Seminar overview & sheaves on $\mathbf{R}$

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1 Seminar overview/logistics

Meeting time. Tuesdays 2-3:30pm

Room. #732

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Subject. Sheaves and microlocal sheaves.

Slogan. 'Think globally, act *micro*locally'.

Ideal seminar output. Produce a 'learner's manual' for sheaves and microlocal sheaves.

Classic reference. Kashiwara and Schapira's text *Sheaves on manifolds* [6].

Recent work. Nadler and Shende's *Sheaf quantization in Weinstein symplectic manifolds* [19]. This is a good target for the seminar.

Table of contents. Below is a tentative list of topics for us to discuss.

(I) Background

- (I.1) Today: sheaves on $\mathbf{R}$.
- (I.2) Setup: dg categories, derived categories of sheaves, …
- (I.3) Operations in sheaves: Grothendieck's six functors
- (I.4) Nearby/vanishing cycles

(II) Microlocal perspective

- (II.1) Symplectic & contact geometry of cotangent bundles
- (II.2) Singular support & involutivity
- (II.3) Non-characteristic propagation (essentially why singular support is defined)
- (II.4) Perverse sheaves

(III) Microlocal sheaves

- (III.1) Definitions, basics

(III.2) μhom

(III.3) Microlocal cutoffs

(III.4) Antimicrolocalization: from microsheaves back to sheaves

(IV) Beyond cotangent bundles

(IV.1) Weinstein manifolds

(IV.2) Homotopical structures

(IV.3) Invariance

(V) 'Beyond sheaves' in topology

Examples (from mirror symmetry and GRT).

(1) The coherent-constructible correspondence [2; 3; 15; 20].

(2) Gammage–Shende: mirror symmetry for affine hypersurfaces [4; 5].

(3) Nadler: mirror symmetry for Landau–Ginzburg models [17; 18].

2 Inviation: sheaves on the real line

2.1 Generalities

2.1 Question. What is a sheaf on a topological space X?

2.2 Answer. In this seminar, we want to take the derived perspective on everything. Write $\mathrm{Open}(X)$ for the poset of open subsets of X ordered by inclusion. A sheaf should mean a 'functor'

$$\mathcal{F}^\bullet \colon \mathrm{Open}(X)^{\mathrm{op}} \to \left(\begin{smallmatrix}\text{dg derived category} \\ \text{of chain complexes}\end{smallmatrix}\right)$$

satisfying descent properties with respect to open coverings. For the sake of concreteness, we'll take our chain complexes to be complexes of $\mathbf{C}$-vector spaces.

The term 'functor' needs to be interpreted in the sense of *differential graded* (*dg*) categories or ∞-categories, and the sheaf condition needs to be interpreted in this setting as well. We'll discuss both of these in the next couple lectures. Here are some resources for these 'derived' perspectives:

(1) A_∞-categories: [7; 8; 9; 11; 10; 13; 21; 22, Chapter I]

(2) dg categories: [HA, §1.3.1; Ker; 1; 12; 14; 23]

(3) Stable ∞-categories: [HA, Chapter 1; SAG, §D.1; 14]

2.3. Note that we can think of the functor $\mathcal{F}^\bullet$ in two ways:

(1) For each open subset $U \subset X$, a complex $\mathcal{F}^\bullet(U)$, and for each inclusion $V \subset U$ compatible restriction maps $\mathcal{F}^\bullet(U) \to \mathcal{F}^\bullet(V)$.

(2) For each integer $n \in \mathbf{Z}$, the data of a sheaf (in the usual sense) $\mathcal{F}^n$ along with maps of sheaves $d \colon \mathcal{F}^n \to \mathcal{F}^{n+1}$ satisfying $d^2 = 0$.

Said differently:

2.4 Slogan. 'Sheaf of complexes = complex of sheaves.'

Both ways of thinking about a sheaf are useful.

2.5 Example. A key example is when $\mathcal{F}^\bullet = \mathrm{C}^\bullet_{\mathrm{sing}}(-)$ is the sheaf of singular cochains on X.

2.6 Example. Let's consider the case $X = \mathbf{R}$ concretely. Since the open intervals (a, b) form a basis for the topology of $\mathbf{R}$, we can equivalently regard a sheaf as an assignment of a complex $\mathcal{F}^\bullet(a, b)$ to every open interval $(a, b) \subset \mathbf{R}$ along with compatible restriction morphisms.

