THE SCHUR–HORN THEOREM

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Abstract. In this paper we give a proof of the Schur–Horn theorem for Hermitian matrices, which describes the possible diagonal entries of a Hermitian matrix with fixed eigenvalues.

0. Overview

Throughout this paper n denotes a positive integer. Recall that an $n \times n$ complex matrix is called ***Hermitian*** if it is equal to its conjugate transpose. By the spectral theorem [1, Cor. 8.6.7] all Hermitian matrices are diagonalizable and have real eigenvalues. In this paper we are interested in determining the possible diagonal entries of $n \times n$ Hermitian matrices with a fixed set of real eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbf{R}$. It is clear that the sum of the diagonal entries of a matrix with pre-specified eigenvalues needs to equal the sum of the eigenvalues, but not much else is immediately obvious. To better understand the problem and the constraints that the Hermitian assumption imposes, let us consider a few examples.

Example A. Consider the 2×2 case. Fix real numbers $\lambda_1 \leq \lambda_2$ and consider an arbitrary 2×2 Hermitian matrix of the form

$$H = \begin{pmatrix} h_1 & c \\ \bar{c} & h_2 \end{pmatrix},$$

with predetermined eigenvalues λ_1 and λ_2. Since H has eigenvalues λ_1 and λ_2, its trace is $\operatorname{tr} H = \lambda_1 + \lambda_2$ and its determinant is $\det H = \lambda_1\lambda_2$. Writing out the trace and determinant of H in terms of the entries of H, we see that $h_1 + h_2 = \lambda_1 + \lambda_2$ and $h_1h_2 - |c|^2 = \lambda_1\lambda_2$. Since $|c|^2$ is a nonnegative real number, in particular we see that $h_1h_2 - \lambda_1\lambda_2 \geq 0$. This inequality along with the fact that $h_1 + h_2 = \lambda_1 + \lambda_2$ imply that $h_1, h_2 \in [\lambda_1, \lambda_2]$.

On the other hand, if we are given two real numbers a_1 and a_2 such that $a_1 + a_2 = \lambda_1 + \lambda_2$ and $a_1, a_2 \in [\lambda_1, \lambda_2]$ these conditions imply that $a_1a_2 - \lambda_1\lambda_2 \geq 0$. Setting $\gamma = a_1a_2 - \lambda_1\lambda_2$ we see that the matrix

$$A = \begin{pmatrix} a_1 & \sqrt{\gamma} \\ \sqrt{\gamma} & a_2 \end{pmatrix}$$

is Hermitian, $\operatorname{tr} A = \lambda_1 + \lambda_2$, and $\det A = \lambda_1\lambda_2$, which implies that A has eigenvalues $\lambda_1 \leq \lambda_2$. Hence real numbers a_1 and a_2 occur as the diagonal entries of a 2×2 Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2$ if and only if $a_1 + a_2 = \lambda_1 + \lambda_2$ and $a_1, a_2 \in [\lambda_1, \lambda_2]$.

To see why the Hermitian assumption makes the problem subtle, consider the following 2×2 example where we relax the Hermitian assumption.

Example B. Consider 2×2 real diagonalizable matrices with real eigenvalues $\lambda_1 \neq \lambda_2$. For any $d_1 \in \mathbf{R}$, the real matrix

$$\begin{pmatrix} d_1 & d_1(\lambda_1 + \lambda_2 - d_1) - \lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 - d_1 \end{pmatrix}$$

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has trace $\lambda_1 + \lambda_2$ and determinant $\lambda_1\lambda_2$, hence has eigenvalues λ_1 and λ_2. This shows that a real diagonalizable 2×2 matrix with real eigenvalues $\lambda_1 \neq \lambda_2$ can have any diagonal entries, as long as they sum to $\lambda_1 + \lambda_2$.

Examples A and B illustrate that the Hermitian assumption greatly limits the possible diagonal entries of a matrix with predetermined eigenvalues. Though the method we used in Example A to solve the 2×2 problem relied on special properties of 2×2 matrices and clearly does not generalize to higher dimensions, in very special cases we can use some tricks to solve the problem, for example, in the case that all of the eigenvalues are the same.

Example C. Since Hermitian matrices are diagonalizable, any Hermitian matrix with only one eigenvalue is a scalar multiple of the identity. Thus there is a single Hermitian matrix with only one eigenvalue.

In the special cases of Examples A and C, the solutions to the problem of determining the possible diagonal entries of a Hermitian matrix with predetermined eigenvalues have simple geometric interpretations: a line segment and a point, respectively. In this paper we show that this is true generally, namely, we show that the set of possible diagonal entries of a Hermitian matrix with fixed eigenvalues is the *convex hull* of a certain finite set.

Definition. Suppose that n is a positive integer and $v_1, \dots, v_m$ are vectors in $\mathbf{R}^n$. A ***convex combination*** of $v_1, \dots, v_m$ is a linear combination $\alpha_1 v_1 + \cdots + \alpha_m v_m$ such that $0 \leq \alpha_i \leq 1$ for each i and $\sum_{i=1}^{m} \alpha_i = 1$. The ***convex hull*** of a finite subset $S \subset \mathbf{R}^n$ is the set of all convex combinations of elements of S.

The main point of this paper is to present a complete "from scratch" proof of a generalization of Example A to arbitrary dimension, originally due to Alfred Horn in 1954. This result is known as the *Schur–Horn theorem*.

