Lecture 2: Interlude on $D(R)$

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Recall from the end of the previous lecture that we want to remove these non-intuitive and non-canonical injective resolutions from the theory of sheaf cohomology. We do this by working in the ∞-category of sheaves with values in $D(\mathbb{Z})$, where $D(\mathbb{Z})$ is the derived ∞-category of abelian groups.

We will see a construction of $D(\mathbb{Z})$ later. For now let's take an axiomatic approach, inspired by the Schwede-Shipley theorem (see their paper "Stable model categories are categories of modules", or in the present language Lurie's Theorem 7.1.2.1 in Higher Algebra). Actually we'll work with an arbitrary ring R.

Definition 1. *Let R be a ring. The ∞-category $D(R)$ is a stable ∞-category with all colimits, generated (as a cocomplete stable ∞-category) by a distinguished compact object $\mathbf{1}$ with a distinguished identification $\pi_0 \operatorname{Map}(\mathbf{1},\mathbf{1}) = R^{op}$ as a ring, such that $\pi_0 \operatorname{Map}(\Sigma^d \mathbf{1},\mathbf{1}) = 0$ for $d \in \mathbb{Z} \setminus \{0\}$.*

Let us unpack this. A stable ∞-category is a pointed ∞-category $\mathcal{C}$ with all pushouts and pullbacks such that the suspension functor $\Sigma(X) = \operatorname{colim}(0 \leftarrow X \to 0)$ is inverse to its adjoint, the loop functor $\Omega(X) = \lim(0 \leftarrow X \to 0)$. We therefore also denote $\Sigma^{-1}(X) = \Omega(X)$, and we can similarly define $\Sigma^d(X)$ for all $d \in \mathbb{Z}$. It follows (nonobviously) from the axioms that $\mathcal{C}$ is additive; see Lurie's Higher Algebra for this and all other basic facts about stable ∞-categories I use. This additivity is what makes $\pi_0 \operatorname{Map}(\mathbf{1},\mathbf{1})$ a priori a ring and therefore makes sense of the condition that it identify with R^{op} as a ring. Furthermore, it also follows that a commutative square is a pullback if and only if it is a pushout. Basically, it's a nice symmetric set of conditions which identify certain finite colimit diagrams with the dual finite limit diagrams. In fact, since arbitrary finite colimits are built from pushouts (and $\varnothing$) and arbitrary finite limits are built from pullbacks (and $*$), we can express each in terms of the other somehow.

An important special case of a commutative square is a null-composite sequence

$$X \to Y \to Z.$$

(Beware that the nullhomotopy of the composite is part of the data!) It follows that a null-composite sequence is a cofiber sequence if and only if it is a fiber sequence. An example of such a fiber-cofiber sequence is

$$X \to 0 \to \Sigma X,$$

and by mapping our first cofiber sequence to this one and using functoriality we produce a boundary map $\partial : Z \to \Sigma X$. Moreover all of the "rotated" sequences such as $Y \to Z \to \Sigma X$ have canonical null-composites and are also fiber-cofiber sequences. In particular, fiber-cofiber sequences of the form $X \to ? \to Z$, which are also called extensions of Z by X, are classified by maps $Z \to \Sigma X$; we recover $?$ as the fiber.

For X and Y in $\mathcal{C}$, we define $[X,Y] = \pi_0 \operatorname{Map}(X,Y)$, and more generally $[X,Y]_d = [\Sigma^d X, Y] = [X, \Sigma^{-d} Y]$ for $d \in \mathbb{Z}$. If $d \geq 0$, this is also equal to $\pi_d \operatorname{Map}(X,Y)$. Because $\mathcal{C}$ is additive, these are abelian groups. Moreover, given a fiber-cofiber sequence $X \to Y \to Z$ and an arbitrary $A \in \mathcal{C}$, we deduce that there are associated long exact sequences

$$\ldots \to [A,X]_d \to [A,Y]_d \to [A,Z]_d \to [A,X]_{d-1} \to \ldots$$

and

$$\ldots [X,A]_{d+1} \to [Z,A]_d \to [Y,A]_d \to [X,A]_d \to \ldots$$

Every pushout-pullback square can be converted into a null-composite sequence by the standard $A \to B \oplus C \to D$, so we see that pushout-pullback squares induce long exact sequences of Mayer-Vietoris type both on mapping in and on mapping out to an arbitrary object.

