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THE HOMOTOPY-INVARIANCE OF CONSTRUCTIBLE SHEAVES

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(*communicated by Emily Riehl*)

Abstract

In this paper we show that the functor sending a stratified topological space S to the ∞-category of constructible (hyper)sheaves on S with coefficients in a large class of presentable ∞-categories is homotopy-invariant. To do this, we first establish a number of results for locally constant (hyper)sheaves. For example, if X is a locally weakly contractible topological space and $\mathcal{E}$ is a presentable ∞-category, then we give a concrete formula for the constant hypersheaf functor $\mathcal{E} \to \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$, implying that the constant hypersheaf functor is a right adjoint, and is fully faithful if X is also weakly contractible. It also lets us prove a general monodromy equivalence and categorical Künneth formula for locally constant hypersheaves.

0. Introduction

A classical result from sheaf theory says that the functor $S \mapsto \mathrm{LC}(S;\mathbf{Set})$ sending a topological space S to the category of locally constant sheaves of sets on S is homotopy-invariant. More generally, if P is a poset then the functor

$$S \mapsto \mathrm{Cons}_P(S;\mathbf{Set})$$

sending a P-stratified topological space S to the category of sheaves of sets on S that are constructible with respect to the stratification $S \to P$ is invariant under stratified homotopy equivalences. Lurie's work on the topological exodromy equivalence (see [**16**, Theorems A.1.15 & A.4.19]) generalizes these results by considering sheaves with values in the ∞-category of *spaces*, provided that we restrict to the following classes of well-behaved (stratified) topological spaces:

(1) For locally constant sheaves, we take topological spaces S that are *locally of singular shape*.

(2) For constructible sheaves, we take stratified topological spaces $S \to P$ for which the poset P is *Noetherian*, the stratification is *conical*, S is *paracompact*, and all of the strata of S are *locally of singular shape*.

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The goal of this paper is to establish the homotopy-invariance result for the ∞-categories of locally constant and constructible sheaves with coefficients in the ∞-category of spaces *removing all of the above hypotheses*.

In the higher-categorical world, alongside with sheaves, it is often important to also consider hypersheaves. Depending on the situation, one is better behaved than the other (see the discussion in [**15**, §6.5.4]). In the main body of the paper, we prove two versions of the homotopy-invariance theorem: one in the setting of hypersheaves and one in the setting of sheaves. The hypersheaf one is stronger, requiring fewer assumptions than its sheaf-theoretic counterpart. The precise statements are given later in this introduction, but the main advantages can be summarized as follows:

(1) Working with hypersheaves, we establish invariance not only with respect homotopy equivalences, but to a large class of *weak* homotopy equivalences (a result that seems new even for sheaves of *sets*).

(2) Working with hypersheaves, we can drop the Noetherianity assumption on the poset P.

We expect both of these statements to fail in the sheaf-theoretic setting. Furthermore, both facts have interesting consequences. The first is needed, *at this level of generality*, in the companion paper of Porta–Teyssier [**21**] concerning a strengthening of the exodromy equivalence of [**16**, Theorem A.9.3]. The second lets us apply the homotopy-invariance theorem to key examples like infinite Grassmannians or the *Ran space* of a manifold [**3**, §3.7], [**6**], [**7**], [**15**, §5.5.1], [**18**], whose natural stratification is not Noetherian. This was one of the motivations behind Lejay's work on the exodromy equivalence [**14**].

Finally, we do not limit ourselves to sheaves of spaces. Rather, our results apply to more general presentable (not necessarily compactly generated) ∞-categories: the methods of this paper are explicit enough that we can handle any stable presentable ∞-category, and any ∞-topos without any added difficulty.

Statement of results

Before giving the precise statements of the main homotopy-invariance results of this paper, let us be precise about what we mean by *homotopy-invariance*. Fix a poset P, that we regard as a topological space via the Alexandroff topology (where the open subsets are the upwards-closed subsets, see Notation 5.1). A *P-stratified topological space* is the data of a topological space S together with a continuous map $S \to P$. When $P = *$, a P-stratified space is just a topological space. Given $S \in \mathbf{Top}_{/P}$ and $X \in \mathbf{Top}$, we regard $S \times X$ as a P-stratified space via the projection $S \times X \to S \to P$. Consider the following definition:

Definition 0.1. Let P be a poset. A functor $C\colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is *homotopy-invariant* if for each P-stratified space S, the functor

$$C(\mathrm{pr}_S)\colon C(S) \to C(S \times [0,1])$$

induced by the projection $\mathrm{pr}_S\colon S \times [0,1] \to S$ is an equivalence of ∞-categories. A functor $C\colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is *strongly homotopy-invariant* if for each P-stratified space S and each weakly contractible and locally weakly contractible[1] topological space X,

[1] Starting from here on we shorten "weakly contractible and locally weakly contractible" to *wclwc*.

the induced functor $C(\mathrm{pr}_S)\colon C(S) \to C(S \times X)$ is an equivalence of ∞-categories.

Let $\mathcal{E}$ be a presentable ∞-category and S a topological space. We write $\mathrm{LC}(S;\mathcal{E})$ for the ∞-category of locally constant $\mathcal{E}$-valued sheaves on S, and write $\mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E})$ for the hypersheaf variant of this ∞-category (see §§ 1.1 and 1.3). Beware that, in general, the notions of local constancy for sheaves and hypersheaves are not the same and $\mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E})$ is not a subcategory of $\mathrm{LC}(S;\mathcal{E})$.

Theorem 0.2 (Theorem 3.17 & Corollary 4.13)**.** *The functors*

$$\mathrm{LC}(-;\mathcal{E}),\ \mathrm{LC}^{\mathrm{hyp}}(-;\mathcal{E})\colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}_\infty$$

are homotopy-invariant. Moreover, $\mathrm{LC}^{\mathrm{hyp}}(-;\mathcal{E})$ *is strongly homotopy invariant*

Passing to global sections, Theorem 0.2 implies that cohomology with coefficients in a locally constant sheaf valued in any presentable ∞-category is homotopy-invariant.

Remark 0.3. The same kind of techniques involved in the proof of Theorem 0.2 allow to show that the functor $\mathrm{LC}^{\mathrm{hyp}}(-;\mathcal{E})$ inverts all weak homotopy equivalences between locally weakly contractible topological spaces (see Observation 3.8). However, the functors $\mathrm{LC}(-;\mathcal{E})$ and $\mathrm{LC}^{\mathrm{hyp}}(-;\mathcal{E})$ do *not* invert all weak homotopy equivalences between *arbitrary* topological spaces: sheaf cohomology with constant coefficients is not an invariant of the weak homotopy type of a topological space. Indeed, for paracompact spaces, Čech cohomology and sheaf cohomology agree [**9**, Théorème 5.10.1]. Now, the Warsaw circle is weakly contractible, but the quotient map from it to the circle induces an isomorphism on Čech cohomology, hence sheaf cohomology. Note that this doesn't fall into the setting of Theorem 0.2: the Warsaw circle is not even locally path-connected.

Fix a poset P. Given a P-stratified topological space $S \to P$, we write $\mathrm{Cons}_P(S;\mathcal{E})$ for the ∞-category of constructible $\mathcal{E}$-valued sheaves on S, and write $\mathrm{Cons}_P^{\mathrm{hyp}}(S;\mathcal{E})$ for the hypersheaf variant of this ∞-category (see § 5.1 for precise definitions). Since constructible sheaves are locally constant along a stratification, as long as the poset P and coefficients $\mathcal{E}$ allow to check equivalences after pulling back to strata, then Theorem 0.2 implies that constructible sheaves are homotopy-invariant. We offer two ways of checking this:

Theorem 0.4 (Corollaries 5.13 and 5.19)**.** *Consider the functors*

$$\mathrm{Cons}_P(-;\mathcal{E}),\ \mathrm{Cons}_P^{\mathrm{hyp}}(-;\mathcal{E})\colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty.$$

(0.4.1) *If $\mathcal{E}$ is compactly generated, then the functor* $\mathrm{Cons}_P^{\mathrm{hyp}}(-;\mathcal{E})$ *is strongly homotopy-invariant.*

(0.4.2) *If P is Noetherian and $\mathcal{E}$ is compactly generated, stable, or an ∞-topos, then the functor* $\mathrm{Cons}_P(-;\mathcal{E})$ *is homotopy-invariant, and* $\mathrm{Cons}_P^{\mathrm{hyp}}(-;\mathcal{E})$ *is strongly homotopy-invariant.*

Remark 0.5. Theorem 0.4 holds under much weaker assumptions than Lurie's exodromy equivalence [**16**, Theorem A.9.3]. For instance, it holds for stratified spaces that are not necessarily conical.

Again, passing to global sections, Theorem 0.4 implies that (under the above hypotheses) sheaf cohomology with coefficients in a constructible sheaf is homotopy-invariant. Also note that Theorem 0.4 generalizes the following existing results about the homotopy-invariance of constructible sheaves:

(1) In the setting of topologically stratified spaces in the sense of Goresky–MacPherson [**10**, §1.1], Treumann showed that constructible sheaves with values in the 2-category $\mathbf{Cat}_1$ of 1-categories is homotopy-invariant [**24**, Theorem 3.11].

(2) When P is Noetherian, Clausen–Ørsnes Jansen proved that $\mathrm{Cons}_P(-;\mathbf{Spc})$ is homotopy-invariant [**8**, Proposition 3.2]. Our proof for $\mathrm{Cons}_P(-;\mathcal{E})$ is a mild extension of their work.

One of the key ingredients of the proofs of Theorems 0.2 and 0.4 is the notion of *topological family of locally hyperconstant hypersheaves*. Concretely, if X is a wclwc topological space (e.g. $X = [0,1]$), and S is any topological space, we are led to consider the full subcategory

$$\mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E}) \subset \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$$

spanned by those hypersheaves that are, locally on X, pulled back from hypersheaves on S (see Definition 1.17 for the precise definition). When $\mathcal{E} = \mathbf{Spc}$, this definition recovers the usual notion of *foliated* hypersheaf (see § 2.4), but it is better behaved for general coefficients. The main theorem concerning these objects is the following:

Theorem 0.6 (Proposition 2.5 and Theorem 2.12)**.** *Let X be a locally weakly contractible topological space and let $\mathcal{E}$ be a presentable ∞-category. Then for every topological space S, the pullback functor*

$$\mathrm{pr}_S^{*,\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$$

admits a left adjoint. Moreover, if X is also weakly contractible, then $\mathrm{pr}_S^{*,\mathrm{hyp}}$ is fully faithful with essential image $\mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E})$.

It is easy to explain the idea behind this theorem when $S = *$ and $\mathcal{E} = \mathbf{Spc}$. Let $\Gamma_X \colon X \to *$ be the unique map, and let

$$\Pi_\infty \colon \mathrm{Sh}^{\mathrm{hyp}}(X) \to \mathbf{Spc}$$

be the left Kan extension of the functor sending an open $U \subset X$ to its underlying homotopy type $\Pi_\infty(U)$. For formal reasons, Π_∞ admits a right adjoint, that we denote Π^∞. Given $K \in \mathbf{Spc}$, the hypersheaf $\Pi^\infty(K)$ is given by the assignment

$$U \mapsto \Pi^\infty(K)(U) := \mathrm{Map}_{\mathbf{Spc}}(\Pi_\infty(U), K).$$

There is a natural comparison map $\Gamma_X^{*,\mathrm{hyp}} \to \Pi^\infty$, and the fact that we are working in the hypercomplete setting and that X is locally weakly contractible implies that this map is an equivalence (see Proposition 2.5). Note that if X is also weakly contractible, then the full faithfulness part follows from the assumption that $\Pi_\infty(X) \simeq *$.

Besides Theorems 0.2 and 0.4, Theorem 0.6 has many other consequences; we discuss them in § 3. Among them are: a general form of the monodromy equivalence (Corollary 3.6), a Künneth formula for locally hyperconstant hypersheaves (Corollary 3.14), and the comparison between sheaf and singular cohomology on locally weakly contractible spaces (Corollary 3.29). We also establish the following handy recognition criterion:

Corollary 0.7 (Proposition 3.1)**.** *Let X be a locally weakly contractible topological space and let $\mathcal{E}$ be a presentable ∞-category. For a hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$, the following statements are equivalent:*

(0.7.1) *The hypersheaf F is locally hyperconstant.*

(0.7.2) *For every pair of weakly contractible open subsets $U \subset V$ of X, the restriction map $F(V) \to F(U)$ is an equivalence in $\mathcal{E}$.*

In particular, it immediately follows that if X is locally weakly contractible, then locally hyperconstant hypersheaves are closed under arbitrary limits in $\mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$.

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1. Sheaf-theoretic background

In this section we explain our sheaf-theoretic conventions and notation as well as recall some background on hypersheaves and locally (hyper)constant (hyper)sheaves.

1.1. Background on sheaves & hypersheaves

We fix an ∞-site $(\mathcal{C}, \tau)$ and a presentable ∞-category $\mathcal{E}$.

Notation 1.1. We write

$$\mathrm{PSh}(\mathcal{C};\mathcal{E}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{E})$$

for the ∞-category of $\mathcal{E}$-valued presheaves on $\mathcal{C}$. We also write $\mathrm{Sh}_\tau(\mathcal{C};\mathcal{E}) \subset \mathrm{PSh}(\mathcal{C};\mathcal{E})$ for the full subcategory spanned by $\mathcal{E}$-valued presheaves that satisfy τ-descent. When $\mathcal{E} = \mathbf{Spc}$, we simply write

$$\mathrm{PSh}(\mathcal{C}) := \mathrm{PSh}(\mathcal{C};\mathbf{Spc}) \qquad \text{and} \qquad \mathrm{Sh}_\tau(\mathcal{C}) := \mathrm{Sh}_\tau(\mathcal{C};\mathbf{Spc}).$$

1.2. The ∞-categories $\mathrm{PSh}(\mathcal{C};\mathcal{E})$ and $\mathrm{Sh}_\tau(\mathcal{C};\mathcal{E})$ are naturally identified with the tensor products of presentable ∞-categories $\mathrm{PSh}(\mathcal{C}) \otimes \mathcal{E}$ and $\mathrm{Sh}_\tau(\mathcal{C}) \otimes \mathcal{E}$ [**17**, Remark 1.3.1.6 & Proposition 1.3.1.7]. We refer the reader to [**16**, §4.8.1] for a thorough treatment of the tensor product of presentable ∞-categories. As both points of view have their own advantages, in this paper we use both descriptions interchangeably.

