LIFTING ENHANCED FACTORIZATION SYSTEMS TO FUNCTOR 2-CATEGORIES (PRELIMINARY VERSION)

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Abstract. In this paper we give sufficient conditions for lifting an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ on a 2-category $\boldsymbol{K}$ to the functor 2-category $\boldsymbol{K}^C$, where C is a small category. Due to work of Lack, this result gives coherence results for 2-monads on functor 2-categories. These coherence results are of immediate interest to work in progress of Guillou, May, Merling, and Osorno on *equivariant infinite loopspace theory.*

0. Overview

The motivation for this work comes from the theory of coherence results for 2-monads, specifically 2-monads of functor 2-categories, as this immediate interest to current work in progress of Guillou, May, Merling, and Osorno in *equivariant infinite loopspace theory.* The basic problem is the following: Given a 2-category $\boldsymbol{K}$ and a 2-monad T on $\boldsymbol{K}$, is every pseudo-T-algebra equivalent (as a pseudo-T-algebra) to a strict T-algebra? In other words, this question asks if every pseudo-T-algebra be "strictified" to a strict T-algebra. Power [4, Cor. 3.5] proved if T is a 2-monad on the 2-category **Cat**, of categories, functors, and natural transformations, and T preserves all functors which are bijective-on-objects, then every pseudo-T-algebra is equivalent to a strict T-algebra. In fact, Power's result is more general: he showed that if S is a small set (regarded as a discrete 2-category) and T is a 2-monad on the functor 2-category $\mathbf{Cat}^S$ that preserves 1-cells in $\mathbf{Cat}^S$ which are S-indexed sets of bijective-on-objects functors, then every pseudo-T-algebra is equivalent to a strict T-algebra. There are two ideas employed here; the first is to lift the result pointwise to the functor 2-category, and the second is that the preservation of a certain class of 1-cells is a sufficient condition for every pseudo-T-algebra to be "strictified".

Lack [3, §4.2] used *enhanced factorization systems,* developed by Kelly [1], to generalize Power's result to an arbitrary 2-category $\boldsymbol{K}$ with an enhanced factorization system. What we are interested in is a generalization of Power's result to the functor 2-category $\mathbf{Cat}^C$, or to $\boldsymbol{K}^C$, where C is a small category, and $\boldsymbol{K}$ is a 2-category. The goal of this paper is to give conditions to lift an enhanced factorization system on a 2-category $\boldsymbol{K}$ to the functor 2-category $\boldsymbol{K}^C$, which provides a setting where Lack's generalized coherence result can be applied.

In § 1 we review enhanced factorization systems, as well as precisely state Lack's generalization of Power's result, which requires an additional condition which we call *rigidity* of the enhanced factorization system. We also show how to lift the factorization of 1-cells of an enhanced factorization system on a 2-category $\boldsymbol{K}$ to the functor 2-category $\boldsymbol{K}^C$, where C is a small category. In §2 we analyze the enhanced factorization system on **Cat** which Power inherently employed in his coherence result in order to determine sufficient conditions for lifting an enhanced factorization system on a 2 category $\boldsymbol{K}$ to $\boldsymbol{K}^C$. We conclude §2 by stating the main result of this paper, which says that under mild assumptions, an enhanced factorization system on $\boldsymbol{K}$ can be lifted to $\boldsymbol{K}^C$, and that if the enhanced factorization system on

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$\boldsymbol{K}$ is rigid, the lifted enhanced factorization system if also rigid. Section 3 is dedicated to proving the "lifting" part of the main result, while §4 is dedicated to showing that the lift of a rigid enhanced factorization system is rigid, under mild hypotheses.

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1. Enhanced Factorization Systems

We begin by reviewing *enhanced factorization systems.*

1.1. **Notation.** Throughout this paper, we adopt the following notational conventions.

(1.1.a) We write **Cat** for the 2-category of small categories, functors, and natural transformations.

(1.1.b) Throughout $\boldsymbol{K}$ denotes a 2-category and C denotes an ordinary category.

(1.1.c) We denote the objects of an ordinary category C by lowercase Roman letters c, c', etc., and a morphism $c \longrightarrow c'$ by a lowercase Roman letter such as f or g.

(1.1.d) Since we are mostly interested in the case of functor 2-categories of the form $\boldsymbol{K}^C$, unless otherwise specified we write capital Roman letters F, G, etc. for objects, lowercase Greek letters α, β, etc. for 1-cells, and uppercase Greek letters Ψ, Φ, etc. for 2-cells.

(1.1.e) We use calligriphic letters such as $\mathcal{E}$ and $\mathcal{M}$ to denote a distinguished class of 1-cells in a 2-category.

(1.1.f) We use a double-struck arrow "$\Longrightarrow$" to denote a 2-cell in a 2-category, or, when noted, a natural transformation.

1.2. **Definition** ([3, § 4.2]). An ***enhanced factorization system*** on a 2-category $\boldsymbol{K}$ consists of a pair of classes of 1-cells $(\mathcal{E}, \mathcal{M})$, both containing all isomorphisms, satisfying the following properties.

(1.2.a) Every 1-cell α of $\boldsymbol{K}$ factors as a composite $\bar{\alpha} \circ \tilde{\alpha}$, where $\tilde{\alpha} \in \mathcal{E}$ and $\bar{\alpha} \in \mathcal{M}$.

