AN INTRODUCTION TO GOODWILLIE CALCULUS

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Abstract. We introduce the main definitions and structural theorems from Goodwillie Calculus. Most of the material in these notes is taken from [3, §§6.1.1–6.1.4]. These notes are rough — use at own risk!

Corrections welcome.

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1. The Analogy from Calculus

1.1. **Idea.** If $f\colon \mathbf{R} \to \mathbf{R}$ is a smooth function, for each $n \geq 0$ Taylor's formula gives an identity

$$f(x) = c_0 + c_1 x + \cdots + c_n x^n + u(x) x^{n+1} ,$$

where $c_m = f^{(m)}(0)/m!$ and $u \in C^\infty(\mathbf{R})$. The polynomial $p_n(x) \coloneqq c_0 + \cdots + c_n x^n$ is the n^{th} ***Taylor approximation*** of f (at $0 \in \mathbf{R}$).

Moreover, the polynomial p_n is uniquely characterized by the following properties:

(1.1.a) p_n has degree at most n.
(1.1.b) The difference $f - p_n$ vanishes to order n at $0 \in \mathbf{R}$.

The idea behind Goodwillie calculus is to try to approximate functors between suitable ∞-categories by functors that are "polynomial" in a suitable sense and satisfy an appropriate analog of conditions (1.1.a) and (1.1.b).

Part of the goal is for Goodwillie calculus to be applicable in very general contexts, and in particular to the case where the categories involved are not ∞-topoi and tools like Postnikov towers are not available to us.

1.2. **Definition.** Let n be a nonnegative integer. The category $\mathrm{Poly}^n(\mathbf{Sp}, \mathbf{Sp})$ of ***polynomial functors of degree at most*** n is the smallest full subcategory of $\mathrm{Fun}(\mathbf{Sp}, \mathbf{Sp})$ closed under translation, small colimits, equivalence, and containing all of the functors $(-)^{\wedge m}\colon \mathbf{Sp} \to \mathbf{Sp}$ for $0 \leq m \leq n$.

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1.3. **Observation.** For each $m \geq 0$, the functor $(-)^{\wedge m} \colon \mathbf{Sp} \to \mathbf{Sp}$ preserves filtered colimits. Then since colimits commute and the translation functors on **Sp** are equivalences, by construction every functor in $\mathrm{Poly}^n(\mathbf{Sp}, \mathbf{Sp})$ preserves filtered colimits. In Proposition 2.10 we see that the preservation of filtered colimits is one of the two properties that characterizes polynomial functors of degree at most n.

1.4. **Example** ([3, Ex. 6.1.0.3])**.** For all tuples $C_0, \ldots, C_n \in \mathbf{Sp}$, the functor given by the assignment

$$X \mapsto \bigvee_{0 \leq m \leq n} C_m \wedge X^{\wedge m}$$

is polynomial of degree at most n. However, not every polynomial functor is of this form.

1.5. **Issues.** There are a few issues with polynomial functors:

(1.5.a) There is no obvious way to check that a functor is polynomial.

(1.5.b) This definition depends on some structural features of **Sp** (namely the existence of a monoidal structure), and is not very general.

(1.5.c) Polynomial functors automatically preserve filtered colimits. The preservation of filtered colimits is a finiteness condition that we did not a priori try to impose on polynomial functors, and it is quite restrictive.

1.6. **Analogies** ([3, p. 1011])**.** The following is a table of analogies between Goodwillie calculus and differential calculus to keep in mind as we set up the machinery behind Goodwillie calculus.

Differential Calculus	*Functor Calculus*
Smooth Manifold M	Compactly generated ∞-category C
Smooth function $M \to N$	Functor $C \to D$ preserving filtered colimits
Point $x \in M$	Object $c \in C$
Real vector space	Stable ∞-category
$\mathbf{R}$	$\mathbf{Sp}$
Linear map of vector spaces	Exact functor between stable ∞-categories
$T_{M,x}$	$\mathbf{Sp}(C_{/c})$
Differential of a smooth map	Excisive approximation of a functor
Polynomial	Excisive functor
Homogeneous polynomial	Homogeneous functor
Subtraction of polynomials	Fiber of a map of excisive functors

2. Cubes & n-Excisive Functors

2.1. **Notation.** For a finite set S, write $P(S)$ for the poset of subsets of S ordered by inclusion. Given an integer $i \leq \#S$, write $P_{\leq i}(S)$ for the full sub-poset of $P(S)$ spanned by those subsets of S of cardinality at most i.