Schur–Horn Theorem ([3, Th. 5]). *Let $d_1, \dots, d_n$ and $\lambda_1, \dots, \lambda_n$ be real numbers. There is an $n \times n$ Hermitian matrix with diagonal entries $d_1, \dots, d_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$ if and only if the vector $(d_1, \dots, d_n)$ lies in the convex hull of the set of vectors whose coordinates are all possible permutations of $(\lambda_1, \dots, \lambda_n)$.*

Example A proved this in the case that $n = 2$, and Example C showed a special case of this when all of the eigenvalues are the same, but the methods used in these examples were quite ad hoc. For more insight into the problem, consider the following rank one example which illustrates a simplified version of our proof of the Schur–Horn theorem.

Example D. Suppose that H is an $(n+1)\times(n+1)$ Hermitian matrix with 2 distinct eigenvalues; an eigenvalue of 1 with multiplicity 1, and an eigenvalue of 0 with multiplicity n. By the spectral theorem there exists a unitary matrix U such that $H = U\Lambda U^\star$, where $U^\star$ is the conjugate transpose of U and Λ is the diagonal matrix of eigenvalues of H, with the eigenvalue 1 in the upper-left hand corner. Computing the product $U\Lambda U^\star$ and comparing its diagonal entries to the diagonal entries of H, we see that $h_{i,i} = |u_{i,1}|^2$. Since U is unitary, the rows and columns of U have length 1, so $\sum_{i=1}^{n} |u_{i,1}|^2 = 1$. This shows that the vector $(h_{1,1}, \dots, h_{n+1,n+1})$ of diagonal entries of H lies in the ***standard n-simplex***

$$\Delta^n = \left\{(\alpha_1, \dots, \alpha_{n+1}) \in \mathbf{R}^{n+1} \;\middle|\; \alpha_i \geq 0 \text{ and } \textstyle\sum_{i=1}^{n+1} \alpha_i = 1\right\}$$

from topology. Notice that Δ^n is the convex hull of the set of standard basis vectors for $\mathbf{R}^{n+1}$.

Conversely, given an element $(\alpha_1, \dots, \alpha_{n+1})$ of Δ^n, choose a unitary matrix U whose first column has entries $\sqrt{\alpha_1}, \dots, \sqrt{\alpha_{n+1}}$. Notice that choosing such a unitary matrix U is equivalent to choosing an orthonormal basis for $\mathbf{C}^{n+1}$ with $(\sqrt{\alpha_1}, \dots, \sqrt{\alpha_{n+1}})$ as a basis vector, which is trivially possible. Thus it is always possible to choose such a unitary matrix. By simple matrix multiplication we see that the matrix $U\Lambda U^*$ has the desired diagonal entries.

Our proof of the Schur–Horn theorem goes as follows. In §1, we show that the vector of diagonal entries of a Hermitian matrix can be written as the product of a *bistochastic matrix* (i.e., a matrix of nonnegative real numbers where each row and each column sums to 1) with the vector of eigenvalues; this generalizes the proof given in Example D for the rank one case. We then finish the proof of this implication of the Schur–Horn theorem by applying the *Birkhoff–von Neumann theorem*, which characterizes the set of bistochastic matrices as the convex hull of the set of permutation matrices. In §2 we turn our attention toward dealing with the other implication of the Schur–Horn theorem. To prove the remaining implication we provide an algebraic characterization of elements of the convex hull of the set of vectors whose coordinates are all possible permutations of a given vector.

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1. All Diagonals lie in the Permutation Polytope

In this section we prove "half" of the Schur–Horn theorem by proving that the vector of diagonal entries of an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ lies in the convex hull of of the set of vectors whose coordinates are some permutation of $(\lambda_1, \dots, \lambda_n)$. One of the key players in our approach to this problem is the action of the group Π_n of $n \times n$ permutation matrices on $\mathbf{R}^n$ by left multiplication of column vectors. This action allows us to define a polytope constructed from the orbit of a fixed vector in $\mathbf{R}^n$. We use this polytope to relate the eigenvalues of a Hermitian matrix to its possible diagonal entries. We then use this action to help transform the cumbersome geometry of this polytope into more tangible algebra.

1.1. **Notation.** Suppose that n is a positive integer. We write Π_n for the group of $n \times n$ permutation matrices.

1.2. **Definition.** Suppose that n is a positive integer and x is a column vector in $\mathbf{R}^n$. Write O_x for the orbit of x under the action of Π_n on $\mathbf{R}^n$, that is, the set of all points of the form σx for $\sigma \in \Pi_n$. We call the convex hull of O_x the ***permutation polytope generated by*** x, denoted by P_x.

1.3. **Example.** The permutation polytope generated by any one of the standard basis vectors in $\mathbf{R}^{n+1}$ is the standard n-simplex Δ^n (of Ex. D).

To simplify our notation and language we make the following convenient conventions, some of which we have already alluded to.

1.4. **Conventions.**

(1.4.a) We regard elements of $\mathbf{R}^n$ as column vectors, but for notational simplicity we write them as $(x_1, \dots, x_n)$ with no additional decoration.

(1.4.b) We often need to write the entries of a matrix out explicitly — we write $A = (a_{i,j})$ to signify that A is the matrix whose (i, j) entry is $a_{i,j}$.

(1.4.c) Given an $n \times n$ matrix $A = (a_{i,j})$, we call the vector $(a_{1,1}, \dots, a_{n,n})$ the ***diagonal*** of A.

(1.4.d) Given a complex matrix A, we write A^* for the conjugate transpose of A.

Nonnegative matrices with the property that the sum along any row or column is equal to 1, called *bistochastic matrices*, play key roles in many of our arguments throughout this section.

1.5. **Definition.** We call an $n \times n$ matrix $A = (a_{i,j})$ ***bistochastic*** if A has nonnegative real entries, and, in addition, $\sum_{i=1}^{n} a_{i,j} = 1$ for all $1 \leq j \leq n$ and $\sum_{j=1}^{n} a_{i,j} = 1$ for all $1 \leq i \leq n$.

