The homotopy-invariance of constructible sheaves

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Abstract

We explain why the functor that sends a stratified topological space S to the ∞-category of constructible (hyper)sheaves on S with coefficients in a large class of presentable ∞-categories is homotopy-invariant.

Contents

0 Introduction

A classical result from sheaf theory says that the functor

$$S \mapsto \mathrm{LC}(S; \mathbf{Set})$$

that sends a topological space S to the category of locally constant sheaves of sets on S is homotopy-invariant. More generally, if P is a poset then the functor

$$S \mapsto \mathrm{Cons}_P(S; \mathbf{Set})$$

that sends a P-stratified topological space S to the of sheaves of sets on S that are constructible with respect to the stratification $S \to P$ is invariant under stratified homotopy equivalences. In this note we explain how to generalize this result to the setting of sheaves with values in the ∞-category of *spaces*, or, more generally, a large class of presentable ∞-categories.

There are two potential ways to generalize these classical results about the homotopy-invariance of locally constant and constructible sheaves. In the higher-categorical world, not all sheaves satisfy descent with respect to *hyper*covers, so one can work with the ∞-category $\mathrm{Sh}(S)$ sheaves of spaces or the full subcategory $\mathrm{Sh}^{\mathrm{hyp}}(S) \subset \mathrm{Sh}(S)$ spanned by the *hypersheaves*. Both of these generalizations are relevant and each has its advantage.

The larger ∞-category of sheaves generally has better categorical properties than the subcategory of hypersheaves; see [HTT, §6.5.4]. For example, Lurie's nonabelian refinement of the Proper Basechange Theorem in topology holds for sheaves of spaces, but fails when the ∞-categories of sheaves of spaces are replaced by their subcategories of hypersheaves; see [HTT, Counterexample 6.5.4.2; 5, Remark 0.8]. On the other hand, one cannot generally check equivalences in $\mathrm{Sh}(S)$ on stalks, and $\mathrm{Sh}^{\mathrm{hyp}}(S) \subset \mathrm{Sh}(S)$ is the largest subcategory on which equivalences are checked stalkwise.

There is also a reason why we are interested in hypersheaves in particular. Part of our motivation for this work is ongoing work of Lejay [7] and Porta–Teyssier [9] to generalize Lurie's *exodromy equivalence* in topology; homotopy-invariance is a key tool in these generalizations. Lurie's higher-categorical generalization of the classical *monodromy equivalence* says that if S is a sufficiently nice topological space, then there is a natural equivalence of ∞-categories

$$\mathrm{LC}(S) \xrightarrow{\sim} \mathrm{Fun}(\Pi_\infty(S), \mathbf{Spc}) \tag{0.1}$$

from ∞-category of locally constant sheaves of spaces on S to space-valued representations of the fundamental ∞-groupoid $\Pi_\infty(S)$ of S [HA, Theorems A.1.15 & A.4.19]. Building of work of MacPherson and Treumann [10], given a sufficiently nice stratification of S by a poset P *satisfying the ascending chain condition*, Lurie extended the monodromy equivalence to take into account sheaves that are constructible with respect to the stratification $S \to P$. Specifically, Lurie provided a natural *exodromy equivalence* of ∞-categories

$$\mathrm{Cons}_P(S) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Exit}_P(S), \mathbf{Spc}) \tag{0.2}$$

from ∞-category of P-constructible sheaves of spaces on S to space-valued representations of the *exit-path ∞-category* $\mathrm{Exit}_P(S)$ of S [HA, Theorem A.9.3]. For the present discussion it is not necessary to know the definitions of the ∞-categories $\Pi_\infty(S)$ and $\mathrm{Exit}_P(S)$; the important point is that Lurie expressed the ∞-categories of locally constant and constructible sheaves as presheaf ∞-categories.

There are two directions in which to generalize the exodromy equivalence. First, to remove the ascending chain condition from the stratifying poset: this allows for key examples like the *Ran space* of a manifold [HTT, §5.5.1; 2, §3.7], which does satisfy the hypotheses of Lurie's result. Second, to generalize the coefficients space-valued sheaves to sheaves with values in more general presentable ∞-categories.[1] Recall that every object of a presheaf ∞-category is the limit of its Postnikov tower, and every sheaf that is the limit of its Postnikov tower is automatically a hypersheaf. Thus, in order for the equivalences (0.1) and (0.2) to hold, all locally constant and constructible sheaves on S are necessarily hypersheaves. Said differently, one should generalize Lurie's result by providing an equivalence between constructible *hypersheaves* and representations of the exit-path ∞-category; it is then a question whether or not every constructible sheaf is hypercomplete.

Statement of results

Let $\mathcal{E}$ be a presentable ∞-category and S a topological space. We write $\mathrm{LC}(S;\mathcal{E})$ for the ∞-category of locally constant $\mathcal{E}$-valued sheaves on S, and write $\mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E})$ for the hypersheaf variant of this

[1] For sheaves valued in a compactly generated ∞-category, this generalization is immediate; see [8, Appendix B].

∞-category (see § 1.1 for precise definitions). Here it is important to highlight the subtle point that the notions of local constancy for sheaves and hypersheaves are not the same and $\mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E})$ is not a subcategory of $\mathrm{LC}(S;\mathcal{E})$.

0.3 Theorem (Corollary 3.7)**.** *For any presentable ∞-category $\mathcal{E}$, the functors*

$$\mathrm{LC}(-;\mathcal{E}), \mathrm{LC}^{\mathrm{hyp}}(-;\mathcal{E})\colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}_{\infty}$$

are homotopy-invariant.

Passing to global sections, Theorem 0.3 implies that cohomology with coefficients in a locally constant sheaf valued in any presentable ∞-category is homotopy-invariant.

Since constructible sheaves are sheaves that are locally constant along a stratification, as long as the stratifying poset P and coefficients $\mathcal{E}$ are such that we can check equivalences after pulling back to strata, then Theorem 0.3 implies that constructible sheaves are homotopy-invariant. One way to ensure this is to work with hypersheaves as assume that $\mathcal{E}$ is compactly generated; in this setting the stalk functors $\mathrm{Sh}(S;\mathcal{E}) \to \mathcal{E}$ are jointly conservative. In particular, equivalences can be checked after pulling back to strata. As second way is to assume that $\mathcal{E}$-valued sheaves glue over open-closed decompositions and that P satisfies the *ascending chain condition*. Then one can proceed by Noetherian induction to show that equivalences can be checked on strata.

Given a poset P and P-stratified topological space $S \to P$, we write $\mathrm{Cons}_P(S;\mathcal{E})$ for the ∞-category of constructible $\mathcal{E}$-valued sheaves on S, and write $\mathrm{Cons}_P^{\mathrm{hyp}}(S;\mathcal{E})$ for the hypersheaf variant of this ∞-category (see §1.2 for precise definitions).

0.4 Theorem (Corollaries 3.10 and 3.16)**.** *Let P be a poset an $\mathcal{E}$ a presentable ∞-category.*

(0.4.1) *If $\mathcal{E}$ is compactly generated, then the functor $\mathrm{Cons}_P^{\mathrm{hyp}}(-;\mathcal{E})\colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_{\infty}$ is homotopy-invariant.*

(0.4.2) *If P is satisfies the ascending chain condition and $\mathcal{E}$ is compactly generated, stable, or an ∞-topos, then the functors*

$$\mathrm{Cons}_P(-;\mathcal{E}), \mathrm{Cons}_P^{\mathrm{hyp}}(-;\mathcal{E})\colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_{\infty}$$

are homotopy-invariant.

Again, passing to global sections, Theorem 0.4 implies that (under the above hypotheses) sheaf cohomology with coefficients in a constructible sheaf is homotopy-invariant.

Linear overview

In § 1, we recall some background from sheaf theory and explain the subtle differences between the locally constant and constructible sheaves and hypersheaves that we use in this work. To prove Theorems 0.3 and 0.4, we need to show that the pullback functor

$$\mathrm{pr}_S^*\colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S \times [0,1];\mathcal{E}) \tag{0.5}$$

is an equivalence when restricted to the appropriate subcategory of locally constant or constructible sheaves. In §2, we recall why pr_S^* is fully faithful and admits a left adjoint $\mathrm{pr}_{S,!}$ (§2.1). Thus our work is in showing that for a locally constant or constructible sheaf F on $S \times [0,1]$, the unit $F \to \mathrm{pr}_S^* \mathrm{pr}_{S,!}(F)$ is an equivalence. We then follow the method of proof that Clausen–Ørsnes Jansen used to show that when P satisfies the ascending chain condition, the functor $S \mapsto \mathrm{Cons}_P(S;\mathbf{Spc})$ is homotopy-invariant [4, §3]. We first show that $\mathrm{pr}_{S,!}$ is compatible with pullbacks in the S variable (§2.2). Section 3 is dedicated to showing that $\mathrm{pr}_{S,!}$ preserves constructibility and using this to deduce Theorems 0.3 and 0.4. Section 4 is dedicated to analyzing the essential image of pr_S^*; this can also be used to prove (0.4.1).

