ON THE K-THEORY OF FINITE FIELDS

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Abstract. In this talk we introduce higher algebraic K-groups via Quillen's plus construction. We then give a brief tour of algebraic K-theory and its relation to stable homotopy theory and number theory — in particular noting that a K-theory computation of the integers is equivalent to the Kummer–Vandiver conjecture. We next state Quillen's computations of the K-theory of $\mathbf{F}_q$ and $\overline{\mathbf{F}}_q$. The remainder of the talk focuses on how Quillen is able to execute these K-theory computations by using the Adams operations to relate K-theory to modular representation theory. We finish by discussing the Brauer lift to — the main tool that makes Quillen's computation go — and how Quillen uses the Brauer lift to show that $B\mathrm{GL}(\mathbf{F}_q)^+$ is homotopy equivalent to a space $F\psi^q$, defined via the Adams operations, whose homotopy groups are easily computable using Bott periodicity.

Contents

1. Algebraic K-theory via perfect groups and the Plus Construction

In this section we define higher algebraic K-theory via Quillen's plus construction. To motivate this, we recall how the group K_1 was first defined via elementary matrices.

1.1. **Convention.** For us R always denotes a commutative (unital) ring.

1.2. **Recollection.** Let n be a positive integer, and consider the group $\mathrm{GL}_n(R)$ of $n \times n$ matrices. Given $1 \leq i, j \leq n$ and $r \in R$, let $\delta_{i,j}$ by the $n \times n$ matrix whose only nonzero entry is 1 in the (i, j) position, and let $\varepsilon_{i,j}(r) \coloneqq \mathrm{id}_n + r\delta_{i,j}$. The matrices $\varepsilon_{i,j}(r)$ are called ***elementary matrices***. Let $E_n(R)$ denote the subgroup of $\mathrm{GL}_n(R)$ *generated* by the elementary matrices.

The ***infinite general linear group*** is the colimit

$$\mathrm{GL}(R) \coloneqq \mathrm{colim}\left(\mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \cdots \right).$$

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Note: Items marked with a " * " are extra remarks that we likely did not have time to mention in the talk.

Similarly, define

$$E(R) \coloneqq \operatorname{colim}\left(E_1(R) \hookrightarrow E_2(R) \hookrightarrow E_2(R) \hookrightarrow \cdots \right).$$

1.3. **Remark.** If R is a field, then $E_n(R) = \mathrm{SL}_n(R)$.

1.4. **Preliminary Definition.** Let R be a commutative ring. The ***first algebraic K-group*** of R is the quotient

$$K_1(R) \coloneqq \mathrm{GL}(R)/E(R).$$

As algebraic topologists, we prefer definitions to have a homotopy-theoretic flavor. In particular, we might try to define $K_1(R)$ in terms of homotopy groups. We already know that $\pi_1(B\mathrm{GL}(R)) \cong \mathrm{GL}(R)$. Moreover, $B\mathrm{GL}(R)$ has an explicit CW-structure. One idea to produce K_1 homotopy–theoretically is to add cells to $B\mathrm{GL}(R)$ to create a space whose fundamental group is $K_1(R)$ — is is precisely what the plus construction will do for us. Before we do this, let us take a closer look at $E(R)$ to see exactly what sort of group-theoretic properties $E(R)$ has.

1.5. **Notation.** Let G be a group. Write $[G, G]$ for the commutator subgroup of G and write $G_{ab} \cong G/[G, G]$ for the abelianization of G.

1.6. **Example.** For $n \geq 3$, the subgroup $E_n(R)$ is a normal subgroup of $\mathrm{GL}_n(R)$ with the property that $[E_n(R), E_n(R)] = E_n(R)$. Moreover, $[E(R), E(R)] = E(R)$.

1.7. **Definition.** A group G is ***perfect*** if $G_{ab} = 0$.

1.8. **Example.** The alternating group $A_5 \lhd \Sigma_5$ is the smallest nontrivial perfect group.

1.9. **Example.** The ***infinite symmetric group*** is the colimit

$$\Sigma_\infty \coloneqq \operatorname{colim}\left(\Sigma_1 \hookrightarrow \Sigma_2 \hookrightarrow \Sigma_2 \hookrightarrow \cdots \right).$$

The ***infinite alternating group*** is the colimit

$$A_\infty \coloneqq \operatorname{colim}\left(A_1 \hookrightarrow A_2 \hookrightarrow A_2 \hookrightarrow \cdots \right).$$

The index of A_∞ in Σ_∞ is 2, and A_∞ is a perfect normal subgroup of Σ_∞.

Acyclic spaces generate a class of examples of perfect groups.