The main point of this section of the paper is to provide a complete "from scratch" proof of the following proposition, which is one of the implications of the Schur–Horn theorem. The last step of our proof of this proposition uses the *Birkhoff–von Neumann theorem*. Because our application of the Birkhoff–von Neumann theorem is very straightforward, we prove the proposition and defer the proof of the Birkhoff–von Neumann theorem for later.

1.6. **Proposition.** *The diagonal of an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ lies in the permutation polytope generated by $(\lambda_1, \ldots, \lambda_n)$.*

Proof. Suppose that $H = (h_{i,j})$ is an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and write $\lambda \coloneqq (\lambda_1, \ldots, \lambda_n)$. Let Λ denote the diagonal matrix with diagonal λ. By the spectral theorem there exists a unitary matrix $U = (u_{i,j})$ so that $U\Lambda U^\star = H$. Carrying out the multiplication $U\Lambda U^\star$, we see that the diagonal entries of H can be expressed as

$$h_{i,i} = \sum_{j=1}^{n} \lambda_j |u_{i,j}|^2,$$

for $1 \leq i \leq n$. Let $B = (b_{i,j})$ denote the $n \times n$ matrix with entries $b_{i,j} \coloneqq |u_{i,j}|^2$. Writing d for the diagonal of H, we see that $B\lambda = d$. Since U is unitary, the rows and columns of U each have length 1, so for each $1 \leq j \leq n$ we have $\sum_{i=1}^{n} |u_{i,j}|^2 = 1$, and, similarly, for each $1 \leq i \leq n$ we have $\sum_{j=1}^{n} |u_{i,j}|^2 = 1$. Thus B is bistochastic. The Birkhoff–von Neumann theorem says that all bistochastic matrices can be written as a convex combination of permutation matrices, so, in particular, B is a convex combination of permutation matrices. Then by the definition of the action of Π_n on $\mathbf{R}^n$, we see that $B\lambda$ is a convex combination of elements of the orbit of λ, so $B\lambda$ lies in the permutation polytope generated by λ. Thus, the equation $B\lambda = d$ shows that d lies in the permutation polytope generated by λ. $\square$

Now we turn our attention to proving the Birkhoff–von Neumann theorem which characterizes bistochastic matrices.

1.7. **Theorem** (Birkhoff–von Neumann theorem)**.** *Any $n \times n$ bistochastic matrix lies in the convex hull of the group of permutation matrices Π_n.*

As is turns out, the driving force behind the proof of the Birkhoff-von Neumann theorem is *Hall's perfect matching theorem* (also known as *Hall's marriage theorem*, or simply *Hall's theorem*) for *bipartite graphs*. In order to state Hall's theorem, we first recall a little bit of terminology from graph theory, as well as make a few notational conventions.

1.8. **Recollection.** Recall that a ***bipartite*** graph is a graph $G = (V, E)$ where V is partitioned into two disjoint nonempty sets $V = R \sqcup C$ such that every edge contains a vertex in R and a vertex in C. We write $G = (R \sqcup C, E)$ to indicate that a graph is bipartite. A ***perfect matching*** of a graph $G = (V, E)$ is a subset $M \subset E$ of edges of G such that every vertex of G is incident to exactly one edge in M.

1.9. **Notation.** For a set S we write $\#S$ for the *size (cardinality)* of S. For a nonempty subset T of vertices of a graph $G = (V, E)$, we write $N(T)$ for the *(open) neighborhood of T in G*, i.e., the subset of vertices in $V \smallsetminus T$ which are adjacent to some element of T.

1.10. **Theorem** (Hall's perfect matching theorem)**.** *A finite bipartite graph $G = (R \sqcup C, E)$, where R and C have the same size, has a perfect matching if and only if for every subset S of vertices in R we have $\#S \leq \#N(S)$.*

Suppose that M is a bistochastic matrix. The idea behind the proof of Theorem 1.7 is to take M and repeatedly subtract positive multiples of permutation matrices from it until we are left with the zero matrix. Then M is equal to the sum of the matrices that we subtracted off from M, which shows that M can be written as a linear combination of permutation matrices. Moreover, we show that it is possible to perform this subtraction procedure in such a way that the linear combination is actually a convex combination. This motivates the following lemma.

1.11. **Lemma.** *Let A be an $n \times n$ bistochastic matrix. Then there exists a permutation matrix $\sigma \in \Pi_n$ and a real number $t \in (0, 1]$ so that all of the entries of $A - t\sigma$ are nonnegative. Moreover, we may take t to be the largest such real number, in which case $A - t\sigma$ has at least one more zero entry than A does.*

Proof. The second statement is obvious from the first statement, so we just prove the first statement. Notice that if A has (some set of) n nonzero entries which all lie in distinct rows and columns, then if we let σ be the permutation matrix with a 1 in the positions of these n nonzero elements of A, and t be any number between 0 and the minimum of these n chosen entries (which is at most 1 since A is bistochastic), the matrix $A-t\sigma$ has nonnegative entries. Thus it suffices to find n nonzero entries of A which all lie in distinct rows and columns.

Amazingly, it is now possible to reformulate our understanding of this lemma in terms of perfect matchings. To do this, we construct a bipartite graph from the matrix A and apply Theorem 1.10. Let $R := \{r_1, \dots, r_n\}$ and $C := \{c_1, \dots, c_n\}$ be n-element sets. The idea is to think about the element r_i of R as the i^{th} row of A and c_i as the i^{th} column of A. Define a bipartite graph G in the following manner: the vertex set of G is $R \sqcup C$, and there is an edge between r_i and c_j if and only if the entry $a_{i,j}$ of A is nonzero. Finally, note that by the construction of the bipartite graph G, choices of n nonzero elements of A that all lie in distinct rows and columns correspond bijectively to choices of n edges of the graph G which yield a perfect matching of G. Thus it suffices to demonstrate a perfect matching in the graph G. For an example of this in the 3×3 case, see Figure 1.