(1.2.b) For a diagram in $\boldsymbol{K}$ of the form

$$\begin{tikzcd} F \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & F' \arrow[d, "\alpha'"] \arrow[dl, phantom, "\swarrow\Psi"] \\ G \arrow[r, "\mu"'] & G' \end{tikzcd}$$

where $\varepsilon \in \mathcal{E}$, $\mu \in \mathcal{M}$, and Ψ is an invertible 2-cell, there is a unique pair $(\delta, \widetilde{\Psi})$ consisting of a 1-cell $\delta \colon F' \longrightarrow G$ and an invertible 2-cell $\widetilde{\Psi} \colon \alpha' \Longrightarrow \mu\delta$ so that we have a factorization

$$\begin{tikzcd} F \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & F' \arrow[d, "\alpha'"] \arrow[dl, phantom, "\swarrow\Psi"] \\ G \arrow[r, "\mu"'] & G' \end{tikzcd} \quad = \quad \begin{tikzcd} F \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] & F' \arrow[d, "\alpha'"] \arrow[dl, "\delta"'] \\ G \arrow[r, "\mu"'] \arrow[ur, phantom, "\overset{\Longleftarrow}{\widetilde{\Psi}}"] & G' \end{tikzcd} .$$

By the uniqueness, if Ψ is the identity, then $\gamma\delta = \alpha'$ and $\widetilde{\Psi}$ is the identity.

(1.2.c) Suppose that we are given 1-cells $\varepsilon\colon F \longrightarrow F'$ in $\mathcal{E}$, $\mu\colon G \longrightarrow G'$ in $\mathcal{M}$, and two pairs $\alpha_1, \alpha_2\colon F \longrightarrow G$ and $\alpha_1', \alpha_2'\colon F' \longrightarrow G'$ in $\boldsymbol{K}$, so that the squares

$$\begin{tikzcd} F \arrow[r, "\varepsilon"] \arrow[d, "\alpha_1"'] & F' \arrow[d, "\alpha_1'"] \\ G \arrow[r, "\mu"'] & G' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r, "\varepsilon"] \arrow[d, "\alpha_2"'] & F' \arrow[d, "\alpha_2'"] \\ G \arrow[r, "\mu"'] & G' \end{tikzcd}$$

commute, and 2-cells Ψ and Ψ' such that $\mu\Psi = \Psi'\varepsilon$. Let δ_1 and δ_2 denote the unique 1-cells so that $\alpha_1 = \delta_1\varepsilon$ and $\alpha_2 = \delta_2\varepsilon$ given by (1.2.b). Then there exists a unique 2-cell $\Delta\colon \delta_1 \Longrightarrow \delta_2$ so that $\Delta\varepsilon = \Psi$ and $\mu\Delta = \Psi'$.

We say that an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ is ***rigid*** if the following additional property holds.

(1.2.d) For any 1-cell $\mu\colon F \longrightarrow G$ in $\mathcal{M}$ and any 1-cell $\alpha\colon G \longrightarrow F$ in $\boldsymbol{K}$, if $\mu\alpha \cong \mathrm{id}_G$, then $\alpha\mu \cong \mathrm{id}_F$.

1.3. **Examples.** The following classical examples of enhanced factorization systems are essentially due to Power [4].

(1.3.a) The 2-category **Cat** has an rigid enhanced factorization system $(\mathcal{B}, \mathcal{F})$ where $\mathcal{B}$ is the class of functors which are bijective-on-objects, and $\mathcal{F}$ is the class of fully faithful functors.

(1.3.b) If S is a small set, the 2-category $\mathbf{Cat}^S$ has an rigid enhanced factorization system $(\mathcal{B}^S, \mathcal{F}^S)$ where $\mathcal{B}^S$ is the class of S-indexed sets of functors, all of which are bijective-on-objects, and $\mathcal{F}^S$ is the class of S-indexed sets of functors, all of which are fully faithful functors.

(1.3.c) More generally, if S is a small set, and $\boldsymbol{K}$ is a 2-category with a rigid enhanced factorization system $(\mathcal{E}, \mathcal{M})$, then $\boldsymbol{K}^S$ has an rigid enhanced factorization system $(\mathcal{E}^S, \mathcal{M}^S)$ where $\mathcal{E}^S$ is the class of S-indexed sets of 1-cells in $\mathcal{E}$, and $\mathcal{M}^S$ is the class of S-indexed sets of 1-cells in $\mathcal{M}$.

Now let us explain Lack's coherence result [3, §4.2]. We assume familiarity with 2-monads, algebras for 2-monads, and pseudo-algebras for 2-monads. We do not review these notions because the main results and proofs in this paper do not actually require any knowledge of 2-monads, although the motivation for this work does. The unfamiliar reader should consult [2, §§3.1–3.2; 3, §1; 4, §2].

1.4. **Notation.** Suppose that $\boldsymbol{K}$ is a 2-category and that T is a 2-monad on $\boldsymbol{K}$. Write $\mathbf{Alg}_T^{ps}$ for the 2-category of pseudo-T-algebras, morphisms, and algebra 2-cells.

1.5. **Definition.** Suppose that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$. We say that a 2-monad T on $\boldsymbol{K}$ ***preserves*** $\mathcal{E}$ if $\varepsilon \in \mathcal{E}$ implies that $T\varepsilon \in \mathcal{E}$.

Lack's generalization of Power's result is the following.