1.10. **Definition.** A space X is ***acyclic*** if $\widetilde{H}_*(X; \mathbf{Z}) = 0$.

1.11. **Lemma.** *Let X be an acyclic space. Then X is connected and $\pi_1(X)$ is perfect.*

Proof. Since X is acyclic, $H_0(X) \cong \mathbf{Z}$, hence X is connected. Similarly, since X is acyclic $H_1(X) = 0$, but the Hurewicz homomorphism exhibits $H_1(X)$ as an abelianization of $\pi_1(X)$, hence $\pi_1(X)_{ab} = 0$. □

1.12. **Definition.** Let X and Y be pointed connected CW-complexes. A pointed map

$$f : X \longrightarrow Y$$

is ***acyclic*** if $\mathrm{hofib}(f)$ is acyclic.

*1.13. **Lemma.** *Let X and Y be connected CW-complexes. A map $f : X \longrightarrow Y$ is acyclic if and only if for every local coefficient system $L : \Pi_1(Y) \longrightarrow$ **Ab**, the map f induces an isomorphism*

$$f_\star : H_*(X; L \circ \Pi_1(f)) \xrightarrow{\sim} H_*(Y; L).$$

Proof Sketch. The proof is not very hard — the main tool of is the Serre spectral sequence, and the comparison theorem for cellular maps of fibrations. A proof can be found in [12, Ch. 4 Lem. 1.6]. □

1.14. **Definition.** Let X be a pointed connected CW-complex and $P \lhd \pi_1(X)$ a perfect normal subgroup. An acyclic map $\iota\colon X \longrightarrow X^+$ is called a ***plus construction*** for X (relative to P) if P is the kernel of $\iota_\star\colon \pi_1(X) \longrightarrow \pi_1(X^+)$.

1.15. **Theorem** (Quillen). *Let X be a pointed CW-complex and $P \lhd \pi_1(X)$ a perfect normal subgroup. Then there exists a plus construction $\iota\colon X \longrightarrow X^+$ for X relative to P. Moreover, the plus construction enjoys the following universal property: for every map $f\colon X \longrightarrow Y$ so that $f_\star(P) = 0$, there is a map $f^+\colon X^+ \longrightarrow Y$, unique up to pointed homotopy, so that the triangle*

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "f"'] & \arrow[d, phantom, "\simeq"] & X^+ \arrow[dl, dotted, "f^+"] \\ & Y & \end{tikzcd}$$

commutes up to pointed homotopy.

Idea of the construction of X^+. The idea of the construction of X^+ is to attach 2-cells to X to kill P from $\pi_1(X)$, then attach 3-cells to make sure that we retain the correct homology. Proofs can be found in [6, Prop. 4.40; 9, Thé. 1.1.1]. From this explicit construction, it is possible to use obstruction theory to show that X^+ has the desired universal property, which in turn characterizes X^+ uniquely up to homotopy equivalence. □

The plus construction is functorial in the following sense.

1.15.1. **Corollary.** *Let $\mathbf{CW}_*^{perf}$ denote the category with objects pairs (X, P), where X is a pointed connected CW-complex and $P \lhd \pi_1(X)$ is a perfect normal subgroup, and a morphism*

$$f\colon (X, P) \longrightarrow (X', P')$$

is a map $f\colon X \longrightarrow X'$ so that $f_\star(P) \subset P'$. The plus construction defines a functor

$$(-)^+\colon \mathbf{CW}_*^{perf} \longrightarrow h\mathbf{CW}_* .$$

1.16. **Notation.** For a commutative ring R, let $B\mathrm{GL}(R)^+$ denote the plus construction of $B\mathrm{GL}(R)$ with respect to the perfect normal subgroup $E(R) \lhd \mathrm{GL}(R) \cong \pi_1(B\mathrm{GL}(R))$.

1.17. **Definition.** Let R be a commutative ring and n a positive integer. The n^{th} ***algebraic*** K***-group of*** R is the homotopy group

$$K_n(R) \coloneqq \pi_n(B\mathrm{GL}(R)^+) .$$

1.18. **Example.** By the definition of the plus construction, for any ring R we have

$$K_1(R) \cong \mathrm{GL}(R)/E(R) ,$$

so this definition agrees with the preliminary definition of K_1.

1.19. **Warning.** Notice that we have expressly avoided mentioning anything about K_0! That is because K_0 needs to be defined differently — Jesse will give a more careful analysis of K_0, K_1, and K_2 in his talk on 3 October.

2. A Whirlwind Tour of K-theory

In this section we give one manifestation of the connections between algebraic K-theory and stable homotopy theory, via the Barratt–Priddy–Quillen theorem, as well as indicate how difficult K-theory computations are by stating the the Kummer–Vandiver conjecture from number theory is equivalent to a K-theory computation of the integers.