2.2. **Remark.** If S has cardinality $n+1$, then $P(S) \cong (\Delta^1)^{\times n}$.

2.3. **Definition.** Let S be a finite set and C an ∞-category. The ***category of S-cubes in C*** is the functor category

$$\mathbf{Cb}_S(C) \coloneqq \mathrm{Fun}(P(S), C)\ .$$

2.4. **Definition.** Let S be a finite set and C an ∞-category.

(2.4.a) An S-cube $X \colon P(S) \to C$ is ***Cartesian*** if X is a limit diagram, i.e., the map

$$X(\varnothing) \to \lim_{\varnothing \neq S' \subset S} X(S')$$

is an equivalence in C. ***CoCartesian*** S-cubes are defined dually.

(2.4.b) An S-cube $X: P(S) \to C$ is ***strongly coCartesian*** if X is a left Kan extension of $X|_{P_{\leq 1}(S)}$. ***Strongly Cartesian*** S-cubes are defined dually.

2.5. **Remark.** If S is a finite set and C is an ∞-category admitting finite limits, then saying that an S-cube X is strongly coCartesian means that X can be constructed by iterated pushout from the subdiagram $X|_{P_{\leq 1}(S)}$.

2.6. **Definition.** Let n be a nonnegative integer and C an ∞-category. An ***n-cube*** in C is a $\{0, \ldots, n\}$-cube in C.

The *excision axiom* in the Eilenberg–Steenrod axioms can be phrased by saying that (homotopy) pushouts are sent to (homotopy) pullbacks. The following definition is a generalization of the excision axiom that makes sense in a broader context (see (2.8.b)).

2.7. **Definition.** Let C be an ∞-category admitting finite colimits, D an ∞-category, and n a nonnegative integer. A functor $F: C \to D$ is ***n-excisive*** if F carries *strongly* coCartesian n-cubes in C to Cartesian n-cubes in D.

Writ $\mathrm{Exc}^n(C, D)$ for the full subcategory of $\mathrm{Fun}(C, D)$ spanned by the n-excisive functors.

2.8. **Examples** (Exs. 6.1.1.4 & 6.1.1.5)**.** Let C be an ∞-category.

(2.8.a) A 0-cube in C is just a morphism of C. Every 0-cube is strongly coCartesian, and a 0-cube is Cartesian if and only if the morphism specified by the 0-cube is an equivalence. A functor $F: C \to D$ is 0-excisive if and only if F factors through the maximal sub-∞-groupoid $D^{\simeq} \subset D$.

(2.8.b) A 1-cube in C is a commutative square in C. A 1-cube in C is Cartesian if and only if the square specified by the 1-cube is a pullback square. Dually, a 1-cube in C is coCartesian if and only if the corresponding square is a pushout square. In this case, strongly coCartesian 1-cubes are just coCartesian 1-cubes, so a functor $F: C \to D$ is 1-excisive if and only if F carries pushout squares in C to pullback squares in D.

(2.8.c) In light of (2.8.b), if C has finite colimits, then the identity functor id_C is 1-excisive if and only if every pushout square in C is also a pullback square in C. If C is pointed, every morphism admits a fiber (every morphism automatically admits a cofiber since we assume that C has finite colimits), and id_C is 1-excisive, then C is stable. Moreover, by [3, Prop. 1.1.3.4] if C admits finite limits and colimits, then C is stable if and only if id_C is 1-excisive.

The following proposition follows easily from the definition of an n-excisive functor.

2.9. **Proposition** ([3, Cor. 6.1.1.14])**.** *Let C be an ∞-category admitting finite colimits, let D be an ∞-category admitting finite limits, and let n be a nonnegative integer. If $F: C \to D$ is an n-excisive functor, then F is m-excisive for all $m \geq n$.*

2.10. **Proposition** ([3, Cor. 6.1.4.15])**.** *Let F:* $\mathbf{Sp} \to \mathbf{Sp}$ *be a functor, and let n be a nonnegative integer. Then the following conditions are equivalent:*

(2.10.a) *The functor F is polynomial of degree at most n.*

(2.10.b) *The functor F is a n-excisive and commutes with filtered colimits.*

The following definition is somewhat unmotivated at the moment, and the purpose of these conditions is so that we can construct an n^{th} *excisive approximation* to a functor. We define a *differentiable* ∞*-category* now so that we can state Theorem 2.13, and reason for the definition should become apparent from the construction of the 1^{st} excisive approximation (see Construction 2.16 and the proof outline of Theorem 2.13).