1.6. **Theorem** ([3, Thm. 4.6]). *Suppose that $\boldsymbol{K}$ is a 2-category with a rigid enhanced factorization system $(\mathcal{E}, \mathcal{M})$, and that T is a 2-monad on $\boldsymbol{K}$. If T preserves $\mathcal{E}$, then every pseudo-T-algebra is equivalent in $\mathbf{Alg}_T^{ps}$ to a strict T-algebra.*

For the rest of this section we are concerned with showing that an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ on a 2-category $\boldsymbol{K}$ can be lifted to the functor 2-category $\boldsymbol{K}^C$ to classes of morphisms $\mathcal{E}^C$ and $\mathcal{M}^C$ so that every 1-cell in $\boldsymbol{K}^C$ factors as a composite of a 1-cell in $\mathcal{E}^C$ followed by a 1-cell in $\mathcal{M}^C$, though the other axioms to define an enhanced factorization system on $\boldsymbol{K}^C$ are not generally satisfied without additional assumptions. We first make the following simplifying observation.

1.7. **Observation.** Suppose that C is a small category, and that $\boldsymbol{K}$ is a 2-category. The objects of the functor 2-category $\boldsymbol{K}^C$ are, *a priori*, 2-functors $C \longrightarrow \boldsymbol{K}$, 1-cells are strict 2-natural transformations, and 2-cells are modifications. However, since C is an ordinary category, a 2-functor $C \longrightarrow \boldsymbol{K}$ is just a functor from C to the underlying 1-category $\boldsymbol{K}_1$ of $\boldsymbol{K}$, and a 2-natural transformation between 2-functors $F, G\colon C \longrightarrow \boldsymbol{K}$ is simply a natural transformation. Since modifications do not deal with the 2-naturality of a 2-natural transformation, they have the usual structure [2, §1.4].

1.8. **Proposition.** *Suppose that C is a small category and that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$. Given objects $F, G \in \boldsymbol{K}^C$ and a 1-cell $\alpha\colon F \Longrightarrow G$, then α factors as $\bar{\alpha} \circ \tilde{\alpha}$, where $\tilde{\alpha}_c \in \mathcal{E}$ for each $c \in C$, and $\bar{\alpha}_c \in \mathcal{M}$ for each $c \in C$.*

Proof. Using the enhanced factorization system on $\boldsymbol{K}$, for each $c \in C$, choose a factorization

$$\begin{tikzcd}
F(c) \arrow[rr, "\alpha_c"] \arrow[dr, "\tilde{\alpha}_c"'] & & G(c) \\
& R(c) \arrow[ur, "\bar{\alpha}_c"'] &
\end{tikzcd},$$

where $R(c)$ is an object of $\boldsymbol{K}$, $\tilde{\alpha}_c \in \mathcal{E}$, and $\bar{\alpha}_c \in \mathcal{M}$. Then because $(\mathcal{E}, \mathcal{M})$ is an enhanced factorization system on $\boldsymbol{K}$, for each morphism $f\colon c \longrightarrow c'$ in C, we have a unique factorization

$$\begin{tikzcd}
F(c) \arrow[r, "\tilde{\alpha}_c"] \arrow[d, "F(f)"'] & R(c) \arrow[d, "\bar{\alpha}_c"] \\
F(c') \arrow[d, "\tilde{\alpha}_{c'}"'] & G(c) \arrow[d, "G(f)"] \\
R(c') \arrow[r, "\bar{\alpha}_{c'}"'] & G(c')
\end{tikzcd}
=
\begin{tikzcd}
F(c) \arrow[r, "\tilde{\alpha}_c"] \arrow[d, "F(f)"'] & R(c) \arrow[d, "\bar{\alpha}_c"] \arrow[ddl, dotted, "R(f)"] \\
F(c') \arrow[d, "\tilde{\alpha}_{c'}"'] & G(c) \arrow[d, "G(f)"] \\
R(c') \arrow[r, "\bar{\alpha}_{c'}"'] & G(c')
\end{tikzcd}, \tag{1.8.1}$$

so that each of the triangles in the right-hand diagram commute. We want to show that the assignment

$$[f\colon c \longrightarrow c'] \longmapsto [R(f)\colon R(c) \longrightarrow R(c')]$$

defines a functor, that the components $\tilde{\alpha}_c$ and $\bar{\alpha}_c$ assemble into 2-natural transformations. To see this, first notice that by the uniqueness of the factorization (1.8.1), it is clear that $R(\mathrm{id}_c) = \mathrm{id}_{R(c)}$. Then by the commutativity of the triangles in the right-hand diagram of (1.8.1), we get a commutative diagram

$$\begin{tikzcd}
F(c) \arrow[r, "F(f)"] \arrow[d, "\tilde{\alpha}_c"'] & F(c') \arrow[r, "F(g)"] \arrow[d, "\tilde{\alpha}_{c'}"] & F(c'') \arrow[d, "\tilde{\alpha}_{c''}"] \\
R(c) \arrow[r, "R(f)"] \arrow[d, "\bar{\alpha}_c"'] & R(c') \arrow[r, "R(g)"] \arrow[d, "\bar{\alpha}_{c'}"] & R(c'') \arrow[d, "\bar{\alpha}_{c''}"] \\
G(c) \arrow[r, "G(f)"'] & G(c') \arrow[r, "G(g)"'] & G(c'')
\end{tikzcd}, \tag{1.8.2}$$

where each of the sub-squares commutes. Then the commutativity of (1.8.2) along with the functoriality of F and G shows that the diagram

$$\begin{tikzcd} F(c) \arrow[r, "F(gf)"] \arrow[d, "\tilde{\alpha}_c"'] & F(c'') \arrow[d, "\tilde{\alpha}_{c''}"] \\ R(c) \arrow[r, "R(g)R(f)"] \arrow[d, "\bar{\alpha}_c"'] & R(c'') \arrow[d, "\bar{\alpha}_{c''}"] \\ G(c) \arrow[r, "G(gf)"'] & G(c'') \end{tikzcd}$$

commutes. Thus by the uniqueness of $R(gf)$, this implies that $R(gf) = R(g)R(f)$. Then, by construction, the components $\tilde{\alpha}_c$ and $\bar{\alpha}_c$ assemble into 2-natural transformations $\tilde{\alpha}$ and $\bar{\alpha}$, respectively. Moreover, $\alpha = \bar{\alpha}\tilde{\alpha}$ because $\alpha_c = \bar{\alpha}_c\tilde{\alpha}_c$ for all $c \in C$. □