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1 Sheaf-theoretic background

The purpose of this section is to explain our sheaf-theoretic conventions and notation as well as the differences between the notions of local constancy and constructibility for sheaves and hypersheaves.

1.1 Sheaves & hypersheaves

1.1 Notation. Let S be a topological space. We write $\mathrm{Sh}(S)$ for the ∞-topos of sheaves of spaces on S. We write $\mathrm{Sh}^{\mathrm{hyp}}(S) \subset \mathrm{Sh}(S)$ for the full subcategory spanned by the *hypersheaves*, and write $(-)^{\mathrm{hyp}} \colon \mathrm{Sh}(S) \to \mathrm{Sh}^{\mathrm{hyp}}(S)$ for the left exact left adjoint to the inclusion. The functor $F \mapsto F^{\mathrm{hyp}}$ is called *hypercompletion*. The reader unfamiliar with hypercomplete objects and hypercompletion should consult [HTT, §§6.5.2–6.5.4] or [3, §3.11].

Throughout this section, fix a presentable ∞-category $\mathcal{E}$. The ∞-categories $\mathrm{Sh}(S;\mathcal{E})$ and $\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ of *$\mathcal{E}$-valued (hyper)sheaves* on S are given by the tensor product of presentable ∞-categories

$$\mathrm{Sh}(S;\mathcal{E}) \coloneqq \mathrm{Sh}(S) \otimes \mathcal{E} \qquad \text{and} \qquad \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \coloneqq \mathrm{Sh}^{\mathrm{hyp}}(S) \otimes \mathcal{E}\,.$$

The reader unfamiliar with tensor products of presentable ∞-categories should consult [HA, §4.8.1]. Tensoring the fully faithful right adjoint $\mathrm{Sh}^{\mathrm{hyp}}(S) \hookrightarrow \mathrm{Sh}(S)$ with the presentable ∞-category $\mathcal{E}$, we obtain a fully faithful right adjoint $\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \hookrightarrow \mathrm{Sh}(S;\mathcal{E})$. We still denote its left adjoint by $(-)^{\mathrm{hyp}}$.

1.2. The ∞-category $\mathrm{Sh}(S;\mathcal{E})$ is naturally identified with the ∞-category of $\mathcal{E}$-valued presheaves $\mathrm{Open}(S)^{\mathrm{op}} \to \mathcal{E}$ that satisfy descent with respect to open covers [SAG, §1.3.1]. However, expressing $\mathrm{Sh}(S;\mathcal{E})$ as the tensor product $\mathrm{Sh}(S) \otimes \mathcal{E}$ allows us to reduce many claims about $\mathcal{E}$-valued sheaves to claims about space-valued sheaves.

1.3. If there exists an integer $n \geq 0$ such that $\mathcal{E}$ is an n-category, then $\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) = \mathrm{Sh}(S;\mathcal{E})$ [HTT, Lemma 6.5.2.9; HA, Example 4.8.1.22]. In particular, every sheaf of sets is a hypersheaf.

1.4. Sheaves and hypersheaves coincide in many situations in which homotopy-invariance of topological spaces is a well-behaved notion. For example, the ∞-topos of sheaves on a topological space admitting a CW structure is hypercomplete [6].

Now we recall a bit about the functoriality of hypersheaves.

1.5 Recollection. Let $f\colon T \to S$ be a map of topological spaces. Then the pushforward functor $f_*\colon \mathrm{Sh}(T;\mathcal{E}) \to \mathrm{Sh}(S;\mathcal{E})$ carries hypersheaves to hypersheaves (see the proof of [HTT, Proposition 6.5.2.13]). However, if $F \in \mathrm{Sh}(S;\mathcal{E})$ is hypercomplete, then the pullback $f^*(F)$ need not be hypercomplete. We write $f^{*,\mathrm{hyp}}$ for the composite

$$f^{*,\mathrm{hyp}}\colon\ \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \xrightarrow{f^*} \mathrm{Sh}(T;\mathcal{E}) \xrightarrow{(-)^{\mathrm{hyp}}} \mathrm{Sh}^{\mathrm{hyp}}(T;\mathcal{E})\ .$$

Note that $f^{*,\mathrm{hyp}}$ is left adjoint to $f_*\colon \mathrm{Sh}^{\mathrm{hyp}}(T;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$.

1.6. If the space-valued sheaf pullback functor $f^*\colon \mathrm{Sh}(S) \to \mathrm{Sh}(T)$ admits a left adjoint, then for every presentable ∞-category $\mathcal{E}$, the pullback functor $f^*\colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(T;\mathcal{E})$ carries hypersheaves to hypersheaves [HA, Lemma A.2.6]. In particular, if $U \subset S$ is an open subset, then the restriction functor $(-)|_U\colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(U;\mathcal{E})$ carries hypersheaves to hypersheaves.

We now turn to the notion of local constancy in the context of sheaves and hypersheaves.

1.7 Notation. Let S be a topological space. We write $\Gamma_S\colon S \to *$ for the unique morphism. Thus $\Gamma_{S,*}\colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(*;\mathcal{E}) \simeq \mathcal{E}$ is the global sections functor and

$$\Gamma_S^*\colon \mathcal{E} \to \mathrm{Sh}(S;\mathcal{E}) \qquad \text{and} \qquad \Gamma_S^{*,\mathrm{hyp}}\colon \mathcal{E} \to \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$$

are the constant sheaf and hypersheaf functors.

1.8 Definition. Let S be a topological space.

(1.8.1) We say that a sheaf $L \in \mathrm{Sh}(S;\mathcal{E})$ is *constant* if L is in the essential image of the constant sheaf functor $\Gamma_S^*\colon \mathcal{E} \to \mathrm{Sh}(S;\mathcal{E})$.

(1.8.2) We say that $L \in \mathrm{Sh}(S;\mathcal{E})$ is *locally constant* if there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of S such that for each $\alpha \in A$, the restriction $L|_{U_\alpha}$ is a constant sheaf on U_α.

(1.8.3) We say that a hypersheaf $L \in \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ is *hyperconstant* if L is in the essential image of the constant hypersheaf functor $\Gamma_S^{*,\mathrm{hyp}}\colon \mathcal{E} \to \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$.

(1.8.4) We say that $L \in \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ is *locally hyperconstant*[2] if there exists an open cover $\{U_\alpha\}_{\alpha\in A}$ of S such that for each $\alpha \in A$, the restriction $L|_{U_\alpha}$ is a constant *hyper*sheaf on U_α.

We write

$$\mathrm{LC}(S;\mathcal{E}) \subset \mathrm{Sh}(S;\mathcal{E}) \qquad \text{and} \qquad \mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E}) \subset \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$$

for the full subcategories spanned by the locally constant sheaves and locally hyperconstant hypersheaves, resepctively.

1.9 Warning. We emphasize that for a given object $E \in \mathcal{E}$, the constant sheaf $\Gamma_S^*(E)$ need not be hypercomplete. Similarly, a hyperconstant hypersheaf need not be a constant sheaf; the notions of constant sheaves and hyperconstant hypersheaves are genuinely different. Also notice that there is a containment

$$\mathrm{LC}(S;\mathcal{E}) \cap \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \subset \mathrm{LC}^{\mathrm{hyp}}(S;\mathcal{E})\ .$$

However, this inclusion is not generally an equality.

[2] This terminology is due to Lejay [7, §1.3]. We originally used different terminology, but find that Lejay's is the most clear.

1.10 Observation. For any map $f\colon T \to S$ of topological spaces, the sheaf pullback functor

$$f^*\colon \mathrm{Sh}(S; \mathcal{E}) \to \mathrm{Sh}(T; \mathcal{E})$$

preserves locally constant sheaves and the hypersheaf pullback functor

$$f^{*,\mathrm{hyp}}\colon \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(T; \mathcal{E})$$

preserves locally hyperconstant hypersheaves. Hence the assignments

$$S \mapsto \mathrm{LC}(S; \mathcal{E}) \qquad \text{and} \qquad S \mapsto \mathrm{LC}^{\mathrm{hyp}}(S; \mathcal{E})$$

define subfunctors of the functors $\mathrm{Sh}(-; \mathcal{E}), \mathrm{Sh}^{\mathrm{hyp}}(-; \mathcal{E})\colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}_\infty$.