2.1. **Theorem** (Barratt–Priddy–Quillen [3]). *Let $B\Sigma_\infty^+$ denote the plus construction of $B\Sigma_\infty$ with respect to A_∞. There is an equivalence*

$$B\Sigma_\infty^+ \simeq \Omega^\infty \Sigma^\infty S^0 = \mathrm{colim}_n\, \Omega^n \Sigma^n S^0 = \mathrm{colim}_n\, \Omega^n S^n \,.$$

In particular, $B\Sigma_\infty^+ \cong \pi_n^S$.

*2.2. **Philosophical Remark.** Now, Σ_∞ is not the infinite general linear group of any ring, but if there were a field $\mathbf{F}_1$ with one element, heuristically $\mathrm{GL}_n(\mathbf{F}_1)$ should be Σ_n, so $\mathrm{GL}(\mathbf{F}_1)$ should be Σ_∞. Then under this heuristic, $K_n(\mathbf{F}_1) \cong \pi_n^S$. This is only philosophy, but later we will see that this can be made precise with the algebraic K-theory of various types of categories. In this setting $K_n(\mathbf{Fin}_*) \cong \pi_n^S$.

2.3. **Observation** (Relating algebraic K-theory to stable homotopy theory). For each positive integer n, there is an inclusion $\Sigma_n \hookrightarrow \mathrm{GL}_n(\mathbf{Z})$ given by the permutation representation, hence this induces an inclusion $\Sigma_\infty \hookrightarrow \mathrm{GL}(\mathbf{Z})$. The elementary integer matrices generate $\mathrm{SL}_n(\mathbf{Z})$, and the matrix of an even permutation is 1, so we have compatible inclusions

$$\begin{tikzcd} A_\infty \arrow[r, hook] \arrow[d, hook] & \mathrm{SL}(\mathbf{Z}) \arrow[d, hook] \\ \Sigma_\infty \arrow[r, hook] & \mathrm{GL}(\mathbf{Z}) \,, \end{tikzcd}$$

so by the functoriality of the plus construction we get an induced map $B\Sigma_\infty^+ \longrightarrow B\mathrm{GL}(\mathbf{Z})^+$. Taking homotopy groups and using the Barratt–Priddy–Quillen theorem gives a map

$$\pi_n^S \longrightarrow K_n(\mathbf{Z}) \,.$$

2.4. **Observation.** Since $\mathbf{Z}$ is the initial ring, every ring receives a unique map $\phi_R \colon \mathbf{Z} \longrightarrow R$. Since taking the general linear group is a functor and the image of an elementary matrix is an elementary matrix, by the universal property of the plus construction, ϕ_R induces map $B\mathrm{GL}(\mathbf{Z})^+ \longrightarrow B\mathrm{GL}(R)^+$, and applying homotopy groups yields a map $K_n(\mathbf{Z}) \longrightarrow K_n(R)$. Hence by Observation 2.3, stable homotopy maps to the K-theory of every ring.

*2.5. **Proposition** (Kurihara [8]). *We have that $K_{4n}(\mathbf{Z}) = 0$ for all positive integers n if and only if the Kummer–Vandiver conjecture holds.*

*2.6. **Conjecture** (Kummer–Vandiver). *For all primes p, the prime p does not divide the class number of the maximal real subfield of the p^{th} cyclotomic field $\mathbf{Q}(\zeta_p)$.*

*2.7. **Remark.** The K-theory of $\mathbf{Z}$ is hair-raisingly difficult to compute — a summary of many of the known computations can be found in [12, Ch. 6 §10].

3. The K-theory of finite fields

Our goal is to understand the main ideas of the proof of the following amazing computation of Quillen.

3.1. **Theorem** (Quillen). *Let q be a power of a prime p.*

(3.1.a) *For all $n \geq 1$ we have $K_{2n}(\mathbf{F}_q) = 0$ and $K_{2n-1}(\mathbf{F}_q) \cong \mathbf{Z}/(q^n - 1)$.*

(3.1.b) *If* $\iota\colon \mathbf{F}_q \hookrightarrow \mathbf{F}_{q^m}$ *is a field extension, then* $\iota_\star\colon K_*(\mathbf{F}_q) \longrightarrow K_*(\mathbf{F}_{q^m})$ *is injective.*

