Splitting free loop spaces

Peter J. Haine

April 17, 2021

Overview

The purpose of this note is to give a modern account of the classical fact that given a space X equipped with a unital multiplication that has inverses in an appropriate sense, the free loop space $\mathrm{L}X$ of X splits as a product $\mathrm{L}X \simeq X \times \Omega X$ of X with the loop space of X based at the unit for the multiplication. Specifically, given an A_2-algebra M in an ∞-category with finite limits, only using basic manipulations with limits we produce a natural morphism

$$c_M \colon M \times \Omega M \to \mathrm{L}M$$

(Construction 3.1). If, in addition, the *shear map* $M \times M \to M \times M$ informally described by $(x, y) \mapsto (x, xy)$ is an equivalence, then the morphism $c_M \colon M \times \Omega M \to \mathrm{L}M$ is an equivalence (Proposition 3.3). One example that fits into this framework is the 7-sphere with A_2-multiplication induced by the multiplication on the octonions (Example 2.6).

In Section 1, we recall the definition of a free loop object in an ∞-category with finite limits and prove two results relating free loop objects to based loop objects (Lemma 1.4 and Corollary 1.5). In Section 2, we give a reminder on A_2-structures and provide an example showing that an A_2-structure is not enough to guarantee that the free loop space fibration splits (Example 2.6). Section 3 states and proves the main splitting result of this note. We conclude with a result of Agudé [1] and Ziller [4, p. 21] that characterizes the spheres for which the free loop space fibration splits (Theorem 3.7).

The reader familiar with A_2-structures and free loop objects in this generality is advised to take a look at Definition 2.8, then skip straight to Section 3; once the basics are set up, the proof of Proposition 3.3 is immediate.

1 Recollections on free loop objects

Let X be a space. Since the circle S^1 is the pushout $* \sqcup^{\mathrm{S}^0} *$ in the ∞-category **Spc**, the free loop space $\mathrm{L}X = \mathrm{Map}(\mathrm{S}^1, X)$ is given by the pullback $X \times_{X \times X} X$. The morphisms $X \to X \times X$ appearing in the pullback are both the diagonal. We use this description to define free loop objects in any ∞-category with finite limits.

1.1 Notation. Let C be an ∞-category with finite products and $X \in C$.

(1) We write $1_C \in C$ for the terminal object.

(2) We write $\Delta_X \colon X \to X \times X$ for the diagonal morphism.

(3) Given a point $x \colon 1_C \to X$, we write $(x,x) \colon 1_C \to X \times X$ for the point defined by the composite

$$1_C \xrightarrow{x} X \xrightarrow{\Delta_X} X \times X .$$

1.2 Definition. Let C be an ∞-category with finite limits and $X \in C$. The *free loop* object on X is the pullback

$$\begin{array}{ccc} \mathrm{L}X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \Delta_X \\ X & \xrightarrow[\Delta_X]{} & X \times X . \end{array}$$

1.3 Observation. The natural commutative square

$$\begin{array}{ccc} X & = & X \\ \| & & \downarrow \Delta_X \\ X & \xrightarrow[\Delta_X]{} & X \times X . \end{array}$$

provides a natural section $s_X \colon X \to \mathrm{L}X$ of the projections $\mathrm{L}X \to X$.

The following description of the based loop object of a pointed object in terms of the diagonal allows us to relate the free loop object to the based loop object.

1.4 Lemma. *Let C be an ∞-category with finite limits, $X \in C$, and $x \colon 1_C \to X$ a point of X. There is a natural equivalence*

$$\mathrm{fib}_{(x,x)}(\Delta_X \colon X \to X \times X) \simeq \Omega_x X .$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} 1_C & = & 1_C & = & 1_C \\ x \downarrow & & \| & & \downarrow x \\ X & \longrightarrow & 1_C & \longleftarrow & X \\ \| & & \uparrow & & \| \\ X & = & X & = & X . \end{array}$$

Taking pullbacks vertically then horizontally yields

$$\lim\left(1_C \xrightarrow{x} X \xleftarrow{x} 1_C \right) = \Omega_x X ,$$

and taking pullbacks horizontally then vertically yields

$$\lim\left(1_C \xrightarrow{(x,x)} X \times X \xleftarrow{\Delta_X} X \right) = \mathrm{fib}_{(x,x)}(\Delta_X) .$$

The claim follows from the fact that limits commute. □

1.5 Corollary. *Let C be an ∞-category with finite limits, $X \in C$, and $x \colon 1_C \to X$ a point of X. There is a natural pullback square*

$$\begin{tikzcd} \Omega_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{L}X \arrow[d] \\ 1_C \arrow[r, "x"'] & X \end{tikzcd}$$

Proof. Express $\mathrm{fib}_{(x,x)}(\Delta_X)$ as the iterated pullback

$$\begin{tikzcd} \mathrm{fib}_{(x,x)}(\Delta_X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{L}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\Delta_X"] \\ 1_C \arrow[r, "x"'] & X \arrow[r, "\Delta_X"'] & X \times X \end{tikzcd}$$

and apply Lemma 1.4. □

2 Reminder on A_2-algebras

We now recall the definition of an A_2-*algebra* and explain the variant of an A_2-structure needed for the free loop space fibraion to split.