1.2 Stratified spaces & constructible sheaves

We now turn to sheaves that are locally constant along the strata of a stratification.

1.11 Notation. Let P be a poset. We also write P for the set P equipped with the *Alexandroff topology* in which a subset $U \subset P$ is open if and only if U is upwards-closed. Given an element $p \in P$, we write

$$P_{\geq p} \coloneqq \{q \in P \,|\, q \geq p\} \qquad \text{and} \qquad P_{>p} \coloneqq P_{\geq p} \smallsetminus \{p\}\ .$$

The category of *P-stratified topological spaces* is the overcategory $\mathbf{Top}_{/P}$. Given a P-stratified topological space $\sigma\colon S \to P$ and $p \in P$, we write $S_p \coloneqq \sigma^{-1}(p)$ and call S_p the *p-th stratum* of S. We also write

$$S_{\geq p} \coloneqq \sigma^{-1}(P_{\geq p}) \qquad \text{and} \qquad S_{>p} \coloneqq \sigma^{-1}(P_{>p})\ .$$

1.12 Definition. Let P be a poset and $S \to P$ be a P-stratified topological space.

(1.12.1) We say that a sheaf $F \in \mathrm{Sh}(S; \mathcal{E})$ is a *P-constructible* if F for each $p \in P$, the restriction $F|_{S_p}$ is a locally constant sheaf on the stratum S_p.

(1.12.2) We say that a hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E})$ is a *P-hyperconstructible*[3] if F for each $p \in P$, the hypersheaf restriction $(F|_{S_p})^{\mathrm{hyp}}$ is a locally hyperconstant hypersheaf on the stratum S_p.

We write

$$\mathrm{Cons}_P(T; \mathcal{E}) \subset \mathrm{Sh}(T; \mathcal{E}) \qquad \text{and} \qquad \mathrm{Cons}_P^{\mathrm{hyp}}(T; \mathcal{E}) \subset \mathrm{Sh}^{\mathrm{hyp}}(T; \mathcal{E})$$

for the full subcategories spanned by the P-constructible sheaves and P-hyperconstructible hypersheaves, resepctively.

1.13 Warning. There is a containment

$$\mathrm{Cons}_P(S; \mathcal{E}) \cap \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \subset \mathrm{Cons}_P^{\mathrm{hyp}}(S; \mathcal{E})\ ,$$

however, this inclusion need not be an equality. Also note that if F is a P-constructible sheaf, then then $F^{\mathrm{hyp}} \in \mathrm{Cons}_P^{\mathrm{hyp}}(S; \mathcal{E})$.

[3] Again, this terminology is due to Lejay [7, §1.4].

1.14 Observation. For any map $f: T \to S$ of P-stratified spaces, the sheaf pullback functor

$$f^*: \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(T;\mathcal{E})$$

preserves P-constructible sheaves and the hypersheaf pullback functor

$$f^{*,\mathrm{hyp}}: \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(T;\mathcal{E})$$

preserves P-hyperconstructible hypersheaves. Hence the assignments

$$S \mapsto \mathrm{Cons}_P(S;\mathcal{E}) \qquad \text{and} \qquad S \mapsto \mathrm{Cons}_P^{\mathrm{hyp}}(S;\mathcal{E})$$

define subfunctors of the functors $\mathrm{Sh}(-;\mathcal{E}), \mathrm{Sh}^{\mathrm{hyp}}(-;\mathcal{E}): \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$.

1.15 Warning. The hypersheaf pullback $f^{*,\mathrm{hyp}}$ need not carry $\mathrm{Cons}_P(S;\mathcal{E}) \cap \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ to $\mathrm{Cons}_P(T;\mathcal{E}) \cap \mathrm{Sh}^{\mathrm{hyp}}(T;\mathcal{E})$. In particular, it does not appear that there is a way to extend the assignment on objects

$$S \mapsto \mathrm{Cons}_P(S;\mathcal{E}) \cap \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$$

into a functor $\mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$.

The main aim of this paper is to explain when the functors $\mathrm{LC}(-;\mathcal{E})$, $\mathrm{LC}^{\mathrm{hyp}}(-;\mathcal{E})$, $\mathrm{Cons}_P(-;\mathcal{E})$, and $\mathrm{Cons}_P^{\mathrm{hyp}}(-;\mathcal{E})$ satisfy the following homotopy-invariance property.

1.16 Convention. Let P be a poset and $\sigma: S \to P$ be a P-stratified topological space. Write $S \times [0,1]$ for the P-stratified topological space with stratification given by the composite

$$S \times [0,1] \xrightarrow{\mathrm{pr}_S} S \xrightarrow{\sigma} P .$$

1.17 Definition. Let P be a poset. A functor $C: \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is *homotopy-invariant* if for each P-stratified space S, the induced functor

$$C(\mathrm{pr}_S): C(S) \to C(S \times [0,1])$$

is an equivalence of ∞-categories.

1.18 Remark. Recall that a map $f: T \to S$ of P-stratified spaces is called a *stratified homotopy equivalence* if there exists a map of P-stratified spaces $g: S \to T$ and P-stratified homotopies between gf and id_T as well as between fg and id_S. By the 2-of-3 property, a functor $C: \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ is homotopy-invariant in the sense of Definition 1.17 if and only if for each P-stratified homotopy equivalence $f: T \to S$, the functor $C(f): C(S) \to C(T)$ is an equivalence of ∞-categories.

2 The exceptional pushforward

In order to explain the contents of this section, let us outline a strategy to prove Theorems 0.3 and 0.4. For the purposes of this discussion we consider only locally constant and constructible sheaves, and suppress their hypercomplete variants.

2.1 Proof Outline. We want to understand when the pullback functors

$$\mathrm{pr}_S^*\colon \mathrm{LC}(S;\mathcal{E}) \to \mathrm{LC}(S\times[0,1];\mathcal{E}) \qquad \text{and} \qquad \mathrm{pr}_S^*\colon \mathrm{Cons}_P(S;\mathcal{E}) \to \mathrm{Cons}_P(S\times[0,1];\mathcal{E})$$

are equivalences. For this, it helps to consider the pullback on all sheaves: one can show that

$$\mathrm{pr}_S^*\colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S\times[0,1];\mathcal{E})$$

is fully faithful and admits a left adjoint $\mathrm{pr}_{S,!}$ (see Example 2.4). We refer to $\mathrm{pr}_{S,!}$ as the *exceptional pushforward*. As an immediate consequence, the adjunction $\mathrm{pr}_{S,!} \dashv \mathrm{pr}_S^*$ restricts to an equivalence between constant sheaves on S and $S\times[0,1]$. Thus, if we can show that:

(2.1.1) The exceptional pushforward functors $\mathrm{pr}_{S,!}$ commute with pullbacks in the S variable.

(2.1.2) The functor $\mathrm{pr}_{S,!}\colon \mathrm{Sh}(S\times[0,1];\mathcal{E}) \to \mathrm{Sh}(S;\mathcal{E})$ preserves locally constant sheaves.

then

(2.1.3) The exceptional pushforward $\mathrm{pr}_{S,!}$ preserves constructible sheaves.

(2.1.4) The adjunction $\mathrm{pr}_{S,!} \dashv \mathrm{pr}_S^*$ restricts to an equivalence $\mathrm{LC}(S;\mathcal{E}) \simeq \mathrm{LC}(S\times[0,1];\mathcal{E})$.

Moreover, if equivalences of constructible sheaves can be checked on strata, then (2.1.1), (2.1.3), and (2.1.4) together imply the homotopy-invariance of constructible sheaves.

In § 2.1, we introduce the exceptional pushforward $\mathrm{pr}_{S,!}$. Subsection 2.2 is dedicated to proving the compatibility of the exceptional pushforward with pullbacks (2.1.1). We treat steps (2.1.2)–(2.1.4) as well as the question of when equivalences of constructible sheaves can be checked on strata in § 3.

2.1 Preliminary observations

Now we introduce the exceptional pushforward $\mathrm{pr}_{S,!}$. The easiest way to prove properties of the exceptional pushforward is to express $\mathrm{Sh}(S\times[0,1])$ as the tensor product of presentable ∞-categories $\mathrm{Sh}(S)\otimes\mathrm{Sh}([0,1])$ in order to reduce to the case $S=*$. We make these observations a bit more generally since we also have occasion to replace $[0,1]$ by open or half-open intervals.