Theorem 1.10 gives us the conditions under which a perfect matching of a bipartite graph exists, so if we can show that the hypotheses for Theorem 1.10 hold for the graph G, then the lemma follows. In particular, to apply Theorem 1.10 we must show that for each nonempty subset S of R we have $\#S \leq \#N(S)$. Expanding $\#S$ in a convenient manner, we see that

$$\#S = \sum_{r_i \in S} 1 = \sum_{r_i \in S} \sum_{j=1}^{n} a_{i,j}$$

Now, notice that if $c_j \notin N(S)$, then $a_{i,j} = 0$ for any $r_i \in S$ Thus, we see that we can write $\#S$ as

$$\#S = \sum_{r_i \in S} \sum_{c_j \in N(S)} a_{i,j}.$$

Switching the order of summation we see that

$$\#S = \sum_{c_j \in N(S)} \sum_{r_i \in S} a_{i,j} \leq \sum_{c_j \in N(S)} \sum_{i=1}^{n} a_{i,j} = \sum_{c_j \in N(S)} 1 = \#N(S),$$

which is the condition for Theorem 1.10 to hold. Thus G has a perfect matching, so there exists a choice of n nonzero entries of A which all lie in distinct rows and columns. $\square$

Now we are prepared to prove the Birkhoff–von Neumann theorem.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \leftrightsquigarrow$$

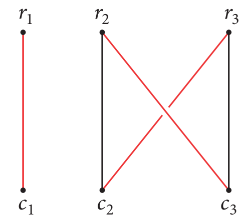

FIGURE 1. Given a bistochastic matrix, the corresponding bipartite graph has one edge for each nonzero entry in the matrix. In this 3×3 example, a choice of 3 nonzero entries in different rows and columns, as well as their corresponding edges in a perfect matching of the corresponding bipartite graph are colored red.

Proof of Theorem 1.7. Let M be an $n \times n$ bistochastic matrix. The goal of this proof is to show that M can be written as a convex combination of permutation matrices. To do this we make rigorous the idea of repeatedly subtracting positive multiples of permutation matrices from M by applying *strong* induction on the number of nonzero entries of M.

Because a bistochastic matrix must have a nonzero entry in every row and column, the base case is the case where the number of nonzero entries is exactly n. To prove the base case, notice that in order for a bistochastic matrix M to have exactly n nonzero entries, those entries must all lie in distinct rows and columns. Then since M is bistochastic, each of the nonzero entries of M is necessarily equal to 1, so M is a permutation matrix.

Now we are ready for the induction step. Suppose that M is a bistochastic matrix with $m + 1$ nonzero entries, with $m \geq n$. The (strong) inductive hypothesis is that every bistochastic matrix with at most m nonzero entries can we be written as a convex combination of permutation matrices. If such a matrix M exists[†], then it follows from the Lemma 1.11 that there exists a permutation matrix $\sigma \in \Pi_n$ and real number $t \in (0, 1]$ so that all of the entries of $M - t\sigma$ are nonnegative and at most m of the entries of $M - t\sigma$ are nonzero. It is obvious that each row and column of $M - t\sigma$ sums to $1 - t$, thus, we have two cases to consider.

Case 1 ($t = 1$)**.** If $t = 1$, then since $M - \sigma$ is a matrix of nonnegative entries, and each of the rows and columns of $M - \sigma$ sum to 0, it follows that $M - \sigma = 0$. Hence $M = \sigma$, so M is a permutation matrix.

Case 2 ($0 < t < 1$)**.** If $t \neq 1$, then it follows that $M - t\sigma$ may be written as $(1 - t)B$, where B is a bistochastic matrix. Since B has at most m nonzero entries, by the inductive hypothesis we can express B as a convex combination of permutation matrices, say $B = \sum_{\alpha \in \Pi_n} c_\alpha \alpha$, where the c_α are nonnegative real numbers such that $\sum_{\alpha \in \Pi_n} c_\alpha = 1$. Then it follows that

$$M = t\sigma + (1 - t) \sum_{\alpha \in \Pi_n} c_\alpha \alpha,$$

which expresses M as a convex combination of permutation matrices.

This completes the induction step, completing the proof of Theorem 1.7. □

[†]It turns out that for a given value of i, there need not exist an $n \times n$ bistochastic matrix with i nonzero entries. For example, there is no 3×3 bistochastic matrix with 4 nonzero entries. We ignore such cases in our proof, since if for a given i there are no bistochastic matrices, then it vacuously follows that they may all be written as convex combinations of the desired form.

This finishes off all of the technical details of the proof of Proposition 1.6, and one whole implication of the Schur–Horn theorem. For the rest of this paper we are concerned with the remaining implication of the Schur–Horn theorem, that is, the converse statement to Proposition 1.6.

One might wonder if we can simply reverse the proof of Proposition 1.6 to prove the remaining implication of the Schur–Horn theorem — unfortunately it is not that simple. The main obstruction to doing this comes from the fact that we cannot reverse one of the final steps of our proof. We were able to apply the Birkhoff–von Neumann theorem by expressing the diagonal of a Hermitian matrix as the product of a bistochastic matrix with the vector of eigenvalues. It is easy to reverse this step; by the definition of the permutation polytope, any element of the permutation polytope generated by the vector of eigenvalues can be written in this way. However, the bistochastic matrix B from our proof of Proposition 1.6 is rather special as its entries are $b_{i,j} = |u_{i,j}|^2$ for some unitary matrix $U = (u_{i,j})$. A bistochastic matrix which arises in this way is called ***unistochastic***. In order to reverse the proof all bistochastic matrices would have to be unistochastic. As it turns out, in general this is not the case.