2.11. **Definition** ([3, Def. 6.1.1.6]). An ∞-category D is ***differentiable*** if D satisfies the following conditions.

(2.11.a) D admits finite limits.
(2.11.b) D admits sequential colimits.
(2.11.c) The functor $\operatorname{colim}\colon \operatorname{Fun}(\mathbf{Z}_{\geq 0}, D) \to D$ is left exact (informally, sequential colimits commute with finite limits).

2.12. **Examples** ([3, Exs. 6.1.1.7 & 6.1.1.8]).

(2.12.a) Stable ∞-categories admitting countable coproducts are differentiable; in particular, **Sp** is differentiable.
(2.12.b) If $\mathcal{X}$ is an ∞-topos, then $\mathcal{X}$ is differentiable; in particular, **Spc** is differentiable.

2.13. **Theorem** ([3, Thm. 6.1.1.10]). *Let C be an ∞-category admitting finite colimits and a final object and D be a differentiable ∞-category. Then:*

(2.13.a) *The inclusion* $\operatorname{Exc}^n(C, D) \hookrightarrow \operatorname{Fun}(C, D)$ *admits a left adjoint*

$$P_n\colon \operatorname{Fun}(C, D) \to \operatorname{Exc}^n(C, D) .$$

(2.13.b) *The functor P_n is left exact, i.e., P_n preserves finite limits.*

We call P_n the n^{th} ***excisive approximation*** *functor.*

As motivation for the construction of excisive approximations, we construct the 0^{th} and 1^{st} excisive approximations P_0 and P_1 "by hand". The 0^{th} excisive approximation is essentially trivial to construct.

2.14. **Notation.** From now on we more or less always assume that C is an ∞-category admitting finite colimits and a final object and D is a differentiable ∞-category, though we always state this explicitly in the hypotheses of our theorems. Sometimes we need more structure, namely that D be pointed.

2.15. **Construction** (of P_0). Let C be an ∞-category admitting finite colimits and a final object and D an ∞-category. By (2.8.a), a 0-excisive functor $C \to D$ factors through $D^{\simeq}$. Since C has a final object, C is connected and for every object $X \in C$, the mapping space $\operatorname{Map}_C(X, *)$ is contractible, where $*$ denotes a final object of C. Hence if $G\colon C \to D$ is 0-excisive, then G is canonically equivalent to the constant functor at $G(*)$.

Define $P_0\colon \operatorname{Fun}(C, D) \to \operatorname{Exc}^0(C, D)$ by sending a functor $F\colon C \to D$ to the constant functor at $F(*)$. Then P_0 comes equipped with a natural transformation $\phi\colon \mathrm{id} \to P_0$ coming from the essentially unique natural transformation $\mathrm{id}_C \to *$. Thus if G is 0-excisive, then $\phi_G\colon G \to P_0(G)$ an equivalence. From what we know it is easy to see that P_0 is a left adjoint to the inclusion $\operatorname{Exc}^0(C, D) \hookrightarrow \operatorname{Fun}(C, D)$. Since a natural transformation to a constant functor is a cone under that diagram, for $F \in \operatorname{Fun}(C, D)$ and $G \in \operatorname{Exc}^0(C, D)$, we see that we have natural equivalences

$$\begin{aligned}
\operatorname{Map}_{\operatorname{Fun}(C,D)}(F, G) &\simeq \operatorname{Map}_{\operatorname{Fun}(C,D)}(F, P_0(G)) \simeq \operatorname{Map}_D(F(*), G(*)) \\
&\simeq \operatorname{Map}_{\operatorname{Fun}(C,D)}(P_0(F), P_0(G)) \simeq \operatorname{Map}_{\operatorname{Fun}(C,D)}(P_0(F), G) \\
&= \operatorname{Map}_{\operatorname{Exc}^0(C,D)}(P_0(F), G) .
\end{aligned}$$

2.16. **Construction** (of P_1). We describe the construction of P_1 as it is simple and illuminating. We begin with an observation. If $F\colon C \to D$ is 1-excisive, then for every $X \in C$ the functor F sends the pushout square

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma_C(X) \end{array}$$

defining the suspension of X to a pullback square

$$\begin{array}{ccc} F(X) & \longrightarrow & F(*) \\ \downarrow & & \downarrow \\ F(*) & \longrightarrow & F(\Sigma_C(X)) \, . \end{array}$$

In the case that F is *reduced*, i.e., $F(*) \simeq *$, this requirement says that $F(X) \simeq \Omega_D F(\Sigma_C(X))$.