2.2 Recollection. Let S and X be topological spaces. There is a natural geometric morphism of ∞-topoi

$$\mathrm{Sh}(S\times X) \to \mathrm{Sh}(S)\otimes\mathrm{Sh}(X)$$

[HA, Example 4.8.1.19]. Moreover, if X is locally compact, then this natural geometric morphism $\mathrm{Sh}(S\times X) \to \mathrm{Sh}(S)\otimes\mathrm{Sh}(X)$ is an equivalence [HTT, Proposition 7.3.1.11].

2.3 Recollection. Let S be a topological space and $\mathcal{E}$ a presentable ∞-category. Let X be a locally compact topological space locally of constant shape in the sense of [HA, Definition A.1.5]. For example, $X=[0,1]$. Then:

(2.3.1) The constant sheaf functor $\Gamma_X^*\colon \mathbf{Spc} \to \mathrm{Sh}(X)$ admits a left adjoint $\Gamma_{X,!}$ [HA, Proposition A.1.8].

(2.3.2) By tensoring with $\mathrm{Sh}(S)$ and applying Recollection 2.2, we deduce that the pullback functor

$$\mathrm{pr}_S^*\colon\ \mathrm{Sh}(S) \simeq \mathrm{Sh}(S)\otimes\mathrm{Sh}(*) \xrightarrow{\mathrm{id}_{\mathrm{Sh}(S)}\otimes\Gamma_X^*} \mathrm{Sh}(S)\otimes\mathrm{Sh}(X) \simeq \mathrm{Sh}(S\times X)$$

admits a left adjoint

$$\mathrm{pr}_{S,!} : \mathrm{Sh}(S \times X) \simeq \mathrm{Sh}(S) \otimes \mathrm{Sh}(X) \xrightarrow{\mathrm{id}_{\mathrm{Sh}(S)} \otimes \Gamma_{X,!}} \mathrm{Sh}(S) \otimes \mathrm{Sh}(*) \simeq \mathrm{Sh}(S) .$$

In particular, $\mathrm{pr}_S^* : \mathrm{Sh}(S) \to \mathrm{Sh}(S \times X)$ preserves hypersheaves (1.6).

(2.3.3) By further tensoring with $\mathcal{E}$, we deduce that the pullback functor $\mathrm{pr}_S^* : \mathrm{Sh}(S; \mathcal{E}) \to \mathrm{Sh}(S \times X; \mathcal{E})$ preserves hypersheaves and admits a left adjoint $\mathrm{pr}_{S,!}$.

(2.3.4) The functor $\mathrm{pr}_S^* : \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E})$ is right adjoint to the composite

$$\mathrm{pr}_{S,!}^{\mathrm{hyp}} : \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E}) \xrightarrow{\mathrm{pr}_{S,!}} \mathrm{Sh}(S; \mathcal{E}) \xrightarrow{(-)^{\mathrm{hyp}}} \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) .$$

(2.3.5) Since $\mathrm{pr}_S^* : \mathrm{Sh}(S; \mathcal{E}) \to \mathrm{Sh}(S \times X; \mathcal{E})$ is given by applying the tensor product of presentable ∞-categories to the right adjoint functor $\Gamma_X^* : \mathbf{Spc} \to \mathrm{Sh}(X)$, if Γ_Y^* is fully faithful, then pr_S^* is also fully faithful.

2.4 Example. Let S be a topological space, let $\mathcal{E}$ be a presentable ∞-category, and let $I \subset \mathbf{R}$ be an open, closed, or half-open interval. Combining Recollection 2.3 with [HA, Lemma A.2.2 & Remark A.2.3], we see that the pullback functor

$$\mathrm{pr}_S^* : \mathrm{Sh}(S; \mathcal{E}) \to \mathrm{Sh}(S \times I; \mathcal{E})$$

is fully faithful, preserves hypersheaves, and admits a left adjoint $\mathrm{pr}_{S,!}$.

2.2 Compatibility with pullbacks

Now we prove that the exceptional pushforwards $\mathrm{pr}_{S,!}$ and $\mathrm{pr}_{S,!}^{\mathrm{hyp}}$ are compatible with pullbacks. To do so, we prove the equivalent statement for right adjoints as it is more straightforward.

2.5 Lemma. *Let $f : T \to S$ be a map of topological spaces, let X be a locally compact topological space locally of constant shape, and let $\mathcal{E}$ be a presentable ∞-category. Then there are canonically commutative squares of ∞-categories*

$$\begin{array}{ccc} \mathrm{Sh}(T; \mathcal{E}) & \xrightarrow{f_*} & \mathrm{Sh}(S; \mathcal{E}) \\ \downarrow{\scriptstyle \mathrm{pr}_T^*} & & \downarrow{\scriptstyle \mathrm{pr}_S^*} \\ \mathrm{Sh}(T \times X; \mathcal{E}) & \xrightarrow[(f \times \mathrm{id}_X)_*]{} & \mathrm{Sh}(S \times X; \mathcal{E}) \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathrm{Sh}^{\mathrm{hyp}}(T; \mathcal{E}) & \xrightarrow{f_*} & \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \\ \downarrow{\scriptstyle \mathrm{pr}_T^*} & & \downarrow{\scriptstyle \mathrm{pr}_S^*} \\ \mathrm{Sh}^{\mathrm{hyp}}(T \times X; \mathcal{E}) & \xrightarrow[(f \times \mathrm{id}_X)_*]{} & \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E}) . \end{array}$$

Proof. The commutativity of the right-hand square follows from the commutativity of the left-hand square and the fact that all functors in the square preserve hypersheaves. To see that the left-hand square is commutative, note that since all functors involved are right adjoints, by tensoring with the presentable ∞-category $\mathcal{E}$ it suffices to prove the claim for sheaves of spaces. In this case, since X is locally compact, Recollection 2.2 imples that the claim is equivalent to showing that the square

$$\begin{array}{ccc} \mathrm{Sh}(T) \otimes \mathrm{Sh}(*) & \xrightarrow{f_* \otimes \mathrm{id}_{\mathrm{Sh}(*)}} & \mathrm{Sh}(S) \otimes \mathrm{Sh}(*) \\ \downarrow{\scriptstyle \mathrm{id}_{\mathrm{Sh}(T)} \otimes \Gamma_X^*} & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{Sh}(S)} \otimes \Gamma_X^*} \\ \mathrm{Sh}(T) \otimes \mathrm{Sh}(X) & \xrightarrow[f_* \otimes \mathrm{id}_{\mathrm{Sh}(X)}]{} & \mathrm{Sh}(S) \otimes \mathrm{Sh}(X) \end{array}$$

commutes. This is immediate: since f_* and Γ_X^* are right adjoints, both composites are identified with the tensor product $f_* \otimes \Gamma_X^*$ (see [5, Observation 1.15]). ☐

Passing to left adjoints we deduce:

2.6 Corollary. *In the notation of Lemma 2.5, there are canonically commutative squares of ∞-categories*

$$\begin{array}{ccc} \mathrm{Sh}(S \times X; \mathcal{E}) & \xrightarrow{(f \times \mathrm{id}_X)^*} & \mathrm{Sh}(T \times X; \mathcal{E}) \\ {\scriptstyle \mathrm{pr}_{S,!}} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_{T,!}} \\ \mathrm{Sh}(S; \mathcal{E}) & \xrightarrow[f^*]{} & \mathrm{Sh}(T; \mathcal{E}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Sh}^{\mathrm{hyp}}(S \times X; \mathcal{E}) & \xrightarrow{(f \times \mathrm{id}_X)^{*,\mathrm{hyp}}} & \mathrm{Sh}^{\mathrm{hyp}}(T \times X; \mathcal{E}) \\ {\scriptstyle \mathrm{pr}_{S,!}^{\mathrm{hyp}}} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_{T,!}^{\mathrm{hyp}}} \\ \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) & \xrightarrow[f^{*,\mathrm{hyp}}]{} & \mathrm{Sh}^{\mathrm{hyp}}(T; \mathcal{E}) \, . \end{array}$$

We conclude this section by using Corollary 2.6 to show that one can check that the unit morphism $u_F \colon F \to \mathrm{pr}_S^* \, \mathrm{pr}_{S,!}(F)$ is an equivalence locally. In fact, we prove a slightly more general result that applies to the strata of a suitable stratification:

2.7 Lemma. *Let S be a topological space, $\{f_\alpha \colon S_\alpha \to S\}_{\alpha \in A}$ a collection of maps of topological spaces, and $\mathcal{E}$ a presentable ∞-category.*

(2.7.1) *Assume that the pullback functors*

$$\{(f_\alpha \times \mathrm{id}_{[0,1]})^* \colon \mathrm{Sh}(S \times [0,1]; \mathcal{E}) \to \mathrm{Sh}(S_\alpha \times [0,1]; \mathcal{E})\}_{\alpha \in A}$$

are jointly conservative. Given $F \in \mathrm{Sh}(S \times [0,1]; \mathcal{E})$, the unit $u_F \colon F \to \mathrm{pr}_S^* \, \mathrm{pr}_{S,!}(F)$ is an equivalence if and only if for each $\alpha \in A$, the unit

$$(f_\alpha \times \mathrm{id}_{[0,1]})^*(F) \to \mathrm{pr}_{S_\alpha}^* \, \mathrm{pr}_{S_\alpha,!}(f_\alpha \times \mathrm{id}_{[0,1]})^*(F)$$

is an equivalence.

