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Orientation on Enhancing Derived Categories

Reminder. I'm very happy to talk to anyone about these things.

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Recall from last time. R ring, $A \in Ch(R)$. The square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

is a pushout.

- However, Dennis showed us that if we weaken the definition of a pushout to only require the square to commute up to chain homotopy, then we recover the **shift** of complexes!

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\simeq"] & 0 \arrow[d] \\ 0 \arrow[r] & A[1] \end{tikzcd}$$

> As a complex with this property, $M[1]$ is unique up to quasi-isomorphism.

Question. How unique is this quasi-isomorphism?

Slogan. '$\underbrace{\text{Contractibility}}_{\text{acyclicity}}$ as uniqueness' ← Riehl from homotopy theory

Note. We're already familiar with this principle from category theory! (Co)limits are not unique, there is a contractible groupoid of them.

Key Idea of dg categories. We should record/remember **how** complexes are quasi-isomorphic by encoding a **Hom complex** between objects.

> Can then formulate 'universal properties' with Hom complexes replacing

uniqueness $\rightsquigarrow$ acyclicity.

Key Features of the dg derived category $D(R)$.

(0) Objects are chain complexes of R-modules. Between any two complexes A and B, there is a Hom complex

$$\mathrm{Hom}_{D(R)}(A,B).$$

(1) $D(R)$ has a zero object 0:

$$\mathrm{Hom}_{D(R)}(A,0) \quad \text{and} \quad \mathrm{Hom}_{D(R)}(0,A)$$

are acyclic

(2) Every morphism in $D(R)$ has a cone/cocone:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}$$

'homotopy pushout'

$\rightsquigarrow$ triangulated structure

(3) Stability: Given a square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd} \quad \text{in } D(R)$$

$$\mathrm{Cone}(f) \xrightarrow{\sim} C$$

iff

$$A \xrightarrow{\sim} \mathrm{Cocone}(g)$$

⚠ In an ordinary category, this only happens in trivial situations!

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

so $A \xrightarrow{\sim} 0 \times_0 0 = 0$.

(4) $D(R)$ has homotopy direct sums and they're computed in by the direct sum of chain complexes

(4') Ditto for products

(5) $D(R)$ has all homotopy limits and colimits

(6) $D(R)$ is generated under $\underbrace{\text{Cones and direct sums}}_{\text{Equivalently, homotopy colims}}$ by $R[0]$.

(7) There is a (derived) tensor product

$$(-) \otimes_R (-) : D(R) \times D(R) \longrightarrow D(R)$$

'uniquely' characterized by the requirements:

(a) $R[0]$ is the unit

(b) $\otimes_R$ commutes with homotopy colimits separately in each variable.

In addition, if $M, N \in Mod(R)$ and either is flat, then

$$M[0] \otimes_R N[0] \simeq (\underbrace{M \otimes^{\heartsuit}_R N}_{\substack{\text{Tensor product of ordinary} \\ R\text{-modules (i.e., in } Mod(R) = D(R)^{\heartsuit})}})[0].$$

(8) For $A, B, C \in D(R)$,

$$Hom_{D(R)}(A \otimes_R B, C) \underset{qis}{\simeq} Hom_{D(R)}(A, Hom_{D(R)}(B, C)).$$