Exodromy in topology & Applications

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joint with
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arxiv.org/abs/2401.12825

Monodromy

Monodromy V1 If X is a locally simply connected top space, then there is an equivalence of categories

$$\begin{array}{ccc} LC(X;\mathrm{Set}) & \xrightarrow{\sim} & \mathrm{Fun}(\Pi_1(X),\mathrm{Set}) \\ L & \longmapsto & [x \longmapsto L_x] \end{array}$$

locally constant sheaves of sets on X

Monodromy V∞ If X is a locally weakly contractible top space, then there is an equivalence of ∞-categories

$$LC(X;\mathrm{Spc}) \xrightarrow{\sim} \mathrm{Fun}(\Pi_\infty(X),\mathrm{Spc})$$

∞-category of spaces

underlying homotopy type / ∞-groupoid

Moreover, $LC(X;\mathrm{Spc}) \subset Sh(X;\mathrm{Spc})$ is closed under lims + colims

Stratified Spaces

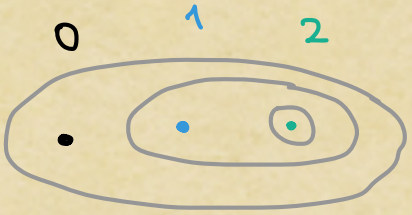
Def $(P, \leq)$ poset

The **Alexandroff topology** on P has

$$[U \subset P \text{ open}] \iff [p \in U \text{ and } q > p \Rightarrow q \in U]$$

U is upwards closed

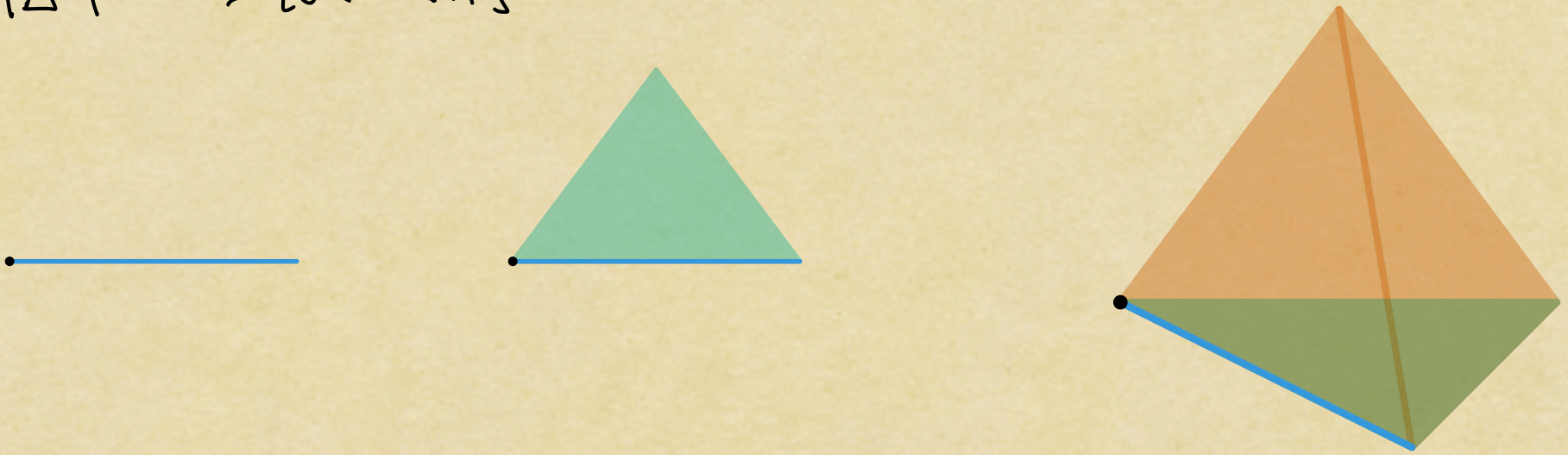
Ex $P = \{0 < 1 < \cdots < n\}$



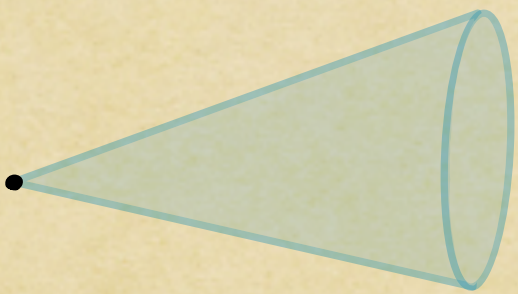
Def A **P-Stratification** of a space X is a continuous map $s : X \longrightarrow P$.