1.3. Crucial to the current paper is the notion of *hypersheaf*. When $\mathcal{E}$ is the ∞-category of spaces, hypersheaves can be defined intrinsically in the ∞-topos $\mathrm{Sh}_\tau(\mathcal{C})$ as *hypercomplete objects*, that is, objects that are local with respect to ∞-connected morphisms. Hypersheaves thus form a full subcategory $\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) \subset \mathrm{Sh}_\tau(\mathcal{C})$. It is then possible to *define* hypersheaves with coefficients in $\mathcal{E}$ as the tensor product

$$\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C};\mathcal{E}) := \mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) \otimes \mathcal{E}.$$

Each of the inclusions

$$\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C}) \qquad \text{and} \qquad \mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C}) \subset \mathrm{Sh}_\tau(\mathcal{C})$$

admits a left adjoint. We refer to both left adjoints as the *hypercompletion functors*, and we denote them by $(-)^{\mathrm{hyp}}$. Functoriality of the tensor product of presentable ∞-categories produces functors

$$(-)^{\mathrm{hyp}} \colon \mathrm{PSh}(\mathcal{C};\mathcal{E}) \to \mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C};\mathcal{E}) \qquad \text{and} \qquad (-)^{\mathrm{hyp}} \colon \mathrm{Sh}_\tau(\mathcal{C};\mathcal{E}) \to \mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C};\mathcal{E}).$$

Both these functors still admit fully faithful right adjoints. We refer the reader unfamiliar with hypercomplete objects and hypercompletion to [**15**, §§6.5.2–6.5.4] or [**4**, §3.11] for further reading on the subject.

1.4. If there exists an integer $n \geqslant 0$ such that $\mathcal{E}$ is an n-category, then $\mathrm{Sh}_\tau^{\mathrm{hyp}}(\mathcal{C};\mathcal{E}) = \mathrm{Sh}_\tau(\mathcal{C};\mathcal{E})$ [**15**, Lemma 6.5.2.9] [**16**, Example 4.8.1.22]. In particular, every sheaf of sets is a hypersheaf.

Notation 1.5. Let S be a topological space. We write $\mathrm{Open}(S)$ the poset of open subsets of S, ordered by inclusion. We regard $\mathrm{Open}(S)$ as a site with the covering families given by open covers. We write

$$\mathrm{PSh}(S;\mathcal{E}) := \mathrm{PSh}(\mathrm{Open}(S);\mathcal{E}), \quad \mathrm{Sh}(S;\mathcal{E}) := \mathrm{Sh}(\mathrm{Open}(S);\mathcal{E}),$$
$$\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) := \mathrm{Sh}^{\mathrm{hyp}}(\mathrm{Open}(S);\mathcal{E}).$$

1.6. Sheaves and hypersheaves on topological spaces coincide in many situations in which homotopy-invariance is a well-behaved notion. For example, the ∞-topos of sheaves on a topological space admitting a CW structure is hypercomplete [**12**].

Recollection 1.7. Let S be a topological space. Then the stalk functors

$$\{s^* \colon \mathrm{Sh}^{\mathrm{hyp}}(S) \to \mathrm{Sh}(\{s\}) \simeq \mathbf{Spc}\}_{s\in S}$$

are jointly conservative [**16**, Lemma A.3.9]. Since the stalk functors are left exact, [**11**, Lemma 2.8] shows that for any compactly generated ∞-category $\mathcal{E}$, the stalk functors

$$\{s^* \colon \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}(\{s\};\mathcal{E}) \simeq \mathcal{E}\}_{s\in S}$$

are jointly conservative.

1.8. Let S be a topological space and $\mathcal{E}$ a compactly generated ∞-category. Then the subcategory $\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \subset \mathrm{Sh}(S;\mathcal{E})$ is the localization obtained by inverting all morphisms that induce equivalences on stalks.

1.2. Hypersheaves and bases

Definition 1.9. Let $(\mathcal{C}, \tau)$ be an ∞-site. A *basis* of $(\mathcal{C}, \tau)$ is a full subcategory $\mathcal{B}$ of $\mathcal{C}$ such that every object $U \in \mathcal{C}$ admits a τ-covering $\{U_i\}_{i \in I}$ where for each $i \in I$, we have $U_i \in \mathcal{B}$.

Example 1.10. Let S and X be topological spaces. Write

$$\mathrm{Open}_{\times}(S \times X) \subset \mathrm{Open}(S \times X)$$

for the subposet spanned by the open subsets of the form $V \times U$, where $V \in \mathrm{Open}(S)$ and $U \in \mathrm{Open}(X)$. Then $\mathrm{Open}_{\times}(S \times X)$ is a basis of $\mathrm{Open}(S \times X)$. We write

$$\mathrm{Open}_{\mathrm{all,ctr}}(S \times X) \subset \mathrm{Open}_{\times}(S \times X)$$

for the subposet spanned by the open subsets of the form $V \times U$, where U is a weakly contractible open subset of X. When $S = *$, we simply write $\mathrm{Open}_{\mathrm{ctr}}(X)$ instead of $\mathrm{Open}_{\mathrm{all,ctr}}(* \times X)$. If X is locally weakly contractible, then $\mathrm{Open}_{\mathrm{all,ctr}}(S \times X)$ is also basis of $\mathrm{Open}(S \times X)$.

Let $(\mathcal{C}, \tau)$ be an ∞-site and $\mathcal{B}$ be a basis for $(\mathcal{C}, \tau)$. Write $j \colon \mathcal{B}^{\mathrm{op}} \hookrightarrow \mathcal{C}^{\mathrm{op}}$ for the inclusion. Right Kan extension along j defines a fully faithful functor

$$j_* \colon \mathrm{PSh}(\mathcal{B}; \mathcal{E}) \hookrightarrow \mathrm{PSh}(\mathcal{C}; \mathcal{E})$$

with left adjoint $j^* \colon \mathrm{PSh}(\mathcal{C}; \mathcal{E}) \to \mathrm{PSh}(\mathcal{B}; \mathcal{E})$ given by restriction of presheaves.

Definition 1.11. We will say that an $\mathcal{E}$-valued presheaf $F \in \mathrm{PSh}(\mathcal{B}; \mathcal{E})$ on $\mathcal{B}$ is a *τ-hypersheaf* if $j_*(F)$ belongs to $\mathrm{Sh}_{\tau}^{\mathrm{hyp}}(\mathcal{C}; \mathcal{E})$. We write $\mathrm{Sh}_{\tau}^{\mathrm{hyp}}(\mathcal{B}; \mathcal{E}) \subset \mathrm{PSh}(\mathcal{B}; \mathcal{E})$ for the full subcategory spanned by τ-hypersheaves.

The key fact we need is that hypersheaves on a site and a basis agree:

Proposition 1.12 ([2, Theorem A.6] [4, Proposition 3.12.11])**.** *Let $(\mathcal{C}, \tau)$ be an ∞-site and $\mathcal{B} \subset \mathcal{C}$ a basis. Then:*

(1.12.1) *For every $F \in \mathrm{Sh}_{\tau}^{\mathrm{hyp}}(\mathcal{C}; \mathcal{E})$, the unit transformation $u \colon F \to j_* j^*(F)$ is an equivalence.*

(1.12.2) *The functor $j_* \colon \mathrm{Sh}_{\tau}^{\mathrm{hyp}}(\mathcal{B}; \mathcal{E}) \to \mathrm{Sh}_{\tau}^{\mathrm{hyp}}(\mathcal{C}; \mathcal{E})$ is an equivalence with inverse given by the presheaf-theoretic restriction j^*.*

Remark 1.13. Let $F \in \mathrm{PSh}(\mathcal{C}; \mathcal{E})$. It follows directly from Proposition 1.12 that if $j^*(F)$ is a hypersheaf in the sense of Definition 1.11, then the unit $F \to j_* j^*(F)$ exhibits $j_* j^*(F)$ as hypersheafification of F.

1.3. Background on locally (hyper)constant sheaves

We limit our discussion of locally (hyper)constancy and the functoriality of sheaves to the setting of topological spaces. Fix a presentable ∞-category $\mathcal{E}$.

Recollection 1.14. Let $f \colon X \to Y$ be a map of topological spaces. We write

$$f_* \colon \mathrm{PSh}(X; \mathcal{E}) \to \mathrm{PSh}(Y; \mathcal{E})$$

for the *pushforward* functor defined by the formula $f_*(F)(V) := F(f^{-1}(V))$. Recall that the pushforward functor f_* carries sheaves to sheaves and hypersheaves to hypersheaves (for the latter statement, see the proof of [15, Proposition 6.5.2.13]). We write

$$f^{-1}\colon \mathrm{PSh}(Y) \to \mathrm{PSh}(X)$$

for *presheaf pullback* functor; f^{-1} is the left adjoint to $f_*\colon \mathrm{PSh}(Y;\mathcal{E}) \to \mathrm{PSh}(X;\mathcal{E})$. In general, the functor f^{-1} preserves neither sheaves nor hypersheaves. We write

$$f^*\colon \mathrm{Sh}(Y;\mathcal{E}) \to \mathrm{Sh}(X;\mathcal{E}) \qquad \text{and} \qquad f^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(Y;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$$

for the composites of $f^{-1}\colon \mathrm{Sh}(Y;\mathcal{E}) \to \mathrm{PSh}(X;\mathcal{E})$ with (hyper)sheafification. It follows formally that $f^{*,\mathrm{hyp}} \simeq (-)^{\mathrm{hyp}} \circ f^*$. By construction, there are adjunctions $f^* \dashv f_*$ and $f^{*,\mathrm{hyp}} \dashv f_*$.

Notation 1.15. If $f\colon X \hookrightarrow Y$ is the inclusion of a subspace, we simply write

$$(-)|_X := f^* \qquad \text{and} \qquad (-)|_X^{\mathrm{hyp}} := f^{*,\mathrm{hyp}}.$$

If the space-valued sheaf pullback functor $f^*\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ admits a left adjoint, then for every presentable ∞-category $\mathcal{E}$, the pullback functor $f^*\colon \mathrm{Sh}(Y;\mathcal{E}) \to \mathrm{Sh}(X;\mathcal{E})$ carries hypersheaves to hypersheaves [16, Lemma A.2.6]. In particular, if $U \subset Y$ is an open subset, then the functor $(-)|_U\colon \mathrm{Sh}(Y;\mathcal{E}) \to \mathrm{Sh}(U;\mathcal{E})$ carries hypersheaves to hypersheaves.

Notation 1.16. Let S and X be topological spaces. We denote by

$$\mathrm{pr}_S\colon S \times X \to S \qquad \text{and} \qquad \mathrm{pr}_X\colon S \times X \to X$$

the projections. When $S = *$ we write Γ_X instead of pr_*. Thus

$$\Gamma_{X,*}\colon \mathrm{Sh}(X;\mathcal{E}) \to \mathrm{Sh}(*;\mathcal{E}) \simeq \mathcal{E}$$

is the global sections functor and Γ_X^{-1} is the constant *pre*sheaf functor. Moreover, the functors

$$\Gamma_X^*\colon \mathcal{E} \to \mathrm{Sh}(X;\mathcal{E}) \qquad \text{and} \qquad \Gamma_X^{*,\mathrm{hyp}}\colon \mathcal{E} \to \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$$

are the constant sheaf and hypersheaf functors, respectively. Analogously, for every S we refer to the functor pr_S^{-1} (resp. pr_S^*, $\mathrm{pr}_S^{*,\mathrm{hyp}}$) as the *$S$-constant presheaf* (resp. *sheaf*, *hypersheaf*) *functor*.

Definition 1.17. Let S and X be topological spaces and let $\mathcal{E}$ be a presentable ∞-category.

(1.17.1) We say that a sheaf $L \in \mathrm{Sh}(S \times X;\mathcal{E})$ is *constant relative to S* (or *S-constant*) if L is in the essential image of the S-constant sheaf functor pr_S^*.

(1.17.2) We say that $L \in \mathrm{Sh}(S \times X;\mathcal{E})$ is *locally constant relative to S* (or *locally S-constant*) if there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that for each $\alpha \in A$, the restriction $L|_{S \times U_\alpha}$ is a S-constant sheaf on $S \times U_\alpha$.

(1.17.3) We say that a hypersheaf $L \in \mathrm{Sh}^{\mathrm{hyp}}(S \times X;\mathcal{E})$ is *hyperconstant relative to S* (or *S-hyperconstant*) if L is in the essential image of the constant hypersheaf functor $\mathrm{pr}_S^{*,\mathrm{hyp}}$.

(1.17.4) We say that $L \in \mathrm{Sh}^{\mathrm{hyp}}(S \times X;\mathcal{E})$ is *locally hyperconstant relative to S* (or *locally S-hyperconstant*) if there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that for each $\alpha \in A$, the restriction $L|_{S\times U_\alpha}$ is a S-hyperconstant *hyper*sheaf on $S \times U_\alpha$.

We write

$$\mathrm{LC}_S(S\times X;\mathcal{E}) \subset \mathrm{Sh}(S\times X;\mathcal{E}) \qquad \text{and} \qquad \mathrm{LC}_S^{\mathrm{hyp}}(S\times X;\mathcal{E}) \subset \mathrm{Sh}^{\mathrm{hyp}}(S\times X;\mathcal{E})$$

for the full subcategories spanned by the locally S-constant sheaves and the locally S-hyperconstant hypersheaves, respectively. When $S = *$ we denote these ∞-categories by $\mathrm{LC}(X;\mathcal{E})$ and $\mathrm{LC}^{\mathrm{hyp}}(X;\mathcal{E})$, respectively.

Warning 1.18. We emphasize that for a given object $E \in \mathcal{E}$, the constant sheaf $\Gamma_X^*(E)$ need not be hypercomplete. Similarly, a hyperconstant hypersheaf need not be a constant sheaf; the notions of constant sheaves and hyperconstant hypersheaves are genuinely different. Also notice that there is a containment

$$\mathrm{LC}(X;\mathcal{E}) \cap \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \subset \mathrm{LC}^{\mathrm{hyp}}(X;\mathcal{E}).$$

However, this inclusion is not generally an equality.

Remark 1.19. If X is a topological space *locally of singular shape* in the sense of [**16**, Definition A.4.15], then $\mathrm{LC}^{\mathrm{hyp}}(X) = \mathrm{LC}(X) \cap \mathrm{Sh}^{\mathrm{hyp}}(X) = \mathrm{LC}(X)$ See [**14**, Proposition 2.1] [**16**, Corollary A.1.17].