The point of this is that the enhanced factorization system on $\boldsymbol{K}$ yields a way of factoring 1-cells in $\boldsymbol{K}^C$ pointwise.

1.9. **Definition.** Suppose that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$, and that C is a small category. Let $\mathcal{E}^C$ denote the collection of 1-cells in $\boldsymbol{K}^C$, all of whose components are in $\mathcal{E}$, and $\mathcal{M}^C$ denote the collection of 1-cells in $\boldsymbol{K}^C$, all of whose components are in $\mathcal{M}$.

Given a 1-cell $\alpha \in \boldsymbol{K}^C$, we call the factorization $\alpha = \bar{\alpha} \circ \tilde{\alpha}$, where $\tilde{\alpha} \in \mathcal{M}^C$ and $\bar{\alpha} \in \mathcal{E}^C$ (described in Proposition 1.8) the ***pointwise factorization*** of α.

2. The Enhanced Factorization System on **Cat**

In this section we analyze the enhanced factorization system $(\mathcal{B}, \mathcal{F})$ on **Cat** of bijective-on-objects and fully faithful functors. Most if not all of the results in this section are well-known, but we provide proofs of them as they are the motivation for the definitions that we give at the end of the section, and we do not have explicit references for them. Moreover, they also provide us with an example of our main result, which can be easily generalized to the 2-category of categories internal to a cartesian monoidal category, which is of interest to Guillou, May, Merling, and Osorno. The following few results will use more classical categorical notation.

2.1. **Notation.** In Lemmas 2.2 and 2.5 only, we change notation, using classical categorical notation as we find it is more familiar. In Lemmas 2.2 and 2.5, we write:

(2.1.a) C, D, and E for categories,
(2.1.b) B for a functor which is bijective-on-objects,
(2.1.c) F for a fully faithful functor,
(2.1.d) G and H for arbitrary functors,
(2.1.e) and lowercase Greek letters such as η and λ for natural transformations.

2.2. **Lemma.** *Suppose that we have categories and functors as displayed below*

$$\begin{tikzcd} C \arrow[r, shift left, "G"] \arrow[r, shift right, "H"'] & D \arrow[r, "F"] & E, \end{tikzcd}$$

where F is fully faithful.

(2.2.a) *If $\eta, \eta' : G \Longrightarrow H$ are natural transformations such that $F\eta = F\eta'$, then $\eta = \eta'$.*

(2.2.b) *A natural transformation* $\eta\colon FG \Longrightarrow FH$ *lifts to a unique natural transformation* $\hat{\eta}\colon G \Longrightarrow H$ *with the property that* $F\hat{\eta} = \eta$.
(2.2.c) *With notation as in (2.2.b), if* η *is a natural isomorphism, so is* $\hat{\eta}$.
(2.2.d) *If* $G(c) = H(c)$ *for all* $c \in C$ *and* $FG = FH$*, then* $G = H$*. Thus, if* $B\colon C' \longrightarrow C$ *is bijective-on-objects,* $GB = BH$ *and* $FG = FH$*, then* $G = H$.

Proof. Notice that (2.2.a) follows immediately from the fact that F is fully faithful and for all $c \in C$ the components η_c and η'_c have the same domain and codomain.

Second, suppose that we have a natural transformation $\eta\colon FG \Longrightarrow FH$. Since F is fully faithful, for each $c \in C$, there exists a unique morphism $\hat{\eta}_c\colon G(c) \longrightarrow H(c)$ in D so that $F(\hat{\eta}_c) = \eta_c$. To see that the morphisms $\hat{\eta}_c$ assemble into a natural transformation $\hat{\eta}\colon G \longrightarrow H$, suppose that $f : c \longrightarrow c'$ consider the square

$$\begin{tikzcd} G(c) \arrow[r, "G(f)"] \arrow[d, "\hat{\eta}_c"'] & G(c') \arrow[d, "\hat{\eta}_{c'}"] \\ H(c) \arrow[r, "H(f)"'] & H(c')\,. \end{tikzcd} \tag{2.2.1}$$

To see that (2.2.1) commutes, notice that

$$F(\hat{\eta}_{c'} \circ G(f)) = \eta_{c'} \circ FG(f) = FH(f) \circ \eta_c = F(H(f) \circ \hat{\eta}_c),$$

by the functoriality of F and naturality of η. Since F is fully faithful and $\hat{\eta}_{c'} \circ G(f)$ and $H(f) \circ \hat{\eta}_c$ have the same source and target, this implies that $\hat{\eta}_{c'} \circ G(f) = H(f) \circ \hat{\eta}_c$, so (2.2.1) commutes. Hence the morphisms $\hat{\eta}_c$ assemble into a natural transformation $\hat{\eta}\colon G \Longrightarrow H$ with the property that $F\hat{\eta} = \eta$.

