AN OVERVIEW OF MOTIVIC COHOMOLOGY

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ABSTRACT. In this talk we give an overview of some of the motivations behind and applications of motivic cohomology. We first present a K-theoretic timeline converging on motivic cohomology, then briefly discuss algebraic K-theory and its more concrete cousin Milnor K-theory. We then present a geometric definition of motivic cohomology via Bloch's higher Chow groups and outline the main features of motivic cohomology, namely, its relation to Milnor K-theory. In the last part of the talk we discuss the role of motivic cohomology in Voevodsky's proof of the Bloch–Kato conjecture. The statement of the Bloch–Kato conjecture is elementary and predates motivic cohomolgy, but its proof heavily relies on motivic cohomology and motivic homotopy theory.

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1. A TIMELINE CONVERGING TO MOTIVIC COHOMOLOGY

In this section we give a timeline of events converging on motivic cohomology. We first say a few words about what motivic cohomology *is*.

- Motivic cohomology is a bigraded cohomology theory $H^p(-; \mathbf{Z}(q)) \colon \mathbf{Sm}_{/k}^{op} \to \mathbf{Ab}$.
- Voevodsky won the fields medal because "he defined and developed motivic cohomology and the $\mathbf{A}^1$-homotopy theory of algebraic varieties; he proved the Milnor conjectures on the K-theory of fields." Moreover, Voevodsky's proof of the Milnor conjecture makes extensive use of motivic cohomology.
- Later Voevodsky went on to prove the (Milnor K-theory) Bloch–Kato conjecture using motivic homotopy theory.
- There are many applications of motivic cohomology, even in classical settings. For example, motivic cohomology has been used by Asok–Fasel [1] to classify vector bundles.
- Motivic cohomology is relatively new, so there is a lot of work to be done in this field and there are many conjectures.

1.1. Timeline (*K*-theoretic timeline converging to motivic cohomology [13, 28]). There were various motivations for and timelines converging to motivic cohomology. We present only one of them to remain concise.

> **1957:** Grothendieck defines algebraic *K*-theory to prove the Grotendieck–Riemann–Roch theorem. One of Grothendieck's motivations for defining *K*-theory was his dislike for the Chow groups and

1.2. Definition. Let *k* be a field and *X* be a smooth *k*-variety[†]. The 0th *algebraic K-group* $K_0(X)$ of *X* is the quotient of the free abelian group on the set of algebraic vector bundles on *X* by the relation $[V] = [V'] + [V'']$ if there is a short exact sequence of algebraic vector bundles on *X*

$$0 \longrightarrow V' \rightarrowtail V \longrightarrow\!\!\!\!\!\rightarrow V'' \longrightarrow 0\,.$$

Note that the tensor product of algebraic vector bundles endows $K_0(X)$ with the structure of a commutative ring.

1.3. Remarks.

(1.3.a) Historically K_0 was simply written *K* and $K_0(X)$ was called the algebraic *K*-theory of *X*.

(1.3.b) The letter "K" stands for the German word for "class".

1.4. Example. If *k* is a field, then the category of algebraic vector bundles on *k* is equivalent to the category $\mathbf{Vect}_k$ of *finite-dimensional k*-vector spaces. Since every short exact sequence of vector bundles splits, if *V* is an *n*-dimensional *k*-vector space, we see (by induction) that $[V] = n[k]$. From this observation it easily follows that $K_0(\mathrm{Spec}(k)) \cong \mathbf{Z}$.

A consequence of the Grothendieck–Riemann–Roch Theorem is the following.

1.5. Corollary. *Let X be a smooth quasiprojective variety. Then the reational Chern character*

$$\mathrm{ch} \otimes \mathbf{Q} \colon K_0(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathrm{CH}^*(X) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is an isomorphism of rings, where $\mathrm{CH}^*(X)$ *denotes the Chow ring of X (see Definition 4.4).*