Observation 1.20. Let S be a topological space and $f\colon X \to Y$ a map of topological spaces. Write $f_S := \mathrm{id}_S \times f$. Then the functors

$$f_S^*\colon \mathrm{Sh}(S\times Y;\mathcal{E}) \to \mathrm{Sh}(S\times X;\mathcal{E}) \;\text{ and }\; f_S^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(S\times Y;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S\times X;\mathcal{E})$$

preserve locally S-constant and S-hyperconstant sheaves. Hence the assignments

$$Y \mapsto \mathrm{LC}_S(S\times Y;\mathcal{E}) \qquad \text{and} \qquad Y \mapsto \mathrm{LC}_S^{\mathrm{hyp}}(S\times Y;\mathcal{E})$$

define subfunctors of the functors

$$\mathrm{Sh}(S\times -;\mathcal{E}), \mathrm{Sh}^{\mathrm{hyp}}(S\times -;\mathcal{E})\colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}_\infty.$$

Moreover, they are hypercomplete sheaves with respect to the open topology on **Top**.

Observation 1.21. Let X be a topological space and $g\colon S \to T$ a map of topological spaces. Write $g_X := g \times \mathrm{id}_X$. Then the functors

$$g_X^*\colon\ \mathrm{Sh}(T\times X;\mathcal{E}) \to \mathrm{Sh}(S\times X;\mathcal{E}), \quad g_X^{*,\mathrm{hyp}}\colon\ \mathrm{Sh}^{\mathrm{hyp}}(T\times X;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S\times X;\mathcal{E})$$

carry locally T-constant sheaves to locally S-constant sheaves and locally T-hyperconstant hypersheaves to locally S-hyperconstant hypersheaves. In particular, objects of $\mathrm{LC}_S^{\mathrm{hyp}}(S\times X;\mathcal{E})$ can be seen as *families* of objects in $\mathrm{LC}^{\mathrm{hyp}}(X;\mathcal{E})$ parametrized by the points of S.

2. Sheaves on locally weakly contractible topological spaces

In this section we prove Theorem 2.12. The proof relies on an alternative characterization of $\mathrm{pr}_S^{*,\mathrm{hyp}}$ which is discussed in § 2.1. In § 2.4 we reinterpret our ∞-category $\mathrm{LC}_S^{\mathrm{hyp}}(S\times X;\mathcal{E})$ in terms of *foliated* hypersheaves.

2.1. Formula for the hypersheaf pullback

Fix topological spaces S and X, and a presentable ∞-category $\mathcal{E}$. Our first goal is to show that if X is locally weakly contractible, then the functor $\mathrm{pr}_S^{*,\mathrm{hyp}}$ admits a left adjoint. For the following constructions, recall the notations for posets of open subsets introduced in Notation 1.5 and Example 1.10.

Construction 2.1. Consider the functor

$$\Pi_\infty(-/S)\colon \mathrm{PSh}(\mathrm{Open}_\times(S\times X)) \to \mathrm{Sh}^{\mathrm{hyp}}(S)$$

left Kan extended from the functor $\mathrm{Open}_\times(S\times X) \to \mathrm{Sh}^{\mathrm{hyp}}(S)$ sending $V\times U$ to $V\otimes \Pi_\infty(U)$. This functor admits a right adjoint

$$\Pi^\infty(-/S)\colon \mathrm{Sh}^{\mathrm{hyp}}(S) \to \mathrm{PSh}(\mathrm{Open}_\times(S\times X))$$

given by the assignment

$$G \mapsto [W \mapsto \mathrm{Map}_{\mathrm{Sh}^{\mathrm{hyp}}(S)}(\Pi_\infty(W/S), G)].$$

By [**16**, Proposition A.3.2 & Lemma A.3.10], the functor $\Pi_\infty(-/S)$ takes hypercover diagrams to colimits in $\mathrm{Sh}^{\mathrm{hyp}}(S)$, and it therefore factors through

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathrm{Open}_\times(S\times X)) \simeq \mathrm{Sh}^{\mathrm{hyp}}(S\times X).$$

(The above equivalence follows from the fact that $\mathrm{Open}_\times(S\times X)$ is a basis for the opens of $S\times X$; see Proposition 1.12.) Consequently, it follows that $\Pi^\infty(-/S)$ factors through $\mathrm{Sh}^{\mathrm{hyp}}(S\times X)$. We use the same notation for the resulting adjunction

$$\Pi_\infty(-/S)\colon \mathrm{Sh}^{\mathrm{hyp}}(S\times X) \rightleftarrows \mathrm{Sh}^{\mathrm{hyp}}(S) :\Pi^\infty(-/S).$$

Given a presentable ∞-category $\mathcal{E}$, write $\Pi_\infty^{\mathcal{E}}(-/S)\colon \mathrm{Sh}^{\mathrm{hyp}}(S\times X;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ for the tensor product

$$\mathrm{Sh}^{\mathrm{hyp}}(S\times X;\mathcal{E}) \simeq \mathrm{Sh}^{\mathrm{hyp}}(S\times X)\otimes\mathcal{E} \xrightarrow{\Pi_\infty(-/S)\otimes \mathrm{id}_{\mathcal{E}}} \mathrm{Sh}^{\mathrm{hyp}}(S)\otimes\mathcal{E} \simeq \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}).$$

We write

$$\Pi^\infty_{\mathcal{E}}(-/S)\colon \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S\times X;\mathcal{E})$$

for the right adjoint of $\Pi_\infty^{\mathcal{E}}(-/S)$. Concretely, $\Pi^\infty_{\mathcal{E}}(-/S)$ is defined by sending any $G\in \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ to the $\mathcal{E}$-valued hypersheaf

$$V\times U \mapsto G(V)^{\Pi_\infty(U)};$$

here the exponential notation denotes the cotensoring of $\mathcal{E}$ over **Spc**.

Observation 2.2. Let X be a topological space and $g\colon S\to T$ a map of topological spaces. Write $g_X := g\times \mathrm{id}_X$. For $V\times U\in \mathrm{Open}_\times(T\times X)$ there are canonical and functorial identifications

$$\begin{aligned} g^{*,\mathrm{hyp}}(\Pi_\infty(V\times U/T)) &\simeq g^{*,\mathrm{hyp}}(V\otimes\Pi_\infty(U)) \\ &\simeq g^{*,\mathrm{hyp}}(V)\otimes\Pi_\infty(U) \\ &\simeq \Pi_\infty(g_X^{*,\mathrm{hyp}}(V\times U)/S). \end{aligned}$$

This implies that the diagram of left adjoints

$$\begin{array}{ccc} \mathrm{Sh}^{\mathrm{hyp}}(T\times X) & \xrightarrow{g_X^{*,\mathrm{hyp}}} & \mathrm{Sh}^{\mathrm{hyp}}(S\times X) \\ {\scriptstyle \Pi_\infty(-/T)}\downarrow & & \downarrow{\scriptstyle \Pi_\infty(-/S)} \\ \mathrm{Sh}^{\mathrm{hyp}}(T) & \xrightarrow[g^{*,\mathrm{hyp}}]{} & \mathrm{Sh}^{\mathrm{hyp}}(S) \end{array} \tag{2.3}$$

is canonically commutative. Given a presentable ∞-category $\mathcal{E}$, tensoring the diagram (2.3) with $\mathcal{E}$, we see that the same commutativity holds with coefficients in $\mathcal{E}$.

We now compare the functor $\Pi^\infty_{\mathcal{E}}(-/S)$ to the hypersheaf pullback $\mathrm{pr}_S^{*,\mathrm{hyp}}$.

Construction 2.4. Fix $G \in \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ and $V \in \mathrm{Open}(S)$. The unique final map $\Pi_\infty(X) \to *$ induces a map

$$G(V) \simeq G(V)^* \longrightarrow G(V)^{\Pi_\infty(X)} \simeq \Pi^\infty_{\mathcal{E}}(G/S)(V\times X) \simeq \mathrm{pr}_{S,*}(\Pi^\infty_{\mathcal{E}}(G/S))(V).$$

By adjunction, this corresponds to a map $\alpha_G\colon \mathrm{pr}_S^{*,\mathrm{hyp}}(G) \to \Pi^\infty_{\mathcal{E}}(G/S)$. These maps assemble together into a natural transformation

$$\alpha\colon \mathrm{pr}_S^{*,\mathrm{hyp}} \to \Pi^\infty_{\mathcal{E}}(-/S)$$

of functors $\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S\times X;\mathcal{E})$.

Proposition 2.5. *Let S and X be topological spaces. Assume that X is locally weakly contractible. Then the natural transformation*

$$\alpha\colon \mathrm{pr}_S^{*,\mathrm{hyp}} \to \Pi^\infty_{\mathcal{E}}(-/S)$$

is an equivalence. In particular, the functor $\mathrm{pr}_S^{*,\mathrm{hyp}}$ is right adjoint to $\Pi^{\mathcal{E}}_\infty(-/S)$.

Proof. First we treat the case where $\mathcal{E} = \mathbf{Spc}$. Let

$$j\colon \mathrm{Open}_{\mathrm{all,ctr}}(S\times X)^{\mathrm{op}} \hookrightarrow \mathrm{Open}(S\times X)^{\mathrm{op}}, \quad i\colon \mathrm{Sh}^{\mathrm{hyp}}(S\times X) \hookrightarrow \mathrm{PSh}(S\times X)$$

denote the inclusions. Write $u\colon \mathrm{pr}_S^{-1} \to i\,\mathrm{pr}_S^{*,\mathrm{hyp}}$ for the unit. Write

$$\widetilde{\alpha}\colon \mathrm{pr}_S^{-1} \to i\Pi^\infty(-/S)$$

for the composite of $i(\alpha)$ with u. Fix $F \in \mathrm{Sh}^{\mathrm{hyp}}(S)$ and let $V\times U \in \mathrm{Open}_{\mathrm{all,ctr}}(S\times X)$. Unraveling the definitions shows that

$$\mathrm{pr}_S^{-1}(F)(V\times U) \simeq F(V) \qquad \text{and} \qquad i\Pi^\infty(F/S)(V\times U) \simeq F(V)^{\Pi_\infty(U)}.$$

Moreover, the map $\widetilde{\alpha}$ is induced by the unique map $\Pi_\infty(U) \to *$. Since U is weakly contractible, we deduce that for every $F \in \mathrm{Sh}^{\mathrm{hyp}}(S)$, the map $j^*(\widetilde{\alpha})$ is an equivalence.

Since $\Pi^\infty(F/S)$ is a hypersheaf, it follows from (1.12.1) that $j^*(\mathrm{pr}_S^{-1}(F))$ is a hypersheaf on $\mathrm{Open}_{\mathrm{all,ctr}}(S\times X)$. By Remark 1.13 we the unit $\mathrm{pr}_S^{-1}(F) \to i\,\mathrm{pr}_S^{*,\mathrm{hyp}}(F)$ is identified with the unit

$$\mathrm{pr}_S^{-1}(F) \to j_*j^*(\mathrm{pr}_S^{-1}(F)) \simeq j_*j^*(i\Pi^\infty(F/S)).$$

Using (1.12.1) once more shows that the unit map $i\Pi^\infty(F/S) \to j_*j^*(i\Pi^\infty(F/S))$ is an equivalence, as desired.

Now we treat the case where $\mathcal{E}$ is any presentable ∞-category. Since we just showed that $\Pi^\infty(-/S)$ is equivalent to $\mathrm{pr}_S^{*,\mathrm{hyp}}$; it follows that $\Pi^\infty(-/S)$ commutes with colimits. The functoriality of tensor product of presentable ∞-categories implies therefore that $\Pi^\infty_{\mathcal{E}}(-/S) \simeq \Pi^\infty(-/S) \otimes \mathrm{id}_{\mathcal{E}}$. Since the same holds for the functor $\mathrm{pr}_S^{*,\mathrm{hyp}}$ and the map α respects such decomposition, the conclusion follows. □

Remark 2.6 (truncated coefficients). Let $n \geqslant 1$ be an integer and let $\mathcal{E}$ be a presentable n-category. In this setting, to prove Proposition 2.5, we only need to assume that X is *locally weakly* $(n-1)$*-connected* in the following sense: there is a basis of opens $U \subset X$ such that $\pi_0(U) = *$ and all of the homotopy groups of U in degrees $\leqslant n-1$ vanish. In this case, in Construction 2.1 we replace the underlying homotopy type $\Pi_\infty(U)$ by the fundamental $(n-1)$-groupoid $\Pi_{n-1}(U)$. That is, we use the $(n-1)$-truncation of $\Pi_\infty(U)$. In particular, when $\mathcal{E} = \mathbf{Set}$ and $S = *$, the constant sheaf functor $\mathbf{Set} \to \mathrm{Sh}(X;\mathbf{Set})$ is given by sending a set E to the sheaf

$$U \mapsto \mathrm{Map}_{\mathbf{Set}}(\pi_0(U), E).$$

For $n = 2$ and $S = *$, these results (essentially) recover results of Polesello–Waschkies [**20**, §§2.1–2.2].

All of the results in the rest of the paper can be formulated with coefficients in a presentable n-category replacing assumptions of (local) weak contractibility with assumptions of (local) weak $(n-1)$-connectedness. The proofs are exactly the same, replacing Π_∞ by Π_{n-1}. Since we are most interested in ∞-categories that are *not* truncated, we will not explicitly highlight this generalization in the rest of the text.

2.2. The exceptional pushforward

Before moving on to the main result of this section, we need a brief digression about the exceptional pushforward whose existence is guaranteed by Proposition 2.5:

Notation 2.7. Let S and X be topological spaces and assume that X is locally weakly contractible. In light of Proposition 2.5, we write

$$\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(S \times X;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$$

for the left adjoint to $\mathrm{pr}_S^{*,\mathrm{hyp}}$. We refer to $\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}$ as the *exceptional pushforward*.

Corollary 2.8. *Let $g \colon S \to T$ be a map of topological spaces, let X be a locally weakly contractible topological space, and let $\mathcal{E}$ be a presentable ∞-category. Then the squares*

$$\begin{array}{ccc} \mathrm{Sh}^{\mathrm{hyp}}(T;\mathcal{E}) & \xrightarrow{g^{*,\mathrm{hyp}}} & \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \\ {\scriptstyle \mathrm{pr}_S^{*,\mathrm{hyp}}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_T^{*,\mathrm{hyp}}} \\ \mathrm{Sh}^{\mathrm{hyp}}(T \times X;\mathcal{E}) & \xrightarrow[g_X^{*,\mathrm{hyp}}]{} & \mathrm{Sh}^{\mathrm{hyp}}(S \times X;\mathcal{E}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Sh}^{\mathrm{hyp}}(S \times X;\mathcal{E}) & \xrightarrow{g_{X,*}} & \mathrm{Sh}^{\mathrm{hyp}}(T \times X;\mathcal{E}) \\ {\scriptstyle \mathrm{pr}_{S,*}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{T,*}} \\ \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) & \xrightarrow[g_*]{} & \mathrm{Sh}^{\mathrm{hyp}}(T;\mathcal{E}) \end{array}$$

are vertically left adjointable. In particular, taking $T = *$, it follows that whenever $F \in \mathrm{Sh}^{\mathrm{hyp}}(S \times X;\mathcal{E})$ is a S-hyperconstant hypersheaf on $S \times X$, then $\mathrm{pr}_{X,*}(F)$ is a hyperconstant hypersheaf on X.