2.1 Recollection. Let C be an ∞-category with finite products. An A_2-*algebra* in C consists of the following data:

(1) An object $M \in C$.

(2) A *multiplication* morphism $m \colon M \times M \to M$.

(3) A *unit* morphism $u \colon 1_C \to M$.

(4) Choices of 2-morphisms filling the diagrams

$$\begin{tikzcd} M \times 1_C \arrow[r, "\mathrm{pr}_1", "\sim"'] \arrow[d, "\mathrm{id} \times u"'] & M \\ M \times M \arrow[ur, "m"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 1_C \times M \arrow[r, "\mathrm{pr}_2", "\sim"'] \arrow[d, "u \times \mathrm{id}"'] & M \\ M \times M \arrow[ur, "m"'] & \end{tikzcd} .$$

2.2 Remark. In the classical literature, an A_2-algebra in **Spc** is called an H-*space*.

2.3 Convention. Let C be an ∞-category with finite limits and let M be an A_2-algebra in C. We write $\Omega M \coloneqq \Omega_u M$ for the loop object of M based at the unit $u \colon 1_C \to M$.

A counterexample

In the classical literature, various authors claim that the free loop space fibration of an A_2-algebra in spaces splits. In this subsection we provide a counterexample to this claim (Example 2.6). The point is that for the free loop space fibration of M to split, at very least the based loop spaces of M at each point need to be equivalent. However, an A_2-algebra structure is not enough to guarantee that all of the connected components of M have the same homotopy type: we can always adjoint a new point 0 that acts as an 'absorbing element' for the multiplication. That is, we can generalize the following construction for magmas (i.e., A_2-sets).

2.4 Construction. Let $(M, \cdot_M)$ be a magma. Write M_+ for the magma with underlying set $M_+ \coloneqq M \sqcup \{0\}$ with mutiplicattion defined by

$$x \cdot_{M_+} y \coloneqq \begin{cases} x \cdot_M y, & x, y \in M \\ 0, & x = 0 \text{ or } y = 0 \,. \end{cases}$$

The following is the generalization of Construction 2.4 from sets to spaces:

2.5 Construction. Write $(-)_+ \colon \mathbf{Spc} \to \mathbf{Spc}_*$ for the left adjoint to the forgetful functor $\mathbf{Spc}_* \to \mathbf{Spc}$. Given spaces $X_1, \ldots, X_n$, there is a natural splitting

$$\begin{aligned} \prod_{i=1}^{n} X_{i,+} &\simeq \bigvee_{\varnothing \neq I \subset \{1,\ldots,n\}} \Big(\prod_{i \in I} X_i\Big)_+ \\ &\simeq \Big(\prod_{i=1}^{n} X_i\Big)_+ \vee \bigvee_{\varnothing \neq I \subsetneq \{1,\ldots,n\}} \Big(\prod_{i \in I} X_i\Big)_+ \,. \end{aligned}$$

Using this splitting, one can give the functor $(-)_+ \colon \mathbf{Spc} \to \mathbf{Spc}_*$ a lax-monoidal structure with respect to the product. The structure morphisms

$$\Big(\prod_{i=1}^{n} X_i\Big)_+ \vee \bigvee_{\varnothing \neq I \subsetneq \{1,\ldots,n\}} \Big(\prod_{i \in I} X_i\Big)_+ \longrightarrow \Big(\prod_{i=1}^{n} X_i\Big)_+$$

are given by the identity on the first factor and the constant map at the point on the second factor.[1]

Since the forgetful functor $\mathbf{Spc}_* \to \mathbf{Spc}$ is naturally symmetric monoidal with respect to the product, this gives the composite $(-)_+ \colon \mathbf{Spc} \to \mathbf{Spc}$ a lax-monoidal structure with respect to the product. In particular, the functor $(-)_+ \colon \mathbf{Spc} \to \mathbf{Spc}$ preserves algebras over any operad. Hence, for each $1 \leq n \leq \infty$ the functor $(-)_+$ lifts to functors

$$(-)_+ \colon \mathrm{Alg}_{\mathrm{A}_n}(\mathbf{Spc}) \to \mathrm{Alg}_{\mathrm{A}_n}(\mathbf{Spc}) \qquad \text{and} \qquad (-)_+ \colon \mathrm{Alg}_{\mathrm{E}_n}(\mathbf{Spc}) \to \mathrm{Alg}_{\mathrm{E}_n}(\mathbf{Spc}) \,.$$