2.7 Definition. Let X be a topological space and $\mathcal{F}^\bullet$ a sheaf on X. For each $i \in \mathbf{Z}$, the *i-th cohomology sheaf* is the quotient

$$\mathrm{H}^i(\mathcal{F}^\bullet) \coloneqq \frac{\ker(d^i : \mathcal{F}^i \to \mathcal{F}^{i+1})}{\mathrm{im}(d^{i-1} : \mathcal{F}^{i-1} \to \mathcal{F}^i)} \ .$$

We write $\mathrm{H}^\bullet(\mathcal{F}^\bullet)$ for the complex of sheaves with trivial differential $\bigoplus_{i \in \mathbf{Z}} \mathrm{H}^i(\mathcal{F}^\bullet)[i]$.

2.8 Definition. Let X be a topological space. A sheaf $\mathcal{F}^\bullet$ on X is *locally constant* if the cohomology sheaf $\mathrm{H}^\bullet(\mathcal{F}^\bullet)$ is locally constant. That is, $\mathrm{H}^\bullet(\mathcal{F}^\bullet)$ is locally isomorphic to a sum of constant sheaves.

2.2 Sheaves on R

2.9 Exercise. What is the category of locally constant sheaves on the real line $\mathbf{R}$?

2.10 Answer. The (dg derived) category of chain complexes! The point is that if a sheaf is constant on overlapping intervals, then it is constant on their union. An inductive argument then shows that the sheaf has to be constant on all of $\mathbf{R}$.

Write $\mathrm{D}(\mathbf{C})$ for the dg derived category of chain complexes of $\mathbf{C}$-vector spaces and $\mathrm{LC}(\mathbf{R})$ for the dg category of locally constant sheaves on $\mathbf{R}$. Write $\mathbf{C}_\mathbf{R}$ for the constant sheaf with value $\mathbf{C}$. More precisely, taking global sections defines an equivalence

$$\begin{aligned} \mathrm{LC}(\mathbf{R}) &\xrightarrow{\sim} \mathrm{D}(\mathbf{C}) \\ \mathcal{F}^\bullet &\mapsto \Gamma(\mathbf{R}; \mathcal{F}^\bullet) \end{aligned}$$

The inverse $\mathrm{D}(\mathbf{C}) \xrightarrow{\sim} \mathrm{LC}(\mathbf{R})$ is the constant sheaf functor, which can be described by sending a complex $C^\bullet$ to the sheaf $C^\bullet \otimes \mathbf{C}_\mathbf{R}$.

Note also that given a locally constant sheaf $\mathcal{F}^\bullet$ on $\mathbf{R}$, the global sections $\Gamma(\mathbf{R}; \mathcal{F}^\bullet)$ agree with the stalk of $\mathcal{F}^\bullet$ at any point.

2.11 Exercise. What is the category of sheaves on $\mathbf{R}$ that are locally constant on $\mathbf{R}_{\neq 0}$?

2.12 Answer. The relevant category of sheaves is dg modules over the quiver $\bullet \leftarrow \bullet \rightarrow \bullet$.

To see this, note that since $\mathbf{R}_{>0}$ and $\mathbf{R}_{<0}$ are homeomorphic to $\mathbf{R}$, every locally constant sheaf on $\mathbf{R}_{>0}$ or $\mathbf{R}_{<0}$ is constant. Hence a sheaf $\mathcal{F}^\bullet$ on $\mathbf{R}$ that is locally constant on $\mathbf{R}_{\neq 0}$ is completely determined by the three sections $\Gamma(\mathbf{R}_{>0}; \mathcal{F}^\bullet)$, $\Gamma(\mathbf{R}_{<0}; \mathcal{F}^\bullet)$, and $\Gamma(\mathbf{R}; \mathcal{F}^\bullet)$, along with restriction maps

$$\Gamma(\mathbf{R}_{<0}; \mathcal{F}^\bullet) \leftarrow \Gamma(\mathbf{R}; \mathcal{F}^\bullet) \rightarrow \Gamma(\mathbf{R}_{>0}; \mathcal{F}^\bullet) \ .$$

Note that $\Gamma(\mathbf{R}_{>0}; \mathcal{F}^\bullet)$ agrees with the stalk of $\mathcal{F}^\bullet$ at $+1$, $\Gamma(\mathbf{R}_{<0}; \mathcal{F}^\bullet)$ agrees with the stalk of $\mathcal{F}^\bullet$ at -1, and $\Gamma(\mathbf{R}; \mathcal{F}^\bullet)$ agrees with the stalk of $\mathcal{F}^\bullet$ at 0. Hence another way of presenting this calculation is that the data of the sheaf $\mathcal{F}^\bullet$ is equivalent to specifying the stalks $\mathcal{F}^\bullet_1$, $\mathcal{F}^\bullet_{-1}$, and $\mathcal{F}^\bullet_0$ along with *specialization maps*

$$\mathcal{F}^\bullet_{-1} \xleftarrow{r_-} \mathcal{F}^\bullet_0 \xrightarrow{r_+} \mathcal{F}^\bullet_1 \ .$$

2.13 Idea. Part of the motivation of microlocal sheaf theory is that Answer 2.12 is a *lousy* presentation of this category: this presentation misses symmetries visible from the microlocal persepctive.