Notice that the claim (2.2.c) follows immediately from (2.2.b) and the fact that fully faithful functors reflect isomorphisms.

To prove (2.2.d) all that needs to be verified is that $G(f) = H(f)$ for all morphisms f of C. Since $G(c) = H(c)$ for all $c \in C$, the morphisms $G(f)$ and $H(f)$ have the same source and target. Since $FG(f) = FH(f)$, and F is fully faithful, this implies that $G(f) = H(f)$. □

The following definition is a generalization of items (2.2.b) and (2.2.c) of Lemma 2.2.

2.3. **Definition.** Suppose that $\boldsymbol{K}$ is a 2-category and suppose that μ is a 1-cell in $\boldsymbol{K}$. We say that ***post-comosition with*** μ ***creates invertible* 2*-cells*** if whenever we have 1-cells $\alpha, \beta \in \boldsymbol{K}$ with the same source and target equipped with an invertible 2-cell

$$\Psi\colon \mu\alpha \Longrightarrow \mu\beta,$$

there exists a unique invertible 2-cell $\widehat{\Psi}\colon \alpha \Longrightarrow \beta$ so that $\mu\widehat{\Psi} = \Psi$.

Suppose that $\mathcal{M}$ is a class of 1-cells in a 2-category $\boldsymbol{K}$. We say that ***post-composition with* 1*-cells in*** $\mathcal{M}$ ***creates invertible* 2*-cells***, if post-composition with every 1-cell $\mu \in \mathcal{M}$ creates invertible 2-cells.

The following definition is a generalization of item (2.2.d) of Lemma 2.2.

2.4. **Definition.** We say that an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ on a 2-category $\boldsymbol{K}$ ***separates parallel pairs*** if whenever we are given a parallel pair $\alpha, \alpha'\colon F \longrightarrow G$ so that there exists a 1-cell $\varepsilon \in \mathcal{E}$ such that $\alpha\varepsilon = \alpha'\varepsilon$ and a 1-cell $\mu \in \mathcal{M}$ such that $\mu\alpha = \mu\alpha'$, we have $\alpha = \alpha'$.

Now for a result regarding functors which are bijective-on-objects.

2.5. **Lemma.** *Suppose that we have categories, functors, and natural transformations as displayed below*

$$\begin{tikzcd} C \arrow[r, "B"] & D \arrow[r, bend left, "G"{name=U}] \arrow[r, bend right, "H"'{name=L}] \arrow[r, phantom, "\eta\Downarrow\ \Downarrow\lambda"] & E \end{tikzcd},$$

where B is bijective-on-objects. If $\eta B = \lambda B$, then $\eta = \lambda$.

Proof. Since $\eta B = \lambda B$, for all $c \in C$ we have $\eta_{B(c)} = \lambda_{B(c)}$. Since B is bijective-on-objects, this says that $\eta_d = \lambda_d$ for each $d \in D$, so $\eta = \beta$. □

Representable epimorphisms express the property conclusion of Lemma 2.5 about functors which are bijective-on-objects. *Representable monomorphism* express the dual property of fully faithful functors, which is (2.2.d) of Lemma 2.2.

2.6. **Definition.** We say that a 1-cell $\varepsilon\colon F \longrightarrow G$ in a 2-category $\boldsymbol{K}$ is a ***representable epimporphism*** if whenever we have objects, 1-cells, and 2-cells as displayed below

$$\begin{tikzcd} F \arrow[r, "\varepsilon"] & G \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Psi\Downarrow\ \Downarrow\Phi"] & H \end{tikzcd},$$

if $\Psi\varepsilon = \Phi\varepsilon$, then $\Psi = \Phi$. ***Representable monomorphisms*** are defined dually.

Given an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ on a 2-category K, we say that $\mathcal{E}$ ***consists of representable epimorphisms*** if all 1-cells in $\mathcal{E}$ are representable epimorphisms, and $\mathcal{M}$ ***consists of representable monomorphisms*** if all of the 1-cells in $\mathcal{M}$ are representable monomorphisms. If $\mathcal{E}$ consists of representable epimorphisms and $\mathcal{M}$ consists of representable monomorphisms, we simply say that the enhanced factorization system $(\mathcal{E}, \mathcal{M})$ on $\boldsymbol{K}$ is ***representable.***

Now we are ready to state the main results of this paper.

2.7. **Proposition.** *Suppose that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ and that C is a small category. If $(\mathcal{E}, \mathcal{M})$ separates parallel pairs and $\mathcal{E}$ consists of representable epimorphisms, then the pointwise factorization $(\mathcal{E}^C, \mathcal{M}^C)$ produced in Proposition 1.8 defines an enhanced factorization system on $\boldsymbol{K}^C$.*

2.8. **Theorem** (Main Result). *Suppose that C is a small category, and that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$. If $(\mathcal{E}, \mathcal{M})$ is representable, separates parallel pairs, and post-composition with 1-cells in $\mathcal{M}$ creates invertible 2-cells, then the pointwise factorization $(\mathcal{E}^C, \mathcal{M}^C)$ on $\boldsymbol{K}^C$ defines an rigid enhanced factorization system on $\boldsymbol{K}^C$.*

Sections 3 and 4 are dedicated to proving Proposition 2.7 and Theorem 2.8. The point of the conditions stated in Proposition 2.7 and Theorem 2.8 is the following: When we were proving Proposition 1.8, we could use the properties of higher cells in $\boldsymbol{K}$ to show that a collection of 1-cells defined pointwise were "coherent" in the sense that they assembled into a natural transformation. However, when given a collection of 2-cells in $\boldsymbol{K}$, there is not a higher structure to ensure that the 2-cells are "coherent" in the sense that they assemble themselves into a modification. Hence, in order to have such a condition hold, we need some extra underlying structure in the enhanced factorization system.