Of central importance are the following functors.

Definition 2. *For $d \in \mathbb{Z}$, define $H_d : D(R) \to Mod_R$ by*

$$H_d(X) = [\mathbf{1}, X]_d.$$

As a special case of the long exact sequences discussed above, a cofiber-fiber sequence $X \to Y \to Z$ induces a long exact sequence on homology of the usual form. These homology functors give us an important foothold from which we can prove some things about $D(R)$.

Lemma 3. *For each $d \in \mathbb{Z}$, the functor H_d commutes with arbitrary products and coproducts (direct sums). It also commutes with all filtered colimits, and for sequential inverse limits there is a Milnor sequence.*

Proof. The claim about products is immediate because $\pi_0 : \mathcal{S} \to \mathrm{Sets}$ commutes with products. Similarly for the claim about filtered colimits, using that $\mathbf{1}$ is compact. For the claim about coproducts, using filtered colimits we reduce to finite coproducts; but these agree with finite products both in $D(R)$ and in Mod_R by additivity. As for sequential inverse limits, in a general ∞-category there is a version of the Milnor telescope construction expressing a sequential inverse limit as a pullback of countable products, and in the stable case this can be rewritten as a fiber-cofiber sequence

$$\varprojlim_n X_n \to \prod_n X_n \stackrel{\sigma - 1}{\to} \prod_n X_n$$

with σ the shift map, which gives the claim on taking long exact sequence of homology groups after using the claim about products. □

Lemma 4. *A map $f : X \to Y$ in $D(R)$ is an isomorphism if and only if $H_d(f)$ is an isomorphism for all $d \in \mathbb{Z}$.*

Proof. Passing to the cofiber it suffices to show that if $Z \in D(R)$ has $H_d(Z) = 0$ for all d, then $Z = 0$. Consider the full subcategory $\mathcal{C} \subset D(R)$ of those objects A such that $Map(\Sigma^d A, Z) = *$ for all $d \in \mathbb{Z}$, or equivalently (taking homotopy groups) such that $[\Sigma^d A, Z] = 0$ for all $d \in \mathbb{Z}$. By hypothesis $\mathbf{1} \in \mathcal{C}$. By the first condition $\mathcal{C}$ is closed under all colimits, since any limit of $*$ is $*$. By the second condition and the long exact Mayer-Vietoris sequence, $\mathcal{C}$ is closed under pushouts and pullbacks. Hence $\mathcal{C}$ is a stable cocomplete subcategory containing $\mathbf{1}$, so $\mathcal{C} = D(R)$ and hence $Z = 0$ by Yoneda. □

Now we can try to build the canonical filtration on an object $X \in D(R)$.

Lemma 5. *Let $X \in D(R)$. There is an object $Y \in D(R)$ with a map $f : Y \to X$ such that:*

1. $H_d(Y) = 0$ *for* $d < 0$;
2. $H_d(f)$ *is an isomorphism for* $d \geq 0$.

Proof. Let us inductively produce a sequence of objects and maps

$$Y_0 \to Y_1 \to \dots$$

in $D(R)_{/X}$ such that each Y_n satisfies 1 and 2, and $H_d(Y_n \to X)$ is an iso for $0 \leq d < n$ and a surjection for $d = n$. For Y_0, we take a direct sum of copies of $\mathbf{1}$ mapping to X inducing a surjection on H_0. This is possible simply by choosing representatives for a set of generators of $H_0(X)$. For the inductive step, pass to the fiber

$$F \to Y_{n-1} \to X,$$

similarly choose a map $\Sigma^{n-1} \oplus_I \mathbf{1} \to F$ inducing a surjection on H_{n-1}, and define Y_n to be the cofiber of the composite $\Sigma^{n-1} \oplus_I \mathbf{1} \to F \to Y_{n-1}$. Passing to the filtered colimit we get the required. (This is very much like building a CW approximation, or a free resolution.) □

