Derived categories and sheaves

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1 Introduction

These notes are for a seminar run by David Nadler on derived symplectic geometry in Fall 2021. We introduce the basics of derived categories of sheaves.

2 Basics of Sheaves

Given a topological space X and some algebraic category $\mathcal{A}$, like $\mathbf{Mod}_R$, we can define a category of sheaves on X with values in $\mathcal{A}$. For the most part, $\mathcal{A}$ will be some set-based category so we'll be able to talk about elements in the objects of $\mathcal{A}$.

Let's we denote by $\mathbf{Open}(X)$ the category whose objects are open sets in X, and whose morphisms are inclusions of open sets.

First, a preliminary definition.

2.1 Presheaves

Definition 2.1: Presheaf

A presheaf P on X with values in $\mathcal{A}$ is a functor $P : \mathbf{Open}(X)^{\mathrm{op}} \to \mathcal{A}$. Let $\mathbf{Psh}(X; \mathcal{A})$ denote the category of all such presheaves. The morphisms in the category are just morphisms of functors (or, natural transformations).

We also write $\mathbf{Psh}(X)$ if $\mathcal{A}$ is known from context.

Unpacking the definition, this means we must assign to each open $U \subseteq X$ an object $P(U)$ in $\mathcal{A}$ and to each inclusion $V \subseteq U$ a "restriction" $P(U) \to P(V)$, subject to compatibility with the composition of inclusions, ie given $W \subseteq V \subseteq U$ we get that the composition of restrictions $P(U) \to P(V) \to P(W)$ is equal to the direct restriction $P(U) \to P(W)$. Elements $s \in P(U)$ are usually called "sections of P at U", because of the function sheaf and sheaf of sections examples below.

Example 2.1.1: Constant presheaf

Take any topological space X and any object $A \in \mathcal{A}$, just take the constant functor at A, usually denoted by $A : \mathbf{Open}(X)^{\mathrm{op}} \to \mathcal{A}$ which just always evaluates to $A \in \mathcal{A}$. Unpacking, we get that $A(U) = A$ for any $U \subseteq X$. The restriction maps are just the identity on A.

Example 2.1.2: Function sheaf

Take a topological space X, and we'll let $\mathcal{A}$ be $\mathbf{Mod}_{\mathbb{R}}$. Let $F : \mathbf{Open}(X)^{\mathrm{op}} \to \mathbf{Mod}_{\mathbb{R}}$ assign to each open $U \subseteq X$ the $\mathbb{R}$-module of (continuous, smooth, etc) functions $\mathrm{Map}(U, \mathbb{R})$. Then given $V \subseteq U$ the restriction $F(U) \to F(V)$ is literally the restriction of functions. This clearly also works for $\mathbb{C}$ instead of $\mathbb{R}$.

Example 2.1.3: Sheaf of sections

Given a $\mathbb{C}$-vector bundle $E \to B$, we can create a presheaf S on B with values in $\mathbf{Mod}_{\mathbb{C}}$ by assigning to each $U \subseteq B$ the $\mathbb{C}$-vector space of sections $U \to E|_U$. In other words, maps $U \to E|_U$ such that each $x \in U$ is assigned an element of the fiber $E|_x$ (in some continuous manner). This example encapsulates the above one by considering the vector bundle $\mathbb{C} \times X \to X$.

Example 2.1.4: Sheaf of solutions to a differential equation

Given a differential equation (for example, $y'(x) = sin(y)$), we can create a presheaf Sol on $\mathbb{R}$ with values in $\mathbf{Mod}_{\mathbb{R}}$ as follows: for each $U \subseteq \mathbb{R}$ we let $Sol(U)$ be the $\mathbb{R}$-vector space of solutions to the given differential equation on U. Once again, the "restriction" morphisms are actual restriction of solutions to the differential equation.

Example 2.1.5: Integral at most 1 presheaf

Let's create a presheaf on $\mathbb{R}$ with values in $\mathbf{Mod}_{\mathbb{R}}$ by assigning to each open $U \subseteq \mathbb{R}$ the $\mathbb{R}$-vector space of integrable functions $f : U \to \mathbb{R}$ with $\int_U f(x)\,\mathrm{dx} \leq 1$. For the restriction maps, we once again use function restriction just like the function sheaf example above. Notice that if $\int_U f(x)\,\mathrm{dx} \leq 1$, then the restriction of f to a subset $V \subseteq U$ also has integral at most 1.

2.2 Sheaves

Notice that the function sheaf, sheaf of sections, and sheaf of solutions to differential equations are called *sheaves* instead of presheaves. These geometric examples actually satisfy an additional property called the "sheaf condition" or the "descent condition" which basically says that given several different sections $s_i \in P(U_i)$ such that their restrictions agree on intersection, you can glue them together to get a section $s \in P(U)$ where $U = \bigcup_i U_i$.

Definition 2.2: Sheaf

A sheaf F on X with values in $\mathcal{A}$ is a presheaf that satisfies the following condition: Given any open $U \subseteq X$ and any family of opens $U_i \subseteq U$ that covers U, we have:

$$F(U) \xrightarrow{\sim} \lim \left(\prod_i F(U_i) \overset{r_1}{\underset{r_2}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j) \right)$$

The morphism from $F(U)$ to the limit is induced by the restrictions $F(U) \to F(U_i)$ is an isomorphism, and the limit is the equalizer.