The following corollary of Theorem 2.8 is a direct application of Lack's result Theorem 1.6.

2.8.1. **Corollary.** *Suppose that C is a small category, $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$, and that T is a 2-monad on $\boldsymbol{K}^C$. Also suppose that $(\mathcal{E}, \mathcal{M})$ is representable, separates parallel pairs, and post-composition with 1-cells in $\mathcal{M}$ creates invertible 2-cells. Then every pseudo-T-algebra is equivalent in $\mathbf{Alg}_T^{ps}$ to a strict T-algebra.*

2.9. **Example.** For any small category C, the pointwise factorization $(\mathcal{B}^C, \mathcal{F}^C)$ defines a rigid enhanced factorization system on $\mathbf{Cat}^C$. Moreover, if T is a 2-monad on $\mathbf{Cat}^C$ which preserves the natural transformations whose components are bijective-on-objects functors, then every pseudo-T-algebra is equivalent to a strict T-algebra. In particular, this implies Power's result [4, Cor. 3.5].

3. Lifting Enhanced Factorization Systems

This section is dedicated to proving Proposition 2.7. The first step in this is to show that, under these hypotheses, the pointwise factorization satisfies the property Definition 1.2.b defining an enhanced factorization system.

3.1. **Lemma.** *Suppose that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ and that C is a small category. If $(\mathcal{E}, \mathcal{M})$ separates parallel pairs and $\mathcal{E}$ consists of representable epimorphisms, then for every diagram*

$$\begin{tikzcd} F \arrow[r, "\varepsilon"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\swarrow\Psi"] & F' \arrow[d, "\alpha'"] \\ G \arrow[r, "\mu"'] & G' \end{tikzcd}$$

in $\boldsymbol{K}^C$, where Ψ is an invertible 2-cell, $\varepsilon \in \mathcal{E}^C$, and $\mu \in \mathcal{M}^C$, the 2-category $\boldsymbol{K}^C$ has a unique 1-cell $\delta\colon F' \longrightarrow G$ in and a unique modification $\widetilde{\Psi}\colon \alpha' \Longrightarrow \mu\delta$ so that $\delta\varepsilon = \alpha$ and $\widetilde{\Psi}\varepsilon = \Psi$. Moreover, $\widetilde{\Psi}$ is necessarily invertible.

Proof. Since $\varepsilon \in \mathcal{E}^C$, $\mu \in \mathcal{M}^C$, and $(\mathcal{E}, \mathcal{M})$ is an enhanced factorization system on $\boldsymbol{K}$, for each $c \in C$, we have a factorization

$$\begin{tikzcd} F(c) \arrow[r, "\varepsilon_c"] \arrow[d, "\alpha_c"'] \arrow[dr, phantom, "\swarrow\Psi_c"] & F'(c) \arrow[d, "\alpha'_c"] \\ G(c) \arrow[r, "\mu_c"'] & G'(c) \end{tikzcd} = \begin{tikzcd} F(c) \arrow[r, "\varepsilon_c"] \arrow[d, "\alpha_c"'] & F'(c) \arrow[d, "\alpha'_c"] \arrow[dl, "\delta_c"'] \arrow[d, phantom, shift right=20, "\Leftarrow \widetilde{\Psi}_c"] \\ G(c) \arrow[r, "\mu_c"'] & G'(c) \end{tikzcd}.$$

The goal is to show that the 1-cells δ_c assemble into a 2-natural transformation δ and that the 2-cells $\widetilde{\Psi}_c$ assemble into an invertible modification $\widetilde{\Psi}\colon \alpha' \Longrightarrow \mu\delta$. A priori, the 1-cells δ_c for $c \in C$ are not related to one-another, so to show that the 1-cells δ_c are natural, we exploit the fact that the enhanced factorization system on $\boldsymbol{K}$ separates parallel pairs. Since the enhanced factorization system separates parallel pairs, it suffices to show that $f\colon c \longrightarrow c'$ in C we have

$$\delta_{c'} F'(f) \circ \varepsilon_c = G(f)\delta_c \circ \varepsilon_c \quad \text{and} \quad \mu_{c'} \circ \delta_{c'} F'(f) = \mu_{c'} \circ G(f)\delta_c\,.$$

First let us show that $\delta_{c'} F'(f) \circ \varepsilon_c = G(f)\delta_c \circ \varepsilon_c$. Since $\delta_c\varepsilon_c = \alpha_c$ for each $c \in C$, and α and ε are 2-natural transformations, for all morphisms $f\colon c \longrightarrow c'$ in C we have

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\alpha_c"'] & F(c') \arrow[d, "\alpha_{c'}"] \\ G(c) \arrow[r, "G(f)"'] & G(c') \end{tikzcd} = \begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\varepsilon_c"'] & F(c') \arrow[d, "\varepsilon_{c'}"] \\ F'(c) \arrow[d, "\delta_c"'] & F'(c') \arrow[d, "\delta_{c'}"] \\ G(c) \arrow[r, "G(f)"'] & G(c') \end{tikzcd} = \begin{tikzcd} F(c) \arrow[d, "\varepsilon_c"] & \\ F'(c) \arrow[r, "F'(f)"] \arrow[d, "\delta_c"'] & F'(c') \arrow[d, "\delta_{c'}"] \\ G(c) \arrow[r, "G(f)"'] & G(c') \end{tikzcd}.$$

Hence for all $f : c \longrightarrow c'$ in C we have $\delta_{c'} F'(f) \circ \varepsilon_c = G(f)\delta_c \circ \varepsilon_c$.