Proof. We have to prove that the exchange transformations

$$\mathrm{pr}_{T,\sharp}^{\mathrm{hyp}} \circ g_X^{*,\mathrm{hyp}} \to g^{*,\mathrm{hyp}} \circ \mathrm{pr}_{S,\sharp}^{\mathrm{hyp}} \qquad \text{and} \qquad \mathrm{pr}_T^{*,\mathrm{hyp}} \circ g_* \to g_{X,*} \circ \mathrm{pr}_S^{*,\mathrm{hyp}}$$

are equivalences. The one on the right can be deduced from the one on the left by passing to right adjoints. By Proposition 2.5,

$$\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}} \simeq \Pi_\infty^{\mathcal{E}}(-/S) \qquad \text{and} \qquad \mathrm{pr}_{T,\sharp}^{\mathrm{hyp}} \simeq \Pi_\infty^{\mathcal{E}}(-/T).$$

Hence the conclusion follows from Observation 2.2. □

Corollary 2.9. *Let S and X be topological spaces and assume that X is locally weakly contractible. Then the pushforward functor $\mathrm{pr}_{X,*}\colon \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ restricts to a functor*

$$\mathrm{pr}_{X,*}\colon \mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E}) \to \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E}).$$

Proof. Since the formation of $\mathrm{pr}_{X,*}$ is compatible with restriction to an open subset of X, the question is local on X. Thus it is enough to check that if F is a S-hyperconstant hypersheaf, then $\mathrm{pr}_{X,*}(F) \in \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$. This is guaranteed by Corollary 2.8. □

Corollary 2.10. *Let S and X be topological spaces and $\{f_\alpha\colon S_\alpha \to S\}_{\alpha \in A}$ a collection of maps of topological spaces. Assume that X is wclwc and that the hypersheaf pullback functors*

$$\{(f_\alpha \times \mathrm{id}_X)^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S_\alpha \times X; \mathcal{E})\}_{\alpha \in A}$$

are jointly conservative. Then the unit $F \to \mathrm{pr}_S^{*,\mathrm{hyp}} \mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}(F)$ is an equivalence if and only if for each $\alpha \in A$, the unit

$$(f_\alpha \times \mathrm{id}_X)^{*,\mathrm{hyp}}(F) \to \mathrm{pr}_{S_\alpha}^{*,\mathrm{hyp}} \mathrm{pr}_{S_\alpha,\sharp}^{\mathrm{hyp}}(f_\alpha \times \mathrm{id}_X)^{*,\mathrm{hyp}}(F)$$

is an equivalence.

Proof. Corollary 2.8 implies that $(f_\alpha \times \mathrm{id}_X)^{*,\mathrm{hyp}}$ takes the unit of the adjunction $\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}} \dashv \mathrm{pr}_S^{*,\mathrm{hyp}}$ to the unit of the adjunction $\mathrm{pr}_{S_\alpha,\sharp}^{\mathrm{hyp}} \dashv \mathrm{pr}_{S_\alpha}^{*,\mathrm{hyp}}$. □

2.3. Full faithfulness of the hypersheaf pullback

Now we prove Theorem 0.6.

Notation 2.11. We will write $\mathrm{Env}\colon \mathbf{Cat}_\infty \to \mathbf{Spc}$ for the left adjoint to the inclusion $\mathbf{Spc} \subset \mathbf{Cat}_\infty$. For an ∞-category $\mathcal{C}$, the space $\mathrm{Env}(\mathcal{C})$ can be computed as the colimit of the constant functor $\mathcal{C} \to \mathbf{Spc}$ at the terminal object $* \in \mathbf{Spc}$ [**8**, Corollary 2.10].

Theorem 2.12. *Let S and X be topological spaces and assume that X is wclwc. Then:*

(2.12.1) *The functor $\mathrm{pr}_S^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$ is fully faithful.*

(2.12.2) *The essential image of $\mathrm{pr}_S^{*,\mathrm{hyp}}$ coincides with $\mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E})$.*

Remark 2.13. In other words, (2.12.2) asserts that, if X is wclwc, then every locally S-hyperconstant sheaf is automatically globally S-hyperconstant.

Proof. For (2.12.1), note that since $\mathrm{pr}_S^{*,\mathrm{hyp}}$ is left adjoint to $\mathrm{pr}_{S,*}$, it suffices to provide a natural equivalence $\mathrm{pr}_{S,*} \mathrm{pr}_S^{*,\mathrm{hyp}} \simeq \mathrm{id}$ [**5**, Lemma 3.3.1]. Now note that since X is weakly contractible, applying Proposition 2.5 we see that for $G \in \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ and $V \in \mathrm{Open}(S)$ we have natural equivalences

$$\left(\mathrm{pr}_{S,*}\,\mathrm{pr}_S^{*,\mathrm{hyp}}(G)\right)(V) \simeq \left(\Pi_{\mathcal{E}}^{\infty}(G/S)\right)(V \times X) \simeq G(V)^{\Pi_\infty(X)} \simeq G(V)^* \simeq G(V).$$

Now we prove (2.12.2). Let $F \in \mathrm{LC}_S^{\mathrm{hyp}}(S \times X;\mathcal{E})$. It suffices to prove that the counit

$$c\colon\ \mathrm{pr}_S^{*,\mathrm{hyp}}\,\mathrm{pr}_{S,*}(F) \to F$$

is an equivalence. Let $\mathcal{B}_F$ be the full subposet of $\mathrm{Open}_{\mathrm{ctr}}(X)$ formed by those weakly contractible opens U such that $F|_{S\times U}$ is hyperconstant. Since X is locally weakly contractible and F is locally S-hyperconstant, the inclusion

$$\mathrm{Open}(S) \times \mathcal{B}_F \hookrightarrow \mathrm{Open}_{\mathrm{all},\mathrm{ctr}}(S \times X) \hookrightarrow \mathrm{Open}(S \times X)$$

is a basis for $\mathrm{Open}(S \times X)$. Since both the source and target of c are hypersheaves, (1.12.1) shows that it suffices to check that c is an equivalence when restricted to $\mathrm{Open}(S) \times \mathcal{B}_F$. Fix $U \in \mathcal{B}_F$ and write $q_U : S \times U \to S$ for the projection; note that we have a natural identification

$$(\mathrm{pr}_S^{*,\mathrm{hyp}}\,\mathrm{pr}_{S,*}(F))|_{S\times U} \simeq q_U^{*,\mathrm{hyp}}\,\mathrm{pr}_{S,*}(F).$$

Since U is wclwc, statement (2.12.1) implies that the pushforward of $q_U^{*,\mathrm{hyp}}\,\mathrm{pr}_{S,*}(F)$ along q_U canonically coincides with $\mathrm{pr}_{S,*}(F)$. It follows that the counit transformation c evaluated on $V \times U \in \mathrm{Open}(S) \times \mathcal{B}_F$ is identified with the restriction morphism

$$F(V \times X) \to F(V \times U). \tag{2.14}$$

Setting $F_V := \mathrm{pr}_{X,*}(F|_{V\times X}) \in \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$, we are reduced to proving that for every fixed $V \in \mathrm{Open}(S)$ and every $U \in \mathrm{Open}_{\mathrm{ctr}}(X)$, the restriction map $F_V(X) \to F_V(U)$ is an equivalence. Corollary 2.8 implies that $F_V \in \mathrm{LC}^{\mathrm{hyp}}(X;\mathcal{E})$; we are therefore reduced to the case $S = *$.

Let $j\colon \mathcal{B}_F^{\mathrm{op}} \hookrightarrow \mathrm{Open}_{\mathrm{ctr}}(X)^{\mathrm{op}}$ denote the inclusion. Proposition 1.12 guarantees that the unit transformation $F_V \to j_* j^*(F_V)$ is an equivalence. It follows that for every $V \in \mathrm{Open}(S)$, the natural map

$$F_V(X) \to \lim_{U \in \mathcal{B}_F} F_V(U)$$

is an equivalence. We claim that the functor $j^*(F_V) = F_V \circ j$ inverts every morphism in $\mathcal{B}_F$. To see this, let $i\colon W \hookrightarrow U$ be a morphism in $\mathcal{B}_F$. Since $U \in \mathcal{B}_F$, there exists an object $E \in \mathcal{E}$ and an equivalence $\Gamma_U^{*,\mathrm{hyp}}(E) \simeq F_V|_U$. Since $\Gamma_W = \Gamma_U \circ i$, it follows that $F_V|_W \simeq \Gamma_W^{*,\mathrm{hyp}}(E)$. Consider the commutative triangle

$$\begin{array}{ccc} & E & \\ \swarrow & & \searrow \\ \Gamma_{U,*}(\Gamma_U^{*,\mathrm{hyp}}(E)) & \longrightarrow & \Gamma_{W,*}(\Gamma_W^{*,\mathrm{hyp}}(E)). \end{array}$$

The bottom horizontal morphism is naturally identified with the restriction map $F_V(i)\colon F_V(U) \to F_V(W)$. On the other hand, since both W and U are wclwc, (2.12.1) implies that both the diagonal morphisms are equivalences. The 2-of-3 property implies that $F_V(i)$ is an equivalence as well.

Thus, the functor $j^*(F_V)$ factors through $\mathrm{Env}(\mathcal{B}_F)$. Observe that the functor $\Pi_\infty \colon \mathcal{B}_F \to \mathbf{Spc}$ is equivalent to the constant functor sending every object of $\mathcal{B}_F$ to $* \in \mathbf{Spc}$. It follows from Notation 2.11 that

$$\mathrm{Env}(\mathcal{B}_F) \simeq \operatorname*{colim}_{V \in \mathcal{B}_F} \Pi_\infty(V).$$

Van Kampen's Theorem identifies this colimit with $\Pi_\infty(X)$. Since X is weakly contractible, we conclude that $\mathrm{Env}(\mathcal{B}_F) \simeq *$, and therefore that $j^*(F_V)$ is a constant functor. Finally, since $\mathrm{Env}(\mathcal{B}_F)$ is contractible, the restriction maps

$$F_V(X) \simeq \lim_{U \in \mathcal{B}_F} F(U) \to F(U)$$

are equivalences for every $U \in \mathcal{B}_F$. The conclusion follows. □

Corollary 2.15. *Let X be a wclwc topological space while $\mathcal{E}$ is a presentable ∞-category. Then:*

(2.15.1) *The constant hypersheaf functor $\Gamma_X^{*,\mathrm{hyp}} \colon \mathcal{E} \to \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ is fully faithful.*

(2.15.2) *The essential image of $\Gamma_X^{*,\mathrm{hyp}}$ is $\mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$.*

2.4. Foliated hypersheaves

We end this section with an alternative description of locally S-hyperconstant hypersheaves on $S \times X$. The idea is that in order to check local S-hyperconstancy, it suffices to check hyperconstancy on the 'leaves' $\{s\} \times X$.

Definition 2.16. Let S and X be topological spaces and assume that X is wclwc. Let $\mathcal{E}$ be a presentable ∞-category. A hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$ is *foliated* if for each $s \in S$, the restriction $F|^{\mathrm{hyp}}_{\{s\} \times X}$ is a hyperconstant hypersheaf.

Example 2.17. Given $G \in \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E})$, the pullback $\mathrm{pr}_S^{*,\mathrm{hyp}}(G)$ is foliated.

The following generalizes [**16**, Proposition A.2.5] (that deals with the case $X = \mathbf{R}$).

Proposition 2.18. *Let S and X be topological spaces and assume that X is wclwc. Let $\mathcal{E}$ be a* compactly generated *∞-category. For $F \in \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$, the following statements are equivalent:*

(2.18.1) *The hypersheaf F is in the essential image of $\mathrm{pr}_S^{*,\mathrm{hyp}}$.*

(2.18.2) *The hypersheaf F is foliated.*

Proof. The implication (2.18.1) ⇒ (2.18.2) is the content of Example 2.17.

To see that (2.18.2) ⇒ (2.18.1), we need to show that if F is foliated, then the unit $u_F \colon F \to \mathrm{pr}_S^{*,\mathrm{hyp}} \mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}(F)$ is an equivalence. Notice that since $\mathcal{E}$ is compactly generated, the restriction functors

$$\left\{ (-)|^{\mathrm{hyp}}_{\{s\} \times X} \colon \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(\{s\} \times X; \mathcal{E}) \right\}_{s \in S}$$

are jointly conservative (Recollection 1.7). Applying Corollary 2.10, we see that to prove that u_F is an equivalence, it suffices to show that for each $s \in S$, the unit

$$F|^{\mathrm{hyp}}_{\{s\}\times X} \longrightarrow \mathrm{pr}^{*,\mathrm{hyp}}_{\{s\}} \mathrm{pr}^{\mathrm{hyp}}_{\{s\},\sharp} \left(F|^{\mathrm{hyp}}_{\{s\}\times X}\right) \simeq \Gamma^{*,\mathrm{hyp}}_X \Gamma^{\mathrm{hyp}}_{X,\sharp} \left(F|^{\mathrm{hyp}}_{\{s\}\times X}\right)$$

is an equivalence. The claim now follows from the assumption that $F|^{\mathrm{hyp}}_{\{s\}\times X}$ is hyperconstant combined with Corollary 2.15. □

3. Consequences of Theorem 2.12

The relative full faithfulness theorem we proved in the previous section is the cornerstone of this paper. We now explore some of its main consequences. Among others, we prove a general version of the monodromy equivalence, a categorical Künneth formula for locally hyperconstant hypersheaves, and a comparison result for sheaf and singular cohomology on locally weakly contractible spaces. We also prove the hypercomplete part of Theorem 0.2.

3.1. Structural results for locally hyperconstant hypersheaves

We start with the following recognition criterion for the objects of $\mathrm{LC}^{\mathrm{hyp}}_S(S \times X; \mathcal{E})$. We fix topological spaces S and X and a presentable ∞-category $\mathcal{E}$. Furthermore, we assume X to be locally weakly contractible.