Proposition 6. *For an object $X \in D(R)$, the following are equivalent:*

1. $H_d(X) = 0$ *for* $d < 0$;
2. X *is generated by* $\mathbf{1}$ *under colimits;*
3. *there is a sequence of maps $X_0 \to X_1 \to \dots$ with colimit X, where the cofiber of $X_{i-1} \to X_i$ is of the form $\Sigma^i \oplus_I \mathbf{1}$.*

Proof. Suppose X satisfies 1. The previous proof produced a Y satisfying both 3 and 1, and a map $f : Y \to X$ which is an iso on homology in non-negative degrees. It follows f is an iso in all degrees hence an iso, so X satisfies 3. It is clear that 3 implies 2 because by rotation X_i is the cofiber of a map $\Sigma^{i-1} \oplus_I \mathbf{1} \to X_{i-1}$. Finally, to see that 2 implies 1 we recall that an arbitrary colimit can be written in terms of finite colimits and filtered colimits, and a finite colimit can be written in terms of pushouts. Both operations preserve condition 1 by, respectively, commutation of homology with filtered colimits and the Mayer-Vietoris sequence. □

Let $D(X)_{\geq 0}$ denote the full subcategory of $D(X)$ on those objects satisfying the above conditions, and let $D(X)_{<0}$ denote the full subcategory of those satisfying the opposite condition: $H_d(X) = 0$ for $d \geq 0$.

Corollary 7. *For $X \in D(X)_{\geq 0}$ and $Y \in D(X)_{<0}$, we have $Map(X, Y) = *$.*

Proof. This is true for $X = \mathbf{1}$ by definition. On the other hand the set of objects for which it's true is closed under all colimits, because any limit of $*$ is $*$. Thus this follows from criterion 2 of the proposition. □

Corollary 8. *The inclusion $D(R)_{\geq 0} \to D(R)$ admits a right adjoint $\tau_{\geq 0}$, and the inclusion $D(R)_{<0} \to D(R)$ admits a left adjoint $\tau_{<0}$. If $Y \to X$ is as in Lemma 5, then $Y \simeq \tau_{\geq 0} X$ compatibly with map to X. Similarly if $X \to Z$ is an iso on homology in negative degrees and $Z \in D(R)_{<0}$, then $Z \simeq \tau_{>0} X$ compatibly with map from X.*

Proof. Recall that to prove a functor has an adjoint it's enough to prove it has the requisite universal property objectwise, and then the functoriality and so on comes for free. Thus one need only check that a map $f : Y \to Z$ as in Lemma 5 satisfies the condition that $\mathrm{Map}(X,Y) \tilde{\to} \mathrm{Map}(X,Z)$ for all $X \in D(R)_{\geq 0}$. However, the cofiber of f lives in $D(R)_{<0}$ and analyzing the long exact sequence on maps from X we see that this follows from Corollary 7. □

By this universal property, these truncation functors organize into a tower

$$\ldots \to \tau_{\leq n} \to \tau_{\leq n-1} \to \ldots,$$

and similarly

$$\ldots \to \tau_{\geq n} \to \tau_{\geq n+1} \to \ldots.$$

We can also see that these towers are convergent.

Corollary 9. *For $X \in D(R)$, we have*

$$X \simeq \varprojlim_n \tau_{\leq n}(X)$$

and

$$X \simeq \varinjlim_n \tau_{\geq -n}(X).$$

Proof. Check on homology using the fact that homology commutes with filtered colimits, and the Milnor sequence for sequential inverse limits. The point is that in either case the homology of the terms in any fixed degree stabilizes after a given point. □

This is extremely useful for reducing the case of a general object $X \in D(R)$ to one which lives in a bounded range of degrees, which by truncation can be reduced to one living in a single degree.

By the way, If you want to see some of the kind of arguments we're making here concerning these truncation functors done in a more abstract framework, you can look up the theory of t-structures, first introduced by Beilinson-Bernstein-Deligne Asterisque 100 in the triangulated category context, and redone in Lurie's Higher Algebra in the stable ∞-category context.