> **1959:** ‣ Atiyah and Hirzebruch define complex topological *K*-theory $K_{top}^0(X)$ for a space *X*.
> ‣ Bott proves Bott periodicity (i.e., that the homotopy groups of the infinite unitary group are 2-periodic, or that $\Omega^2 U \simeq U$, or that $\Omega^2 BU \simeq \mathbf{Z} \times BU$).
> ‣ Combining their definition of K_{top}^0 with Bott's periodicity theorem, Atiyah and Hirzebruch extended K_{top}^0 to a generalized cohomology theory K_{top}. This cohomology theory has the property that $K_{top}^*(*) \cong \mathbf{Z}[\beta^{\pm 1}]$, where $\beta \in K_{top}^2(*)$ is an element called the *Bott element*.
> ‣ One immediately gets the *Atiyah–Hirzebruch spectral sequence*
> $$\mathrm{H}^*_{sing}(X; K_{top}^*(*)) \Longrightarrow K_{top}^*(X)\,,$$
> an extremely powerful tool for computing topological *K*-theory.

1.6. Question. What about in algebraic geometry? We have only defined K_0 so far.

> **1964:** Bass defined K_1 for rings using matrices. Later some generalizations were made to the settings of symmetric monoidal or abelian categories.
> **1967:** Milnor defines K_2 for rings, again using matrices.

[†]This assumption is a technical one used to insure that this definition agrees with a less naïve definition using *perfect complexes* (see [25, §2]). We could also require *X* to be a quasiprojective scheme over a ring.

1971: Quillen gives definition of $K_n(R)$ for all $n \geq 0$ and rings R using homotopy theory that agrees with the previous definitions of K_0, K_1, and K_2.

1972: Quillen computes $K_*(\mathbf{F}_q)$ [21]; this is essentially the only "complete" K-theory computation.

1973: Quillen defines K_n for all $n \geq 0$ for schemes, and much more generally [22]. Again this requires homotopy theory.

1.7. Question. Now what about an Atiyah–Hirzebruch spectral sequence ? $\implies K_*(X)$ in algebraic K-theory?

- ➤ First, for this to make sense, we remark that the algebraic K-theory of a scheme has a natural filtration, called the *γ-filtration*.
- ➤ People called whatever should be on the E_2-page of an algebraic Atiyah–Hirzebruch spectral sequence converging to algebraic K-theory *motivic cohomology*.
- ➤ Thus motivic cohomology groups should be functors $H^p(-; A(q)) \colon \mathbf{Sm}^{op}_{/k} \to \mathbf{Ab}$, defined for $p, q \in \mathbf{Z}$ and $A \in \mathbf{Ab}$.

1980s: Beilinson, Lichtenbaum, and others make bold conjectures about motivic cohomology and its relation to number theory.

1986: Bloch defines motivic cohomology in the guise of *higher Chow groups* [4], but could neither prove all of Beilinson and Lichtenbaum's conjectures nor produce an Atiyah–Hirzebruch spectral sequence. Suslin later found a crucial mistake in Bloch's paper, which Bloch later fixed in 1994 [5].

~1999: Bloch, Deligne, Friedland, Lichtenbaum, and Deligne construct the Atiyah–Hirzebruch spectral sequence (see [9, Thm. 13.6]).

Early 2000s: Voevodsky completed the proofs of the main bold conjectures that Beilinson and Lichtenbaum made in the 1980s.

2. A Taste of Algebraic K-Theory

2.1. Idea. "Algebraic K-theory is the universal invariant of categories equipped with a suitable notion of exact sequences that splits short exact sequences."

For precise universal properties of algebraic K-theory, see [2, Props. 10.2 & 10.8; 7, Thm. 1.3].

2.2. Example. The category $\mathbf{Fin}_*$ of finite pointed sets has a reasonable notion of exact sequences, and $K_n(\mathbf{Fin}_*)$ is the nth stable homotopy group of spheres.

For our purposes, algebraic K-theory is a collection of functors $K_n \colon \mathbf{Sch}^{op} \to \mathbf{Ab}$ for $n \geq 0$ satisfying a number of properties. We are mostly interested in the affine case, and can restrict to $K_n \colon \mathbf{CRing} \to \mathbf{Ab}$. The following property suggests that algebraic K-theory should have something to do with the homotopy theory of schemes.

2.3. Property ($\mathbf{A}^1$-homotopy invariance [29, Ch. v Thm. 6.3]). If R is a regular ring, then the inclusion $R \hookrightarrow R[t]$ induces an isomorphism $K_*(R) \xrightarrow{\sim} K_*(R[t])$.