1.12. **Counterexample** ([2, Th. 2]). Though all 1×1 and 2×2 bistochastic matrices are trivially unistochastic, in higher dimensions this is not the case. Consider the 3×3 bistochasitc matrix

$$B = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We show that B is not unistochastic. To see this, consider the constraints on a unitary matrix U with the property that $b_{i,j} = |u_{i,j}|^2$ for all i and j. Let U_1 and U_2 denote the first and second column of U, respectively. Then since $|u_{1,1}|^2 = |u_{2,2}|^2 = 0$ we see that

$$\langle U_1, U_2 \rangle = u_{1,1}\overline{u_{1,2}} + u_{2,1}\overline{u_{2,2}} + u_{3,1}\overline{u_{3,2}} = u_{3,1}\overline{u_{3,2}},$$

where $\langle -, - \rangle$ denotes the Hermitian inner product. Then since $|u_{3,1}|^2 = |u_{3,2}|^2 = 1/2$, we see that $\langle U_1, U_2 \rangle$ is nonzero, showing that U_1 and U_2 are not orthogonal, which contradicts the fact that U is unitary. Thus there can be no such unitary matrix, i.e., B is not unistochastic. This example also generalizes to higher dimensions — in the case where $n > 3$ we just take the the block diagonal matrix

$$B_n = \begin{pmatrix} B & 0 \\ 0 & I_{n-3} \end{pmatrix},$$

where I_{n-3} denotes the the $(n-3) \times (n-3)$ identity matrix.

2. All Elements of the Permutation Polytope are Diagonals

In this section we prove the remaining implication of the Schur–Horn theorem, namely that any element of the permutation polytope generated by $(\lambda_1, \ldots, \lambda_n)$ is the diagonal of an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. We prove this by describing the geometry of the permutation polytope through a few elementary algebraic operations which are much more manageable to deal with than the pure geometry. In particular, we show that we can move from one of the vertices of the permutation polytope generated by $(\lambda_1, \ldots, \lambda_n)$ to any other vector in the permutation polytope by a finite sequence of algebraic operations. We then show that each of the vectors we get along the way is the diagonal of some Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Combining these facts we can then prove the remaining implication of the Schur–Horn theorem.

We now turn out attention to describing the permutation polytope generated by a given vector. To simplify language, we introduce the following terminology which is important in this key description of the permutation polytope.

2.1. **Definition.** We say that $(x_1, \dots, x_n) \in \mathbf{R}^n$ is ***weakly decreasing*** if $x_1 \geq x_2 \geq \cdots \geq x_n$.

2.2. **Lemma** ([4, Lm. 5]). *Suppose that $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are weakly decreasing vectors in $\mathbf{R}^n$ and that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Then the following are equivalent.*

(2.2.a) *The vector y is in the permutation polytope P_x.*

(2.2.b) *There are vectors $v_1, \dots, v_n$ such that $v_1 = x$, $v_n = y$, for each $1 \leq m < n$, there is a transposition matrix $\tau_m \in \Pi_n$ and real number $t_m \in [0, 1]$ such that*

$$v_{m+1} = t_m v_m + (1 - t_m)\tau_m v_m,$$

and for $m > 1$ the first m coordinates of v_{m+1} agree with the first m coordinates of y.

The proof of Lemma 2.2 is more technical than one would hope, so we leave it for the appendix, but describe the main ideas of the proof here. In particular, there is an important geometric interpretation of (2.2.b) which motivates the algebraic operation of taking a convex combination of a two vectors in the permutation polytope related by a transposition matrix.

Geometrically a transposition matrix $\tau \in \Pi_n$ acts on $\mathbf{R}^n$ by reflection about a certain hyperplane. The equivalent characterization (2.2.b) of the permutation polytope says that we can obtain any weakly decreasing element of P_x by starting at x, using some transposition matrix τ to go to another vertex of P_x, drawing a line segment between x and τx to change the first coordinate of x to agree with the first coordinate of y, and then iterating this procedure, at each step reflecting about a certain hyperplane, drawing a line between a point and its reflection, and then adjusting a coordinate to agree with a coordinate of y. Example 2.3 provides a more concrete illustration of this.

2.3. **Example.** Let $x = (3, 2, 1)$ and consider the permutation polytope generated by x. The vector

$$y = \frac{1}{3}(3, 2, 1) + \frac{1}{3}(2, 1, 3) + \frac{1}{3}(1, 2, 3) = (2, 2, 2)$$

lies in P_x and is weakly decreasing. To move from x to y in a sequence of steps as in (2.2.b), we first set $v_1 = x$. To construct v_2, we want to find a real number $t_1 \in [0, 1]$ and a transposition τ_1 so that the first coordinate of

$$t_1 x + (1 - t_1)\tau_1 x$$

agrees with the first coordinate of y, as we want to "adjust" the vector x coordinate-by-coordinate to agree with y. Since the second coordinate of x already agrees with the first coordinate of y, if we let τ_1 be the transposition matrix which interchanges the first and second coordinates, and $t_1 = 0$ so that

$$t_1 x + (1 - t_1)\tau_1 x = (2, 3, 1).$$

Now we set $v_2 = (2, 3, 1)$ and repeat this process. The first coordinates of y and v_2 already agree, so we just need to adjust the second coordinate of v_2 to agree with y, and then the last coordinate will be strictly determined by the first two coordinates. Since $2 = (3 + 1)/2$, it is easy to see that if we let $t_2 = 1/2$ and τ_2 be the transposition matrix which interchanges the second and third coordinates, then

$$t_2 v_2 + (1 - t_2)\tau_2 v_2 = \frac{1}{2}(2, 3, 1) + \frac{1}{2}(2, 1, 3) = (2, 2, 2) = y.$$

Geometrically all that we have done is drawn a line segment between v_2 and $\tau_2 v_2$ and determined where y lies on this line segment, as illustrated in Figure 2.

