On the homotopy theory of stratified spaces

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Trying to understand invariants of stratified topological spaces, such as intersection cohomology, naturally leads to the question of what the correct homotopy theory of stratified topological spaces is. Just as in the classical setting, we would like a 'homotopy hypothesis' for stratified spaces

$$\begin{pmatrix}\text{A homotopy theory of} \\ \text{stratified topological spaces}\end{pmatrix} \simeq \begin{pmatrix}\text{Purely homotopical} \\ \text{objects}\end{pmatrix} ,$$

where

(1) The 'purely homotopical' side is simple to define and has excellent formal properties (e.g., is a presentable ∞-category).
(2) The 'topological' side is also simple to define and captures all examples of differential-topological interest (e.g., *topologically stratified spaces* in the sense of Goresky–MacPherson [5, §1.1]).
(3) The equivalence is given by MacPherson's *exit-path* construction.

Though many have attempted to construct such a homotopy theory, notably Henriques [6, 7], Ayala and Francis with Rozenblyum [1] and Tanaka [2], and Nand-Lal [11], a homotopy theory of stratified topological spaces satisfying (1)–(3) does not yet exist. We report on our recent preprint [8] where we define a new homotopy theory of stratified topological spaces satisfying these criteria, and show that all existing homotopy theories of stratified topological spaces embed into ours.

STRATIFIED TOPOLOGICAL SPACES & EXIT-PATHS

Definition. The *Alexandroff topology* on a poset P is the topology on the underlying set of P in which a subset $U \subset P$ is open if and only if $x \in U$ and $y \geq x$ implies that $y \in U$. We simply write $P \in \mathbf{Top}$ for the set P equipped with the Alexandroff topology.

The category of *P-stratified topological spaces* is the overcategory $\mathbf{Top}_{/P}$. If $s \colon T \to P$ is a P-stratified topological space, for each $p \in P$ we write $T_p := s^{-1}(p)$ for the p^{th} *stratum* of T.

MacPherson had the idea that the 'stratified homotopy type' of a P-stratified topological space T should be determined by its 'exit-path ∞-category' $\mathrm{Exit}_P(T)$ with:

(0) Objects: points of T.
(1) 1-morphisms: *exit-paths*, that is, paths in T that flow from lower to higher strata, and once they exit a stratum are not allowed to return.
(2) 2-morphisms: homotopies between exit-paths respecting stratifications.

$$\vdots$$

It is difficult to make a construction of $\mathrm{Exit}_P(T)$ that is both precise and useful, but the takeaway is that $\mathrm{Exit}_P(T)$ should be an ∞-category with a functor to the poset P with strata ∞-groupoids. Another way of saying this is that the functor

$\mathrm{Exit}_P(T) \to P$ is conservative. This idea informs what the 'purely homotopical' side of a stratified homotopy hypothesis should be:

Definition. The ∞-category of *abstract P-stratified homotopy types* is the ∞-category

$$\mathbf{Str}_P := \mathbf{Cat}^{\mathrm{cons}}_{\infty,/P} \subset \mathbf{Cat}_{\infty,/P}$$

of ∞-categories C over P with conservative structure morphism $C \to P$.

The following 'exit-path simplicial set' construction due to Henriques [7] and Lurie [9, §A.6] is an attempt to make MacPherson's idea precise.

Construction. Let P be a poset. There is a natural stratification

$$\pi_P \colon |N(P)| \to P$$

of the geometric realization of the nerve $N(P)$ of P by the Alexandroff space P extended from the natural $[n]$-stratification $|\Delta^n| \to [n]$ of the standard topological n-simplex defined by the assignment

$$(t_0, \ldots, t_n) \mapsto \max\{i \in [n] \,|\, t_i \neq 0\} \ .$$

If X is a simplicial set over $N(P)$, then we can stratify the geometric realization $|X|$ by composing the structure morphism $|X| \to |N(P)|$ with π_P. This defines a left adjoint functor $|-|_P : s\mathbf{Set}_{/N(P)} \to \mathbf{Top}_{/P}$ with right adjoint $\mathrm{Sing}_P \colon \mathbf{Top}_{/P} \to s\mathbf{Set}_{/N(P)}$ computed by the pullback of simplicial sets

$$\mathrm{Sing}_P(T) := N(P) \times_{\mathrm{Sing}(P)} \mathrm{Sing}(T) \ ,$$

where the morphism $N(P) \to \mathrm{Sing}(P)$ is adjoint to π_P.

