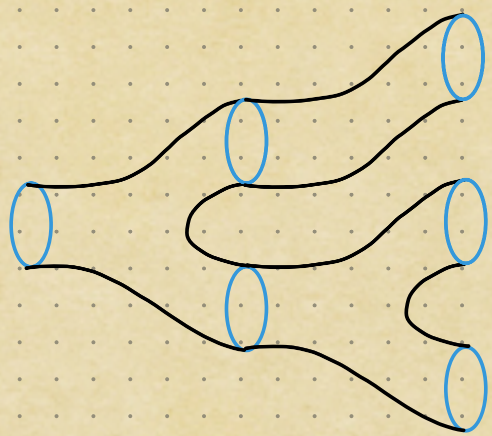


$a \cdot (b \cdot c)$ $(a \cdot b) \cdot c$

Coassociativity Relation



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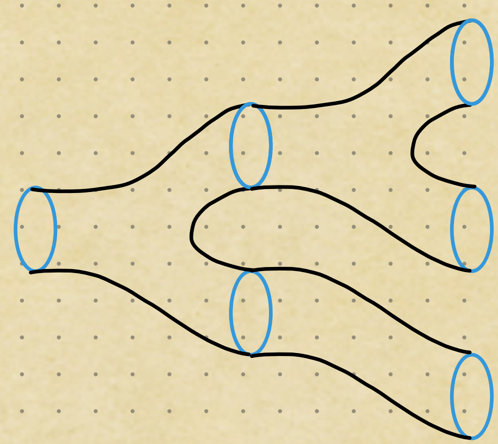
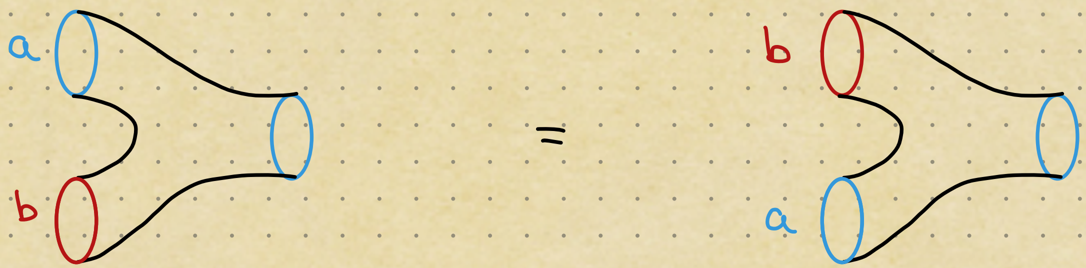
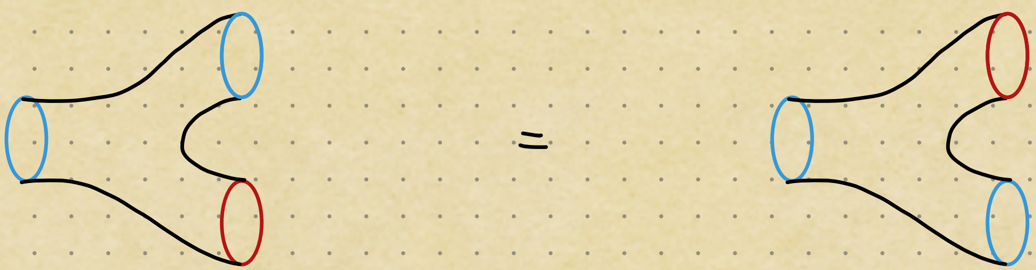