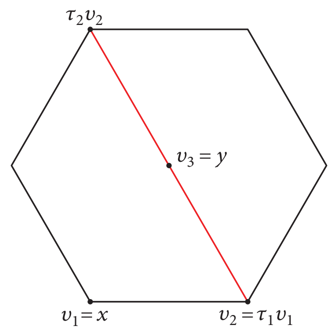

Figure 2. The hexagon represents the permutation polytope generated by $(3, 2, 1)$ of Example 2.3. The red line segment through the center of the hexagon represents the line segment drawn between v_2 and $\tau_2 v_2$ to construct v_3.

A very important point to notice about Lemma 2.2 is that with some (cumbersome) adjustments to the statement of the lemma, we can phrase the result without the assumption that x and y are weakly decreasing. This simply comes from the fact that if y is in the permutation polytope of any $x \in \mathbf{R}^n$, then σy is also in the permutation polytope of x for any $\sigma \in \Pi_n$. Since there is always a permutation matrix $\bar{\sigma}$ so that $\bar{\sigma} y$ is weakly decreasing we can first permute coordinates to get something that is weakly decreasing, and then work with that. Similarly, for any vector x, any element of the orbit of x generates the same permutation polytope, so if x is not weakly decreasing to begin with, we can replace x with some weakly decreasing element of its orbit.

Now we want to relate Lemma 2.2 to Hermitian matrices. Specifically, given n real numbers $\lambda_1 \geq \cdots \geq \lambda_n$ and a weakly decreasing vector y in the permutation polytope generated by $(\lambda_1, \ldots, \lambda_n)$, we want to show that each of the vectors $v_1, \ldots, v_n$ from Lemma 2.2 (2.2.b) is the diagonal of a Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Each of these vectors is related to the previous one by the same algebraic operation, namely, taking a convex combination of two vectors related by a transposition matrix. If we can show that the set of possible diagonals of a Hermitan matrix with a fixed set of eigenvalues is closed under this algebraic operation, then we can apply a simple induction argument to show that each of the vectors $v_1, \ldots, v_n$ does indeed occur as the diagonal of some $n \times n$ Hermitan matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. The following lemma shows that the set of possible diagonals is closed under this operation.

2.4. **Lemma.** *Suppose that $d = (d_1, \ldots, d_n)$ occurs as the diagonal of an $n \times n$ Hermitian matrix H with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then for any real number $t \in [0, 1]$ and any transposition matrix $\tau \in \Pi_n$, there exists a Hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and diagonal $td + (1-t)\tau d$.*

Proof. In the case that $n = 1$ this is trivial, so suppose that $n > 1$. Write $H = (h_{i,j})$ and suppose that $\tau \in \Pi_n$ transposes the k^{th} and ℓ^{th} coordinates. The idea behind this proof is to construct a unitary matrix U so that $UHU^\star$ has $td + (1-t)\tau d$ as a diagonal. This matrix will have the same eigenvalues as H because we are simply changing basis. We find this matrix U by reducing to the case where $n = 2$.

When $n > 2$, by conjugating H by an appropriate permutation matrix P, without loss of generality we can assume that $k = 1$ and $\ell = 2$. Thus we have reduced to the case of finding a Hermitian matrix with diagonal $(td_1 + (1-t)d_2, td_2 + (1-t)d_1, d_3, \dots, d_n)$. Let U be a 2×2 unitary matrix and consider the $n \times n$ block-diagonal unitary matrix

$$V = \begin{pmatrix} U & 0 \\ 0 & I_{n-2} \end{pmatrix},$$

where I_{n-2} denotes the $(n-2) \times (n-2)$ identity matrix. We see that the diagonal entries of $VHV^\star$ are exactly the diagonal entries of the matrix

$$U \begin{pmatrix} d_1 & h_{1,2} \\ h_{2,1} & d_2 \end{pmatrix} U^\star$$

followed by $h_{3,3} = d_3, \dots, h_{n,n} = d_n$. In light of this, the problem of finding a Hermitian matrix with diagonal $(td_1 + (1-t)d_2, td_2 + (1-t)d_1, d_3, \dots, d_n)$ reduces to the case $n = 2$. Thus we may then assume that

$$H = \begin{pmatrix} d_1 & h_{1,2} \\ h_{2,1} & d_2 \end{pmatrix},$$

where $h_{1,2} = \overline{h_{2,1}}$ since H is Hermitian.

Define a complex number ζ by

$$\zeta \coloneqq \begin{cases} i\overline{h_{1,2}}/|h_{1,2}|, & h_{1,2} \neq 0 \\ 1, & \text{otherwise} . \end{cases}$$

Then

$$\overline{\zeta h_{1,2}} = -\zeta h_{1,2}$$

and $|\zeta| = 1$. Now let

$$U = \begin{pmatrix} \zeta\sqrt{t} & -\sqrt{1-t} \\ \zeta\sqrt{1-t} & \sqrt{t} \end{pmatrix}.$$

It is clear by the definitions of the complex number ζ and the entries of U that U is unitary. The matrix $A = UHU^\star$ has the same eigenvalues as H, and, moreover the diagonal of A is $(td_1 + (1-t)d_2, td_2 + (1-t)d_1)$, as desired. □

We are now ready to prove the remaining implication of the Schur–Horn theorem, expressed in the following proposition.