Algebraic K-theory has numerous deep relations to algebraic geometry and number theory. The Kummer–Vandiver conjecture gives a taste of the deep relationship between algebraic K-theory and number theory.

2.4. Conjecture (Kummer–Vandiver). *For all primes p, the prime p does not divide the class number of the maximal real subfield of the pth cyclotomic field $\mathbf{Q}(\zeta_p)$.*

2.5. Proposition (Kurihara [16]). *We have that $K_{4n}(\mathbf{Z}) = 0$ for all positive integers n if and only if the Kummer–Vandiver Conjecture 2.4 holds.*

2.6. **Remark.** The K-theory of $\mathbf{Z}$ is hair-raisingly difficult to compute — a summary of many of the known computations can be found in [29, Ch. 6 §10].

3. Milnor K-theory

The following is a classical computation of low K-groups for fields, done *before* Quillen defined K_n for $n \geq 3$.

3.1. **Computation** ([11, p. 1]). If k is a field, then we have

$$\begin{cases} K_0(k) \cong \mathbf{Z} \\ K_1(k) \cong k^\times \\ K_2(k) \cong (k^\times \otimes_{\mathbf{Z}} k^\times)/\langle a \otimes (1-a) \mid a, 1-a \in k^\times \rangle \, . \end{cases}$$

3.2. **Question** (Milnor, before higher K-theory was defined). What if the relations appearing in Computation 3.1 are the only relations in K-theory?

3.3. **Notation.** For an abelian group G, write $T(G)$ for the tensor algebra $T(G) = \bigoplus_{n \geq 0} G^{\otimes n}$ of G (as a $\mathbf{Z}$-module).

3.4. **Definition** (Milnor 1970 [20, §1]). The ***Milnor K-theory*** of a commutative ring A is the graded abelian group

$$\mathrm{K}^M_*(A) := T(A^\times)/\langle a \otimes (1-a) \mid a, 1-a \in A^\times \rangle \, ,$$

where we write $\langle a \otimes (1-a) \mid a, 1-a \in A^\times \rangle$ for the *homogeneous* ideal of $T(A^\times)$ generated by $a \otimes (1-a)$, where $a, 1-a \in A^\times$.

One of the main results of local class field theory involves Milnor K-theory.

3.5. **Theorem** ([15]). *Let k be an n-dimensional local field. Then there exists a canonical homomorphism*

$$\mathrm{K}^M_n(k) \to \mathrm{Gal}(k_{ab}/k) \cong \mathrm{Gal}(k_{sep}/k)^{ab}$$

inducing an isomorphism $\mathrm{K}^M_n(k)/\mathrm{N}_{L/k}(\mathrm{K}^M_n(L)) \xrightarrow{\sim} \mathrm{Gal}(L/k)$ *for every finite abelian extension* $k \subset L$. *Here* $\mathrm{N}_{L/k} \colon \mathrm{K}^M_n(L) \to \mathrm{K}^M_n(k)$ *is a* norm map *on Milnor K-theory.*

3.6. **Remark.** Though Milnor K-theory may seem somewhat ad hoc, there are good reasons to care about Milnor K-theory. From the perspective of a number theorist, Milnor K-theory is crucial in Kato's *higher class field theory* [8, 15].

4. Bloch's Higher Chow Groups

Now that we have said a few things about K-theory, we need to introduce the other key player — *motivic cohomology*. As motivation, we first remark that on the following topological generalization of due to Atiyah and Hirzebruch.

4.1. **Theorem** (Atiyah–Hirzebruch). *Let X be a space with the homotopy type of a finite CW-complex. Then there exists a topological Chern character giving a natural isomorphism of graded rings*

$$\mathrm{ch}_{top} \colon \bigoplus_{n \in \mathbf{Z}} K^{-n}_{top}(X) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\sim} \bigoplus_{n,q \in \mathbf{Z}} \mathrm{H}^{2q-n}_{sing}(X; \mathbf{Q}) \, .$$

We now recall the definition of *Chow groups* from algebraic geometry.