Here the stratified story diverges from the classical story: the simplicial set $\mathrm{Sing}_P(T)$ generally is *not* a quasicategory. This creates a lot of technical problems if one attempts to prove a stratified homotopy hypothesis by proving a Quillen equivalence between a model structure on $s\mathbf{Set}_{/N(P)}$ presenting $\mathbf{Str}_P$ and a model structure on $\mathbf{Top}_{/P}$.

Nevertheless, when $\mathrm{Sing}_P(T)$ *is* a quasicategory, it has the properties we want out of an exit-path ∞-category. Write $\mathbf{Top}^{\mathrm{ex}}_{/P} \subset \mathbf{Top}_{/P}$ for the full subcategory spanned by those P-stratified topological spaces T for which the exit-path simplicial set $\mathrm{Sing}_P(T)$ is a quasicategory. Let W denote the class of morphisms in $\mathbf{Top}^{\mathrm{ex}}_{/P}$ that are sent to weak equivalences in the Joyal model structure under Sing_P (i.e., equivalences of ∞-categories). The following is our 'stratified homotopy hypothesis', which we regard as a precise form of [1, Conjecture 0.0.4]:

Theorem (H.)**.** *For any poset P, the induced functor*

$$\mathrm{Sing}_P \colon \mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}] \to \mathbf{Str}_P$$

is an equivalence of ∞-categories.

Our proof is somewhat indirect. Using the main result of Chapter 7 of Douteau's thesis [4], which realizes pioneering ideas of Henriques [7], we show that a 'nerve' functor provides an equivalence between an ∞-category obtained from $\mathbf{Top}_{/P}$

by inverting a class of weak equivalences and and a Segal space model for $\mathbf{Str}_P$ introduced in work with Barwick and Glasman [3, §4.2]. This immediately implies that $\mathrm{Sing}_P \colon \mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}] \to \mathbf{Str}_P$ is fully faithful, and a bit more careful analysis shows that it is also essentially surjective.

Comparisons to conically smooth stratified spaces

In work with Tanaka [2, §3], Ayala and Francis introduced *conically smooth structures* on stratified topological spaces, which they further studied in work with Rozenblyum [1]. Their homotopy theory of P-stratified spaces is the ∞-category obtained from the category $\mathbf{Con}_P$ of conically smooth P-stratified spaces by inverting the class H of stratified homotopy equivalences. The functor Sing_P sends stratified homotopy equivalences to equivalences of ∞-categories, hence descends to a functor $\mathbf{Con}_P[H^{-1}] \to \mathbf{Str}_P$. The Ayala–Francis–Rozenblyum 'stratified homotopy hypothesis' states that this functor is fully faithful. Hence we have a commutative triangle of fully faithful functors of ∞-categories

$$\begin{array}{ccc} \mathbf{Con}_P[H^{-1}] & \xhookrightarrow{\mathrm{Sing}_P} & \mathbf{Str}_P \\ \big\downarrow & \nearrow_{\sim}^{\mathrm{Sing}_P} & \\ \mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}] & & \end{array} ,$$

where the vertical functor is induced by the functor $\mathbf{Con}_P \to \mathbf{Top}^{\mathrm{ex}}_{/P}$ forgetting conically smooth structures.

One of the major benefits of the ∞-category $\mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}]$ over $\mathbf{Con}_P[H^{-1}]$ is that all *conically stratified topological spaces* fit into this framework [9, Theorem A.6.4], in particular topologically stratified spaces in the sense of Goresky–MacPherson, and even more particularly Whitney stratified spaces [10, 12], are conically stratified. Thus the ∞-category $\mathbf{Top}^{\mathrm{ex}}_{/P}[W^{-1}]$ captures most, if not all, examples of differential-topological interest. On the other hand, it is still unknown whether or not every Whitney stratified space admits a conically smooth structure [1, Conjecture 0.0.7].

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