2.5. **Proposition.** *Suppose that $d = (d_1, \dots, d_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ are vectors in $\mathbf{R}^n$. If d lies in the permutation polytope P_λ then there exists an $n \times n$ Hermitian matrix with diagonal d and eigenvalues $\lambda_1, \dots, \lambda_n$.*

Proof. Suppose that d lies in the permutation polytope generated by the vector λ. As remarked earlier, if d or λ is not weakly decreasing, we may replace it by a weakly decreasing element, so, without loss of generality, assume that both d and λ are weakly decreasing. Then by the equivalence of (2.2.b) and (2.2.a) given in Lemma 2.2, there exist vectors $v_1, \dots, v_n \in P_\lambda$ with $v_1 = \lambda$, $v_n = d$, and for each integer $1 \leq m < n$,

$$v_{m+1} = t_m v_m + (1-t_m)\tau_m v_m. \tag{2.5.1}$$

for some $t_m \in [0,1]$ and some transposition matrix τ_m. Let V_1 denote the diagonal matrix with diagonal $v_1 = \lambda$. Since V_1 is Hermitian and the vectors v_k satisfy the relation (2.5.1), by repeated application of Lemma 2.4 we see that there are Hermitian matrices $V_2, \dots, V_n$ with diagonals $v_2, \dots, v_n$, respectively. Since $v_n = d$, this shows that V_n is a Hermitian matrix with diagonal d, which proves the result.

It is worthwhile to note that that if the vector d which we started with was not weakly decreasing, a simple reordering of the basis, i.e., conjugating V_n by a permutation matrix, gives us a matrix whose diagonal is this (non-weakly decreasing) vector. Moreover, since the property of being Hermitian is invariant under conjugation by a unitary matrix, and permutation matrices are, in particular, unitary matrices, this new matrix is Hermitian too. □

Proposition 2.5 together with Proposition 1.6 prove the Schur–Horn theorem, restated below.

2.6. **Theorem** ([3, Th. 5]). *Let $d = (d_1, \dots, d_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ be vectors in $\mathbf{R}^n$. There is an $n \times n$ Hermitian matrix with diagonal entries d and eigenvalues $\lambda_1, \dots, \lambda_n$ if and only if d lies in the permutation polytope generated by λ.*

Appendix: The Proof of a Technical Result

Here we present a proof of Lemma 2.2. First we recall the statement of the lemma.

A.1. **Lemma** (Lm. 2.2). *Suppose that $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are weakly decreasing vectors in $\mathbf{R}^n$ and that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Then the following are equivalent.*

(A.1.a) *The vector y is in the permutation polytope P_x.*

(A.1.b) *There are vectors $v_1, \dots, v_n$ such that $v_1 = x$, $v_n = y$, for each $1 \le m < n$, there is a transposition matrix $\tau_m \in \Pi_n$ and real number $t_m \in [0, 1]$ such that*

$$v_{m+1} = t_m v_m + (1 - t_m)\tau_m v_m,$$

and for $m > 1$ the first m coordinates of v_{m+1} agree with the first m coordinates of y.

Proof. We first prove that (A.1.b) implies (A.1.a), so suppose that (A.1.b) holds. First notice that by the definition of P_x as the convex hull of the orbit O_x of x, it is clear that for any point $z \in P_x$ and any permutation $\sigma \in \Pi_n$, the element σz is in P_x. Moreover, a convex combination of elements of P_x is in P_x. Since $x \in P_x$, the obvious induction argument shows that $v_i \in P_x$ for all integers $1 \le i \le n$. Since $v_n = y$, this shows that $y \in P_x$.

Now we prove that (A.1.a) implies (A.1.b). We set $v_1 = x$, and construct a vector v_2 such that

$$v_2 = t_1 v_1 + (1 - t_1)\tau_1 v_1 \tag{A.1.1}$$

for some $t_1 \in [0, 1]$ and transposition matrix $\tau_1 \in \Pi_n$ so that the first coordinate of v_2 agrees with the first coordinate of y. Then given vectors $v_1, \dots, v_m$, such that

(a.1) for all $1 \le j < m$,

$$v_{j+1} = t_j v_j + (1 - t_j)\tau_j v_j$$

for some $t_j \in [0, 1]$ and transposition $\tau_j \in \Pi_n$,

(a.2) for all $i < j \le m$, writing $v_j = (v_{j,1}, \dots, v_{j,n})$, we have $v_{j,i} = y_i$ so that the first $j - 1$ coordinates of v_j agree with the first $j - 1$ coordinates of y,

(a.3) and $\sum_{i=1}^k y_i \le \sum_{i=1}^k v_{j,i}$ for all $j \le m$ and $k < n$ and $\sum_{i=1}^n y_i = \sum_{i=1}^n v_{j,i}$,

we construct a vector v_{m+1} such that

(b.1) $v_{m+1} = t_m v_m + (1 - t_m)\tau_m v_m$ for some $t_m \in [0, 1]$ and transposition $\tau_m \in \Pi_n$,

(b.2) for all $i < m + 1$ we have $v_{m+1,i} = y_i$ so that the first m coordinates of v_{m+1} agree with the first m coordinates of y,

(b.3) and $\sum_{i=1}^k y_i \le \sum_{i=1}^k v_{m+1,i}$ for all integers $k < n$ and $\sum_{i=1}^n y_i = \sum_{i=1}^n v_{m+1,i}$