Proposition 3.1. *For $F \in \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$, the following statements are equivalent:*

(3.1.1) *The sheaf F is locally S-hyperconstant.*

(3.1.2) *For every pair of weakly contractible open subsets $V \subset U$ of X and every open subset $W \subset S$, the restriction map $F(W \times U) \to F(W \times V)$ is an equivalence.*

Proof. We first prove that (3.1.1) implies (3.1.2). Write

$$q_V : W \times V \to W \qquad \text{and} \qquad q_U : W \times U \to W$$

for the projections. Since U is weakly contractible, Theorem 2.12 implies that $F|_{W\times U}$ is W-hyperconstant. We can then choose a hypersheaf $G \in \mathrm{Sh}^{\mathrm{hyp}}(W; \mathcal{E})$ and an equivalence $F|_{W\times U} \simeq q_U^{*,\mathrm{hyp}}(G)$. It follows that $F|_{W\times V} \simeq q_V^{*,\mathrm{hyp}}(G)$. Now consider the commutative triangle

$$\begin{array}{ccc} & G & \\ \swarrow & & \searrow \\ q_{U,*}q_U^{*,\mathrm{hyp}}(G) & \longrightarrow & q_{V,*}q_V^{*,\mathrm{hyp}}(G). \end{array}$$

Since U and V are weakly contractible, the full faithfulness part of Theorem 2.12 implies that the diagonal maps are equivalences. Thus the horizontal map is an equivalence. To conclude, note that, unraveling the definitions, this horizontal map coincides with the restriction map $F(W \times U) \to F(W \times V)$.

We now prove that (3.1.2) implies (3.1.1). By choosing an open cover of X by wclwc opens, we are reduced to the case that X is wclwc. Let F be a hypersheaf satisfying assumption (3.1.2). Since X is wclwc, is enough to prove that the counit

$$c_F \colon \mathrm{pr}_S^{*,\mathrm{hyp}} \mathrm{pr}_{S,*}(F) \to F$$

is an equivalence. In the first segment of the proof of (2.12.2) we proved that this is the same as saying that the for every $U \in \mathrm{Open}_{\mathrm{ctr}}(X)$ and $W \in \mathrm{Open}(S)$, the restriction map $F(W \times X) \to F(W \times U)$ is an equivalence. Since X and U are weakly contractible, this is guaranteed by our hypothesis. □

Corollary 3.2. *The full subcategory* $\mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E}) \subset \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$ *is closed under limits and colimits.*

Proof. Let A be a small ∞-category and let $F_\bullet \colon A \to \mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E})$ be a diagram. First we treat the case of limits. By Proposition 3.1, it is enough to prove that for every $V \subset U$ in $\mathrm{Open}_{\mathrm{ctr}}(X)$, and every $W \in \mathrm{Open}(S)$, the restriction map

$$\lim_{\alpha \in A} F_\alpha(W \times U) \to \lim_{\alpha \in A} F_\alpha(W \times V)$$

is an equivalence. Since limits in $\mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$ are computed objectwise, the above map is the limit of the individual restriction maps $F_\alpha(W \times U) \to F_\alpha(W \times V)$. Since each F_α is locally S-hyperconstant, Proposition 3.1 implies that all these maps are equivalences. Thus, the same goes for their limit.

For the case of colimits, we have to check that the colimit $\mathrm{colim}_{\alpha \in A} F_\alpha$ computed in $\mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$ is locally S-hyperconstant. The question is local on X, and we can assume that X is weakly contractible. In this case, Theorem 2.12 shows the functor $\mathrm{pr}_S^{*,\mathrm{hyp}}$ is fully faithful; thus there exists a diagram $F'_\bullet \colon A \to \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E})$ and an equivalence $F_\bullet \simeq \mathrm{pr}_S^{*,\mathrm{hyp}} \circ F'_\bullet$. The fact that $\mathrm{pr}_S^{*,\mathrm{hyp}}$ commutes with colimits completes the proof. □

Corollary 3.3. *Assume that X is wclwc. Then for every hypercover $V_\bullet$ of S, the natural functor*

$$\mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E}) \to \lim_{[n] \in \mathbf{\Delta}} \mathrm{LC}_{V_n}^{\mathrm{hyp}}(V_n \times X; \mathcal{E})$$

is an equivalence.

Proof. Consider the commutative square

$$\begin{array}{ccc} \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) & \longrightarrow & \lim_{[n] \in \mathbf{\Delta}} \mathrm{Sh}^{\mathrm{hyp}}(V_n; \mathcal{E}) \\ \downarrow{\scriptstyle \mathrm{pr}_S^{*,\mathrm{hyp}}} & & \downarrow{\scriptstyle \mathrm{pr}_{V_\bullet}^{*,\mathrm{hyp}}} \\ \mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E}) & \longrightarrow & \lim_{[n] \in \mathbf{\Delta}} \mathrm{LC}_{V_n}^{\mathrm{hyp}}(V_n \times X; \mathcal{E}). \end{array}$$

Since $\mathrm{Sh}^{\mathrm{hyp}}(-; \mathcal{E})$ satisfies hyperdescent, the top horizontal functor is an equivalence. Theorem 2.12 implies that both vertical functors are equivalences. □

3.2. Monodromy equivalence and Künneth formula

Let X be a topological space. There is a natural map from the underlying homotopy type $\Pi_\infty(X)$ of X to the shape of the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(X)$. However, this map is typically *not* an equivalence. Our work in § 2 implies that these invariants agree when X is locally weakly contractible:

Corollary 3.4. *Let X be a locally weakly contractible topological space. Then the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(X)$ is locally of constant shape, and its shape coincides with $\Pi_\infty(X)$.*

Proof. This is a direct consequence of Proposition 2.5 and [**16**, Proposition A.1.8 & Remark A.1.10]. □

Notation 3.5. Write $\mathbf{Top}^{\mathrm{lwc}} \subset \mathbf{Top}$ for the full subcategory spanned by the locally weakly contractible topological spaces.

Corollary 3.6 (monodromy equivalence)**.** *Let X be a locally weakly contractible topological space. Then the functor*

$$\Pi_\infty \colon \mathrm{LC}^{\mathrm{hyp}}(X) \to \mathbf{Spc}_{/\Pi_\infty(X)} \tag{3.7}$$

is an equivalence.

Proof. Proposition 2.5 shows that $\Gamma_X^{*,\mathrm{hyp}}$ is right adjoint to the functor Π_∞. The conclusion follows then from [**16**, Theorem A.1.15] □

Observation 3.8. Unraveling the proof of [**16**, Theorem A.1.15], we see that the inverse to (3.7) is given by sending a map $K \to \Pi_\infty(X)$ to the sheaf

$$U \mapsto \mathrm{Map}_{/\Pi_\infty(X)}(\Pi_\infty(U), K).$$

Straightening/unstraightening puts the monodromy equivalence (3.7) into a more familiar form:

$$\mathrm{LC}^{\mathrm{hyp}}(X) \simeq \mathrm{Fun}(\Pi_\infty(X), \mathbf{Spc}). \tag{3.9}$$

Moreover, the equivalence (3.9) refines to an equivalence of functors of the form $\mathbf{Top}^{\mathrm{lwc,op}} \to \mathbf{Cat}_\infty$. In particular, the functor $\mathrm{LC}^{\mathrm{hyp}} \colon \mathbf{Top}^{\mathrm{lwc,op}} \to \mathbf{Cat}_\infty$ inverts *weak* homotopy equivalences between locally weakly contractible topological spaces.

Observation 3.10. Let $\mathcal{E}$ be a presentable ∞-category. Since restriction of sheaves to an open subset is both a left and a right adjoint, the equivalence

$$\mathrm{Sh}^{\mathrm{hyp}}(X) \otimes \mathcal{E} \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{hyp}}(X; \mathcal{E})$$

restricts to an equivalence

$$\mathrm{LC}^{\mathrm{hyp}}(X) \otimes \mathcal{E} \xrightarrow{\sim} \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E}).$$

Thus tensoring (3.9) with $\mathcal{E}$ provides a monodromy equivalence

$$\mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E}) \simeq \mathrm{Fun}(\Pi_\infty(X), \mathcal{E}) \tag{3.11}$$

for $\mathcal{E}$-valued locally hyperconstant hypersheaves. Also note that the functoriality of the equivalence (3.11) implies that given $L \in \mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E})$, the associated functor $\Pi_\infty(X) \to \mathcal{E}$ carries $x \in X$ to the stalk $x^* L \in \mathcal{E}$.

Remark 3.12 (the classical monodromy equivalence)*.* Write $\Pi_1(X)$ for the fundamental groupoid of X. Since $\Pi_1(X)$ is the homotopy 1-category of $\Pi_\infty(X)$, if $\mathcal{E}$ is a presentable 1-category, then

$$\mathrm{LC}^{\mathrm{hyp}}(X; \mathcal{E}) = \mathrm{LC}(X; \mathcal{E}) \qquad \text{and} \qquad \mathrm{Fun}(\Pi_\infty(X), \mathcal{E}) \simeq \mathrm{Fun}(\Pi_1(X), \mathcal{E}).$$

In particular, Observation 3.10 recovers the classical monodromy equivalence for locally weakly contractible topological spaces.

In the classical monodromy equivalence, only local 1-connectedness is needed, so this seems to use stronger hypotheses than the classical result. However, the truncated variants of our results (see Remark 2.6) recover and generalize the classical monodromy equivalence. Let $n \geqslant 1$ and let $\mathcal{E}$ be a presentable n-category. If X is a locally weakly $(n-1)$-connected topological space, then the constant sheaf functor $\mathcal{E} \to \mathrm{Sh}(X;\mathcal{E})$ admits a left adjoint. In particular, the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(X)$ is *locally* $(n-1)$*-connected* in the sense of [**13**, Definition 3.2]. Write $\Pi_n \mathrm{Sh}^{\mathrm{hyp}}(X)$ for the n-truncation of the shape of the ∞-topos $\mathrm{Sh}^{\mathrm{hyp}}(X)$. Applying [**13**, Theorem 3.13] provides a monodromy equivalence

$$\mathrm{LC}(X;\mathcal{E}) \simeq \mathrm{Fun}(\Pi_n \mathrm{Sh}^{\mathrm{hyp}}(X), \mathcal{E}). \tag{3.13}$$

If X is locally weakly n-connected, then the formula for the constant sheaf functor $\mathbf{Spc}_{\leqslant n} \to \mathrm{Sh}(X;\mathbf{Spc}_{\leqslant n})$ provided by Proposition 2.5 shows that $\Pi_n \mathrm{Sh}^{\mathrm{hyp}}(X)$ coincides with the fundamental n-groupoid $\Pi_n(X)$ of X. (This is the truncated variant of Corollary 3.4.) Hence (3.13) becomes an equivalence

$$\mathrm{LC}(X;\mathcal{E}) \simeq \mathrm{Fun}(\Pi_n(X), \mathcal{E}).$$

Setting $n = 1$ we obtain a generalization of the classical monodromy equivalence to locally *weakly* 1-connected topological spaces.

We conclude this subsection with a *categorical Künneth formula* for locally hyperconstant hypersheaves. Given topological spaces X and Y, note that the functors

$$\mathrm{Sh}(X) \times \mathrm{Sh}(Y) \to \mathrm{Sh}(X \times Y) \qquad \text{and} \qquad \mathrm{Sh}^{\mathrm{hyp}}(X) \times \mathrm{Sh}^{\mathrm{hyp}}(Y) \to \mathrm{Sh}^{\mathrm{hyp}}(X \times Y)$$

given by

$$(F,G) \mapsto \mathrm{pr}_X^*(F) \times \mathrm{pr}_Y^*(G) \qquad \text{and} \qquad (F,G) \mapsto \mathrm{pr}_X^{*,\mathrm{hyp}}(F) \times \mathrm{pr}_Y^{*,\mathrm{hyp}}(G)$$

preserve colimits separately in each variable. Since the coproduct in the ∞-category of ∞-topoi and left exact left adjoints is given by the tensor product of presentable ∞-categories [**1**, Theorem 2.15] [**16**, Example 4.8.1.19], these functors induce left exact colimit-preserving functors

$$\mathrm{Sh}(X) \otimes \mathrm{Sh}(Y) \to \mathrm{Sh}(X \times Y) \qquad \text{and} \qquad \mathrm{Sh}^{\mathrm{hyp}}(X) \otimes \mathrm{Sh}^{\mathrm{hyp}}(Y) \to \mathrm{Sh}^{\mathrm{hyp}}(X \times Y).$$

In general, neither of these functors is an equivalence.[2] Nonetheless, locally hyperconstant hypersheaves on $X \times Y$ *do* decompose as a tensor product:

Corollary 3.14 (Künneth formula)**.** *Let X and Y be a locally weakly contractible topological spaces. The natural functor* $\mathrm{LC}^{\mathrm{hyp}}(X) \times \mathrm{LC}^{\mathrm{hyp}}(Y) \to \mathrm{LC}^{\mathrm{hyp}}(X \times Y)$ *induces an equivalence of ∞-categories*

$$\mathrm{LC}^{\mathrm{hyp}}(X) \otimes \mathrm{LC}^{\mathrm{hyp}}(Y) \xrightarrow{\sim} \mathrm{LC}^{\mathrm{hyp}}(X \times Y).$$

Proof. Since Π_∞ preserves finite products while $\mathrm{Fun}(-, \mathbf{Spc})$ carries products of ∞-categories to tensor products of presentable ∞-categories, the conclusion follows from the monodromy equivalence (3.9). □

[2] If X or Y is locally compact Hausdorff, then the functor $\mathrm{Sh}(X) \otimes \mathrm{Sh}(Y) \to \mathrm{Sh}(X \times Y)$ is an equivalence [**15**, Proposition 7.3.1.11].

3.3. Homotopy-invariance for locally hyperconstant hypersheaves

Our next goal is to use Theorem 2.12 to prove the hypercomplete part of Theorem 0.2. We need the following two preliminary results:

Lemma 3.15. *Let S and X be topological spaces, and assume that X is wclwc. Then*

$$\mathrm{LC}^{\mathrm{hyp}}(S \times X; \mathcal{E}) \subset \mathrm{LC}^{\mathrm{hyp}}_S(S \times X; \mathcal{E}).$$

Proof. Applying Corollary 3.3, we see that for every hypercover $V_\bullet$ of S, the natural functor

$$\mathrm{LC}^{\mathrm{hyp}}_S(S \times X; \mathcal{E}) \to \lim_{[m]\in\mathbf{\Delta}} \mathrm{LC}^{\mathrm{hyp}}_{V_m}(V_m \times X; \mathcal{E})$$

is an equivalence. On the other hand, for every hypercover $U_\bullet$ of X where each U_n is wclwc, the natural functor

$$\mathrm{LC}^{\mathrm{hyp}}_{V_m}(V_m \times X; \mathcal{E}) \to \lim_{[n]\in\mathbf{\Delta}} \mathrm{LC}^{\mathrm{hyp}}_{V_m}(V_m \times U_n)$$

is an equivalence. Thus, a hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$ belongs to the full subcategory $\mathrm{LC}^{\mathrm{hyp}}_S(S \times X; \mathcal{E})$ if and only if we can find a hypercover $V_\bullet \times U_\bullet$ of $S \times X$ such that, for every $([n],[m]) \in \mathbf{\Delta} \times \mathbf{\Delta}$,

the restriction $F|_{V_m \times U_n}$ belongs to $\mathrm{LC}^{\mathrm{hyp}}_{V_m}(V_m \times U_n; \mathcal{E})$.