(3.1.c) *Let* $\mathrm{Fr}^p\colon \mathbf{F}_q \xrightarrow{\sim} \mathbf{F}_q$ *denote the absolute Frobenius automorphism. Then for all* $n \geq 1$*, the induced homomorphism*

$$\mathrm{Fr}^p_\star\colon K_{2n-1}(\mathbf{F}_q) \longrightarrow K_{2n-1}(\mathbf{F}_q)$$

is given by multiplication by p^n*.*

*3.1.1. **Corollary** (Quillen). *Let* p *be a prime.*

(3.1.1.a) *For all* $n \geq 1$ *we have* $K_{2n}(\overline{\mathbf{F}}_p) = 0$ *and*

$$K_{2n-1}(\overline{\mathbf{F}}_p) \cong \bigoplus_{\substack{\ell \neq p \\ \text{prime}}} \mathbf{Q}_\ell/\mathbf{Z}_\ell\,.$$

(3.1.1.b) *The automorphism of* $K_{2n-1}(\overline{\mathbf{F}}_p)$ *induced by the Frobenius automorphism is given by multiplication by* p^n*.*

(3.1.1.c) *If* $\mathbf{F}_q \subset \overline{\mathbf{F}}_p$ *is a subfield, then the canonical map*

$$K_*(\mathbf{F}_q) \longrightarrow K_*(\overline{\mathbf{F}}_p)^{\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_q)}$$

is an isomorphism.

4. The Adams Operations via λ-rings

In this section we recall the properties of the Adams operations on K-theory in the setting of λ-*rings* as the representation ring of a finite group also has the structure of a λ-ring, and in both cases Adams operations can be constructed using only the λ-ring structure, as well as an additional augmentation given by the dimension.

We motivate λ-rings by considering the extra structure on $\mathrm{VB}_{\mathbf{C}}(X)$ given by taking exterior powers.

4.1. **Observation** (Extra structure on $\mathrm{VB}_{\mathbf{C}}(X)$). Notice that if E and E' are complex vector bundles over a finite CW-complex X, we can perform the following constructions, which give extra structure to $\mathrm{VB}_{\mathbf{C}}(X)$.

(4.1.a) $\Lambda^0(E) = X \times \mathbf{C}$.

(4.1.b) $\Lambda^1(E) = E$.

(4.1.c) $\Lambda^n(E \oplus E') \cong \bigoplus_{i+j=n} \Lambda^i(E) \otimes \Lambda^j(E')$.

However, notice that these exterior power operations are *not* semiring homomorphisms.

4.2. **Observation.** Let G be a finite group and k a field. Notice that the set $\mathrm{Rep}_k(G)$ of *isomorphism classes* of finite-dimensional k-representations of G has the structure of a commutative semiring, with the additive structure given by direct sum and the multiplicative structure given by tensor product — this structure is analogus to the extra structure on $\mathrm{VB}_{\mathbf{C}}(X)$.

4.3. **Definition.** A λ-***structure*** on a commutative semiring R is a collection of set-maps $\{\lambda^n\colon R \longrightarrow R\}_{n\geq 0}$ called λ-***operations*** so that

(4.3.a) λ^0 is the constant map at 1.

(4.3.b) $\lambda^1 = \mathrm{id}_R$.

(4.3.c) The map $\lambda_t\colon R \longrightarrow R[\![t]\!]^\times$ given by $\lambda_t(r) \coloneqq \sum_{n\geq 0} \lambda^n(r)$ is a monoid map.

We call a semiring equipped with a λ-structure a λ-***semiring***.

4.4. **Remark.** Condition (4.3.c) is equivalent to saying that for all $r, r' \in R$ we have

$$\lambda^n(r + r') = \sum_{i+j=n} \lambda^i(r)\lambda^j(r') \,,$$

which is an abstraction of (4.1.c).

*4.5. **Remark** (on terminology)**.** The terminology in the literature on λ-rings is quite inconsistent. Some call what we have called a λ-ring a *pre-λ-ring*, and insist that λ-rings satisfy three additional properties. Two of these properties encode the expressions of exterior powers $\Lambda^n(\Lambda^n(V))$ and $\Lambda^n(V \otimes V')$ of vector spaces V and V' as integral polynomials in the exterior powers $\Lambda^1(V), \ldots, \Lambda^{mn}(V)$ and $\Lambda^1(V), \ldots, \Lambda^n(V), \Lambda^1(V'), \ldots, \Lambda^n(V')$, respectively. The other property is that $\lambda^k(1) = 0$ if $k > 1$, which encodes the fact that $\Lambda^k(L) = 0$ for $k > 1$ if L is a 1-dimensional vector space. Others take our convention and call a λ-ring with this extra structure a *special λ-ring*.

We have chosen this approach to simplify terminology, though we actually agree that a λ-ring really should satisfy these additional properties. Moreover, all of the λ-rings that we need to consider do satisfy these properties.