4.2. Definition. Let X be a variety. The ***group of cycles*** on X is the free abelian group

$$Z^*(X) \coloneqq \mathbf{Z}\langle\text{closed subvarieties } Y \subset X\rangle \, ,$$

which is naturally a graded group, graded by codimension. Elements of $Z^*(X)$ are called ***cycles*** on X.

4.3. Idea. We would like for $Z^*(X)$ to be a graded *ring* under intersection, but two cycles may not intersect in the correct codimension: in general

$$\text{codim}_X(Y \cap Y') \neq \text{codim}_X(Y) + \text{codim}_X(Y') \, .$$

4.4. Definition. Let X be a k-variety. A ***rational equivalence*** between cycles $\alpha, \beta \in Z^*(X)$ is a cycle $\gamma \in Z^*(X \times \mathbf{A}_k^1)$ such that $\gamma \cap (X \times \{0\}) = \alpha$ and $\gamma \cap (X \times \{1\}) = \beta$.

The ***Chow group*** of X is the graded group

$$\text{CH}^*(X) \coloneqq Z^*(X)/\sim_{rat}$$

of cycles modulo rational equivalence.

4.5. Remark. The Chow group $\text{CH}^*(X)$ is a ring under intersection of rational equivalence classes of cycles. To prove this, one has to prove a *moving lemma*, showing that given a pair of cycles α and β, there is a rationally equivalent cycle $\beta' \sim_{rat} \beta$ so that the terms in α and β intersect in the correct codimensions, i.e., we can "move" cycles to rationally equivalent ones that intersect properly.

4.6. Idea (of higher Chow groups)**.** In homotopy theory we have the following general principle

"Quotienting by an equivalence relation $\iff$ Taking π_0 of some space"

(4.6.a) If $\alpha, \beta \in Z^*(X)$ and γ is a rational equivalence from α to β, we should think "γ is a path from α to β."

(4.6.b) We could also consider "homotopies between paths" and so on.

(4.6.c) Bloch [4, §1]: there is a space (i.e., simplicial abelian group) $Z^*(X, \cdot)$ of cycles on X with a natural identification

$$\pi_0 Z^*(X, \cdot) \cong \text{CH}^*(X) \, .$$

4.7. Definition. Let X be a variety. The ***higher Chow groups*** of X are the homotopy groups

$$\text{CH}^*(X, i) \coloneqq \pi_i Z^*(X, \cdot)$$

of the space of cycles on X (defined for $i \geq 0$).

4.8. Remark (on notation)**.** Each higher Chow group $\text{CH}^*(X, i)$ has a natural grading coming from a grading on $Z^*(X, \cdot)$ given by codimension of cycles, which is why we write "$\text{CH}^*(X, i)$" rather than "$\text{CH}(X, i)$".

4.9. Upshot. The higher Chow groups of a scheme X are geometrically-defined groups that can be computed as the homology of a (graded) chain complex which we abusively also write as $Z^*(X, \cdot)$, with $Z^*(X, n)$ defined as a the subgroup of the group of cycles $Z^*(X \times \Delta^n)$ generated by those cycles which meet all faces $X \times \Delta^m \subset X \times \Delta^n$ properly (i.e., in the correct codimension), where

$$\Delta^n \coloneqq \text{Spec}(k[x_0, \dots, x_n]/\langle x_0 + \cdots + x_n - 1\rangle)$$

is the standard algebraic n-simplex. The boundary maps are defined via the *Dold–Kan correspondence* (see [17, §1.2.3] for the relation between simplicial abelian groups and chain complexes).

A theorem of Voevodsky [26, Cor. 2] allows us to take the following as a *definition* of motivic cohomology.

4.10. **Definition.** Let X be a smooth variety. The ***motivic cohomology groups*** $\mathrm{H}^p(X; \mathbf{Z}(q))$ for $p, q \in \mathbf{Z}$ are given by

$$\mathrm{H}^p(X; \mathbf{Z}(q)) \coloneqq \begin{cases} \mathrm{CH}^q(X, 2q - p), & q \geq 0 \text{ and } 2q - p \geq 0 \\ 0, & \text{otherwise}. \end{cases}$$

The index q in $\mathrm{H}^p(X; \mathbf{Z}(q))$ is referred to as the ***weight*** of the motivic cohomology group.