Now let us show that the diagram

$$\begin{tikzcd} F'(c) \arrow[r, "F'(f)"] \arrow[d, "\delta_c"'] & F'(c') \arrow[d, "\delta_{c'}"] & \\ G(c) \arrow[r, "G(f)"'] & G(c') \arrow[r, "\mu_{c'}"] & G'(c')\,. \end{tikzcd} \tag{3.1.1}$$

commutes. To see this consider the invertible 2-cell

$$\widetilde{\Psi}_{c'} F'(f) G'(f) \widetilde{\Psi}_c^{-1} : G'(f)\mu_c\delta_c \Longrightarrow \mu_{c'}\delta_{c'}F'(f).$$

A priori, we do not know that that $\widetilde{\Psi}_{c'} F'(f) G'(f) \widetilde{\Psi}_c^{-1}$ is the identity. To see that this is true we use the fact that $\mathcal{E}$ consists of representable epimorphisms. Whiskering with ε_c we see that

$$G'(f)\widetilde{\Psi}_c^{-1}\varepsilon_c = G'(f)\Psi_c^{-1} = \Psi_{c'}^{-1}F(f)$$

as Ψ^{-1} is a modification. Similarly, notice that by the naturality of ε, we have

$$\widetilde{\Psi}_{c'}^{-1} F'(f)\varepsilon_c = \widetilde{\Psi}_{c'}^{-1}\varepsilon_{c'}F(f) = \Psi_{c'}^{-1}F(f).$$

Then since ε_c is a representable epimorphism, we see that

$$\widetilde{\Psi}_{c'}^{-1}F'(f) = G'(f)\widetilde{\Psi}_c^{-1}\,,$$

so $\widetilde{\Psi}_{c'} F'(f) G'(f) \widetilde{\Psi}_c^{-1}$ is the identity. Thus the diagram

$$\begin{tikzcd} F'(c) \arrow[r, "F'(f)"] \arrow[d, "\delta_c"'] & F'(c') \arrow[d, "\delta_{c'}"] \\ G(c) \arrow[d, "\mu_c"'] & G(c') \arrow[d, "\mu_{c'}"] \\ G'(c) \arrow[r, "G'(f)"'] & G'(c') \end{tikzcd}$$

commutes. The naturality of μ implies that the diagram (3.1.1) commutes, as desired. Then since $\delta_{c'}F'(f) \circ \varepsilon_c = G(f)\delta_c \circ \varepsilon_c$ and the enhanced factorization system on $\boldsymbol{K}$ distinguishes between parallel pairs, we see that $G(f)\delta_c = \delta_{c'}F'(f)$, so the 1-cells $\delta_c : F'(c) \longrightarrow G(c)$ assemble into a unique 2-natural transformation δ.

Now let us show that the invertible 2-cells $\widetilde{\Psi}_c$ in $\boldsymbol{K}$ assemble into an invertible modification. Suppose that $f : c \longrightarrow c'$ is a morphism in C. A priori it is not clear that $\widetilde{\Psi}_{c'}F'(f) = G'(f)\widetilde{\Psi}_c$; to see this, we exploit the fact that $\mathcal{E}$ consists of representable epimorphisms. Precomposing $\widetilde{\Psi}_{c'}F'(f)$ with ε_c we see that

$$\widetilde{\Psi}_c F'(f)\varepsilon_c = \widetilde{\Psi}_{c'}\varepsilon_{c'}F(f) = \Psi_{c'}F(f) = G'(f)\Psi_c\,,$$

by the naturality of ε and the fact that Ψ is a modification. Moreover, $\widetilde{\Psi}_c$ has the property that $G'(f)\widetilde{\Psi}_c\varepsilon_c = G'(f)\Psi_c$. Since ε is a representable epimorphism this implies that $\widetilde{\Psi}_{c'}F'(f) = G'(f)\widetilde{\Psi}_c$. Hence the 2-cells $\widetilde{\Psi}_c$ assemble into an invertible modification $\widetilde{\Psi} : \alpha' \Longrightarrow \mu\delta$, which is unique because the $\widetilde{\Psi}_c$ are. □

3.2. **Lemma.** *Suppose that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ and that C is a small category. If $(\mathcal{E}, \mathcal{M})$ separates parallel pairs and $\mathcal{E}$ consists of representable epimorphisms, then the pointwise factorization $(\mathcal{E}^C, \mathcal{M}^C)$ satisfies the last condition (1.2.c) of Definition 1.2 to define an enhanced factorization system on $\boldsymbol{K}^C$.*