We begin by making the following observation which is necessary to construct v_2 from v_1. We want to construct the convex combination (A.1.1) so that the first coordinate of v_2 as a convex combination of the first and ${\ell_1}^{\text{th}}$ coordinates of v_1 for some integer ℓ_1, and then let τ_1 be the transposition which interchanges the first and ${\ell_1}^{\text{th}}$ coordinates. It suffices to find an integer ℓ such that $x_\ell \leq y_1$. Since x is weakly decreasing, for any integer k, with $1 \leq k \leq n$, and any permutation $\sigma \in \Pi_n$,

$$\sum_{i=1}^{k} (\sigma x)_i \leq \sum_{i=1}^{k} x_i.$$

Therefore, if $z = (z_1, \ldots, z_n)$ is a convex combination of elements of the orbit O_x of x, for any integer k, with $1 \leq k \leq n$,

$$\sum_{i=1}^{k} z_i \leq \sum_{i=1}^{k} x_i.$$

In particular, this is true for y since $y \in P_x$. Using this observation we can show that there exists an integer ℓ, with $2 \leq \ell \leq n$, such that $x_\ell \leq y_1$. To see this, suppose, for the sake of contradiction, that there exists no such integer. Then

$$\sum_{i=1}^{n} y_i < \sum_{i=1}^{n} x_i,$$

which contradicts the assumption that $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$. Let ℓ_1 be the least greater than 1 such that $x_{\ell_1} \leq y_1$.

Now we are ready to construct v_2. Since $x_{\ell_1} \leq y_1 \leq x_1$, there exists some real number $t_1 \in [0, 1]$ such that

$$y_1 = t_1 x_1 + (1 - t_1) x_{\ell_1}.$$

Let $\tau_1 \in \Pi_n$ be the transposition matrix which interchanges the first and ${\ell_1}^{\text{th}}$ coordinates. Set $v_1 \coloneqq x$ and define

$$v_2 \coloneqq t_1 v_1 + (1 - t_1) \tau_1 v_1.$$

Write the components of v_2 as $v_2 = (v_{2,1}, \ldots, v_{2,n})$. The key feature of v_2 is that $v_{2,1} = y_1$. Then by the construction of v_2 and the minimality of ℓ_1, for every integer $1 \leq k \leq \ell_1$,

$$\sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} v_{2,i}.$$

Moreover, by the construction of v_2, the fact that $v_1 = x$, and the assumptions of the lemma we see that

$$\sum_{i=1}^{n} v_{2,i} = \sum_{i=1}^{n} v_{1,i} = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i.$$

Now suppose that we have constructed vectors $v_1, \ldots, v_m \in P_x$ satisfying conditions (a.1)–(a.3). We need to construct a vector $v_{m+1} \in P_x$ such that (b.1)–(b.3) hold. In particular from the construction of $v_1, \ldots, v_m$ we see that

$$\sum_{i=1}^{m} y_i \leq \sum_{i=1}^{m} v_{m,i} = \sum_{i=1}^{m-1} y_i + v_{m,m},$$

so subtracting $\sum_{i=1}^{m-1} y_i$ shows that $y_m \leq v_{m,m}$. Moreover, since both $\sum_{i=1}^{n} v_{m,i} = \sum_{i=1}^{n} y_i$ and $\sum_{i=1}^{n-1} y_i \leq \sum_{i=1}^{n-1} v_{m,i}$, we see that $v_{m,n} \leq y_n \leq y_m$.

Similarly to the construction of v_2 from v_1, we want to find a coordinate of v_m so that a convex combination of this coordinate with the m^{th} coordinate of v_m is equal to the m^{th} coordinate of y. Notice that since $\sum_{i=1}^{n} v_{m,i} = \sum_{i=1}^{n} y_i$ there exists an integer $\ell > m$ such that

$v_{m,\ell} \leq y_m$, otherwise $\sum_{i=1}^n v_{m,i} < \sum_{i=1}^n y_i$ which contradicts (a.3). Let ℓ_m be the least such integer. Then

$$y_j \leq y_m \leq v_{m,j} \tag{A.1.2}$$

for $m+1 \leq j \leq \ell_m - 1$. Since $v_{m,\ell_m} \leq y_m \leq v_{m,m}$, there is a real number $t_m \in [0,1]$ such that

$$y_m = t_m v_{m,m} + (1-t_m) v_{m,\ell_m}.$$

Let $\tau_m \in \Pi_n$ be the transposition matrix which interchanges the m^{th} and ${\ell_m}^{\text{th}}$ coordinates. Define

$$v_{m+1} \coloneqq t_m v_m + (1-t_m)\tau_m v_m.$$

Writing $v_{m+1} = (v_{m+1,1}, \dots, v_{m+1,n})$ we see that $v_{m+1,j} = y_j$ for $1 \leq j \leq m$. Thus v_{m+1} satisfies conditions (b.1) and (b.2) by construction.

Now all that remains to be shown is that v_{m+1} satisfies condition (b.3). When $1 \leq k \leq m$, this is obvious because $v_{m+1,j} = y_k$ for $1 \leq j \leq m$. If $m+1 \leq k < \ell_m$, then by (A.1.2) we see that

$$\sum_{i=1}^k y_i \leq \sum_{i=1}^m y_i + \sum_{i=m+1}^k v_{m,i} = \sum_{i=1}^k v_{m+1,i},$$

so the inequality holds in this range. If $\ell_m \leq k < n$, then

$$\sum_{i=1}^k y_i \leq \sum_{i=1}^k v_{m,i} = \sum_{i=1}^k v_{m+1,i}.$$

Lastly, by the construction of v_m and the hypothesis (a.3), we see that

$$\sum_{i=1}^n v_{m+1,i} = \sum_{i=1}^n v_{m,i} = \sum_{i=1}^n y_i,$$

which completes the proof that (b.3) holds.

Finally notice that when $m+1 = n$ we have $v_{n,j} = y_j$ for all $1 \leq j < n$, and the fact that $\sum_{i=1}^n v_{n,i} = \sum_{i=1}^n y_i$ shows that $v_{n,n} = y_n$, so $v_n = y$. $\square$

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