*4.6. **Remark.** The λ-operations in some sense are very unnatural as they are not ring homomorphisms. One might ask where these operations come from — the answer is *plethories* and the *plethistic algebra* of Borger and Wieland [4].

4.7. **Example.** The semirings $\mathrm{VB}_{\mathbf{C}}(X)$ and $\mathrm{Rep}_k(G)$ are both λ-semirings with the λ-operations given by taking exterior powers.

4.8. **Example.** The nonnegative integers **N** are a λ-semiring with λ-operations given by

$$\lambda^k(n) = \binom{n}{k} \,.$$

*4.9. **Example.** A **Q**-algebra A is a λ-ring with λ-operations given by

$$\lambda^k(a) = \frac{1}{k!} a(a-1)\cdots(a-k-1) \,.$$

4.10. **Notation.** Given a commutative monoid M, write M^{gp} for the group completion of M.

4.11. **Lemma.** *If R is a λ-semiring. Then R^{gp} has a canonical λ-ring structure with λ operations extending the λ-operations on R.*

Proof. Since the map $\lambda_t \colon R \longrightarrow R[\![t]\!]^\times$ is a monoid map and $R^{gp}[\![t]\!]^\times$ is a group completion for $R[\![t]\!]^\times$, by the universal property of the group completion, there exists a unique monoid map λ_t^{gp} making the square

$$\begin{tikzcd} R \arrow[r, "\lambda_t"] \arrow[d, hook] & R[\![t]\!]^\times \arrow[d, hook] \\ R^{gp} \arrow[r, dotted, "\lambda_t^{gp}"'] & R^{gp}[\![t]\!]^\times \end{tikzcd}$$

commute. The coefficents of λ_t^{gp} define a λ-ring structure on R^{gp} extending the λ-ring structure on R. □

4.12. **Example.** Applying Lemma 4.11 to Example 4.8, we see that the integers **Z** are a λ-ring with λ-operations given by

$$\lambda^k(n) = \binom{n}{k} \frac{1}{k!} n(n-1)\cdots(n-k-1) \,.$$

*4.13. **Remark** (Witt vectors). The abelian group $R[\![t]\!]^\times = 1 + tR[\![t]\!]$ naturally has the structure of a commutative ring, which we will not describe here, and is often called the ring of ***big Witt vectors***, and denoted by $\mathbf{W}(R)$. Moreover, $\mathbf{W}(R)$ is naturally a λ-ring, and the condition that a λ-ring R be a special λ ring can be concisely phrased by requiring that the group homomorphism $R \longrightarrow \mathbf{W}(R)$ be a λ-ring homomorphism.

4.14. **Example.** If X is a finite CW-complex, then $K^0(X)$ is a λ-ring with the λ-structure given by taking exterior powers.

4.15. **Example.** Let G be a finite group and k a field. The ***representation ring*** of G is the group completion $R_k(G) \coloneqq \mathrm{Rep}_k(G)^{gp}$. Hence $R_k(G)$ is a λ-ring. An element of $R_k(G)$ is called a ***virtual representation*** of G.

4.16. **Remark.** The category of λ-rings has objects λ-rings, and a morphism $\phi\colon R \longrightarrow R'$ is a ring homomorphism so that $\lambda^n_{R'} \circ \phi = \phi \circ \lambda^n_R$ for all $n \geq 0$.

4.17. **Definition.** Let R be a λ-ring. An ***augmentation*** of R is a λ-subring $A \subset R$ equipped with a λ-ring surjection $\varepsilon\colon R \longrightarrow A$ so that $\varepsilon|_A = \mathrm{id}_A$.

4.18. **Example.** Both $K^0(X)$ and $R_{\mathbf{C}}(G)$ have augmentations to $\mathbf{Z}$ given by taking (virtual) dimension. Explicitly, the λ-subrings of $K^0(X)$ and $R_{\mathbf{C}}(G)$ isomorphic to $\mathbf{Z}$ providing the augmentation are simply the subrings given by integer multiples of the identity in each case (i.e., the trivial virtual bundles and trivial virtual representations, respectively). The fact that these are actually λ-semiring homomorphisms follows from the fact that

$$\dim(\Lambda^k(V)) = \binom{\dim(V)}{k} .$$

4.19. **Definition** (Adams operations). Let (R, ε) be an augmented λ-ring. There exist natural set-maps $\psi^k\colon R \longrightarrow R$ for all $k \geq 0$, called ***Adams operations***, defined as follows. Let ψ_t denote the formal power series defined by

$$\psi_t(r) \coloneqq \varepsilon(r) - t\frac{d}{dt}\log(\lambda_{-t}(r)) .$$

Then $\psi^k(r)$ is the coefficent of t^k in $\psi_t(r)$.