4.11. **Remark.** Here are some reasons for the choice of grading relationship between the higher Chow groups and motivic cohomology: first, the higher Chow groups have their own natural grading, and second, we want to have a rational isomorphism between algebraic K-theory and motivic cohomology that resembles the Atiyah–Hirzebruch isomorphism of Theorem 4.1.

To relate motivic cohomology to something more concrete, we recall the definition of the *zeta function* of a finite type $\mathbf{Z}$-scheme.

4.12. **Definition.** Let X be a finite type $\mathbf{Z}$-scheme. The ***zeta function*** of X is the Euler product

$$\zeta_X(s) \coloneqq \prod_{\substack{x \in X \\ \text{closed}}} \frac{1}{1 - |\kappa(x)|^{-s}},$$

which converges when $\mathrm{Re}(s) > \dim(X)$.

4.13. **Examples.**
(4.13) If $X = \mathrm{Spec}(\mathbf{Z})$, then ζ_X is the classical *Riemann zeta function*.
(4.13) More generally, if K is an algebraic number field, $O_K \subset K$ is the ring of integers of K, and $X = \mathrm{Spec}(O_K)$, then ζ_X is the *Dedekind zeta function* of K.

4.14. **Conjecture** (Soulé, but in more generality [4, p. 271; 24, Conj. 2.2]). *If X is regular and proper over* $\mathrm{Spec}(\mathbf{Z})$*, then for an integer $n \in \mathbf{Z}$ we have*

$$\mathrm{ord}_{s=n} \zeta_X(s) = -\sum_{i \geq 0} (-1)^i \mathrm{rank}(\mathrm{H}_i(X; \mathbf{Z}(n))),$$

where the groups $\mathrm{H}_i(X; \mathbf{Z}(n))$ are motivic homology groups*, which by a Poincaré duality result are dual to (specific) motivic cohomology groups. That is, the orders of the zero of ζ_X at n is given (up to a sign) by the* weight n motivic Euler characteristic *of X.*

4.15. **Remark.** A special case of Soulé's conjecture for elliptic curves implies part of the Birch–Swinnerton-Dyer conjecture.

4.16. **Idea.** Euler characteristics are tools for *decatigorification*. The classical Euler characteristic decategorifies homology groups. Since zeta functions are much more elementary than motivic cohomology, we might want to think of motivic cohomology as a *categorification* of the zeta function.

5. Properties of Motivic Cohomology

5.1. **Properties** (properties of motivic cohomology [18, p. viii]). Let k be a field, A an abelian group, and $X \in \mathbf{Sm}_{/k}$. Motivic cohomology for smooth k-schemes satisfies the following properties.
(5.1.a) We have $\mathrm{H}^p(X; A(q)) = 0$ if $q < 0$.

(5.1.b) If X is connected,

$$\mathrm{H}^p(X; A(0)) \cong \begin{cases} A, & p = 0 \\ 0, & p \neq 0 \,. \end{cases}$$

(5.1.c) We have

$$\mathrm{H}^p(X; \mathbf{Z}(1)) \cong \begin{cases} O_X^\times(X), & p = 1 \\ \mathrm{Pic}(X), & p = 2 \\ 0, & p \neq 1, 2 \,. \end{cases}$$

(5.1.d) We have $\mathrm{H}^p(\mathrm{Spec}(k); A(p)) \cong \mathrm{K}_p^M(k) \otimes_{\mathbf{Z}} A$.

(5.1.e) If X is a strictly Henselian local k-scheme and $n \in \mathbf{Z}$ is prime to $\mathrm{char}(k)$, we have

$$\mathrm{H}^p(X; (\mathbf{Z}/n)(q)) \cong \begin{cases} \mu_n^{\otimes q}(S), & p = 0 \\ 0, & p \neq 0 \,. \end{cases}$$

6. The Bloch–Kato Conjecture

6.1. **The Kummer Sequence.** If X is a scheme and $\ell \in \mathbf{Z}$ is relatively prime to $\mathrm{char}(X)$, then the we have a *Kummer exact sequence* [19, Ch. II Ex. 2.18(b) & Ch. III Prop. 4.11; 29, Ch. III Ex. 6.10.1]

$$(6.1) \qquad 0 \longrightarrow \mu_\ell \lhook\joinrel\longrightarrow \mathbf{G}_m \xrightarrow{(-)^\ell} \mathbf{G}_m \longrightarrow 0$$

of étale sheaves on X (though usually *not* an exact sequence of Zariski sheaves).