Define a functor $T_1 \colon \operatorname{Fun}(C, D) \to \operatorname{Fun}(C, D)$ by setting $T_1(F)(X)$ to be the pullback

$$\begin{array}{ccc} T_1(F)(X) & \longrightarrow & F(*) \\ \downarrow & & \downarrow \\ F(*) & \longrightarrow & F(\Sigma_C(X)) \, . \end{array}$$

Thus if F is reduced, then $T_1(F) \simeq \Omega_D \circ F \circ \Sigma_C$.

The functor T_1 comes equipped with a natural transformation $\theta \colon \mathrm{id} \to T_1$ induced by the commutative squares

$$\begin{array}{ccc} F(X) & \longrightarrow & F(*) \\ \downarrow & & \downarrow \\ F(*) & \longrightarrow & F(\Sigma_C(X)) \, . \end{array}$$

Define a functor $P_1 \colon \operatorname{Fun}(C, D) \to \operatorname{Fun}(C, D)$ as the colimit

$$P_1 \coloneqq \operatorname{colim} \left(\, \mathrm{id} \xrightarrow{\theta} T_1 \xrightarrow{\theta T_1} T_1^2 \xrightarrow{\theta T_1^2} \cdots \, \right) .$$

Thus if F is reduced, then $P_1(F) \simeq \operatorname{colim}_m \Omega_D^m \circ F \circ \Sigma_C^m$. The functor P_1 is the 1^{st} ***excisive approximation functor***.

2.17. **Example** (relation to classical stable homotopy theory). In Construction 2.16, consider the case where $C = D$ and F is the identity functor. Then since the identity functor is reduced,

$$P_1(\mathrm{id}) \simeq \operatorname*{colim}_m \Omega_C^m \Sigma_C^m \simeq \Omega_C^\infty \Sigma_C^\infty \, ,$$

So if $C = \mathbf{Spc}_*$, then $\pi_k P_1(\mathrm{id})(S^0) \cong \pi_k \mathbf{S}$.

This explains why later on we will be extremely interested in the the excisive approximations of the identity functor.

2.18. **Warning.** In the analogy with differential calculus, $P_1(F)$ corresponds to the first Taylor approximation of a function $\mathbf{R} \to \mathbf{R}$. The identity $\mathbf{R} \to \mathbf{R}$ is the homogeneous polynomial x, so the first Taylor approximation of $\mathrm{id}_{\mathbf{R}}$ is $\mathrm{id}_{\mathbf{R}}$ itself. However Example 2.17 illustrates that the first excisive approximation of the identity on $\mathbf{Spc}_*$ are highly nontrivial. This illustrates one of the important differences between differential calculus and Goodwillie calculus: in differential calculus the derivative of the identity is trivial, but the Goodwillie derivative of the identity is generally interesting.

This results from the failure of the identity on $\mathbf{Spc}_*$ to be 1-excisive. Notice, however, that if C is a *stable* differentiable ∞-category, then id_C is 1-excisive by (2.8.c). In particular, the identity on the ∞-category of spectra is 1-excisive.

Outline of the proof of Theorem 2.13. The construction of the n^{th} excisive approximation P_n is completely analogous to the construction of P_1 from Construction 2.16, once we have construed a suitable functor T_n to take the role of T_1.

(2.13.i) Construct a functor $T_n \colon \mathrm{Fun}(C,D) \to \mathrm{Fun}(C,D)$ so that $T_n(F)$ send *certain* strongly coCartesian n-cubes to Cartesian n-cubes. The construction of T_n is not completely obvious, but is simple. In order to construct P_n as we constructed P_1, we also want T_n to be left exact and admit a natural transformation $\theta \colon \mathrm{id} \to T_n$.

(2.13.ii) The functor T_n does not enforce n-excisivity, so we take the colimit

$$P_n \coloneqq \operatorname{colim}\left(\mathrm{id} \xrightarrow{\theta} T_n \xrightarrow{\theta T_n} T_n^2 \xrightarrow{\theta T_n^2} \cdots \right).$$

Then since finite limits commute with sequential colimits in D and T_n is left exact, the functor P_n is left exact.