> $p \in P$, the **pth-Stratum** of X is $X_p := s^{-1}(p)$.

Ex $|\Delta^n| \longrightarrow \{0 < \cdots < n\}$



Ex Cone(S^1)



Exodromy (MacPherson, Treumann, Lurie, Porta-Teyssier)

If $s: X \to P$ is a conically stratified space with lwc strata, then there is an ∞-category

$\mathrm{Exit}(X,P)$: the exit-path ∞-category of (X,P)

along with an equivalence of ∞-categories

$$\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(\mathrm{Exit}(X,P), \mathrm{Spc}).$$

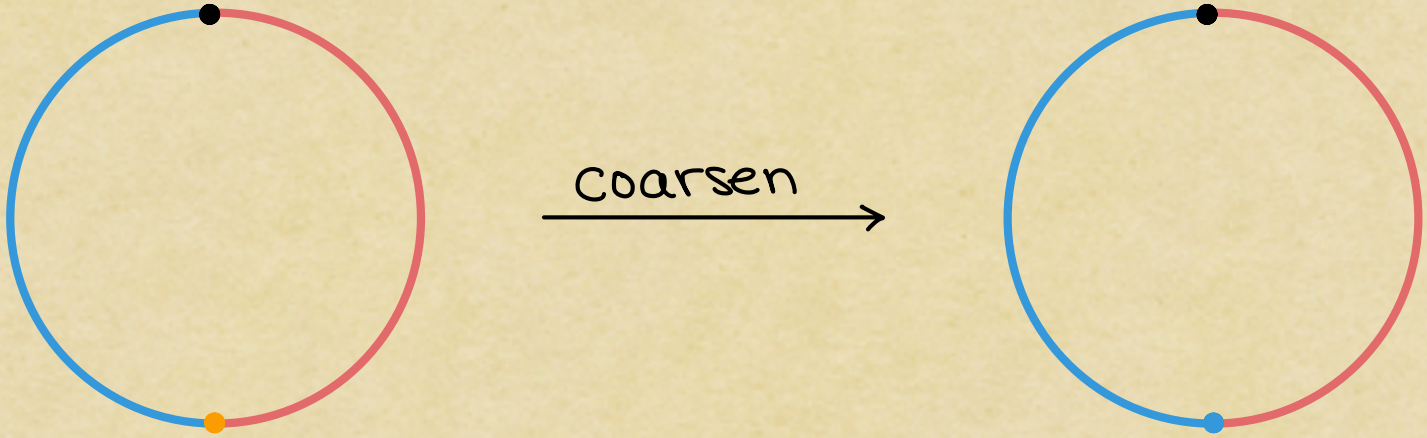
Moreover:

(1) $\mathrm{Cons}_P(X) \subset \mathrm{Sh}(X)$ is closed under limits & colimits.

(2) $s^*: \mathrm{Fun}(P,\mathrm{Spc}) \simeq \mathrm{Sh}(P) \longrightarrow \mathrm{Sh}(X)$ preserves limits.

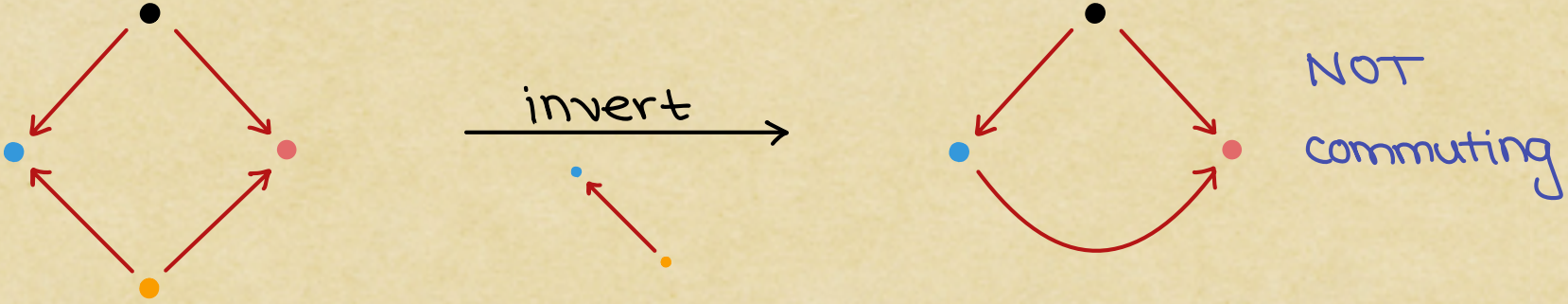
$\iff$ induced by a functor $\mathrm{Exit}(X,P) \to P$

A Simple example



Conical

NOT conical



Exit-path category

Exit-path category
Should be

Key Idea

Make the conclusion of the exodromy theorem into a definition.