Let us examine the Kummer sequence in the case that X is the spectrum of a field — in this case the Kummer sequence has an especially nice description. Let k be a field and we write $G_k := \mathrm{Gal}(k_{sep}/k)$ for the absolute Galois group of k, then the category of discrete G_k-modules is equivalent to the category of étale sheaves on $\mathrm{Spec}(k)$ [19, Ch. II Thm. 1.9].

6.2. **Notation.** For a field k we write $G_k := \mathrm{Gal}(k_{sep}/k)$ for the absolute Galois group of k.

6.3. **Recollection.** Let k be a field. The absolute Galois group is a profinite group, with presentation given by the diagram

$$G_k \cong \varprojlim_{\substack{k \subset K \subset k_{sep} \\ \text{finite}}} \mathrm{Gal}(k_{sep}/K) \,.$$

6.4. **Recollection** ([23, Ch. I §2.1]). Let G be a profinite group. A G-module M is *discrete* if the action map $G \times M \to M$ is continuous, where the abelian group M is given the discrete topology. Write $\mathbf{Mod}_G^{\mathrm{disc}} \subset \mathbf{Mod}_G$ for the full subcategory of $\mathbf{Mod}_G$ spanned by the discrete G-modules.

6.5. **Recollection** ([19, Ch. II Thm. 1.9 & Ch. III Ex. 1.7]). Let k be a field. There is an explicit equivalence of categories

$$\mathbf{Mod}_{G_k}^{\mathrm{disc}} \simeq \mathbf{Shv}_{\text{ét}}(\mathrm{Spec}(k); \mathbf{Ab}) \,,$$

with explicit quasi-inverse. Moreover, this equivalence identifies the G_k-invariants functor $(-)^{G_k} : \mathbf{Mod}_{G_k}^{\mathrm{disc}} \to \mathbf{Ab}$ with the global sections functor

$$\Gamma(\mathrm{Spec}(k); -) : \mathbf{Shv}_{\text{ét}}(\mathrm{Spec}(k); \mathbf{Ab}) \to \mathbf{Ab} \,.$$

Since étale cohomology is the right derived functor cohomology of the left exact functor $\Gamma(\mathrm{Spec}(k); -)$ and Galois cohomology is the right derived functor cohomology of the left exact functor $(-)^{G_k}$, if $\mathcal{F}$ is an étale sheaf on $\mathrm{Spec}(k)$ with corresponding G_k-module $M_{\mathcal{F}}$, then for all $n \geq 0$ we have an identification

$$\mathrm{H}_{\text{ét}}^n(\mathrm{Spec}(k); \mathcal{F}) \cong \mathrm{H}_{\mathrm{Gal}}^n(k; M_{\mathcal{F}}) \,.$$

6.6. Remark. Galois cohomology is very classical and can be computed by an explicit cochain complex (see [23, Ch. I §2.2 & Ch. II §1.1]).

If one writes out this equivalence explicitly for, the exactness of the Kummer sequence for $\mathrm{Spec}(k)$ is equivalent to the exactness of the sequence

$$0 \longrightarrow \boldsymbol{\mu}_\ell(k_{sep}) \lhook\joinrel\longrightarrow k_{sep}^\times \xrightarrow{(-)^\ell} k_{sep}^\times \longrightarrow 0$$

of discrete G_k-modules.