4.20. **Theorem.** *For any finite CW-complex X and any finite group G, the Adams operations on $K^0(X)$ and $R_{\mathbf{C}}(G)$ satisfy the following properties.*

(4.20.a) *For all $k \geq 0$, the set-map ψ^k is a ring homomorphism.*
(4.20.b) *For all $k, \ell \geq 0$ we have $\psi^k\psi^\ell = \psi^{k\ell}$.*
(4.20.c) *If $\dim(L) = 1$, then $\psi^k(L) = L^{\otimes k}$.*

Idea of the proof. In the K-theory case, the idea is to use the splitting principle for K-theory. As it turns out, there is also a splitting principle for $R_{\mathbf{C}}(G)$ [2, Thm. 1.5(ii)] — the fundamental observation is to identify $R_{\mathbf{C}}(G)$ with the ring of complex virtual characters. There is a notion of a splitting principle for λ-rings, and if this is satisfied, then (4.20.a) and (4.20.b) hold [12, Ch. 2 Prop. 4.4] The last condition is obviously something more particular to these settings where we have a notion of dimension. □

4.21. **Remark.** The Adams operations on $R_{\mathbf{C}}(G)$ induce operations on the characters given by $\psi^k(\chi(g)) = \chi(g^k)$, where χ is a character and $g \in G$.

4.22. **Observation.** By the naturality of the Adams operation, ψ^k restricts to an operation $\widetilde{K}^0 \longrightarrow \widetilde{K}^0$ which we also denote by ψ^k

4.23. **Recollection.** Let n be a nonnegative integer. Then $\widetilde{K}^0(S^{2n}) \cong \mathbf{Z}$.

In our computation of the K-theory of finite fields we need the following fact about the Adams operations.

4.24. **Lemma.** *Let n be a nonnegative integer. The Adams operation $\psi^k \colon \widetilde{K}^0(S^{2n}) \longrightarrow \widetilde{K}^0(S^{2n})$ is given by multiplication by k^n.*

Proof sketch. The proof is elementary — for $n = 1$, it follows immediately by looking at the canonical generator of $\widetilde{K}^0(S^2)$, and the general case follows by induction using Bott periodicity. A complete argument can be found in [7, Prop. 2.21]. □

5. The Brauer Lift

Now we explain and exploit an incredible relation between K-theory and representation theory of finite groups via the Adams operations. Throughout G denotes a finite group, but we are particularly interested in the case when $G = \mathrm{GL}_n(\mathbf{F}_q)$.

5.1. **Definition.** Let G be a finite group. Define a ring homomorphism

$$\phi_G \colon R_{\mathbf{C}}(G) \longrightarrow K^0(BG)$$

first on $\mathrm{Rep}_{\mathbf{C}}(G)$ by sending an isomorphism class $[V]$ of representations to the class of the balanced product $EG \times_G V$, and then extending to $R_{\mathbf{C}}(G)$ via the universal property of the group completion.

5.2. **Observation.** Since the λ-operations are defined on $R_{\mathbf{C}}(G)$ and $K^0(BG)$ via taking exterior powers, it is not hard to believe that ϕ_G is a morphism of λ-rings, and indeed this is true. This morphism of λ-rings respects the augmentations of $R_{\mathbf{C}}(G)$ and $K^0(BG)$ defined in Example 4.18, so by the naturality of the Adams operations, for all $q \geq 0$, the square

$$\begin{tikzcd} R_{\mathbf{C}}(G) \arrow[r, "\phi_G"] \arrow[d, "\psi^q"'] & K^0(BG) \arrow[d, "\psi^q"] \\ R_{\mathbf{C}}(G) \arrow[r, "\phi_G"'] & K^0(BG) \end{tikzcd}$$

commutes.

Composing with the homomorphism

$$K^0(BG) \cong [BG, \mathbf{Z} \times BU] \longrightarrow [BG, BU] \cong \widetilde{K}^0(BG)$$

induced by the inclusion $BU \hookrightarrow \mathbf{Z} \times BU$ as $\{0\} \times BU$, we get a homomorphism

$$R_{\mathbf{C}}(G) \longrightarrow \widetilde{K}^0(BG) ,$$

again natural with respect to the Adams operations.

5.3. **Convention.** Fix, once and for all, an embedding $\rho \colon \overline{\mathbf{F}}_q^\times \hookrightarrow \mathbf{C}^\times$.

5.4. **Remark.** Everything from here on will depend on ρ. This dependence will not be problematic and is well understood through the lens of étale cohomology.