If $F \in \mathrm{LC}^{\mathrm{hyp}}(S \times X; \mathcal{E})$, there exists a hypercover $V_\bullet \times U_\bullet$ such that, for every $([n],[m]) \in \mathbf{\Delta} \times \mathbf{\Delta}$, the restriction $F|_{V_m \times U_n}$ is hyperconstant, hence V_m-hyperconstant. The conclusion follows. □

Lemma 3.16. *Let S and X be topological spaces, and assume that X is wclwc. Then the pushforward $\mathrm{pr}_{S,*} \colon \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E})$ preserves locally hyperconstant hypersheaves.*

Proof. Let $F \in \mathrm{LC}^{\mathrm{hyp}}(S \times X; \mathcal{E})$. By Lemma 3.15, we know that $F \in \mathrm{LC}^{\mathrm{hyp}}_S(S \times X; \mathcal{E})$. Thus there exists a hypersheaf G on S and an equivalence $F \simeq \mathrm{pr}_S^{*,\mathrm{hyp}}(G)$. Since $\mathrm{pr}_S^{*,\mathrm{hyp}}$ is fully faithful (Theorem 2.12), the unit defines an equivalence $G \xrightarrow{\sim} \mathrm{pr}_{S,*}(F)$. Hence our goal is to show that G is locally hyperconstant.

Since $F \in \mathrm{LC}^{\mathrm{hyp}}(S \times X; \mathcal{E})$, there exists an open cover $\{V_\alpha \times U_\alpha\}_{\alpha \in A}$ of $S \times X$ such that for each $\alpha \in A$, the hypersheaf $F|_{V_\alpha \times U_\alpha}$ is hyperconstant. Since X is locally weakly contractible, we can furthermore assume that every U_α is weakly contractible. Write $q_\alpha \colon V_\alpha \times U_\alpha \to V_\alpha$ for the projection. Since $F \simeq \mathrm{pr}_S^{*,\mathrm{hyp}}(G)$, we see that

$$F|_{V_\alpha \times U_\alpha} \simeq q_\alpha^{*,\mathrm{hyp}}(G|_{V_\alpha}).$$

Since U_α is weakly contractible, using Theorem 2.12 again we see that the unit

$$G|_{V_\alpha} \to q_{\alpha,*}(F|_{V_\alpha \times U_\alpha})$$

is an equivalence. We can therefore replace S and X by U_α and V_α, respectively. Equivalently, we can assume from the beginning that F is globally hyperconstant. We can therefore write $F \simeq \Gamma^{*,\mathrm{hyp}}_{S \times X}(E)$, for some object $E \in \mathcal{E}$. In this case, we obtain equivalences

$$\mathrm{pr}_S^{*,\mathrm{hyp}}(G) \simeq F \simeq \Gamma_{S\times X}^{*,\mathrm{hyp}}(E) \simeq \mathrm{pr}_S^{*,\mathrm{hyp}}(\Gamma_S^{*,\mathrm{hyp}}(E)).$$

Applying Theorem 2.12 once more, we deduce that

$$G \simeq \mathrm{pr}_{S,*}(F) \simeq \Gamma_S^{*,\mathrm{hyp}}(E).$$ □

Theorem 3.17. *Let S and X be topological spaces, and assume that X is wclwc. Then the functors*

$$\mathrm{pr}_S^{*,\mathrm{hyp}} \colon \mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E}) \rightleftarrows \mathrm{LC}^{\mathrm{hyp}}(S\times X;\mathcal{E}) : \mathrm{pr}_{S,*}$$

are inverse equivalences of ∞-categories. In particular, it follows that the functor $\mathrm{LC}^{\mathrm{hyp}}(-;\mathcal{E})\colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is strongly homotopy-invariant.

Remark 3.18. When S is itself locally weakly contractible, Theorem 3.17 is a consequence of the monodromy equivalence (see Observation 3.8). The strength of Theorem 2.12 is that we have no assumptions on S.

Proof of Theorem 3.17. In virtue of Lemma 3.15, we can consider the following commutative square:

$$\begin{array}{ccc} \mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E}) & \hookrightarrow & \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \\ \downarrow{\scriptstyle \mathrm{pr}_S^{*,\mathrm{hyp}}} & & \downarrow{\scriptstyle \mathrm{pr}_S^{*,\mathrm{hyp}}} \\ \mathrm{LC}^{\mathrm{hyp}}(S\times X;\mathcal{E}) & \hookrightarrow & \mathrm{LC}_S^{\mathrm{hyp}}(S\times X;\mathcal{E}). \end{array}$$

Theorem 2.12 implies that the right vertical functor is an equivalence. Since the horizontal functors are fully faithful, Lemma 3.16 implies that this square is vertically right adjointable. The conclusion follows. □

3.4. Exceptional pushforward on locally hyperconstant hypersheaves

We now prove that the exceptional pushforward preserves locally hyperconstant hypersheaves. We start with the following observations:

Observation 3.19. Let X be a topological space and let $j\colon U \to X$ be a local homeomorphism. Then $j^{-1}\colon \mathrm{PSh}(X;\mathcal{E}) \to \mathrm{PSh}(U;\mathcal{E})$ preserves (hyper)sheaves. Furthermore, the functor

$$j^{-1}\colon \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(U;\mathcal{E})$$

commutes with arbitrary limits, hence admits a left adjoint $j_\sharp^{\mathrm{hyp}}$. Observe that if j is an open immersion, then $j_\sharp^{\mathrm{hyp}}$ coincides with the hypersheafification of the usual extension by zero.

Observation 3.20. Let X be a topological space and let $U_\bullet$ be a hypercover of X. For every $[n] \in \mathbf{\Delta}$, denote by $j_n\colon U_n \to X$ the canonical morphism. Hyperdescent implies that the natural functor

$$j_\bullet^*\colon \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E}) \to \lim_{[n]\in\mathbf{\Delta}} \mathrm{Sh}^{\mathrm{hyp}}(U_n;\mathcal{E})$$

is an equivalence. In particular, $j_\bullet^*$ admits a left adjoint, that we denote $j_{\bullet,\sharp}^{\mathrm{hyp}}$. Using [**22**, §8.2], the left adjoint $j_{\bullet,\sharp}^{\mathrm{hyp}}$ can be described as the functor sending a descent datum $\{F_n\}_{n\geqslant 0}$ to

$$j^{\mathrm{hyp}}_{\bullet,\sharp}(\{F_n\}_{n\geqslant 0}) \simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} j^{\mathrm{hyp}}_{n,\sharp}(F_n).$$

In particular, for every hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(X;\mathcal{E})$, there is a natural equivalence

$$F \simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}^{\mathrm{op}}} j^{\mathrm{hyp}}_{n,\sharp}(j_n^*(F)). \tag{3.21}$$

Notation 3.22. For the remainder of this section, we fix topological spaces S and X, as well as a presentable ∞-category $\mathcal{E}$. Furthermore, we assume that X is locally weakly contractible.

Lemma 3.23. *Let $U_\bullet$ be a hypercover of X. For every $[n] \in \mathbf{\Delta}$, let $j_n \colon U_n \to X$ be the canonical morphism and set $p_n := \mathrm{pr}_S \circ (\mathrm{id}_S \times j_n)$. Then for every $F \in \mathrm{Sh}^{\mathrm{hyp}}(S\times X;\mathcal{E})$, one has a natural equivalence*

$$\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}(F) \simeq \operatorname*{colim}_{[n]\in\mathbf{\Delta}} p^{\mathrm{hyp}}_{n,\sharp}(\mathrm{id}_S \times j_n)^*(F).$$

Proof. Since $\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}$ preserves colimits, the claim follows from applying $\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}$ to the equivalence (3.21), combined with the natural equivalence

$$p^{\mathrm{hyp}}_{n,\sharp} \simeq \mathrm{pr}^{\mathrm{hyp}}_{S,\sharp} \circ (\mathrm{id}_S \times j_n)^{\mathrm{hyp}}_{\sharp}.$$ ☐

Notation 3.24. We denote by

$$\chi\colon\ \mathrm{pr}^{\mathrm{hyp}}_{S,\sharp} \circ \mathrm{pr}^{*,\mathrm{hyp}}_S \to \mathrm{pr}_{S,*} \circ \mathrm{pr}^{*,\mathrm{hyp}}_S$$

the composition of the counit $\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp} \circ \mathrm{pr}^{*,\mathrm{hyp}}_S \to \mathrm{id}$ with the unit $\mathrm{id} \to \mathrm{pr}_{S,*} \circ \mathrm{pr}^{*,\mathrm{hyp}}_S$.

Lemma 3.25. *In addition to the hypotheses made in Notation 3.22, assume that X is weakly contractible. Then for each $F \in \mathrm{LC}^{\mathrm{hyp}}_S(S\times X;\mathcal{E})$, the natural transformation χ induces an equivalence $\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}(F) \xrightarrow{\sim} \mathrm{pr}_{S,*}(F)$.*

Proof. Since X is weakly contractible, Theorem 2.12 guarantees the existence of a hypersheaf $G \in \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ and an equivalence $F \simeq \mathrm{pr}^{*,\mathrm{hyp}}_S(G)$. Since $\mathrm{pr}^{*,\mathrm{hyp}}_S$ is fully faithful (again by Theorem 2.12), the morphism χ applied to G is the composite equivalence

$$\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}(F) \simeq \mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}\mathrm{pr}^{*,\mathrm{hyp}}_S(G) \xrightarrow{\ \sim\ } G \xrightarrow{\ \sim\ } \mathrm{pr}_{S,*}\mathrm{pr}^{*,\mathrm{hyp}}_S(G) \simeq \mathrm{pr}_{S,*}(F).$$ ☐

Corollary 3.26. *In addition to the hypotheses made in Notation 3.22, assume that one of the following hypotheses is satisfied:*

(3.26.1) *The topological space X is weakly contractible.*

(3.26.2) *The topological space S is locally weakly contractible.*

Then the functor $\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}$ preserves locally hyperconstant hypersheaves.

Proof. Let $F \in \mathrm{LC}^{\mathrm{hyp}}(S\times X;\mathcal{E})$. To prove the claim under assumption (3.26.1), using Lemma 3.15, we see that F belongs to $\mathrm{LC}^{\mathrm{hyp}}_S(S\times X;\mathcal{E})$. Lemma 3.25 implies that

$$\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}(F) \simeq \mathrm{pr}_{S,*}(F).$$

The conclusion follows from Lemma 3.16.

To prove the claim under assumption (3.26.2), using Corollary 3.2 we see that that $\mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E})$ is closed under small colimits in $\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$. Using Lemma 3.23, we can reduce to the case where X is weakly contractible, in which case the conclusion follows from (3.26.1). ☐

3.5. Comparison of sheaf and singular cohomology

Now we explain why our work implies that for locally weakly contractible spaces, singular and sheaf cohomology agree.

Notation 3.27. Let R be a ring and X a topological space. Write $\mathrm{D}(R)$ for the derived ∞-category of R, and write $\mathrm{C}_*(X;R) \in \mathrm{D}(R)$ for the complex of singular chains on X. Given an object $M \in \mathrm{D}(R)$, the cotensor $M^{\Pi_\infty(X)}$ is given by the internal Hom complex

$$\mathrm{C}^{-*}(X;M) := \mathrm{RHom}_R(\mathrm{C}_*(X;R), M).$$

If M is an ordinary R-module, then $\mathrm{C}^{-*}(X;M)$ is what is usually referred to as the complex of singular cochains on X with values in M.

3.28. The functor $\Pi^{\infty}_{\mathrm{D}(R)} \colon \mathrm{D}(R) \to \mathrm{Sh}^{\mathrm{hyp}}(X;\mathrm{D}(R))$ is given by $M \mapsto \mathrm{C}^{-*}(-;M)$.

The following is an immediate consequence of Proposition 2.5:

Corollary 3.29. *Let R be a ring and X a locally weakly contractible topological space. Then:*

(3.29.1) *The functor $\mathrm{D}(R) \to \mathrm{Sh}^{\mathrm{hyp}}(X;\mathrm{D}(R))$ given by the formula $M \mapsto \mathrm{C}^{-*}(-;M)$ is the constant hypersheaf functor.*

(3.29.2) *For each $M \in \mathrm{D}(R)$, there is a natural equivalence $\mathrm{R}\Gamma(X;M) \simeq \mathrm{C}^{-*}(X;M)$ from the derived global sections of the constant hypersheaf at M to the complex of singular cochains on X with values in M.*

(3.29.3) *For each ordinary R-module M, there is a natural isomorphism from sheaf cohomology to singular cohomology $\mathrm{H}^*_{\mathrm{sheaf}}(X;M) \simeq \mathrm{H}^*_{\mathrm{sing}}(X;M)$.*

Hence, sheaf cohomology is an invariant of the *weak* homotopy type of locally weakly contractible topological spaces.

Remark 3.30. After work of Sella [**23**], Petersen [**19**, Theorem 1.2] recently proved a comparison for cohomology valued in *ordinary* R-modules. Petersen's comparison is under slightly weaker assumptions on the topological space X, in relation to the chosen R-module.

4. Homotopy-invariance for locally constant sheaves

The purpose of this section is to prove a non-hypercomplete variant of Theorem 3.17, thus completing the proof of Theorem 0.2. The proof follows the same format of Theorem 3.17 expanding on Clausen and Ørsnes Jansen's proof of [**8**, Proposition 3.2].

Remark 4.1. The homotopy-invariance statement we will prove below is with respect to the unit interval rather than a general wclwc space. Indeed, we do not expect that $\mathrm{LC}(-;\mathcal{E})$ is strongly homotopy-invariant (in the sense of Definition 0.1).

4.1. The exceptional pushforward

We now record the existence of the exceptional pushforward in the non-hypercomplete setting as well as its compatibility with basechange. In this section, we are most interested in the case where X is a subinterval of $[0,1]$.

Recollection 4.2. Let S and X be topological spaces. There is a natural geometric morphism of ∞-topoi

$$\mathrm{Sh}(S \times X) \to \mathrm{Sh}(S) \otimes \mathrm{Sh}(X)$$

[**16**, Example 4.8.1.19]. If X is locally compact, then this geometric morphism $\mathrm{Sh}(S \times X) \to \mathrm{Sh}(S) \otimes \mathrm{Sh}(X)$ is an equivalence [**15**, Proposition 7.3.1.11].