(2.7.2) *Assume that the hypersheaf pullback functors*

$$\{(f_\alpha \times \mathrm{id}_{[0,1]})^{*,\mathrm{hyp}} \colon \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S_\alpha \times [0,1]; \mathcal{E})\}_{\alpha \in A}$$

are jointly conservative. Given $F \in \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E})$, the unit $u_F \colon F \to \mathrm{pr}_S^* \, \mathrm{pr}_{S,!}^{\mathrm{hyp}}(F)$ is an equivalence if and only if for each $\alpha \in A$, the unit

$$(f_\alpha \times \mathrm{id}_{[0,1]})^{*,\mathrm{hyp}}(F) \to \mathrm{pr}_{S_\alpha}^* \, \mathrm{pr}_{S_\alpha,!}^{\mathrm{hyp}}(f_\alpha \times \mathrm{id}_{[0,1]})^{*,\mathrm{hyp}}(F)$$

is an equivalence.

Proof. We prove (2.7.1); the proof of (2.7.2) is exactly the same, replacing sheaves by hypersheaves. By Corollary 2.6, we have a natural equivalence

$$(f_\alpha \times \mathrm{id}_{[0,1]})^* \, \mathrm{pr}_S^* \, \mathrm{pr}_{S,!}(F) \simeq \mathrm{pr}_{S_\alpha}^* \, \mathrm{pr}_{S_\alpha,!}(f_\alpha \times \mathrm{id}_{[0,1]})^*(F) \, .$$

Moreover, notice that the pullback

$$(f_\alpha \times \mathrm{id}_{[0,1]})^*(u_F) \colon (f_\alpha \times \mathrm{id}_{[0,1]})^*(F) \to \mathrm{pr}_{S_\alpha}^* \, \mathrm{pr}_{S_\alpha,!}(f_\alpha \times \mathrm{id}_{[0,1]})^*(F)$$

is homotopic to the unit of the adjunction $\mathrm{pr}_{S_\alpha,!} \dashv \mathrm{pr}_{S_\alpha}^*$ applied to the sheaf $(f_\alpha \times \mathrm{id}_{[0,1]})^*(F)$. The claim now follows from the assumption that the functors $\{(f_\alpha \times \mathrm{id}_{[0,1]})^*\}_{\alpha \in A}$ are jointly conservative. ☐

3 Homotopy-invariance

In this section, we prove our main homotopy-invariance results for locally constant sheaves (Theorem 0.3) and constructible sheaves (Theorem 0.4). To do this, in §3.1 we start by verifying the second step (2.1.2) in our proof by checking that the exceptional pushforward $\mathrm{pr}_{S,!}$ preserves locally constant sheaves (see Lemma 3.5). From this we deduce that $\mathrm{pr}_{S,!}$ preserves constructible sheaves (Corollary 3.6) and that locally constant sheaves are homotopy-invariant (Corollary 3.7). In §3.2, we explain how to one can bootstrap the case of locally constant sheaves to prove that constructible sheaves are homotopy-invariant under mild conditions on the stratifying poset P and coefficient ∞-category $\mathcal{E}$. We also treat hypercomplete variants of these statements.

3.1 Homotopy-invariance of locally constant sheaves

In order to show that $\mathrm{pr}_{S,!}$ preserves locally constant sheaves, we use the fact that that every locally constant sheaf on $S \times [0,1]$ can be trivialized on a cover of the form $\{U_\alpha \times [0,1]\}_{\alpha\in A}$ (Lemma 3.3). This follows from the full-faithfulness of the pullback functor $\mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S\times[0,1];\mathcal{E})$ combined with a lemma about the topology of $S\times[0,1]$. One can check the following by using the compactness of $[0,1]$ and the fact that $[0,1]$ has the order topology:

3.1 Lemma. *Let S be a topological space and $\mathcal{U}$ an open cover of $S\times[0,1]$. Then there exist:*

(3.1.1) *An open cover $\{U_\alpha\}_{\alpha\in A}$ of S.*

(3.1.2) *For each $\alpha \in A$, a positive integer n_α and open subintervals $I_{\alpha,1},\dots,I_{\alpha,n_\alpha}$ of $[0,1]$ covering $[0,1]$ such that $I_{\alpha,k}\cap I_{\alpha,\ell} \neq \varnothing$ if and only if $k = \ell \pm 1$.*

Such that $\bigcup_{\alpha\in A}\{U_\alpha\times I_{\alpha,1},\dots,U_\alpha\times I_{\alpha,n_\alpha}\}$ refines the cover $\mathcal{U}$.

3.2 Observation. Let $\mathcal{E}$ be a presentable ∞-category, U a topological space, and $I, J \subset [0,1]$ subintervals with nonempty intersection which are open in $[0,1]$. Consider the following commutative diagram of pullback functors

$$\begin{array}{ccccc}
\mathrm{Sh}(U\times I;\mathcal{E}) & \xrightarrow{(-)|_{U\times(I\cap J)}} & \mathrm{Sh}(U\times(I\cap J);\mathcal{E}) & \xleftarrow{(-)|_{U\times(I\cap J)}} & \mathrm{Sh}(U\times J;\mathcal{E}) \\
 & {\scriptstyle \mathrm{pr}_U^*}\nwarrow & \uparrow {\scriptstyle \mathrm{pr}_U^*} & \nearrow{\scriptstyle \mathrm{pr}_U^*} & \\
 & & \mathrm{Sh}(U;\mathcal{E}) & &
\end{array}$$

Note that if $F_I \in \mathrm{Sh}(U\times I;\mathcal{E})$ and $F_J \in \mathrm{Sh}(U\times J;\mathcal{E})$ are pulled back from U and

$$F_I|_{U\times(I\cap J)} \simeq F_J|_{U\times(I\cap J)} ,$$

then there exists a unique sheaf $G \in \mathrm{Sh}(U;\mathcal{E})$ such that

$$F_I \simeq \mathrm{pr}_U^*(G) \qquad \text{and} \qquad F_J \simeq \mathrm{pr}_U^*(G) .$$

In particular, if $L \in \mathrm{Sh}(U\times(I\cup J);\mathcal{E})$ is a sheaf such that both $L|_{U\times I}$ and $L|_{U\times J}$ are constant, then L is constant. Similarly, if $L \in \mathrm{Sh}^{\mathrm{hyp}}(U\times(I\cup J);\mathcal{E})$ is a hypersheaf such that both $L|_{U\times I}$ and $L|_{U\times J}$ are hyperconstant, then L is hyperconstant.

3.3 Lemma. *Let S be a topolocial space, $\mathcal{E}$ a presentable ∞-category, and $L \in \mathrm{Sh}(S\times[0,1];\mathcal{E})$.*

(3.3.1) *If $L \in \mathrm{LC}(S \times [0,1]; \mathcal{E})$, then there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of S such that for each $\alpha \in A$, the sheaf $L|_{U_\alpha \times [0,1]}$ is constant.*

(3.3.2) *If $L \in \mathrm{LC}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E})$, then there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of S such that for each $\alpha \in A$, the hypersheaf $L|_{U_\alpha \times [0,1]}$ is hyperconstant.*

Proof. We prove (3.3.1); the proof of (3.3.2) is exactly the same, replacing sheaves by hypersheaves. Choose an open cover

$$\bigcup_{\alpha \in A} \{U_\alpha \times I_{\alpha,1}, \dots, U_\alpha \times I_{\alpha,n_\alpha}\}$$

of $S \times [0,1]$ as in Lemma 3.1 such that each restriction $L|_{U_\alpha \times I_{\alpha,k}}$ is constant. We claim that for each $\alpha \in A$, the restriction $L|_{U_\alpha \times [0,1]}$ is constant. To see this, apply Observation 3.2 inductively with $I = I_{\alpha,1} \cup \dots \cup I_{\alpha,m}$ and $J = I_{\alpha,m+1}$. □

From Lemma 3.3 we deduce that the exceptional pushforwards $\mathrm{pr}_{S,!}$ and $\mathrm{pr}^{\mathrm{hyp}}_{S,!}$ preserve locally (hyper)constant (hyper)sheaves.