We denote the category of all such sheaves as $\mathbf{Sh}(X, \mathcal{A})$, or $\mathbf{Sh}(X)$ is $\mathcal{A}$ is known from context. The morphisms are again the natural transformations of functors.

Unpacking, this means that every section in $F(U)$ corresponds uniquely (via the restriction maps) to a family of choices $s_i \in F(U_i)$ such that the restrictions of s_i and s_j to $F(U_i \cap U_j)$ are equal. In other words, this exactly corresponds to the gluing construction mentioned above.

2.3 Scolion: why stop at double intersections?

Given an open cover $U_i \subseteq U$ of U and a presheaf F, we get a cosimplicial structure

$$\prod_{i_0} F(U_{i_0}) \underset{r_1}{\overset{r_0}{\rightrightarrows}} \prod_{i_0,i_1} F(U_{i_0,i_1}) \overset{r_0}{\underset{r_2}{\overset{r_1}{\rightrightarrows}}} \prod_{i_0,i_1,i_2} F(U_{i_0,i_1,i_2}) \qquad \cdots$$

Here U_{i_0,i_1} is the intersection of U_{i_0} and U_{i_1}, etc.

So what happened with the rest of the diagram? Why don't we consider triple intersections, quadruple intersections, and so on? The answer is due to the following ∞-categorical lemma:

Lemma 2.1

Let Δ be the simplex category, and $\Delta_{\leq n}$ be the full subcategory of Δ whose objects are $[0], [1], \ldots, [n]$. Then given an $(n, 1)$-category $\mathcal{C}$ and a ∞ functor $F : \Delta \to \mathcal{C}$, the natural comparison

$$\lim_{\Delta} F \to \lim_{\Delta_{\leq n}} F$$

is an equivalence. In other words, the inclusion $\Delta_{\leq n} \to \Delta$ is an "n-initial" morphism.

Since our presheaves currently take values in 1-categories, we only need to care about the 1-truncation $\Delta_{\leq 1}$ of Δ, which corresponds to only caring about the first two terms!

On the other hand, when gluing sheaves together, one might have seen the need to include a cocycle condition. It's not too difficult to see that this cocycle condition has to do with including back in the third term, in other words using $\Delta_{\leq 2}$. This is because the functor we are considering here is the functor $\mathbf{Sh} : \Delta \to \mathbf{Cat}$, defined by:

$$[n] \mapsto \prod_{i_0,\ldots,i_n} \mathbf{Sh}(U_{i_0,\ldots,i_n})$$

is a functor to the 2-*category* **Cat**! This is why we need the triple intersections.

If you want to think of ∞ sheaves with values in ∞-categories then, you'd need the entire cosimplicial diagram.

2.4 Examples of sheaves

For some examples, the function sheaf 2.1.2, sheaf of sections 2.1.3, and sheaf of solutions of a differential equation 2.1.4 are standard ones, which motivated the definition in the first place.

The constant presheaf 2.1.1 isn't always a sheaf! Note what happens when the underlying space is not connected. The integral at most 1 presheaf 2.1.5 isn't a sheaf either. Every presheaf however presents a sheaf via a procedure called sheafification.

We can define sheafification in (at least) two ways: one using stalks and the other more general method, which can be extended to the context of sheaves on a site. We discuss here the stalk perspective for simplicity.

Definition 2.3: Stalks of presheaves

Given a presheaf F, we define the stalk at a point $x \in X$ as the following filtered colimit

$$F|_x = \operatorname*{colim}_{U \ni x} F(U)$$

where the colimit ranges over opens U containing x.

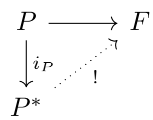
This definition, in the most important examples of the sheaf of functions and solutions to differential equations, corresponds to taking the germs of functions at certain points.

Now we can define the sheafification:

Definition 2.4: Sheafification

Given a presheaf P, we can construct a sheaf P^* called its sheafification as follows: let $P^*(U)$ be the set of "functions" which maps $x \in U$ to $s_x \in P|_x$, such that for all $x \in U$, there's an open $V \subseteq U$ and section $s \in P(V)$ where for all $y \in V$, the stalk of s at y is equal to s_y.

Sheafification is functorial, in that it defines a functor from $\mathbf{Psh}(X) \to \mathbf{Sh}(X)$. It is in fact the left adjoint to the natural inclusion $\mathbf{Sh}(X) \to \mathbf{Psh}(X)$, in that it satisfies the following universal property: Given a presheaf P and a sheaf F, along with a presheaf map $P \to F$, there's a unique morphism from $P^* \to F$ that fills the diagram:



Here i_P is the natural inclusion $P \to P^*$.

Example 2.4.1: Constant sheaf

Take the constant presheaf A^P, which evaluates to $A \in \mathcal{A}$ for any open $U \subseteq X$. If we sheafify, we get what's called the constant sheaf A. For example, let $X = \{a, b\}$ be a discrete set. Then $A(\{a\}) = A(\{b\}) = A$, but by the sheaf condition, $A(\{a, b\}) = A \times A$.