Lemma 4.3. *Let S be a topological space and $\mathcal{E}$ a presentable ∞-category. Let X be a locally compact topological space and assume that the constant sheaf functor $\Gamma_X^* \colon \mathbf{Spc} \to \mathrm{Sh}(X)$ admits a left adjoint $\Gamma_{X,\sharp}$. Then:*

(4.3.1) *The pullback functor $\mathrm{pr}_S^* \colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S \times X;\mathcal{E})$ admits a left adjoint $\mathrm{pr}_{S,\sharp}$.*

(4.3.2) *If $\Gamma_X^* \colon \mathbf{Spc} \to \mathrm{Sh}(X)$ is fully faithful, then $\mathrm{pr}_S^* \colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S \times X;\mathcal{E})$ is also fully faithful.*

Proof. Appealing to Recollection 4.2, this follows by tensoring the chain of adjoints

$$\mathrm{Sh}(X) \underset{\Gamma_{X,*}}{\overset{\Gamma_{X,\sharp}}{\rightleftarrows}} \mathbf{Spc}, \qquad \mathrm{Sh}(X) \xleftarrow{\Gamma_X^*} \mathbf{Spc}.$$

with the presentable ∞-category $\mathrm{Sh}(S;\mathcal{E})$. □

4.4. In the situation of Lemma 4.6, we refer to $\mathrm{pr}_{S,\sharp}$ as the *exceptional pushforward*.

Remark 4.5. In light of (1.6) and Proposition 2.5, if X is a topological space that admits a CW structure, then the hypotheses of Lemma 4.3 are satisfied. Moreover, if X is also contractible, then Γ_X^* is fully faithful.

The following compatibility with basechange is immediate from the definition of $\mathrm{pr}_{S,\sharp}$ as a tensor product [**11**, Observation 1.15]. See also [**25**, Lemma 3.3].

Lemma 4.6. *Let X be topological spaces and $\mathcal{E}$ a presentable ∞-category. Given a map $g \colon T \to S$ of topological spaces, there is a canonically commutative square of ∞-categories*

$$\begin{array}{ccc} \mathrm{Sh}(S \times X;\mathcal{E}) & \xrightarrow{(g\times \mathrm{id}_X)^*} & \mathrm{Sh}(T \times X;\mathcal{E}) \\ {\scriptstyle \mathrm{pr}_{S,\sharp}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{T,\sharp}} \\ \mathrm{Sh}(S;\mathcal{E}) & \xrightarrow[g^*]{} & \mathrm{Sh}(T;\mathcal{E}) \end{array} .$$

Lemma 4.7. *Let S be a topological space, $\{f_\alpha \colon S_\alpha \to S\}_{\alpha\in A}$ a collection of maps of topological spaces, and $\mathcal{E}$ a presentable ∞-category. Assume that the pullback functors*

$$\{(f_\alpha \times \mathrm{id}_{[0,1]})^* \colon \mathrm{Sh}(S \times [0,1];\mathcal{E}) \to \mathrm{Sh}(S_\alpha \times [0,1];\mathcal{E})\}_{\alpha\in A}$$

are jointly conservative. Given $F \in \mathrm{Sh}(S \times [0,1];\mathcal{E})$, the unit $u_F \colon F \to \mathrm{pr}_S^* \mathrm{pr}_{S,\sharp}(F)$

is an equivalence if and only if for each $\alpha \in A$, the unit

$$(f_\alpha \times \mathrm{id}_{[0,1]})^*(F) \to \mathrm{pr}_{S_\alpha}^* \mathrm{pr}_{S_\alpha,\sharp}(f_\alpha \times \mathrm{id}_{[0,1]})^*(F)$$

is an equivalence.

Proof. By Lemma 4.6, we have a natural equivalence

$$(f_\alpha \times \mathrm{id}_{[0,1]})^* \mathrm{pr}_S^* \mathrm{pr}_{S,\sharp}(F) \simeq \mathrm{pr}_{S_\alpha}^* \mathrm{pr}_{S_\alpha,\sharp}(f_\alpha \times \mathrm{id}_{[0,1]})^*(F).$$

Moreover, notice that the pullback

$$(f_\alpha \times \mathrm{id}_{[0,1]})^*(u_F)\colon (f_\alpha \times \mathrm{id}_{[0,1]})^*(F) \to \mathrm{pr}_{S_\alpha}^* \mathrm{pr}_{S_\alpha,\sharp}(f_\alpha \times \mathrm{id}_{[0,1]})^*(F)$$

is homotopic to the unit of the adjunction $\mathrm{pr}_{S_\alpha,\sharp} \dashv \mathrm{pr}_{S_\alpha}^*$ applied to the specific sheaf $(f_\alpha \times \mathrm{id}_{[0,1]})^*(F)$. The claim now follows from the assumption that the functors $\{(f_\alpha \times \mathrm{id}_{[0,1]})^*\}_{\alpha\in A}$ are jointly conservative. □

4.2. Homotopy-invariance of locally constant sheaves

We now show that $\mathrm{pr}_{S,\sharp}$ preserves locally constant sheaves. The compactness of $[0,1]$ and the fact that $[0,1]$ has the order topology imply the following:

Lemma 4.8. *Let S be a topological space and $\mathcal{U}$ an open cover of $S \times [0,1]$. Then there exist:*

(4.8.1) *An open cover $\{U_\alpha\}_{\alpha\in A}$ of S.*

(4.8.2) *For each $\alpha \in A$, a positive integer n_α and open subintervals $I_{\alpha,1}, \ldots, I_{\alpha,n_\alpha}$ of $[0,1]$ covering $[0,1]$ such that $I_{\alpha,k} \cap I_{\alpha,\ell} \neq \emptyset$ if and only if $k = \ell \pm 1$.*

Such that $\bigcup_{\alpha\in A}\{U_\alpha \times I_{\alpha,1}, \ldots, U_\alpha \times I_{\alpha,n_\alpha}\}$ refines the cover $\mathcal{U}$.

Observation 4.9. Let U a topological space, and $I, J \subset [0,1]$ subintervals which are open in $[0,1]$. Assume that the intersection $I \cap J$ is nonempty. Since $\{U \times I, U \times J\}$ is an open cover of $U \times (I \cup J)$, descent and the fact that the pullback functors

$$\mathrm{Sh}(U;\mathcal{E}) \to \mathrm{Sh}(U \times I;\mathcal{E}),\ \ \mathrm{Sh}(U;\mathcal{E}) \to \mathrm{Sh}(U \times J;\mathcal{E}),\ \ \mathrm{Sh}(U;\mathcal{E}) \to \mathrm{Sh}(U \times (I \cap J);\mathcal{E})$$

are fully faithful (Lemma 4.3 and Remark 4.5) implies that if $F_I \in \mathrm{Sh}(U \times I;\mathcal{E})$ and $F_J \in \mathrm{Sh}(U \times J;\mathcal{E})$ are pulled back from U and

$$F_I|_{U\times(I\cap J)} \simeq F_J|_{U\times(I\cap J)},$$

then there exists a *unique* sheaf $G \in \mathrm{Sh}(U;\mathcal{E})$ such that

$$F_I \simeq \mathrm{pr}_U^*(G) \qquad \text{and} \qquad F_J \simeq \mathrm{pr}_U^*(G).$$

In particular, if $L \in \mathrm{Sh}(U \times (I \cup J);\mathcal{E})$ is such that both $L|_{U\times I}$ and $L|_{U\times J}$ are constant, then L is constant.

Lemma 4.10. *Let S be a topological space, and $L \in \mathrm{Sh}(S \times [0,1];\mathcal{E})$. If, in addition, $L \in \mathrm{LC}(S \times [0,1];\mathcal{E})$, then there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of S such that for each $\alpha \in A$, the sheaf $L|_{U_\alpha\times[0,1]}$ is constant.*

Proof. As in Lemma 4.8, choose an open cover $\{U_\alpha \times I_{\alpha,1}, \ldots, U_\alpha \times I_{\alpha,n_\alpha}\}_{\alpha\in A}$ of $S \times [0,1]$ such that each restriction $L|_{U_\alpha\times I_{\alpha,k}}$ is constant. We claim that for each $\alpha \in A$, the restriction $L|_{U_\alpha\times[0,1]}$ is constant. To see this, apply Observation 4.9 inductively with $I = I_{\alpha,1} \cup \cdots \cup I_{\alpha,m}$ and $J = I_{\alpha,m+1}$. □

Observation 4.11. Notice that $\Gamma^*_{S\times[0,1]} \simeq \mathrm{pr}^*_S \Gamma^*_S$. Hence, if $F \in \mathrm{Sh}(S\times[0,1];\mathcal{E})$ is constant, then the exceptional pushforward $\mathrm{pr}_{S,\sharp}(F)$ is constant and the unit map $F \to \mathrm{pr}^*_S \mathrm{pr}_{S,\sharp}(F)$ is an equivalence.

Lemma 4.12. *Let S be a topological space and $\mathcal{E}$ a presentable ∞-category. Then the exceptional pushforward $\mathrm{pr}_{S,\sharp}\colon \mathrm{Sh}(S\times[0,1];\mathcal{E}) \to \mathrm{Sh}(S;\mathcal{E})$ preserves locally constant sheaves.*

Proof. Let $F \in \mathrm{LC}(S\times[0,1];\mathcal{E})$. Using Lemma 4.10, choose an open cover $\{U_\alpha\}_{\alpha\in A}$ of S such that each of the restrictions $F|_{U_\alpha\times[0,1]}$ is constant. By Lemma 4.6 we have $\mathrm{pr}_{S,\sharp}(F)|_{U_\alpha} \simeq \mathrm{pr}_{U_\alpha,\sharp}(F|_{U_\alpha\times[0,1]})$. The conclusion follows from Observation 4.11. □

Corollary 4.13. *Let S be a topological space and $\mathcal{E}$ a presentable ∞-category. Then the functors*

$$\mathrm{pr}_{S,\sharp}\colon \mathrm{LC}(S\times[0,1];\mathcal{E}) \rightleftarrows \mathrm{LC}(S;\mathcal{E}) : \mathrm{pr}^*_S$$

are inverse equivalences of ∞-categories. In particular, it follows that the functor $\mathrm{LC}(-;\mathcal{E})\colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is homotopy-invariant

Proof. Since pr^*_S is fully faithful, it suffices to show that if $F \in \mathrm{LC}(S\times[0,1];\mathcal{E})$, then the unit $F \to \mathrm{pr}^*_S \mathrm{pr}_{S,\sharp}(F)$ is an equivalence. Using Lemma 4.10, choose an open cover $\{U_\alpha\}_{\alpha\in A}$ of S such that each of the restrictions $F|_{U_\alpha\times[0,1]}$ is constant. The claim now follows from Lemma 4.7 and Observation 4.11. □

5. Homotopy-invariance for (hyper)constructible (hyper)sheaves

We now bootstrap our homotopy-invariance results (Theorem 3.17 and Corollary 4.13) from the locally constant setting to the constructible setting. In § 5.1, we review the basics of stratified spaces and (hyper)constructible (hyper)sheaves. In § 5.2, we prove that the exceptional pushforwards $\mathrm{pr}_{S,\sharp}$ and $\mathrm{pr}^{\mathrm{hyp}}_{S,\sharp}$ preserve constructibility (Corollaries 5.6 and 5.7) and give equivalent conditions for homotopy-invariance to hold (Corollary 5.10). Finally, in § 5.3 we use these criteria to show that, in many situations of interest, (hyper)constructible (hyper)sheaves are homotopy-invariant (Corollaries 5.13, 5.15, and 5.19).

5.1. Stratified spaces & constructible sheaves

We first recall the notion of a stratified space:

Notation 5.1. Let P be a poset. We also write P for the set P equipped with the *Alexandroff topology* in which a subset $U \subset P$ is open if and only if U is upwards-closed. Given an element $p \in P$, we write

$$P_{\geqslant p} := \{q \in P \mid q \geqslant p\} \qquad \text{and} \qquad P_{>p} := P_{\geqslant p} \smallsetminus \{p\}.$$

The category of *P-stratified topological spaces* is the overcategory $\mathbf{Top}_{/P}$. Given a P-stratified topological space $\sigma\colon S \to P$ and $p \in P$, we write $S_p := \sigma^{-1}(p)$ and call S_p the *p-th stratum* of S. We also write

$$S_{\geqslant p} := \sigma^{-1}(P_{\geqslant p}) \qquad \text{and} \qquad S_{>p} := \sigma^{-1}(P_{>p}).$$

We write $i_p\colon S_p \to S$ for the inclusion of the p-th stratum.

Definition 5.2. Let P be a poset, $S \to P$ be a P-stratified space, and $\mathcal{E}$ be a presentable ∞-category.

(5.2.1) We say that a sheaf $F \in \mathrm{Sh}(S;\mathcal{E})$ is a *P-constructible* if F for each $p \in P$, the restriction $i_p^*(F)$ is a locally constant sheaf on the stratum S_p.

(5.2.2) We say that a hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ is a *P-hyperconstructible* if F for each $p \in P$, the restriction $i_p^{*,\mathrm{hyp}}(F)$ is a locally hyperconstant hypersheaf on the stratum S_p.

We, respectively, write

$$\mathrm{Cons}_P(T;\mathcal{E}) \subset \mathrm{Sh}(T;\mathcal{E}) \qquad \text{and} \qquad \mathrm{Cons}_P^{\mathrm{hyp}}(T;\mathcal{E}) \subset \mathrm{Sh}^{\mathrm{hyp}}(T;\mathcal{E})$$

for the full subcategories spanned by the P-constructible sheaves and P-hyperconstructible hypersheaves.

Remark 5.3. Let P be a Noetherian poset and let $X \to P$ be a paracompact P-stratified space. Assume that the stratification of X is *conical* in the sense of [**16**, Definition A.5.5] and that all of the strata of X are locally of singular shape. Then

$$\mathrm{Cons}_P^{\mathrm{hyp}}(X) = \mathrm{Cons}_P(X) \cap \mathrm{Sh}^{\mathrm{hyp}}(X) = \mathrm{Cons}_P(X).$$

See [**14**, Proposition 2.11] [**16**, Proposition A.5.9].

Observation 5.4. For any map $f\colon T \to S$ of P-stratified spaces, the sheaf pullback functor f^* preserves P-constructible sheaves and the hypersheaf pullback functor $f^{*,\mathrm{hyp}}$ preserves P-hyperconstructible hypersheaves. Hence the assignments

$$S \mapsto \mathrm{Cons}_P(S;\mathcal{E}) \qquad \text{and} \qquad S \mapsto \mathrm{Cons}_P^{\mathrm{hyp}}(S;\mathcal{E})$$

define subfunctors of the functors $\mathrm{Sh}(-;\mathcal{E}), \mathrm{Sh}^{\mathrm{hyp}}(-;\mathcal{E})\colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$.

Convention 5.5. Let P be a poset and $\sigma\colon S \to P$ be a P-stratified topological space. Let X be a topological space. We write $S \times X$ for the P-stratified topological space with stratification given by the composite $S \times X \to S \to P$.