3.4 Observation. Note that the constant sheaf and hypersheaf functors

$$\Gamma^*_{S \times [0,1]} : \mathcal{E} \to \mathrm{Sh}(S \times [0,1]; \mathcal{E}) \qquad \text{and} \qquad \Gamma^{*,\mathrm{hyp}}_{S \times [0,1]} : \mathcal{E} \to \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E})$$

are given by the composites $\Gamma^*_{S \times [0,1]} \simeq \mathrm{pr}_S^* \, \Gamma_S^*$ and $\Gamma^{*,\mathrm{hyp}}_{S \times [0,1]} \simeq \mathrm{pr}_S^* \, \Gamma_S^{*,\mathrm{hyp}}$. Hence:

(3.4.1) If $F \in \mathrm{Sh}(S \times [0,1]; \mathcal{E})$ is constant, then $\mathrm{pr}_{S,!}(F)$ is constant and the unit $F \to \mathrm{pr}_S^* \, \mathrm{pr}_{S,!}(F)$ is an equivalence.

(3.4.2) If $F \in \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E})$ is hyperconstant, then the hypersheaf $\mathrm{pr}^{\mathrm{hyp}}_{S,!}(F)$ is hyperconstant and the unit $F \to \mathrm{pr}_S^* \, \mathrm{pr}^{\mathrm{hyp}}_{S,!}(F)$ is an equivalence.

3.5 Lemma. *Let S be a topological space and $\mathcal{E}$ a presentable ∞-category. Then:*

(3.5.1) *The functor $\mathrm{pr}_{S,!} : \mathrm{Sh}(S \times [0,1]; \mathcal{E}) \to \mathrm{Sh}(S; \mathcal{E})$ preserves locally constant sheaves.*

(3.5.2) *The functor $\mathrm{pr}^{\mathrm{hyp}}_{S,!} : \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E})$ preserves locally hyperconstant hypersheaves.*

Proof. We prove (3.5.1); the proof of (3.5.2) is exactly the same, replacing sheaves by hypersheaves. Let $F \in \mathrm{LC}(S \times [0,1]; \mathcal{E})$. Using Lemma 3.3, choose an open cover $\{U_\alpha\}_{\alpha \in A}$ of S such that each of the restrictions $F|_{U_\alpha \times [0,1]}$ is constant. By Corollary 2.6 we have

$$\mathrm{pr}_{S,!}(F)|_{U_\alpha} \simeq \mathrm{pr}_{U_\alpha,!}(F|_{U_\alpha \times [0,1]}) .$$

Hence Observation 3.4 shows that the sheaf $\mathrm{pr}_{S,!}(F)|_{U_\alpha}$ is constant. □

The compatibility of the exceptional pushforwards $\mathrm{pr}_{S,!}$ and $\mathrm{pr}^{\mathrm{hyp}}_{S,!}$ with pullbacks shows:

3.6 Corollary. *Let P be a poset, $S \in \mathbf{Top}_{/P}$, and $\mathcal{E}$ a presentable ∞-category. Then:*

(3.6.1) *The functor $\mathrm{pr}_{S,!} : \mathrm{Sh}(S \times [0,1]; \mathcal{E}) \to \mathrm{Sh}(S; \mathcal{E})$ preserves constructible sheaves.*

(3.6.2) *The functor $\mathrm{pr}^{\mathrm{hyp}}_{S,!} : \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E})$ preserves hyperconstructible hypersheaves.*

Proof. Combine Lemma 3.5 with Corollary 2.6. □

The homotopy-invariance of locally constant sheaves now follows from the fact that the adjunction $\mathrm{pr}_{S,!} \dashv \mathrm{pr}_S^*$ restricts to locally constant sheaves.

3.7 Corollary. *Let S be a topological space and $\mathcal{E}$ a presentable ∞-category. Then:*

(3.7.1) *The functors* $\mathrm{pr}_{S,!} \colon \mathrm{LC}(S \times [0,1]; \mathcal{E}) \rightleftarrows \mathrm{LC}(S; \mathcal{E}) : \mathrm{pr}_S^*$ *are inverse equivalences.*

(3.7.2) *The functors* $\mathrm{pr}_{S,!}^{\mathrm{hyp}} \colon \mathrm{LC}(S \times [0,1]; \mathcal{E}) \rightleftarrows \mathrm{LC}(S; \mathcal{E}) : \mathrm{pr}_S^*$ *are inverse equivalences.*

In particular, the functors $\mathrm{LC}(-; \mathcal{E}), \mathrm{LC}^{\mathrm{hyp}}(-; \mathcal{E}) \colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ *are homotopy-invariant*

Proof. We prove (3.7.1); the proof of (3.7.2) is exactly the same, replacing sheaves by hypersheaves. Note that since pr_S^* is fully faithful, it suffices to show that if $F \in \mathrm{LC}(S \times [0,1]; \mathcal{E})$, then the unit

$$u \colon F \to \mathrm{pr}_S^* \mathrm{pr}_{S,!}(F)$$

is an equivalence. Using Lemma 3.3, choose an open cover $\{U_\alpha\}_{\alpha \in A}$ of S such that each of the restrictions $F|_{U_\alpha \times [0,1]}$ is constant. The claim now follows from Lemma 2.7 and Observation 3.4. □

3.2 Pulling back to strata

We now bootstrap Corollary 3.7 to (hyper)constructible (hyper)sheaves. To do so, note that provided that equivalences of constructible sheaves are detected after pulling back to strata, the homotopy-invariance of locally constant sheaves and Lemma 2.7 imply the homotopy-invariance of constructible sheaves. Thus we need to identify which posets P and presentable ∞-categories $\mathcal{E}$ have the property that equivalences of $\mathcal{E}$-valued sheaves on a P-stratified space are detected after pulling back to strata. First we treat the case of hypersheaves.

3.8 Recollection. Let S be a topological space. Then the stalk functors

$$\{s^* : \mathrm{Sh}^{\mathrm{hyp}}(S) \to \mathrm{Sh}(\{s\}) \simeq \mathbf{Spc}\}_{s \in S}$$

are jointly conservative [HA, Lemma A.3.9]. Since the stalk functors are left exact, [5, Lemma 2.8] shows that for any compactly generated ∞-category $\mathcal{E}$, the stalk functors

$$\{s^* : \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \to \mathrm{Sh}(\{s\}; \mathcal{E}) \simeq \mathcal{E}\}_{s \in S}$$

are jointly conservative. As a consequence:

3.9 Lemma. *Let P be a poset, $S \in \mathbf{Top}_{/P}$, and let $\mathcal{E}$ be a compactly generated ∞-category. Then the hypersheaf restriction functors* $\{(-)|_{S_p}^{\mathrm{hyp}} : \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S_p; \mathcal{E})\}_{p \in P}$ *are jointly conservative.*

3.10 Corollary. *Let P be a poset, $S \in \mathbf{Top}_{/P}$, and $\mathcal{E}$ a compactly generated ∞-category. Then the functors*

$$\mathrm{pr}_{S,!}^{\mathrm{hyp}} : \mathrm{Cons}_P^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E}) \rightleftarrows \mathrm{Cons}_P^{\mathrm{hyp}}(S; \mathcal{E}) : \mathrm{pr}_S^*$$

are inverse equivalences of ∞-categories. In particular, the functor $\mathrm{Cons}_P^{\mathrm{hyp}}(-; \mathcal{E}) \colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ *is homotopy-invariant*

Proof. Combine Lemma 2.7 with Corollary 3.7 and Lemma 3.9. □

Corollary 3.10 implies a the following variant for truncated constructible sheaves.

3.11 Notation. Let P be a poset and $S \to P$ be a P-stratified topological space. We write

$$\mathrm{Cons}_P(S)_{<\infty} \subset \mathrm{Cons}_P(S)$$

for the full subcategory spanned by those P-constructible sheaves that are also n-truncated for some integer $n \geq 0$. Since truncated objects of an ∞-topos are hypercomplete, we see that

$$\mathrm{Cons}_P(S)_{<\infty} \subset \mathrm{Cons}_P^{\mathrm{hyp}}(S) .$$

Since left exact functors preserve truncated objects, given a map $f : T \to S$ of P-stratified spaces, the hypersheaf pullback $f^{*,\mathrm{hyp}}$ restricts to the usual sheaf pullback

$$f^* : \mathrm{Cons}_P(S)_{<\infty} \to \mathrm{Cons}_P(T)_{<\infty} .$$

Thus the assignment $S \mapsto \mathrm{Cons}_P(S)_{<\infty}$ defines a subfunctor of $\mathrm{Cons}_P^{\mathrm{hyp}}$.