Commutativity Relation

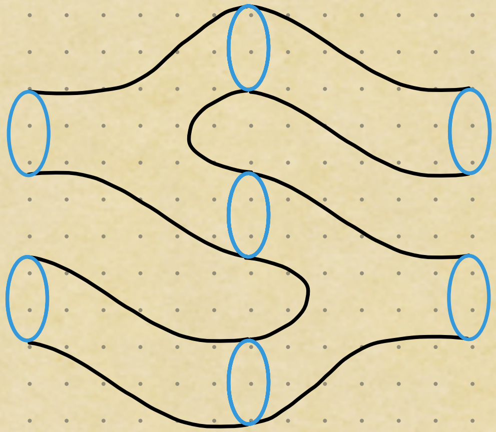


$a \cdot b$ $\qquad$ $b \cdot a$

Cocommutativity Relation

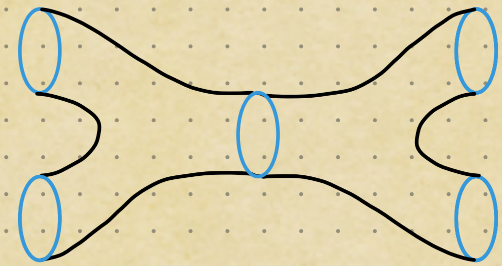


Frobenius Relations



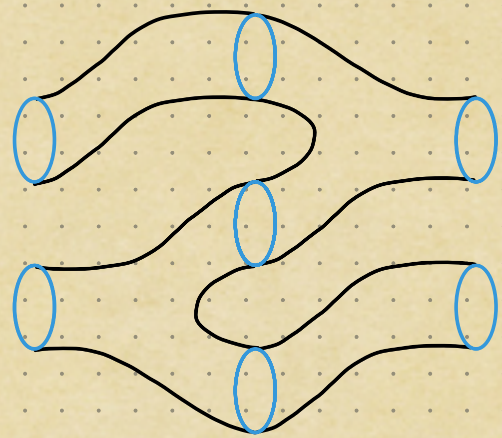
$$Z(S^1)^{\otimes 2} \xrightarrow[\text{comult} \otimes \text{id}]{} Z(S^1)^{\otimes 3} \xrightarrow[\text{id} \otimes \text{mult}]{} Z(S^1)^{\otimes 2}$$

=



$$Z(S^1)^{\otimes 2} \xrightarrow[\text{mult}]{} Z(S^1) \xrightarrow[\text{comult}]{} Z(S^1)^{\otimes 2}$$

=



$$Z(S^1)^{\otimes 2} \xrightarrow[\text{id} \otimes \text{comult}]{} Z(S^1)^{\otimes 3} \xrightarrow[\text{mult} \otimes \text{id}]{} Z(S^1)^{\otimes 2}$$

Commutative Frobenius Algebras

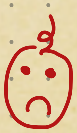
Definition K field

A **commutative Frobenius algebra** over K is a finite dim K-vector space A equipped with linear maps

$$A \otimes_K A \xrightarrow{\text{mult}} A \qquad\qquad A \xrightarrow{\text{comult}} A \otimes_K A$$

$$K \xrightarrow{\text{unit}} A \qquad\qquad A \xrightarrow{\text{counit}} K$$

Satisfying all of the relations we found above.

 This seems very complicated ...

Not obvious

Equivalent Definition

k field

A **commutative Frobenius algebra** over k is a finite dim k-vector space A equipped with

(1) An associative, commutative, and unital multiplication

$$\text{mult}: A \otimes_k A \longrightarrow A \qquad 1_A \in A \iff \begin{array}{ccc} k & \longrightarrow & A \\ 1 & \longmapsto & 1_A \end{array}$$

(2) A linear map $\text{counit}: A \longrightarrow k$ such that the bilinear form

$$\begin{array}{ccccc} A \otimes_k A & \xrightarrow{\text{mult}} & A & \xrightarrow{\text{counit}} & k \\ (a,b) & & \longmapsto & & \text{counit}(ab) \end{array}$$

is nondegenerate.

Examples

(1) $n \times n$ diag. matrices with

$\text{counit} = \text{tr}$

(2) The group algebra $k[G]$ for G an abelian group

$\text{counit}(x) = \text{coeff of } e \in G$

Theorem Evaluation on the circle S^1 defines an equivalence of categories

$$\left\{ \begin{matrix} \text{2d TQFTs} \\ \text{over } k \end{matrix} \right\} = \mathrm{Fun}^{\otimes}(\mathrm{Cob}(2), \mathrm{Vect}_k) \xrightarrow{\sim} \left\{ \begin{matrix} \text{commutative} \\ \text{Frobenius} \\ \text{algebras} / k \end{matrix} \right\}.$$

$$Z \longmapsto Z(S^1)$$

(4) Outlook: The cobordism hypothesis

Question Can n-dim TQFTs also be classified by a single object with algebraic structure?

History

(1) In 1995, Baez-Dolan gave a precise conjecture, called the **cobordism hypothesis**

(2) Requires using **(∞, n)-categories**

Tools (2000s): Joyal, Lurie, Rezk, Barwick, ...

(3) 2008: Lurie sketced a solution

(4) Many others have given alternative solutions:
Ayala-Francis + collaborators introduced **factorization homology**

↑ generalizes Hochschild homology

Applications

Topology Knot invariants, mirror Symmetry

Algebra / Representation Theory Fusion Categories, Ben-Zvi-Nadler Character theory

Relations to geometric Langlands

Further Reading

Kock Frobenius algebras and 2D topological quantum field theories

Atiyah Topological quantum field theory

Baez-Dolan Higher-dimensional algebra and topological quantum field theory

Freed The cobordism hypothesis

Beautifully written expository article

Thanks for Listening!