Proof. Suppose that we are in the situation indicated in the last condition (1.2.c) of Definition 1.2 (since it is lengthy, we will not spell it out again here.) For each $c \in C$, write $\delta_{1,c}$ and $\delta_{1,c}$ for the components of δ_1 at c and δ_2 at c, respectively. Since $(\mathcal{E}, \mathcal{M})$ is an enhanced factorization system on $\boldsymbol{K}$, for each $c \in C$, there exists a unique invertible 2-cell $\Delta_c \colon \delta_{1,c} \Longrightarrow \delta_{2,c}$ in $\boldsymbol{K}$ so that $\Delta_c \varepsilon_c = \Psi_c$ and $\mu_c \Delta_c = \Phi'_c$. To see that the 2-cells Δ_c for $c \in C$ assemble into an invertible modification $\Delta \colon \delta_1 \Longrightarrow \delta_2$, notice that for all morphisms $f \colon c \longrightarrow c'$ in C, since Ψ is a modification we have

$$G(f)\Delta_c\varepsilon_c = G(f)\Psi_c = \Psi_{c'}F(f) .$$

Similarly,

$$\Delta_{c'}F'(f)\varepsilon_c = \Delta_{c'}\varepsilon_{c'}F(f) = \Psi_{c'}F(f)$$

hence $G(f)\Delta_c\varepsilon_c = \Delta_{c'}F'(f)\varepsilon_c$. Then since $\mathcal{E}$ consists of representable epimorphisms, in particular ε_c is a representable epimorphism, so we see that

$$G(f)\Delta_c = \Delta_{c'}F'(f) .$$

Hence the invertible 2-cells Δ_c for $c \in C$ satisfy the necessary conditions to define a modification. □

Combining Proposition 1.8 and Lemmas 3.1 and 3.2 proves Proposition 2.7.

4. Conditions for Rigidity

In this section we analyze conditions on the enhanced factorization system on $\boldsymbol{K}$ which yield a rigid enhanced factorization system on $\boldsymbol{K}^C$, for any small category C.

4.1. **Lemma.** *Suppose that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$. If post-composition with 1-cells in $\mathcal{M}$ creates invertible 2-cells, then the enhanced factorization system on $\boldsymbol{K}$ is rigid.*

Proof. Suppose that we have 1-cells $\mu \colon F \rightleftarrows G : \alpha$, where $\mu \in \mathcal{M}$, and we have an invertible 2-cell $\Psi \colon \mu\alpha \Longrightarrow \mathrm{id}_G$. Then the 2-cell $\Psi\mu \colon \mu\alpha\mu \Longrightarrow \mu$ is invertible. For notational simplicity, write $\Phi \coloneqq \Psi\mu$. Then since post-composition with μ creates invertible 2-cells, there is a unique invertible 2-cell

$$\widehat{\Phi} \colon \alpha\mu \Longrightarrow \mathrm{id}_F$$

so that $\mu\widehat{\Phi} = \Psi\mu$. In particular, $\alpha\mu \cong \mathrm{id}_F$. □

4.2. **Lemma.** *Suppose that $\boldsymbol{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ and that C is a small category. If $\mathcal{M}$ consists of representable monomorphisms and post-composition with 1-cells in $\mathcal{M}$ creates invertible 2-cells, then post-composition with 1-cells in $\mathcal{M}^C$ creates invertible 2-cells in $\boldsymbol{K}^C$.*

Proof. Suppose that we are given 1-cells $\mu \colon F \rightleftarrows G : \alpha$, where $\mu \in \mathcal{M}^C$, and we have an invertible 2-cell $\Psi \colon \mu\alpha \Longrightarrow \mathrm{id}_G$. Then for each $c \in C$, there exists an invertible 2-cell $\Psi_c \colon \mu_c\alpha_c \Longrightarrow \mathrm{id}_{G(c)}$ in $\boldsymbol{K}$. As shown in the proof of Lemma 4.1, there exists a unique invertible 2-cell

$$\widehat{\Phi}_c \colon \alpha_c\mu_c \Longrightarrow \mathrm{id}_{F(c)}$$

so that $\mu_c\widehat{\Phi}_c = \Psi_c\mu_c$. We want to show that the $\widehat{\Phi}_c$ assemble into the components of an invertible modification.

To do this, suppose that $f: c \longrightarrow c'$ is a morphism in C. Writing $\Phi \coloneqq \Psi\mu$, since Φ is a modification, we have

$$\begin{tikzcd} F(c) \arrow[r, bend left, "\mu_c\alpha_c\mu_c"] \arrow[r, bend right, "\mu_c"'] \arrow[r, phantom, "\Phi_c\Downarrow"] & G(c) \arrow[r, "G(f)"] & G(c') \end{tikzcd} \quad = \quad \begin{tikzcd} F(c) \arrow[r, "F(f)"] & F(c') \arrow[r, bend left, "\mu_{c'}\alpha_{c'}\mu_{c'}"] \arrow[r, bend right, "\mu_{c'}"'] \arrow[r, phantom, "\Downarrow\Phi_{c'}"] & G(c') \end{tikzcd} .$$

Hence we see that

$$\mu_{c'}\widehat{\Phi}_{c'}F(f) = \Phi_{c'}F(f) = G(f)\Phi_c .$$

Hy the naturality of μ and the fact that $\mu\widehat{\Phi} = \Phi$ we see that

$$\mu_{c'}F(f)\widehat{\Phi}_c = G(f)\mu_c\widehat{\Phi}_c = G(f)\Phi_c .$$

Then since

$$\mu_{c'}\widehat{\Phi}_{c'}F(f) = \mu_{c'}F(f)\widehat{\Phi}_c$$

and $\mu_{c'}$ is a representable monomorphism, we have $\widehat{\Phi}_{c'}F(f) = F(f)\widehat{\Phi}_c$, hence the components $\widehat{\Phi}_c$ assemble into an invertible modification $\widehat{\Phi}$. $\square$

Hence Proposition 2.7 and Lemmas 4.1 and 4.2 together prove the main result Theorem 2.8.

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