(2.13.iii) Check that P_n lands in n-excisive functors. This is mostly formal, but requires one technical factorization lemma relying on the construction of T_n

(2.13.iv) By the construction of T_n, if $F \in \mathrm{Exc}^n(C,D)$, then the natural map $\theta_F \colon F \to T_n(F)$ is an equivalence. Hence the natural map $\phi_F \colon F \to P_n(F)$ is an equivalence.

(2.13.v) Show that $\phi \colon \mathrm{id} \to P_n$ exhibits P_n as a localization functor by showing that for every $F \in \mathrm{Fun}(C,D)$, the natural maps $P_n(\phi_F), \phi_{P_n(F)} \colon P_n(F) \to P_nP_n(F)$ are equivalences.
- The fact that $P_n(\phi_F)$ is an equivalence is more or less formal [3, Lem. 6.1.1.35].
- The fact that $\phi_{P_n(F)}$ is an equivalence follows from item (2.13.iv) since $P_n(F)$ is n-excisive. □

2.19. **Remark** (Rezk). If C is an ∞-category admitting finite colimits and a final object and $\mathcal{X}$ is an ∞-topos, then $\mathrm{Exc}^n(C,\mathcal{X})$ is a left exact accessible localization of $\mathrm{Fun}(C,\mathcal{X})$. Hence $\mathrm{Exc}^n(C,\mathcal{X})$ is an ∞-topos. This fact can be used to prove a Blakers–Massey theorem and its dual for P_n-equivalences [6, Thms. 4.4.1 & 4.4.2] using the Anel–Biedermann–Finster–Joyal generalized Blakers–Massey theorem for ∞-topoi and its dual [5, Thms. 3.6.1 & 4.1.1].

2.20. **Corollary** ([2, Lem. 2.2]). *Let C be an ∞-category admitting finite colimits and a final object and D be a pointed differentiable ∞-category (so that* $\mathrm{Fun}(C,D)$ *is naturally pointed). Then for any nonnegative integer n, the functor P_n preserves fiber sequences.*

Proof. Since P_n is left exact, P_n preserves the zero object of $\mathrm{Fun}(C,D)$ (i.e., the constant functor at the zero object of D). Hence if

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & H \end{array} \tag{2.21}$$

is a fiber sequence in $\mathrm{Fun}(C,D)$, since P_n is left exact, applying P_n to the fiber sequence (2.21) gives a fiber sequence

$$\begin{array}{ccc} P_n(F) & \longrightarrow & P_n(G) \\ \downarrow & \lrcorner & \downarrow \\ P_n(0) \simeq 0 & \longrightarrow & P_n(H) \end{array}$$

in $\mathrm{Fun}(C,D)$. □

2.22. **Example** (Segal). In this example we explain Segal's construction of the connective spectrum associated to a very special Γ-space, but in a more general context.

Let D be a differentiable ∞-category with all colimits (or at least enough colimits for the relevant left Kan extension to exist). The idea is to use the description of the stabilization of D as $\mathbf{Sp}(D) \simeq \mathrm{Exc}_*(\mathbf{Spc}_*^{fin}, D)$. Let $i\colon \mathbf{Fin}_* \hookrightarrow \mathbf{Spc}_*^{fin}$ be the inclusion of finite pointed sets as discrete spaces. The universal way to extend a Γ-object $X\colon \mathbf{Fin}_* \to D$ to $\mathbf{Spc}_*^{fin}$ is to form the left Kan extension

$$\begin{array}{ccc} \mathbf{Fin}_* & \xrightarrow{X} & D \\ {\scriptstyle i}\big\downarrow & \nearrow{\scriptstyle i_! X} & \\ \mathbf{Spc}_*^{fin} & & \end{array} .$$

Taking the composite of $i_!$ with the first excisive approximation

$$P_1\colon \mathrm{Fun}_*(\mathbf{Spc}_*^{fin}, D) \to \mathrm{Exc}_*(\mathbf{Spc}_*^{fin}, D) \simeq \mathbf{Sp}(D)$$

defines the desired functor

$$P_1 \circ i_!\colon \mathrm{Fun}_*(\mathbf{Fin}_*, D) \to \mathbf{Sp}(D) .$$

Note that this composite actually lands in connective spectra because we first left Kan extend from $\mathbf{Fin}_*$ [3, Rem. 1.4.3.5].