Don't rely on a particular geometric model of $\mathrm{Exit}(X,P)$

[Ayala - Francis - Rozenblyum, Clausen - Ørsnes Jansen]

Def A stratified space $s: X \longrightarrow P$ is **exodromic** if:

(1) $\mathrm{Cons}_P(X) \subset \mathrm{Sh}(X)$ is closed under limits & colimits

(2) $\mathrm{Cons}_P(X)$ is atomically generated.

(3) $s^*: \mathrm{Fun}(P, \mathrm{Spc}) \longrightarrow \mathrm{Sh}(X)$ preserves limits.

Write

$$\mathrm{Exit}(X,P) := \left(\mathrm{Cons}_P(X)^{at}\right)^{op}.$$

Then

$$\mathrm{Cons}_P(X) \simeq \mathrm{Fun}(\mathrm{Exit}(X,P), \mathrm{Spc}).$$

The Stability Theorem

Thm (H.-Porta-Teyssier)

(1) **Stability under pulling back to locally closed subposets**

If (X,P) is exodromic and $S \subset P$ is locally closed, then $(X \times_P S, S)$ is exodromic and

$p<q<r$ and $p,r \in S \Rightarrow q \in S$

$$\mathrm{Exit}(X \times_P S, S) \xrightarrow{\sim} \mathrm{Exit}(X,P) \underset{P}{\times} S.$$

As a consequence, the induced functor

$$\mathrm{Exit}(X,P) \longrightarrow P$$

is conservative.

(2) **Functoriality** $(X,P) \longmapsto \mathrm{Exit}(X,P)$ is functorial in **all maps** of exodromic stratified spaces.

The exodromy equivalence is functorial in **all maps** of exodromic stratified spaces.

(3) **Stability under coarsening** If (X,R) is exodromic, then for any map of posets $\phi: R \to P$, the stratified space (X,P) is exodromic.

Write W_P for those morphisms in $\mathrm{Exit}(X,R)$ that

$$\mathrm{Exit}(X,R) \longrightarrow R \xrightarrow{\phi} P$$

Sends to identities. Then

$$\mathrm{Exit}(X,R)[W_P^{-1}] \xrightarrow{\sim} \mathrm{Exit}(X,P).$$

(4) **van Kampen** The property of being exodromic can be checked locally + colimit formula for $\mathrm{Exit}(-,-)$.

(5) **Stability of finiteness/compactness** the property of an exit-path ∞-category being finite/compact:

> Can be checked on a finite cover

> Is stable under pulling back to locally closed subsets

> Is stable under coarsening

Methods We actually prove everything for **stratified ∞-topoi**.

The results then apply to stratified top. stacks

Examples

Cor If (X,P) locally admits a refinement by a conical stratification with locally weakly contractible strata, then (X,P) is exodromic.

> Deep theorems of Thom, Mather, and Verdier show that such refinements are often available in geometric situations

Ex If (X,P) admits a refinement by a triangulation, then (X,P) is exodromic.

Thm If (X,P) is a real analytic manifold with a locally finite stratification by subanalytic subsets, then:

(1) (X,P) is exodromic

(2) X compact $\Longrightarrow$ $\mathrm{Exit}(X,P)$ is finite

Thm If (X,P) is a stratification of the $\mathbb{R}$-points of an $\mathbb{R}$-variety by Zariski locally closed subsets, then:

(1) (X,P) is exodromic

(2) $\mathrm{Exit}(X,P)$ is finite

Refine results of Lefschetz - Whitehead, Łojasiewicz, & Hironaka

Applications

Application 1 (HPT) Representability results for derived moduli of constructible and perverse sheaves.

> Crucially uses the finiteness of exit-paths!

Application 2 (PT, using App 1) Hall algebras

Application 3 (PT, using App 1) A new approach to **Stokes data** using exit-paths + representability of the derived moduli of Stokes data

> Their results generalize hard theorems of Sabbah.

Beautiful Computations

Ørsnes Jansen computed exit-path ∞-categories for:

(1) Reductive Borel-Serre compactifications

(2) $\overline{M}_{g,n}$ Stratified by stable genus g dual graphs w/ n marked points

moduli of stable genus g nodal curves w/ n marked points