5.5. **Notation.** Let G be a finite group and V a finite-dimensional $\overline{\mathbf{F}}_q$-representation of G. For an element $g \in G$, write $S_V(g)$ for the set of eigenvalues of g. Given an eigenvalue $\lambda \in S_V(g)$, write $\mu(\lambda)$ for the multiplicity of λ.

5.6. **Definition.** Let G be a finite group and $\chi_1, \ldots, \chi_n$ its distinct irreducible complex characters. A ***virtual character*** $\chi\colon G \longrightarrow \mathbf{C}$ is a class function so that $\phi = m_1\chi_1 + \cdots + m_n\chi_n$ for some $m_1, \ldots, m_n \in \mathbf{Z}$.

5.7. **Definition.** Let G be a finite group and V a finite-dimensional $\overline{\mathbf{F}}_q$-representation of G. The ***Brauer character*** of V is the function $\chi_V\colon G \longrightarrow \mathbf{C}$ defined by

$$\chi_V(g) \coloneqq \sum_{\lambda \in S_V(g)} \mu(\lambda)\rho(\lambda)\,.$$

5.8. **Theorem** (Green). *Let G be a finite group and V a finite-dimensional $\overline{\mathbf{F}}_q$-representation of G. The Brauer character χ_V is the character of a unique virtual complex representation V^{br}, called the **Brauer lifting** of V.*

5.9. **Observation.** The Brauer character χ_V is clearly additive in V, that is $\chi_{V\oplus W} = \chi_V + \chi_W$, hence the Brauer lifting is additive as well: $(V \oplus W)^{br} = V^{br} \oplus W^{br}$. Thus the Brauer lifting defines a homomorphism

$$(-)^{br}\colon R_{\overline{\mathbf{F}}_q}(G) \longrightarrow R_{\mathbf{C}}(G)\,.$$

5.10. **Observation.** Extension of scalars defines a homomorphism

$$- \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q\colon R_{\mathbf{F}_q}(G) \longrightarrow R_{\overline{\mathbf{F}}_q}(G)\,.$$

For $V \in R_{\mathbf{F}_q}(G)$, write

$$\overline{V} \coloneqq V \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q\,.$$

Since V is a representation over $\mathbf{F}_q$, the eigenvalues of g acting on $\overline{V}$ are stable under the Frobenius automorphism $x \longmapsto x^q$. Then since ψ^q acts on characters by $\psi^q\chi(g) = \chi(g^q)$, the Adams operation on $R_{\mathbf{C}}(G)$ fixes the Brauer lift of $\overline{V}$, hence we get a homomorphism

$$(-)^{br} \circ (- \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q)\colon R_{\mathbf{F}_q}(G) \longrightarrow R_{\mathbf{C}}(G)^{\psi^q}\,,$$

called the ***Brauer lift***. Composing this with the homomorphism from Observation 5.2 and taking fixed points, we have produced a map

$$\beta_G\colon R_{\mathbf{F}_q}(G) \longrightarrow \widetilde{K}^0(BG)^{\psi^q} \cong [BG, BU]^{\psi^q}\,.$$

6. The space $F\psi^q$ and computing the K-theory of finite fields

6.1. **Convention.** From now on, let q be a fixed nonnegative integer. (We really only care about the case that q is a prime power.)

We have the Adams operations $\psi^q\colon \widetilde{K}^0 \longrightarrow \widetilde{K}^0$ on complex K-theory. Since $\widetilde{K}$ is represented by BU, we would like these to be represented by endomorphisms of BU by just applying the Yoneda lemma, but there is a slight problem. The space BU is not a finite CW-complex, and K-theory is only defined for finite CW-complexes. However, by analyzing an explicit cell structure on BU and using the Milnor exact sequence, it is possible to show that n-ary operations on $\widetilde{K}$ are representable by maps $BU^n \longrightarrow BU$.

6.2. **Notation.** We abusively write $\psi^q\colon BU \longrightarrow BU$ for a map representing the Adams operation ψ^q on $\widetilde{K}^0$.

6.3. **Definition.** The space $F\psi^q$ is the pullback

$$\begin{tikzcd} F\psi^q \arrow[r, "h"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(I, BU) \arrow[d, "(\mathrm{ev}_0, \mathrm{ev}_1)"] \\ BU \arrow[r, "(\mathrm{id}_{BU}, \sigma_q)"'] & BU \times BU . \end{tikzcd}$$

Hence a point the space $F\psi^q$ can be viewed as a pair (x, γ), where $x \in BU$ and γ is a path from x to $\psi^q(x)$. In this sense $F\psi^q$ is a homotopy-theoretic version of the space of fixed points of ψ^q.

6.4. **Recollection.** Recall that a ***simple*** space is a space with abelian fundamental group and the action of the fundamental group on the higher homotopy groups is trivial.