3.12 Corollary. *Let P be a poset. The functor* $\mathrm{Cons}_P(-)_{<\infty} : \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ *is homotopy-invariant.*

Now we turn to the non-hypercomplete case. There are two difficulties to deal with. Recall that a poset P is *Noetherian* if P satisfies the *ascending chain condition*: there does not exist an infinite strictly ascending sequence $p_0 < p_1 < p_2 < \cdots$ of elements of P. The first issue is that there exist non-Noetherian posets P for which the ∞-topos $\mathrm{Sh}(P) = \mathrm{Cons}_P(P)$ is not hypercomplete; see [1, Example A.13]. Said differently, the functors $\mathrm{Sh}(P) \to \mathrm{Sh}(\{p\})$ given by pulling back to strata need not be jointly conservative. Thus we restrict ourselves to Noetherian posets.

The second issue is with the coefficient ∞-category $\mathcal{E}$. Consider the most simple stratification when $P = \{0 < 1\}$ so that a stratification $S \to \{0 < 1\}$ is the data of a closed subspace $Z = S_0$ and its open complement $S \smallsetminus Z = S_1$. Unfortunately, in general the restriction functors

$$(-)|_Z : \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(Z;\mathcal{E}) \qquad \text{and} \qquad (-)|_{S\smallsetminus Z} : \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S \smallsetminus Z;\mathcal{E})$$

need not be jointly conservative. Thus we also assume that our ∞-categories satisfy this joint conservativity property.

3.13 Definition. We say that a presentable ∞-category $\mathcal{E}$ *respects gluing* if for each topological space S and closed subspace $Z \subset S$, the restriction functors

$$(-)|_Z : \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(Z;\mathcal{E}) \qquad \text{and} \qquad (-)|_{S\smallsetminus Z} : \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S \smallsetminus Z;\mathcal{E})$$

and the hypersheaf restriction functors

$$(-)|_Z^{\mathrm{hyp}} : \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(Z;\mathcal{E}) \qquad \text{and} \qquad (-)|_{S\smallsetminus Z} : \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S \smallsetminus Z;\mathcal{E})$$

are jointly conservative.

Luckily, many presentable ∞-categories that arise in nature respect gluing:

3.14 Example. If each ∞-category $\mathrm{Sh}(S;\mathcal{E})$ is the *recollement* of $\mathrm{Sh}(Z;\mathcal{E})$ and $\mathrm{Sh}(S \smallsetminus Z;\mathcal{E})$ in the sense of [HA, Definition A.8.1], and each ∞-category $\mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E})$ is the recollement of $\mathrm{Sh}^{\mathrm{hyp}}(Z;\mathcal{E})$ and $\mathrm{Sh}^{\mathrm{hyp}}(S \smallsetminus Z;\mathcal{E})$, then $\mathcal{E}$ respects gluing. Importantly, this is satisfied if $\mathcal{E}$ is stable or $\mathcal{E} \simeq \mathcal{C} \otimes \mathcal{D}$ where $\mathcal{C}$ is a compactly generated ∞-category and $\mathcal{D}$ is an ∞-topos [5, Corollary 2.12, Proposition 2.20, & Remark 2.24].

3.15 Lemma. *Let P be a Noetherian poset, $S \in \mathbf{Top}_{/P}$, and let $\mathcal{E}$ be a presentable ∞-category that respects gluing. Then:*

(3.15.1) *The pullback functors $\{(-)|_{S_p} \colon \mathrm{Sh}(S;\mathcal{E}) \to \mathrm{Sh}(S_p;\mathcal{E})\}_{p\in P}$ are jointly conservative.*

(3.15.2) *The hypersheaf pullback functors $\{(-)|^{\mathrm{hyp}}_{S_p} \colon \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \to \mathrm{Sh}^{\mathrm{hyp}}(S_p;\mathcal{E})\}_{p\in P}$ are jointly conservative.*

Proof. We prove (3.15.1); the proof of (3.15.2) is exactly the same, replacing sheaves by hypersheaves. Let ϕ be a morphism in $\mathrm{Sh}(S;\mathcal{E})$ that restricts to an equivalence on each stratum; we need to show that ϕ is an equivalence. Since the open subsets $\{S_{\geq p}\}_{p\in P}$ cover S, it suffices to show:

($*$) For each $p \in P$, the restriction $\phi|_{S_{\geq p}}$ is an equivalence in $\mathrm{Sh}(S_{\geq p};\mathcal{E})$.

We prove ($*$) by Noetherian induction on $p \in P$. We need to show that if the restriction $\phi|_{S_{\geq q}}$ is an equivalence for each $q > p$, then $\phi|_{S_{\geq p}}$ is an equivalence. Note that

$$S_{\geq p} \smallsetminus S_p = S_{>p} = \bigcup_{q\in P_{>p}} S_{\geq q} \,.$$

Hence the inductive hypothesis implies that the restriction $\phi|_{S_{>p}}$ is an equivalence. By assumption $\phi|_{S_p}$ is also an equivalence. Since $\mathcal{E}$ respects gluing, the restriction functors

$$(-)|_{S_p} \colon \mathrm{Sh}(S_{\geq p};\mathcal{E}) \to \mathrm{Sh}(S_p;\mathcal{E}) \qquad \text{and} \qquad (-)|_{S_{>p}} \colon \mathrm{Sh}(S_{\geq p};\mathcal{E}) \to \mathrm{Sh}(S_{>p};\mathcal{E})$$

are jointly conservative, completing the proof. □

Finally we deduce the homotopy-invariance of constructible sheaves.

3.16 Corollary. *With the assumptions of Lemma 3.15:*

(3.16.1) *The functors $\mathrm{pr}_{S,!} \colon \mathrm{Cons}_P(S \times [0,1];\mathcal{E}) \rightleftarrows \mathrm{Cons}_P(S;\mathcal{E}) \colon \mathrm{pr}_S^*$ are inverse equivalences.*

(3.16.2) *The functors $\mathrm{pr}^{\mathrm{hyp}}_{S,!} \colon \mathrm{Cons}^{\mathrm{hyp}}_P(S \times [0,1];\mathcal{E}) \rightleftarrows \mathrm{Cons}^{\mathrm{hyp}}_P(S;\mathcal{E}) \colon \mathrm{pr}_S^*$ are inverse equivalences.*

In particular, the functors $\mathrm{Cons}_P(-;\mathcal{E}), \mathrm{Cons}^{\mathrm{hyp}}_P(-;\mathcal{E}) \colon \mathbf{Top}_{/P}^{\mathrm{op}} \to \mathbf{Cat}_\infty$ are homotopy-invariant.

Proof. Combine Lemma 2.7 with Corollary 3.7 and Lemma 3.15. □

4 The classification of foliated sheaves

The goal of this final section is to identify the essential image of the fully faithful embedding

$$\mathrm{pr}_S^* \colon \mathrm{Sh}^{\mathrm{hyp}}(S;\mathcal{E}) \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1];\mathcal{E}) \,,$$

at least when $\mathcal{E}$ is compactly generated.[4] First note a fundamental property of sheaves on $S \times [0,1]$ pulled back from S:

4.1 Observation. Let $G \in \mathrm{Sh}(S;\mathcal{E})$. The pullback $\mathrm{pr}_S^*(G)$ to $S \times [0,1]$ satisfies the following property: for each $s \in S$, the restriction $\mathrm{pr}_S^*(G)|_{\{s\}\times[0,1]}$ is constant with value the stalk of G at s.

[4] One can also use this identification to prove that hyperconstructible hypersheaves with values in a compactly generated ∞-category are homotopy-invariant. However, this approach does not easily generalize beyond compactly generated coefficients.

Thus the natural guess for the essential image of pr_S^* is the subcategory of those sheaves such that the restriction to each interval $\{s\} \times [0,1]$ is constant. For convenience we make the following variant of [HA, Definition A.2.4].

4.2 Definition. Let S be a topological space and $\mathcal{E}$ a presentable ∞-category. We say that a sheaf $F \in \mathrm{Sh}(S \times [0,1]; \mathcal{E})$ is *foliated* if the following conditions are satisfied:

(4.2.1) For each point $s \in S$, the restriction $F|_{\{s\}\times[0,1]}$ is constant.