2.3 A more symmetric viewpoint

So far, we have the following measurements/functionals: the stalks of the sheaf $\mathcal{F}^\bullet$ at -1, 0, and 1.

2.14 Idea. The measurement given by the stalk at 0 is not at the same footing as the stalks at -1 and 1. Since the sheaf $\mathcal{F}^\bullet$ is not required to be locally constant around 0, instead of taking the *static* measurement of the stalk at 0, we should take a *dynamic* measurement of how the sections of F change from $\mathbf{R}_{<0}$ to $\mathbf{R}_{<\varepsilon}$ (or $\mathbf{R}_{>0}$ to $\mathbf{R}_{>-\varepsilon}$).

Taking the stalk at 0 is like ‘‘sticking your nose into the black hole’ of interesting geometry. Generally this should be avoided.

2.15 Definition. Let $\mathcal{F}^\bullet$ be a sheaf on $\mathbf{R}$ locally constant on $\mathbf{R}_{\neq 0}$. The *sheaf of vanishing cycles* for x at 0 is the cone

$$\phi_{x,0}(F) \coloneqq \operatorname{Cone}(r_- : \mathcal{F}_0^\bullet \to \mathcal{F}_{-1}^\bullet)\ .$$

Similarly, the *sheaf of vanishing cycles* for $-x$ at 0 is the cone

$$\phi_{-x,0}(F) \coloneqq \operatorname{Cone}(r_+ : \mathcal{F}_0^\bullet \to \mathcal{F}_1^\bullet)\ .$$

2.16. These vanishing cycles fit into a diagram of exact triangles

$$\begin{tikzcd}
 & & & \phi_{-x,0}(\mathcal{F}^\bullet) \arrow[dl] & \\
\mathcal{F}_{-1}^\bullet \arrow[dr] & & \mathcal{F}_0^\bullet \arrow[ll, "r_-"'] \arrow[rr, "r_+"'] & & \mathcal{F}_1^\bullet \arrow[ul] \\
 & \phi_{x,0}(\mathcal{F}^\bullet) \arrow[ur] & & &
\end{tikzcd} .$$

In this diagram, one can think of the complexes on the horizontal axis as measurements in the $\mathbf{R}$ direction, and the complex on the vertical axis as measurements in cotangent directions. Note that the stalk $\mathcal{F}_0^\bullet$ is recoverable from the stalks $\mathcal{F}_{-1}^\bullet$ and $\mathcal{F}_1^\bullet$, the vanishing cycles $\phi_{-x,0}(\mathcal{F}^\bullet)$ and $\phi_{x,0}(\mathcal{F}^\bullet)$, along with the maps relating them.

2.17 Alternative Answer. While this seems like a more complicated way to answer Exercise 2.11, appropriately keeping track of this data, we can give a better and more symmetric answer.. Sheaves on $\mathbf{R}$ that are locally constant on $\mathbf{R}_{\neq 0}$ are determined by a cycle of complexes

$$\begin{tikzcd}
 & \bullet \arrow[dl, bend right] & \\
\bullet \arrow[dr, bend right] & & \bullet \arrow[ul, bend right] \\
 & \bullet \arrow[ur, bend right] &
\end{tikzcd} \tag{2.18}$$

where:

(1) All pairwise composites are 0.

(2) Each complex is equivalent to the total complex of the others.

One might call this object a ‘totally acyclic cycle’.

This presentation has a 90° rotational symmetry that was not apparent in the presentation in Answer 2.12. Moreover, rotational symmetry is an instance of a *Fourier Transform*! See [16] for more ideas in this direction.

References

HA J. Lurie, *Higher algebra*, Preprint available at math.ias.edu/~lurie/papers/HA.pdf, Sep. 2017.

SAG ______, *Spectral algebraic geometry*, Preprint available at math.ias.edu/~lurie/papers/SAG-rootfile.pdf, Feb. 2018.

Ker ______, *Kerodon*, kerodon.net, 2019.