Due to [3, Ex. 6.1.1.28] we have an explicit formula for the first excisive approximation:

$$P_1(F) \simeq \operatorname*{colim}_n \Omega_D^n \circ F \circ \Sigma^n ,$$

and the identification $\mathrm{Exc}_*(\mathbf{Spc}_*^{fin}, D) \simeq \mathbf{Sp}(D)$ associates to F the Ω-spectrum

$$\{\mathrm{colim}_n\, \Omega_D^n F(\Sigma^n S^m)\}_{m \geq 0} .$$

Setting $D = \mathbf{Spc}$ recovers Segal's construction of the spectrum associated to a very special Γ-space.

3. The Taylor Tower

3.1. **Observation.** Let C be an ∞-category admitting finite colimits and a final object and D be a differentiable ∞-category. Then by Proposition 2.9 we have a sequence of inclusions

$$\mathrm{Fun}(C,D) \supset \cdots \supset \mathrm{Exc}^3(X,D) \supset \mathrm{Exc}^2(X,D) \supset \mathrm{Exc}^1(X,D) \supset \mathrm{Exc}^0(X,D)$$

hence the excisive approximation functors form a tower

$$\mathrm{id} \longrightarrow \cdots \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0$$

of endofunctors of $\mathrm{Fun}(C,D)$. For a functor $F \in \mathrm{Fun}(C,D)$, we call the tower

$$F \longrightarrow \cdots \longrightarrow P_3(F) \longrightarrow P_2(F) \longrightarrow P_1(F) \longrightarrow P_0(F)$$

the ***Taylor tower*** of F.

3.2. **Goal.** Study the differences between successive Taylor approximations.

3.3. **Definition.** Let C be an ∞-category admitting finite colimits and a final object, D be a *pointed* differentiable ∞-category, and n a positive integer. Let $D_n \colon \operatorname{Fun}(C, D) \to \operatorname{Fun}(C, D)$ denote the fiber

$$D_n \coloneqq \operatorname{fib}(P_n \to P_{n-1}) \,.$$

For a functor $F \colon C \to D$, the functor $D_n(F)$ is called the ***n^{th} layer*** of the Taylor tower for F.

Note that if F is reduced so that $P_0(F) = 0$, then $D_1(F) = P_1(F)$.

Thus our goal is to study the layers of the Taylor tower.

3.4. **Definition.** Let C be an ∞-category admitting finite colimits and a final object, D be a differentiable ∞-category, and n a positive integer.

(3.4.a) A functor $F \colon C \to D$ is ***n-reduced*** if $P_{n-1}(F)$ is a final object of $\operatorname{Exc}^{n-1}(X, D)$, i.e., for all $X \in C$ we have $P_{n-1}(F)(X) \simeq *$. Write $\operatorname{Exc}^n_*(C, D)$ for the full subcategory of $\operatorname{Exc}^n(C, D)$ spanned by the 1-reduced functors.

(3.4.b) A functor $F \colon C \to D$ is ***n-homogeneous*** if F is n-reduced and n-excisive. Write $\operatorname{Homog}^n(C, D) \subset \operatorname{Fun}(C, D)$ for the full subcategory spanned by the n-homogenous functors.

3.5. **Lemma.** *Let C be an ∞-category admitting finite colimits and a final object, D be a differentiable ∞-category, and n a positive integer. For any functor $F \colon C \to D$, the n^{th} layer $D_n(F)$ is n-homogeneous.*

Proof. First we show that $P_{n-1}D_n(F) \simeq 0$. To see this, note that by Corollary 2.20, applying P_{n-1} to the defining fiber sequence of $D_n(F)$ we obtain a fiber sequence

$$P_{n-1}D_n(F) \longrightarrow P_{n-1}P_n(F) \longrightarrow P_{n-1}P_{n-1}(F) \,.$$

Since $P_{n-1}(F)$ is $(n-1)$-excisive, the map $P_{n-1}P_n(F) \to P_{n-1}P_{n-1}(F)$ is an equivalence, hence $P_{n-1}D_n(F) \simeq 0$.

To see that $D_n(F)$ is n-excisive, note that by Corollary 2.20 and the fact that P_n is equivalent to the identity on n-excisive functors, applying P_{n-1} to the defining fiber sequence of $D_n(F)$ we obtain a fiber sequence

$$P_nD_n(F) \longrightarrow P_nP_n(F) \longrightarrow P_nP_{n-1}(F) \,.$$

The map $P_nP_n(F) \to P_nP_{n-1}(F)$ is equivalent to the natural map $P_n(F) \to P_{n-1}(F)$, hence $P_nD_n(F) \simeq D_n(F)$. □

We regard the following theorem as saying that the Goodwillie tower is a tower of principal fibrations.