(4.2.2) The sheaf F is hypercomplete.

4.3 Remark. Note that sheaf on $[0,1]$ is locally constant if and only if it is constant (see Lemma 3.3 with $S = *$ or [HA, Proposition A.2.1]).

4.4 Example. Let P be a poset and let S be a P-stratified space. If F is a hyperconstructible hypersheaf on $S \times [0,1]$, then by definition for each $p \in P$ the hypersheaf restriction $(F|_{S_p\times[0,1]})^{\mathrm{hyp}}$ is locally hyperconstant. Hence for each $s \in S$, the restriction $F|_{\{s\}\times[0,1]}$ is constant. That is, every hyperconstructible hypersheaf on $S \times [0,1]$ is foliated.

The following is a variant of [HA, Proposition A.2.5] with $\mathbf{R}$ replaced by $[0,1]$ and with more general coefficients; our proof is much more straightforward than the proof of [HA, Proposition A.2.5].

4.5 Proposition (classification of foliated sheaves)**.** *Let S be a topological space and $\mathcal{E}$ a compactly generated ∞-category. The following are equivalent for a sheaf $F \in \mathrm{Sh}(S \times [0,1]; \mathcal{E})$:*

(4.5.1) *The sheaf F is foliated.*

(4.5.2) *The pushforward $\mathrm{pr}_{S,*}(F)$ is hypercomplete and the counit $\mathrm{pr}_S^* \, \mathrm{pr}_{S,*}(F) \to F$ is an equivalence.*

In particular, a hypersheaf $F \in \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E})$ is in the essential image of the embedding

$$\mathrm{pr}_S^* \colon \mathrm{Sh}^{\mathrm{hyp}}(S; \mathcal{E}) \hookrightarrow \mathrm{Sh}^{\mathrm{hyp}}(S \times [0,1]; \mathcal{E})$$

if and only if F is foliated.

We need the following consequence of Lurie's Proper Basechange Theorem [HTT, §7.3].

4.6 Lemma. *Let $f \colon T \to S$ be a map of topological spaces, let K be a compact Hausdorff space, and let $\mathcal{E}$ be a presentable ∞-category. If $\mathcal{E}$ is compactly generated or stable, the square of ∞-categories of $\mathcal{E}$-valued sheaves*

$$\begin{array}{ccc} \mathrm{Sh}(T \times K; \mathcal{E}) & \xrightarrow{(f \times \mathrm{id}_K)_*} & \mathrm{Sh}(S \times K; \mathcal{E}) \\ {\scriptstyle \mathrm{pr}_{T,*}} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_{S,*}} \\ \mathrm{Sh}(T; \mathcal{E}) & \xrightarrow[f_*]{} & \mathrm{Sh}(S; \mathcal{E}) \end{array}$$

is left adjointable. That is, the exchange morphism $\mathrm{Ex} \colon f^* \, \mathrm{pr}_{S,*} \to \mathrm{pr}_{T,*}(f \times \mathrm{id}_K)^*$ is an equivalence.

Proof. Consider the commutative diagram of ∞-topoi

$$\begin{array}{ccccc} \mathrm{Sh}(T \times K) & \xrightarrow{(f \times \mathrm{id}_K)_*} & \mathrm{Sh}(S \times K) & \xrightarrow{\mathrm{pr}_{K,*}} & \mathrm{Sh}(K) \\ {\scriptstyle \mathrm{pr}_{T,*}} \downarrow & & \downarrow {\scriptstyle \mathrm{pr}_{S,*}} & & \downarrow \\ \mathrm{Sh}(T) & \xrightarrow[f_*]{} & \mathrm{Sh}(S) & \longrightarrow & \mathrm{Sh}(*) \, . \end{array}$$

Since K is locally compact, [HTT, Proposition 7.3.1.11] shows that the right-hand square and outer square are pullback squares of ∞-topoi. Hence the left-hand square is also a pullback square of ∞-topoi. Since K is compact Hausdorff, the global sections geometric morphism $\mathrm{Sh}(K) \to \mathrm{Sh}(*)$ is *proper* in the sense of [HTT, Definition 7.3.1.4]; see [HTT, Corollary 7.3.4.11]. Therefore the left-hand square is left adjointable. Tensoring with the presentable ∞-category $\mathcal{E}$ and applying [5, Example 3.13] completes the proof. □

Now we prove Proposition 4.5. To prove the implication (4.5.1) ⇒ (4.5.2), we check that the counit $\mathrm{pr}_S^* \, \mathrm{pr}_{S,*}(F) \to F$ is an equivalence on stalks by applying Lemma 4.6 in the case where $K = [0,1]$ and f is the inclusion of a point of S.

Proof of Proposition 4.5. The implication (4.5.2) ⇒ (4.5.1) is immediate from Observation 4.1 and the fact that pr_X^* preserves hypercomplete objects (Example 2.4).

Now we prove that (4.5.1) ⇒ (4.5.2). Assume that $F \in \mathrm{Sh}(S \times [0,1]; \mathcal{E})$ is foliated. Since F is hypercomplete, the pushforward $\mathrm{pr}_{S,*}(F)$ is hypercomplete. Since pr_S^* preserves hypercomplete objects, the pullback $\mathrm{pr}_S^* \, \mathrm{pr}_{S,*}(F)$ is also hypercomplete. By Recollection 3.8, to show that the counit

$$c_F \colon \mathrm{pr}_S^* \, \mathrm{pr}_{S,*}(F) \to F$$

is an equivalence, it suffices to show that c_F becomes an equivalence after taking stalks.

Fix $(s,t) \in S \times [0,1]$. To show that the stalk of c_F at (s,t) is an equivalence, consider the commutative diagram of topological spaces

$$\begin{array}{ccccc} \{(s,t)\} & \xrightarrow{i_{(s,t)}} & \{s\} \times [0,1] & \xrightarrow{i_s} & S \times [0,1] \\ & \searrow^{\sim} & \downarrow{\scriptstyle \mathrm{pr}_{\{s\}}} & & \downarrow{\scriptstyle \mathrm{pr}_S} \\ & & \{s\} & \xrightarrow[s]{} & S , \end{array} \tag{4.7}$$

where the horizontal morphisms are the obvious inclusions. The commutativity of (4.7) implies that

$$\begin{aligned} (\mathrm{pr}_S^* \, \mathrm{pr}_{S,*}(F))_{(s,t)} &\simeq i_{(s,t)}^* i_s^* \, \mathrm{pr}_S^* \, \mathrm{pr}_{S,*}(F) \\ &\simeq i_{(s,t)}^* \, \mathrm{pr}_{\{s\}}^* \, s^* \, \mathrm{pr}_{S,*}(F) \, . \end{aligned}$$

Lemma 4.6 applied to the square in (4.7) shows that the exchange morphism

$$\mathrm{Ex} \colon s^* \, \mathrm{pr}_{S,*} \to \mathrm{pr}_{\{s\},*} \, i_s^*$$

is an equivalence. Since F is foliated, the sheaf $i_s^*(F)$ on $\{s\} \times [0,1]$ is constant. Since

$$\mathrm{pr}_{\{s\},*} \colon \mathrm{Sh}(\{s\} \times [0,1]; \mathcal{E}) \to \mathrm{Sh}(\{s\}; \mathcal{E}) \simeq \mathcal{E}$$

restricts to an equivalence on constant sheaves [HA, Proposition A.2.1], the counit

$$\mathrm{pr}_{\{s\}}^* \, \mathrm{pr}_{\{s\},*} \, i_s^*(F) \to i_s^*(F)$$

is an equivalence. Composing the exchange morphism with the counit $\mathrm{pr}_{\{s\}}^* \, \mathrm{pr}_{\{s\},*} \to \mathrm{id}$ thus provides an equivalence

$$(\mathrm{pr}_S^* \, \mathrm{pr}_{S,*}(F))_{(s,t)} \simeq i_{(s,t)}^* \, \mathrm{pr}_{\{s\}}^* \, s^* \, \mathrm{pr}_{S,*}(F) \xrightarrow{\sim} i_{(s,t)}^* \, \mathrm{pr}_{\{s\}}^* \, \mathrm{pr}_{\{s\},*} \, i_s^*(F) \xrightarrow{\sim} F_{(s,t)} \, . \tag{4.8}$$

It follows from the definitions that the composite morphism (4.8) is the stalk of the counit morphism $c_F \colon \mathrm{pr}_S^* \, \mathrm{pr}_{S,*}(F) \to F$ at (s,t), completing the proof. □

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