6.7. Proposition (Kummer isomorphism). *Let k be a field and ℓ an invertible in k. Then there is a canonical isomorphism $\partial\colon k^\times/\ell \to \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}(k); \boldsymbol{\mu}_\ell)$ is an isomorphism.*

Proof. The Kummer sequence (6.1) gives rise to a long exact sequence in étale cohomology

$$(6.8) \quad 0 \longrightarrow \mathrm{H}^0_{\acute{e}t}(\boldsymbol{\mu}_\ell) \longrightarrow \mathrm{H}^0_{\acute{e}t}(\mathbf{G}_m) \xrightarrow{\cdot\ell} \mathrm{H}^0_{\acute{e}t}(\mathbf{G}_m) \xrightarrow{\partial} \mathrm{H}^1_{\acute{e}t}(\boldsymbol{\mu}_\ell) \longrightarrow \mathrm{H}^1_{\acute{e}t}(\mathbf{G}_m) \longrightarrow \cdots ,$$

where we have written $\mathrm{H}^*_{\acute{e}t}(-)$ for $\mathrm{H}^*_{\acute{e}t}(\mathrm{Spec}(k); -)$. By Hilbert's Theorem 90, we have

$$\mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}(k); \mathbf{G}_m) \cong \mathrm{Pic}(k) ,$$

and $\mathrm{Pic}(k) = 0$ since every 1-dimensional k-vector space is isomorphic to k. Hence the long exact sequence (6.8) shows that the boundary map ∂ gives an identification

$$\mathrm{H}^0_{\acute{e}t}(\mathrm{Spec}(k); \mathbf{G}_m)/\ell \cong \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}(k); \boldsymbol{\mu}_\ell) .$$

To conclude, note that $\mathrm{H}^0_{\acute{e}t}(\mathrm{Spec}(k); \mathbf{G}_m) = \mathbf{G}_m(k) = k^\times$. $\square$

6.9. Definition. Let k be a field and ℓ an integer invertible in k. The ***first norm residue map*** or ***Galois symbol*** $\partial\colon k^\times/\ell \to \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}(k); \boldsymbol{\mu}_\ell)$ is the map induced by the boundary map

$$k^\times = \mathrm{H}^0_{\acute{e}t}(\mathrm{Spec}(k); \mathbf{G}_m) \to \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}(k); \boldsymbol{\mu}_\ell)$$

in the long exact sequence in étale cohomology (6.8) coming from the Kummer sequence (6.1).

6.10. Observation. Let k be a field and ℓ an integer invertible in k. Taking the tensor power of the first norm residue map and applying the cup product on étale cohomology defines a map

$$\partial^n\colon (k^\times)^{\otimes n} \xrightarrow{\partial^{\otimes n}} \mathrm{H}^1_{\acute{e}t}(\mathrm{Spec}(k); \boldsymbol{\mu}_\ell)^{\otimes n} \xrightarrow{(-)^{\smile n}} \mathrm{H}^n_{\acute{e}t}(k; \boldsymbol{\mu}_\ell^{\otimes n}) .$$

One can easily check that the map ∂^n factors through the nth Milnor K-group $\mathrm{K}^M_n(k)$, i.e., that if $a, 1 - a \in k^\times$, then

$$\partial^n(b_1 \otimes \cdots \otimes a \otimes \cdots \otimes (1 - a) \otimes \cdots \otimes b_{n-2}) = 0$$

for $b_1, \ldots, b_{n-2} \in k^\times$.

Moreover, because étale cohomology with coefficients in a $\mathbf{Z}/\ell$-module is an ℓ-torsion group, the map ∂^n factors through $\mathrm{K}^M_n(k)/\ell$, defining the nth ***norm residue map*** map

$$\partial^n\colon \mathrm{K}^M_n(k)/\ell \to \mathrm{H}^n_{\acute{e}t}(\mathrm{Spec}(k); \boldsymbol{\mu}_\ell^{\otimes n}) .$$

6.11. Conjecture (Bloch–Kato). *Let k be a field. If ℓ is a prime that is invertible in k, then for all $n \geq 0$ the norm residue map $\partial^n\colon \mathrm{K}^M_n(k)/\ell \to \mathrm{H}^n_{\acute{e}t}(k; \boldsymbol{\mu}_\ell^{\otimes n})$ is an isomorphism.*

6.12. Remarks.

(6.12.a) The case where $\ell = 2$ is known as the *Milnor conjecture*. Voevodsky's proof of the Milnor conjecture was part of his fields medal citation.

(6.12.b) The case where $n = 2$ is known as the *Merkurjev–Suslin theorem*.