[1] There are several ways to make this precise. One is to give the functor $(-)_+ \colon \mathbf{Set}^{\mathrm{fin}} \to \mathbf{Set}_*^{\mathrm{fin}}$ a lax-monoidal structure and take the lax-monoidal left Kan extension of the composite functor $(-)_+ \colon \mathbf{Set}^{\mathrm{fin}} \to \mathbf{Set}_*^{\mathrm{fin}} \hookrightarrow \mathbf{Spc}_*$.

2.6 Example. The free loop space fibration for the A_2-space S^1_+ does not split. To see this, note that

$$\begin{aligned} LS^1_+ &\simeq LS^1 \sqcup L\{0\} \\ &\simeq S^1 \times \Omega S^1 \sqcup \{0\} \\ &\simeq S^1 \times \mathbf{Z} \sqcup \{0\} \,. \end{aligned}$$

On the other hand,

$$\begin{aligned} S^1_+ \times \Omega S^1 &\simeq S^1 \times \Omega S^1 \sqcup \Omega S^1 \\ &\simeq S^1 \times \mathbf{Z} \sqcup \mathbf{Z} \,. \end{aligned}$$

Notice that no amount of commutativity helps here. Since S^1 has an E_∞-structure, S^1_+ also has an E_∞-structure. The problem here is that the multiplication on an A_2-space M needs to have *inverses* in order for the connected components of M to be equivalent.

The necessary structure

To formulate what it means for an A_2-algebra in an ∞-category with finite limits to 'have inverses', we make the following observation.

2.7 Observation. Let M be a monoid. Then M is a group if and only if the *shear maps* $M \times M \to M \times M$ defined by

$$(x, y) \mapsto (x, xy) \qquad \text{and} \qquad (x, y) \mapsto (xy, y)$$

are bijections.

In fact, for the free loop space fibration to split, the we do not need a full A_2-structure, but just a right (or left) unital multiplication 'with inverses'.

2.8 Definition. Let C be an ∞-category with finite limits. A *right* $A_{1½}$*-algebra*[2] in C consists of the following data:

(1) An object $M \in C$.

(2) A *multiplication* morphism $m \colon M \times M \to M$.

(3) A *unit* morphism $u \colon 1_C \to M$.

(4) A choice of 2-morphism filling the diagram

$$\begin{array}{ccc} M \times 1_C & \xrightarrow[\sim]{\mathrm{pr}_1} & M \\ {\scriptstyle \mathrm{id} \times u}\downarrow & \nearrow_{m} & \\ M \times M & & \end{array} \,.$$

2.9 Definition. Let C be an ∞-category with finite limits and let M be a right $A_{1½}$-algebra in C. The *shear map* associated to M is the morphism

$$\mathrm{sh} \coloneqq (\mathrm{pr}_1, m) \colon M \times M \to M \times M \,.$$

[2] We would be grateful to know if there is a better name for this structure.

3 Splitting the fiber sequence

Given an $\mathrm{A}_{1½}$-algebra M, we now easily construct a morphism $M \times \Omega M \to \mathrm{L}M$ that is an equivalence if the shear map associated to M is.

3.1 Construction. Let C be an ∞-category with finite limits and let M be an right $\mathrm{A}_{1½}$-algebra in C. The right $\mathrm{A}_{1½}$-algebra structure on M defines a commutative diagram

$$\begin{array}{ccccc} M \times 1_C & \xrightarrow{\mathrm{id} \times u} & M \times M & \xleftarrow{\mathrm{id} \times u} & M \times 1_C \\ {\scriptstyle \mathrm{pr}_1}\downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \mathrm{sh}} & & {\scriptstyle \wr}\downarrow{\scriptstyle \mathrm{pr}_1} \\ M & \xrightarrow[\Delta_M]{} & M \times M & \xleftarrow[\Delta_M]{} & M . \end{array}$$

We denote the induced morphism on pullbacks by $c_M \colon M \times \Omega M \to \mathrm{L}M$. Note that the morphism c_M is natural in morphisms of right $\mathrm{A}_{1½}$-algebras.

3.2 Observation. By construction, the composite of the section $M \simeq M \times 1_C \to M \times \Omega M$ given by the basepoint of ΩM with c_M is the section $s_M \colon M \to \mathrm{L}M$ of Observation 1.3.