3.6. **Theorem** (Goodwillie, [3, Thm. 6.1.2.4])**.** *Let C be an ∞-category admitting finite colimits and a final object and D be a differentiable ∞-category. For all positive integers n, there is a pullback square*

$$\begin{array}{ccc} P_n & \longrightarrow & P_{n-1} \\ \big\downarrow & \lrcorner & \big\downarrow \\ K & \longrightarrow & R \end{array}$$

of endofunctors of $\operatorname{Fun}_*(C, D)$ *with the following properties:*

(3.6.a) *If $F\colon C \to D$ is reduced, then $K(F)$ carries the final object of C to the final object of D.*
(3.6.b) *If $F\colon C \to D$ is reduced, then $R(F)$ is n-homogeneous.*
(3.6.c) *$R\colon \mathrm{Fun}_*(C,D) \to \mathrm{Fun}_*(C,D)$ is left exact.*
(3.6.d) *If $F \in \mathrm{Exc}_*^{n-1}(C,D)$, then $R(F)$ carries each object of C to a final object of D.*

3.7. **Theorem** ([3, Thm. 6.1.2.5]). *Let C be an ∞-category admitting finite colimits and a final object, D be a differentiable ∞-category, and n a positive integer. Then every reduced n-excisive functor $F\colon C \to D$ can be written essentially uniquely as the fiber of a natural transformation $E \to H$, where E is reduced $(n-1)$-excisive and H is n-homogeneous.*

3.8. **Proposition** ([3, Cor. 6.1.2.8]). *Let C be an ∞-category admitting finite colimits and a final object, D be a pointed differentiable ∞-category, and n a positive integer. Then the ∞-category $\mathrm{Homog}^n(C,D)$ is stable.*

3.9. **Corollary** ([2, Thm. 2.3; 3, Cor. 6.1.2.9]). *Let C be an ∞-category admitting finite colimits and a final object, D be a pointed differentiable ∞-category, and n a positive integer. Then post-composition with Ω_D^∞ defines an equivalence of ∞-categories*

$$\Omega_{D,\star}^\infty\colon \mathrm{Homog}^n(C,\mathbf{Sp}(D)) \xrightarrow{\sim} \mathrm{Homog}^n(C,D)$$

In particular, if $F\colon C \to D$ is n-homogeneous, then there is a naturally-defined n-homogeneous functor $\widetilde{F}\colon C \to \mathbf{Sp}(D)$ such that $F \simeq \Omega_D^\infty \widetilde{F}$.

4. The Classification of Homogeneous Functors

We state (a somewhat vague form of) Goodwillie's classification of homogeneous functors to give a taste of the kinds of results that we can get our hands on. We are particularly interested in Corollary 4.2.

4.1. **Theorem** (Goodwillie's classification of homogeneous functors [3, Thm. 6.1.4.7]). *Let C be an ∞-category admitting finite colimits and a final object, D be a pointed differentiable ∞-category, and $n \in \mathbf{Z}_{>0}$. An n-homogeneous functor $H\colon C \to D$ can be written essentially uniquely as*

$$H(X) = \Omega_D^\infty(h(X,\dots,X)_{\Sigma_n}),$$

where $h\colon C^{\times n} \to \mathbf{Sp}(D)$ is a functor that is 1-homogeneous in each variable and symmetric in its arguments, and $(-)_{\Sigma_n}$ denotes the orbits under the Σ_n action on $h(X,\dots,X)$.

4.2. **Corollary** ([2, Cor. 2.5]). *Let $F\colon \mathbf{Sp} \to \mathbf{Sp}$ be a functor.*

(4.2.a) *If X is a finite spectrum, then $D_n(F)(X) \simeq (\partial_n(F) \wedge X^{\wedge n})_{\Sigma_n}$, where $\partial_n(F)$ is naïve Σ_n-spectrum independent of X.*
(4.2.b) *If F preserves filtered colimits, then $D_n(F) \simeq (\partial_n(F) \wedge (-)^{\wedge n})_{\Sigma_n}$, where $\partial_n(F)$ is the naïve Σ_n-spectrum from (4.2.a).*

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