6.13. **Conjecture** (Beilinson–Lichtenbaum). *Let k be a field. If ℓ is a prime that is invertible in k, then for all smooth k-varieties X, for $p \leq q$ there is a naturally-defined isomorphism*

$$\mathrm{H}^p(X; (\mathbf{Z}/\ell)(q)) \cong \mathrm{H}^p_{\acute{e}t}(X; \boldsymbol{\mu}_\ell^{\otimes q}) \,.$$

6.14. **Theorem** (Voevodsky). *The Bloch–Kato Conjecture 6.11 is equivalent to the Beilinson–Lichtenbaum Conjecture 6.13.*

6.15. **Theorem** (Voevodsky, Rost, Weibel, Haesemeyer, …[12, Thm. A]). *The Bloch–Kato Conjecture 6.11 is true.*

This theorem is incredibly difficult and involves deep theorems in homotopy theory and algebraic geometry (namely, resolution of singularities in characteristic 0). See [12] for an expository account. We give a sweeping overview of the proof of the Bloch–Kato Conjecture 6.11.

Sweeping overview of the proof of the Bloch–Kato Conjecture 6.11 (see [12, §§1.1–1.2]).

(6.15.a) Using a transfer argument, reduce the problem to fields containing all ℓ^{th} roots of unity.

(6.15.b) Using a transfer argument, reduce the proving the result for fields of characteristic 0.

(6.15.c) Translate the problem into motivic language.

6.16. **Definition.** Let n and $\ell \neq 0$ be integers. We say that H90(n) **holds** if

$$\mathrm{H}^{n+1}_{\acute{e}t}(\mathrm{Spec}(k); \mathbf{Z}_{\langle \ell \rangle}(n)) = 0$$

for every field k with ℓ invertible.

6.17. **Remark.** Note that H90(1) holds because H90(1) is equivalent to the classical Hilbert's Theorem 90.

6.18. **Theorem** ([12, Thms. 1.6 & 2.38]). *Let n be a nonnegative integer and ℓ a prime. Then $\partial^n \colon \mathrm{K}^M_n(k)/\ell \xrightarrow{\sim} \mathrm{H}^n_{\acute{e}t}(\mathrm{Spec}(k); \boldsymbol{\mu}_\ell^{\otimes n})$ for all fields k with ℓ invertible if and only if H90(n) holds.*

(6.15.e) Induct on n.

    ➤ To do the induction, one needs to construct a *Rost variety*, a smooth projective variety whose function field satisfies certain specific conditions. This is *very* difficult.

6.19. **Theorem** ([12, Thm. 1.10]). *Suppose that H90($n-1$) holds. Then for every field k of characteristic 0 and every nonzero element $\boldsymbol{a} \in \mathrm{K}^M_n(k)/\ell$ there is a smooth projective variety $X_{\boldsymbol{a}}$ whose function field $K_{\boldsymbol{a}} \coloneqq \kappa(X_{\boldsymbol{a}})$ satisfies the following:*

(6.19.a) *The element $\boldsymbol{a}$ vanishes in $\mathrm{K}^M_n(K_{\boldsymbol{a}})/\ell$.*

(6.19.b) *The map $\mathrm{H}^{n+1}_{\acute{e}t}(\mathrm{Spec}(k); \mathbf{Z}_{\langle \ell \rangle}(n)) \to \mathrm{H}^{n+1}_{\acute{e}t}(K_{\boldsymbol{a}}; \mathbf{Z}_{\langle \ell \rangle}(n))$ is an injection.*

    ➤ The Rost variety needs to have an additional property, namely that it has an associated *Rost motive*. As input, this requires a motivic cohomology class μ, and to show that everything works out, we need to show that a certain motivic cohomology operation associated with μ agrees with βP^i (where i is an integer related to the bidegree of μ). This requires a complete understanding of the motivic Steenrod algebra (in characteristic 0), as per the following theorem.

6.20. **Theorem** (Voevodsky in characteristic 0 [27], Hoyois–Kelly–Østvær in general [14, Thm. 1.1]). *Let k be a field and ℓ a prime different from* char(k). *Then*

$$A^{*,*}_{mot,\mathbf{Z}/\ell} \cong A^*_{\mathbf{Z}/\ell} \otimes \mathrm{H}^*(\mathrm{Spec}(k); (\mathbf{Z}/\ell)(*)) \,. \qquad \square$$

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