The following is immediate:

3.3 Proposition. *Let C be an ∞-category with finite limits and let M be a right $\mathrm{A}_{1½}$-algebra in C. If the shear map $\mathrm{sh} \colon M \times M \to M \times M$ is an equivalence, then the natural morphism $c_M \colon M \times \Omega M \to \mathrm{L}M$ is an equivalence.*

3.4 Remark. Note that Construction 3.1 and Proposition 3.3 have variants for objects with a left unital multiplication (i.e., 'left $\mathrm{A}_{1½}$-algebras').

Free loop spaces of spheres

We conclude by characterizing the spheres for which the free loop space fibration splits.

3.5 Example. Let $n = 0, 1,$ or 3. Then the n-sphere S^n has an E_1-group structure induced by regarding S^n as the norm 1 real numbers ($n = 0$), complex numbers ($n = 1$), or quaternions ($n = 3$). Hence Proposition 3.3 implies that the free loop space fibration for S^n splits.

3.6 Example. The 7-sphere $\mathrm{S}^7 \in \mathbf{Spc}$ has an A_2-structure given by regarding the topological 7-sphere as the norm 1 octonions. Since the octonions $\mathbf{O}$ form an *alternative division algebra* over $\mathbf{R}$, the shear map

$$\mathbf{O} \times \mathbf{O} \to \mathbf{O} \times \mathbf{O} , \quad (x, y) \mapsto (x, xy)$$

is a homeomorphism. Hence, the shear map $\mathrm{S}^7 \times \mathrm{S}^7 \to \mathrm{S}^7 \times \mathrm{S}^7$ is an equivalence. Proposition 3.3 provides a splitting

$$\mathrm{LS}^7 \simeq \mathrm{S}^7 \times \Omega \mathrm{S}^7 .$$

The following characterization of the spheres for which the free loop space fibration splits follows from combining work of Aguadé [1] and Ziller [4, p. 21]. We summarize the proof below.

3.7 Theorem. *Let n be a positive integer. The following conditions are equivalent:*

(1) $n = 0, 1, 3,$ *or* 7.

(2) *The free loop space fiber sequence* $\Omega S^n \to LS^n \to S^n$ *splits.*

(3) *There exists an equivalence of spaces* $S^n \times \Omega S^n \xrightarrow{\sim} LS^n$.

3.8 Observation. Let $n \geq 0$ be an integer. The Serre spectral sequence computing the homology of ΩS^n and the Künneth formula show that for each positive integer k, the homology group

$$H_k(S^n \times \Omega S^n; \mathbf{Z}) \cong \bigoplus_{i+j=k} H_i(S^n; \mathbf{Z}) \otimes_{\mathbf{Z}} H_j(\Omega S^n; \mathbf{Z})$$

is a free abelian group.

Proof Summary. The implication (1)⇒(2) is the content of Examples 3.5 and 3.6, and the implication (2)⇒(3) is clear. Note that the remaining implication (3)⇒(1) is equivalent to the claim:

($*$) If $n \notin \{0, 1, 3, 7\}$, then there does not exist an equivalence $S^n \times \Omega S^n \xrightarrow{\sim} LS^n$.

To prove ($*$), it suffices to assume that $n \geq 2$. We treat the cases of n even and odd separately. If n is even, then Ziller [4, p. 21] shows that for each $m \geq 1$ we have

$$H_{2m(n-1)}(LS^n; \mathbf{Z}) \cong \mathbf{Z}/2 .$$

Thus Observation 3.8 shows there does not exist an equivalence $S^n \times \Omega S^n \xrightarrow{\sim} LS^n$.

For n odd, Ziller [4, p. 21] shows that

$$H_*(S^n \times \Omega S^n; \mathbf{Z}) \cong H_*(LS^n; \mathbf{Z}) ,$$

hence a different method is needed to complete the proof of ($*$). Aguadé [1] shows that for odd n, there exists an equivalence $f\colon S^n \times \Omega S^n \xrightarrow{\sim} LS^n$ if and only if a certain map of spheres associated to f has Hopf invariant 1. Aguadé then uses the solution to the Hopf invariant 1 problem to complete the proof of ($*$). □

References

1. J. Aguadé, *On the space of free loops of an odd sphere*, Publ. Sec. Mat. Univ. Autònoma Barcelona, no. 25, pp. 87–90, 1981.
2. *Math Overflow Question 207844: When does the free loop space fibration split?* MO:207844, 2015.
3. *Math Overflow Question 332943: The free loop space of spheres*, MO:332943, 2019.
4. W. Ziller, *The free loop space of globally symmetric spaces*, Invent. Math., vol. 41, no. 1, pp. 1–22, 1977. DOI: 10.1007/BF01390161.