Remark 10. *For purpose of comparing with the canonical filtration on cochain complexes in the previous lecture, note that we've switched to homological indexing here, so $\tau_{\geq n} = \tau^{\geq -n}$ and $H_n = H^{-n}$.*

Since we can use the truncation functors to reduce to considering objects living in a single degree, the natural question is, what are such objects?

Proposition 11. *Let $D(R)_0$ denote the full subcategory of those objects X with $H_d(X) = 0$ for $d \neq 0$. Then the functor $H_0 : D(R)_0 \to Mod_R$ is an equivalence of ∞-categories (to the* ordinary *category of R-modules.)*

Proof. One readily checks using $[D(R)_{\geq 0}, D(R)_{<0}] = 0$ that $\pi_d \mathrm{Map}(X,Y) = 0$ for $X, Y \in D(R)_0$ and $d > 0$, so that $D(R)_0$ is equivalent to a category. In fact, it is even an abelian category: if $f : X \to Y$, then the kernel of f is $\tau_{\geq 0}$ of the fiber, and the cokernel of f is $\tau_{\leq 0}$ of the cofiber. At the same time one sees that H_0 is exact, and we already know it commutes with direct sums. Moreover, every object in $D(R)_0$ is the cokernel of a map between direct sums of copies of $\mathbf{1}$, by applying $\tau_{\leq 0}$ to the map $Y_1 \to X$ constructed in the proof of Lemma 5. Therefore in both $D(R)_0$ and Mod_R every object is the cokernel of a direct sum of copies of the generator. As the functor preserves all that, to see that the hom sets match up in $D(R)_0$ and Mod_R it suffices to argue that $\mathbf{1} \in D(R)_0$ is compact and projective, as then the maps will be formally determined by the maps from $\mathbf{1}$ to $\mathbf{1}$ which are the same as in Mod_R by definition.

It's compact by assumption. For projectivity, suppose $X \to \mathbf{1}$ is an epimorphism in $D(R)_0$. Then the fiber F also lies in $D(R)_0$, and the short exact sequence $F \to X \to \mathbf{1}$ is classified by a map $\mathbf{1} \to \Sigma F$, or up to homotopy by an element of $H_{-1}(F) = 0$. Hence our extension is classified by the zero map. But the extension classified by the zero map is the split extension by a general fact about stable ∞-categories, so it follows that $X \to \mathbf{1}$ has a splitting, as desired. □

As an exercise you will furthermore see that if $M, N \in Mod_R$, viewed as objets in $D(R)_0$, then $[M,N]_d \simeq Ext_R^{-d}(M,N)$ for all $d \in \mathbb{Z}$. Combined with the previous facts about truncations, this means $D(R)$ does behave exactly as expected of a derived category of R-modules, and in fact for many purposes one can work with $D(R)$ as discussed here and forget all about complexes.

But we're interested in de Rham cohomology, and one the fundamental features of de Rham cohomology is that it is, by definition, represented by a natural complex. So, in terms of the ∞-category $D(R)$, what is the intrinsic meaning of representing an object by a complex? We've already seen hints of this in the previous lecture and in Proposition 6 above, but let's spell it out.

Definition 12. *Let $\mathbb{Z}_{\leq}$ denote the poset of integers under $\leq$. A filtered object of $D(R)$ is an object in the functor ∞-category $Fun(\mathbb{Z}_{\leq}, D(R))$. A filtered object F is called convergent if*

$$\varprojlim_n F(n) = 0;$$

in this case we say it converges to $F(\infty) := \varinjlim_n F(n)$, or that $F(\infty)$ is the underlying object. The n^{th} associated graded $gr_n F$ is by definition the cofiber of $F(n-1) \to F(n)$, also written as $F(n)/F(n-1)$ for short.

Remark 13. *There is an asymmetry here in that we say that we converge to the colimit and the limit has to be zero, but in fact we can just as easily switch it around. Indeed*

$$F(\infty) = \varprojlim_n G(n)$$

and

$$\varinjlim_n G(n) = 0$$

with $G(n) = cofib(F(-n) \to F(\infty))$ is a functor $\mathbb{Z}_{\geq} \to D(R)$. This lets us switch back and forth between the above notion of a convergent filtration and its dual.