1. L. Cohn, *Differential graded categories are k-linear stable ∞-categories*, Preprint available at arXiv:1308.2587, Sep. 2016.
2. B. Fang, C.-C. M. Liu, D. Treumann, and E. Zaslow, *T-duality and homological mirror symmetry for toric varieties*, Adv. Math., vol. 229, no. 3, pp. 1875–1911, 2012. DOI: 10.1016/j.aim.2011.10.022.
3. ______, *The coherent-constructible correspondence for toric Deligne–Mumford stacks*, Int. Math. Res. Not. IMRN, no. 4, pp. 914–954, 2014. DOI: 10.1093/imrn/rns235.
4. B. Gammage and V. Shende, *Homological mirror symmetry at large volume*, Preprint available at arXiv:2104.11129, Apr. 2021.
5. ______, *Mirror symmetry for very affine hypersurfaces*, Preprint available at arXiv:1707.02959, Jan. 2021.
6. M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1994, vol. 292, pp. x+512, With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original, ISBN: 3-540-51861-4.
7. B. Keller, *Bimodule complexes via strong homotopy actions*, in, 4, vol. 3, Special issue dedicated to Klaus Roggenkamp on the occasion of his 60th birthday, 2000, pp. 357–376. DOI: 10.1023/A:1009954126727.
8. ______, *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl., vol. 3, no. 1, pp. 1–35, 2001. DOI: 10.4310/hha.2001.v3.n1.a1.
9. ______, *A-infinity algebras in representation theory*, in *Representations of algebra. Vol. I, II*, Beijing Norm. Univ. Press, Beijing, 2002, pp. 74–86.
10. ______, *Addendum to: "Introduction to A-infinity algebras and modules" [Homology Homotopy Appl.* **3** *(2001), no. 1, 1–35; MR1854636 (2004a:18008a)]*, Homology Homotopy Appl., vol. 4, no. 1, pp. 25–28, 2002. DOI: 10.4310/hha.2002.v4.n1.a2.
11. ______, *A-infinity algebras, modules and functor categories*, in *Trends in representation theory of algebras and related topics*, Contemp. Math. Vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 67–93. DOI: 10.1090/conm/406/07654.
12. ______, *On differential graded categories*, in *International Congress of Mathematicians. Vol. II*, Eur. Math. Soc., Zürich, 2006, pp. 151–190.
13. M. Kontsevich and Y. Soibelman, *Notes on A_∞-algebras, A_∞-categories and non-commutative geometry*, in *Homological mirror symmetry*, Lecture Notes in Phys. Vol. 757, Springer, Berlin, 2009, pp. 153–219.
14. A. Mazel-Gee, *An invitation to higher algebra*, Book project available at etale.site/teaching/w21/math-128-lecture-notes.pdf, Jul. 2021.
15. D. Nadler, *Microlocal branes are constructible sheaves*, Selecta Math. (N.S.), vol. 15, no. 4, pp. 563–619, 2009. DOI: 10.1007/s00029-009-0008-0.
16. ______, *Cyclic symmetries of A_n-quiver representations*, Adv. Math., vol. 269, pp. 346–363, 2015. DOI: 10.1016/j.aim.2014.10.006.
17. ______, *A combinatorial calculation of the Landau-Ginzburg model $M = \mathbf{C}^3$, $W = z_1 z_2 z_3$*, Selecta Math. (N.S.), vol. 23, no. 1, pp. 519–532, 2017. DOI: 10.1007/s00029-016-0254-x.
18. ______, *Mirror symmetry for the Landau-Ginzburg A-model $M = \mathbf{C}^n$, $W = z_1 \cdots z_n$*, Duke Math. J., vol. 168, no. 1, pp. 1–84, 2019. DOI: 10.1215/00127094-2018-0036.

19. D. Nadler and V. Shende, *Sheaf quantization in Weinstein symplectic manifolds*, Preprint available at arXiv:2007.10154, Feb. 2021.

20. D. Nadler and E. Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc., vol. 22, no. 1, pp. 233–286, 2009. DOI: 10.1090/S0894-0347-08-00612-7.

21. Y.-G. Oh and H. L. Tanaka, A_∞*-categories, their* ∞*-category, and their localizations*, Preprint available at arXiv:2003.05806, May 2021.

22. P. Seidel, *Fukaya categories and Picard–Lefschetz theory*, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008, pp. viii+326, ISBN: 978-3-03719-063-0. DOI: 10.4171/063.

23. B. Toën, *Lectures on dg-categories*, in *Topics in algebraic and topological K-theory*, Lecture Notes in Math. Vol. 2008, Springer, Berlin, 2011, pp. 243–302. DOI: 10.1007/978-3-642-15708-0.