The main goal of this section is to explain when the functors $\mathrm{Cons}_P(-;\mathcal{E})$, and $\mathrm{Cons}_P^{\mathrm{hyp}}(-;\mathcal{E})$ are homotopy-invariant in the sense of Definition 0.1.

5.2. Formal homotopy-invariance

Bootstrapping off of the results of §§ 2 and 4, we can provide a first, formal version of our homotopy-invariance result.

Corollary 5.6. *Let $\mathcal{E}$ be a presentable ∞-category and let P be a poset. Let S be a P-stratified space and let X be a wclwc topological space. Then the exceptional hypersheaf pushforward $\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(S \times X;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ preserves hyperconstructible hypersheaves.*

Proof. Let $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(S \times X;\mathcal{E})$ be a hyperconstructible hypersheaf on $S \times X$. We have to prove that for every $p \in P$, the restriction $i_p^{*,\mathrm{hyp}}\,\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}(F)$ is a locally hyperconstant hypersheaf on S_p. By the compatibility of the exceptional pushforward with pullbacks (Corollary 2.8), there is a natural equivalence

$$i_p^{*,\mathrm{hyp}}(\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}(F)) \simeq \mathrm{pr}_{S_p,\sharp}^{\mathrm{hyp}}((i_p \times \mathrm{id}_X)^{*,\mathrm{hyp}}(F)).$$

Since $(i_p \times \mathrm{id}_X)^{*,\mathrm{hyp}}(F)$ is locally hyperconstant on $S_p \times X$, Corollary 3.26 completes the proof. □

Corollary 5.7. *For $S \in \mathbf{Top}_{/P}$, the functor $\mathrm{pr}_{S,\sharp}\colon \mathrm{Sh}(S \times [0,1]; \mathcal{E}) \to \mathrm{Sh}(S; \mathcal{E})$ preserves constructible sheaves.*

Proof. As in the proof of Corollary 5.6, combine Lemma 4.12 with Lemma 4.6. □

Theorem 5.8. *Under the hypotheses of Corollary 5.6, the essential image of the fully faithful functor*

$$\mathrm{pr}_S^{*,\mathrm{hyp}}\colon\ \mathrm{Cons}_P^{\mathrm{hyp}}(S; \mathcal{E}) \hookrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(S \times X; \mathcal{E}) \tag{5.9}$$

is the intersection $\mathrm{Cons}_P^{\mathrm{hyp}}(S \times X; \mathcal{E}) \cap \mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E})$.

Proof. Since $\mathrm{pr}_S^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \to \mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E})$ is an equivalence of ∞-categories and preserves constructiblity, we immediately see that the essential image of (5.9) is contained in

$$\mathrm{Cons}_P^{\mathrm{hyp}}(S \times X; \mathcal{E}) \cap \mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E}).$$

Conversely, assume that F belongs to this intersection. Since $F \in \mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E})$, Theorem 2.12 and Lemma 3.25 imply that $F \simeq \mathrm{pr}_S^{*,\mathrm{hyp}}(\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}(F))$. Since F belongs to $\mathrm{Cons}_P^{\mathrm{hyp}}(S \times X; \mathcal{E})$, Corollary 5.6 implies that $\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}(F) \in \mathrm{Cons}_P^{\mathrm{hyp}}(S; \mathcal{E})$. Therefore, F belongs to the essential image of (5.9), as desired. □

Corollary 5.10. *Under the hypotheses of Corollary 5.6, the following conditions are equivalent:*

(5.10.1) *The functor $\mathrm{pr}_S^{*,\mathrm{hyp}}\colon\ \mathrm{Cons}_P^{\mathrm{hyp}}(S; \mathcal{E}) \hookrightarrow \mathrm{Cons}_P^{\mathrm{hyp}}(S \times X; \mathcal{E})$ is an equivalence.*

(5.10.2) *For each $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(S \times X; \mathcal{E})$, the unit $F \to \mathrm{pr}_S^{*,\mathrm{hyp}}(\mathrm{pr}_{S,\sharp}^{\mathrm{hyp}}(F))$ is an equivalence.*

(5.10.3) *We have the containment $\mathrm{Cons}_P^{\mathrm{hyp}}(S \times X; \mathcal{E}) \subset \mathrm{LC}_S^{\mathrm{hyp}}(S \times X; \mathcal{E})$ as subcategories of $\mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$.*

(5.10.4) *For each $F \in \mathrm{Cons}_P^{\mathrm{hyp}}(S \times X; \mathcal{E})$, each open subset $W \subset S$, as well as each pair of weakly contractible open subsets $U \subset V$ of X, the restriction map $F(W \times V) \to F(W \times U)$ is an equivalence.*

Proof. The equivalence between (5.10.1), (5.10.2), and (5.10.3) follows from Theorem 5.8. On the other hand, Proposition 3.1 shows that (5.10.3) and (5.10.4) are equivalent. □

In particular, we obtain the following sufficient criterion ensuring that $\mathrm{pr}_S^{*,\mathrm{hyp}}$ is an equivalence:

Corollary 5.11. *In the situation of Theorem 5.8, assume that the hypersheaf restriction functors*

$$\Big\{(-)|^{\mathrm{hyp}}_{S_p\times X}\colon \operatorname{Cons}^{\mathrm{hyp}}_P(S\times X;\mathcal{E})\to \operatorname{LC}^{\mathrm{hyp}}(S_p\times X;\mathcal{E})\Big\}_{p\in P}$$

are jointly conservative. Then the functors

$$\operatorname{pr}^{\mathrm{hyp}}_{S,\sharp}\colon \operatorname{Cons}^{\mathrm{hyp}}_P(S\times X;\mathcal{E})\rightleftarrows \operatorname{Cons}^{\mathrm{hyp}}_P(S;\mathcal{E}) :\operatorname{pr}^{*,\mathrm{hyp}}_S$$

are inverse equivalences.

Proof. Combine Theorem 3.17, Corollary 2.10, and (5.10.2). □

Corollary 5.12. *Let P be a poset, $S\in \mathbf{Top}_{/P}$, and $\mathcal{E}$ a presentable ∞-category. Assume that the restriction functors*

$$\big\{(-)|_{S_p\times[0,1]}\colon \operatorname{Cons}_P(S\times[0,1];\mathcal{E})\to \operatorname{LC}(S_p\times[0,1];\mathcal{E})\big\}_{p\in P}$$

are jointly conservative. Then the functors

$$\operatorname{pr}_{S,\sharp}\colon \operatorname{Cons}_P(S\times[0,1];\mathcal{E})\rightleftarrows \operatorname{Cons}_P(S;\mathcal{E}) :\operatorname{pr}^{*}_S$$

are inverse equivalences.

Proof. Combine Lemma 4.7 and Corollaries 4.13 and 5.7. □

5.3. Detecting equivalences on strata

Corollary 5.11 shows that a sufficient criterion for the functor

$$\operatorname{pr}^{*,\mathrm{hyp}}_S\colon \operatorname{Cons}^{\mathrm{hyp}}_P(S;\mathcal{E})\to \operatorname{Cons}^{\mathrm{hyp}}_P(S\times X;\mathcal{E})$$

to be an equivalence is given by the joint conservativity of the hyperrestrictions to the strata of $S\times X$. We offer two ways of checking this independently of both S and X.

The compactly generated case

The fact that equivalences of hypersheaves on a topological space with values in a compactly generated ∞-category can be checked on stalks implies our first homotopy-invariance result:

Corollary 5.13. *Let P be a poset, $S\in \mathbf{Top}_{/P}$, and let $\mathcal{E}$ be a compactly generated ∞-category. Then:*

(5.13.1) *The restriction functors $\{(-)|^{\mathrm{hyp}}_{S_p}\colon \operatorname{Sh}^{\mathrm{hyp}}(S;\mathcal{E})\to \operatorname{Sh}^{\mathrm{hyp}}(S_p;\mathcal{E})\}_{p\in P}$ are jointly conservative.*

(5.13.2) *The functor $\operatorname{Cons}^{\mathrm{hyp}}_P(-;\mathcal{E})\colon \mathbf{Top}^{\mathrm{op}}_{/P}\to \mathbf{Cat}_\infty$ is strongly homotopy-invariant.*

Proof. Recollection 1.7 immediately implies (5.13.1). For (5.13.2), combine (5.13.1) for the P-stratified space $S\times X$ with Corollary 5.11. □

Notation 5.14. Let P be a poset and $S\to P$ be a P-stratified topological space. Write $\operatorname{Cons}_P(S)_{<\infty}\subset \operatorname{Cons}^{\mathrm{hyp}}_P(S)$ for the full subcategory spanned by those P-constructible sheaves that are also n-truncated for some $n\geqslant 0$. Since left exact functors preserve truncatedness, the assignment $S\mapsto \operatorname{Cons}_P(S)_{<\infty}$ defines a subfunctor of $\operatorname{Cons}^{\mathrm{hyp}}_P$.

Corollary 5.15. *Let P be a poset. The functor $\operatorname{Cons}_P(-)_{<\infty}\colon \mathbf{Top}^{\mathrm{op}}_{/P}\to \mathbf{Cat}_\infty$ is strongly homotopy-invariant.*

The Noetherian case

In order to drop the compact generation assumption on $\mathcal{E}$, there are two difficulties to overcome. Recall that a poset P is *Noetherian* if P satisfies the *ascending chain condition*: there are no infinite strictly ascending sequences $p_0 < p_1 < p_2 < \cdots$ of elements of P. The first issue is that there exist non-Noetherian posets P for which the ∞-topos $\mathrm{Sh}(P) = \mathrm{Cons}_P(P)$ is not hypercomplete; see [**2**, Example A.13]. Said differently, the functors $\mathrm{Sh}(P) \to \mathrm{Sh}(\{p\})$ given by pulling back to strata need not be jointly conservative. Thus we restrict ourselves to Noetherian posets.

The second issue is with the coefficient ∞-category $\mathcal{E}$. Consider the most simple stratification when $P = \{0 < 1\}$, so that a stratification $S \to \{0 < 1\}$ is the data of a closed subspace $Z = S_0$ and its open complement $S \smallsetminus Z = S_1$. Unfortunately, in general the restriction functors

$$(-)|_Z \colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(Z;\mathcal{E}) \qquad \text{and} \qquad (-)|_{S\smallsetminus Z} \colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S \smallsetminus Z;\mathcal{E})$$

need not be jointly conservative. Thus, we have to assume this property:

Definition 5.16. We say that a presentable ∞-category $\mathcal{E}$ *respects gluing* if for each topological space S and closed subspace $Z \subset S$, the restriction functors

$$(-)|_Z \colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(Z;\mathcal{E}) \quad \text{and} \quad (-)|_{S\smallsetminus Z} \colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S \smallsetminus Z;\mathcal{E})$$

and the hypersheaf restriction functors

$$(-)|_Z^{\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(Z;\mathcal{E}) \quad \text{and} \quad (-)|_{S\smallsetminus Z} \colon \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S \smallsetminus Z;\mathcal{E})$$

are jointly conservative.

Luckily, many presentable ∞-categories that arise in nature respect gluing:

Example 5.17. If each ∞-category $\mathrm{Sh}(S;\mathcal{E})$ can be recovered as the *recollement* of $\mathrm{Sh}(Z;\mathcal{E})$ and $\mathrm{Sh}(S \smallsetminus Z;\mathcal{E})$ in the sense of [**16**, Definition A.8.1], and each ∞-category $\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ is the recollement of $\mathrm{Sh}^{\mathrm{hyp}}(Z;\mathcal{E})$ and $\mathrm{Sh}^{\mathrm{hyp}}(S \smallsetminus Z;\mathcal{E})$, then $\mathcal{E}$ respects gluing. Importantly, this is satisfied if $\mathcal{E}$ is stable or $\mathcal{E} \simeq \mathcal{C} \otimes \mathcal{D}$ where $\mathcal{C}$ is a compactly generated ∞-category and $\mathcal{D}$ is an ∞-topos [**11**, Corollary 2.13, Proposition 2.21, & Remark 2.26].

Lemma 5.18. *Let P be a Noetherian poset, $S \in \mathbf{Top}_{/P}$, and let $\mathcal{E}$ be a presentable ∞-category that respects gluing. Then:*

(5.18.1) *The functors $\{(-)|_{S_p} \colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S_p;\mathcal{E})\}_{p\in P}$ are jointly conservative.*

(5.18.2) *The functors $\{(-)|_{S_p}^{\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S_p;\mathcal{E})\}_{p\in P}$ are jointly conservative.*

Proof. We prove (5.18.1); the proof of (5.18.2) is exactly the same, replacing sheaves by hypersheaves. Let ϕ be a morphism in $\mathrm{Sh}(S;\mathcal{E})$ that restricts to an equivalence on each stratum; we need to show that ϕ is an equivalence. Since the open subsets $\{S_{\geqslant p}\}_{p\in P}$ cover S, it suffices to show:

($*$) For each $p \in P$, the restriction $\phi|_{S_{\geqslant p}}$ is an equivalence in $\mathrm{Sh}(S_{\geqslant p};\mathcal{E})$.

We prove (∗) by Noetherian induction on $p \in P$. We need to show that if the restriction $\phi|_{S_{\geqslant q}}$ is an equivalence for each $q > p$, then $\phi|_{S_{\geqslant p}}$ is an equivalence. Note that

$$S_{\geqslant p} \smallsetminus S_p = S_{>p} = \bigcup_{q \in P_{>p}} S_{\geqslant q}.$$

Hence the inductive hypothesis implies that the restriction $\phi|_{S_{>p}}$ is an equivalence. By assumption $\phi|_{S_p}$ is also an equivalence. Since $\mathcal{E}$ respects gluing, the restriction functors $(-)|_{S_p} \colon \mathrm{Sh}(S_{\geqslant p}; \mathcal{E}) \to \mathrm{Sh}(S_p; \mathcal{E})$ and $(-)|_{S_{>p}} \colon \mathrm{Sh}(S_{\geqslant p}; \mathcal{E}) \to \mathrm{Sh}(S_{>p}; \mathcal{E})$ are jointly conservative, completing the proof. □

Finally we deduce the homotopy-invariance of constructible sheaves.

Corollary 5.19. *Let P be a Noetherian poset and let $\mathcal{E}$ be a presentable ∞-category that respects gluing. Then:*

(5.19.1) *The functor* $\mathrm{Cons}_P(-; \mathcal{E}) \colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ *is homotopy-invariant.*

(5.19.2) *The functor* $\mathrm{Cons}_P^{\mathrm{hyp}}(-; \mathcal{E}) \colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ *is strongly homotopy-invariant.*

Proof. Combine Corollaries 5.11 and 5.12 with Lemma 5.18. □

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