6.5. **Theorem** (Dror [5]). *If X and Y are simple spaces and $f\colon X \longrightarrow Y$ induces an isomorphism on integral homology, then f is a weak homotopy equivalence.*

6.6. **Goal.** The goal is to construct a map

$$\theta\colon B\mathrm{GL}(\mathbf{F}_q) \longrightarrow F\psi^q$$

that induces an isomorphism on integral homology. The universal property of the plus construction will then give a map $\theta^+\colon B\mathrm{GL}(\mathbf{F}_q)^+ \longrightarrow F\psi^q$, which is again an isomorphism on integral homology. After showing that $B\mathrm{GL}(\mathbf{F}_q)^+$ and $F\psi^q$ are simply, a simple application of Theorem 6.5 and Whitehead's theorem shows that θ^+ is a homotopy equivalence.

6.7. **Lemma.** *We have that $F\psi^q \simeq \mathrm{hofib}(1 - \psi^q)$.*

6.8. **Lemma.** *The space $F\psi^q$ is simple and we have $\pi_{2n}(F\psi^q) = 0$ and $\pi_{2n-1}(F\psi^q) = \mathbf{Z}/(q^n-1)$.*

Proof sketch. By Lemma 6.7, we have a long exact sequence in homotopy

$$\cdots \longrightarrow \pi_m(BU) \xrightarrow{(1-\psi^q)^*} \pi_m(BU) \xrightarrow{\partial} \pi_{m-1}(F\psi^q) \longrightarrow \pi_{m-1}(BU) \longrightarrow \cdots .$$

By Bott periodicity, $\pi_{2n-1}(BU) = 0$ and $\pi_{2n}(BU) = \widetilde{K}^0(S^{2n}) \cong \mathbf{Z}$. In the latter case, by Lemma 4.24, the map $1 - \psi^q$ is given by multiplication by $1 - q^n$. This immediately gives the computations of the homotopy groups of $F\psi^q$.

The fact that $F\psi^q$ is simple follows immediately from a general result about fibrations, using the fact that BU is simply connected. □

6.9. **Lemma.** *Suppose that X is a CW-complex so that $[X, U] = 0$. Then*

$$\phi_\star\colon [X, F\psi^q] \longrightarrow [X, BU]^{\psi^q}$$

is an isomorphism.

Proof sketch. This lemma follows almost immediately from the universal property of the pullback $F\psi^q$. □

6.10. **Lemma.** *Suppose that G is a finite group. Then $[BG, BU]^{\psi^q} \cong [BG, F\psi^q]$.*

The proof of Lemma 6.10 follows almost immediately from the Atiyah–Segal completion theorem [1, Thm. 2.1] and Lemma 6.9. However, given what we have covered, stating the Atiyah–Segal completion theorem is a bit beyond the scope of the lecture, so we will just take Lemma 6.10 for granted.

6.11. **Construction.** We construct the map θ by constructing maps $\theta_n \colon B\mathrm{GL}_n(\mathbf{F}_q) \longrightarrow F\psi^q$, then passing to the colimit. To construct the maps θ_n, we use the Brauer lift with $G = \mathrm{GL}_n(\mathbf{F}_q)$. By Lemma 6.10, the Brauer lift gives a map

$$\beta_{\mathrm{GL}_n(\mathbf{F}_q)} \colon R_{\mathbf{F}_q}(\mathrm{GL}_n(\mathbf{F}_q)) \longrightarrow [B\mathrm{GL}_n(\mathbf{F}_q), BU]^{\psi^q} \cong [B\mathrm{GL}_n(\mathbf{F}_q), F\psi^q] \ .$$

Let $\theta_n \colon B\mathrm{GL}_n(\mathbf{F}_q) \longrightarrow F\psi^q$ be a map representing the homotopy class given by the image of the standard representation of $\mathrm{GL}_n(\mathbf{F}_q)$ on $\mathbf{F}_q$. These maps θ_n are compatible the with formation of the colimit $\mathrm{colim}_n\, B\mathrm{GL}_n(\mathbf{F}_q)$, hence determine a homotopy class of maps $\theta \colon B\mathrm{GL}(\mathbf{F}_q) \longrightarrow F\psi^q$. Moreover, since $[B\mathrm{GL}(\mathbf{F}_q), BU] \cong \lim_n [B\mathrm{GL}_n(\mathbf{F}_q), BU]$, this class is unique.

The bulk of Quillen's paper [11, §§2–6, §8, §9, §11] is dedicated to computing the homology and cohomology of $B\mathrm{GL}(\mathbf{F}_q)$ and $F\psi^q$ with various coefficients to show that the map θ is an integral homology equivalence.

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