Now we can say what a complex is. You can also look up the discussion of the *Beilinson t-structure* in Bhatt-Morrow-Scholze's paper "Topological Hochschild Homology and p-adic Hodge theory" for a more refined perspective on this.

Proposition 14. *Suppose given a convergent filtration $F : \mathbb{Z}_{\leq} \to D(R)$ with the special property that $gr_n F \in D(R)_n$ for all $n \in \mathbb{Z}$. Let*

$$M_n := H_n(gr_n F) \in Mod_R,$$

and define a map $d : M_n \to M_{n-1}$ by taking H_n of the map $gr_n F \to \Sigma gr_{n-1} F$ classifying the fiber-cofiber sequence

$$gr_{n-1} F \to F(n)/F(n-2) \to gr_n(F).$$

Then $d^2 = 0$, defining a complex of R-modules. Moreover:

1. *The homology of the underlying object $F(\infty)$ canonically identifies with the homology of this complex.*

2. *Let $Fun(\mathbb{Z}_{\geq 0}, D(R))_{cx}$ denote the full subcategory of filtrations as above. Then the functor $Fun(\mathbb{Z}_{\geq 0}, D(R))_{cx} \to Ch_R$ defined above is an equivalence of ∞-categories (to the* ordinary *category of complexes of R-modules).*

Proof. We can prove by induction on the length of the interval $[n,m] \subset \mathbb{Z}$ the more refined claim that giving a filtered object indexed by $[n,m]$ with k^{th} associated graded in degree k is equivalent to giving a chain complex concentrated in degrees $[n,m]$ via the indicated construction, and that the homology is "correct". Passing to the limit is then routine because again the homology stabilizes in any given agree. If $n = m$ this is the previous claim about $D(R)_0$. For the inductive step, we have to be able to grow the interval either to the left or the right. The arguments are dual, so let's just go from $[n,m]$ to $[n,m+1]$. Specifying the extra data needed to extend the filtration to $[n,m+1]$ is equivalent to specifying an object $M_{m+1} \in Mod_R = D(R)_0$ and a classifying map

$$\Sigma^{m+1} M_{m+1} \to \Sigma F(m)$$

for the fiber-cofiber sequence $F(m) \to F(m+1) \to gr^{m+1}F$. Since $\Sigma^{m+1}M_{m+1} \in D(R)_{\geq m+1}$, this is the same as mapping $\Sigma^{m+1}M_{m+1} \to \tau_{\geq m+1}\Sigma F(m)/F(n)$. But by the claim about the homology in the inductive hypothesis, the target is also concentrated in degree $m+1$ with homology there equal to $\ker(d : M_m \to M_{m-1})$. Thus by the fact about $D(R)_0$ again this data is exactly the same as the data of a map

$$M_{m+1} \to \ker(d : M_m \to M_{m-1}),$$

which is the same as extending the complex one more term. The long exact sequence on homology then verifies the homology claim for this extended complex as well. □

Definition 15. *We define a functor $|\cdot| : Ch_R \to D(R)$ by the composition*

$$Ch_R \simeq Fun(\mathbb{Z}_{\geq 0}, D(R))_{cx} \to D(R)$$

where the last functor is $F \mapsto F(\infty)$.

For example, if we look back at Proposition 6 we see that what it's showing in this language is that every object in $D(R)_{\geq 0}$ is represented by a non-negatively graded chain complex of free R-modules.

It is straightforward to verify various desired properties of this functor $|\cdot|$. For example, shifting the complex up by one (homologically) corresponds to Σ in $D(R)$, the filtered object in $Fun(\mathbb{Z}_{\geq 0}, D(R))_{cx}$ associated to the complex C is recovered by applying the functor $|\cdot|$ to the brutal filtration, the canonical filtration of $|C|$ is recovered by applying $|\cdot|$ to the canonical filtration of C in Ch_R, cofibers can be realized by mapping cones, and so on.

Less straightforward is the following example; it follows from Lurie's ∞-version of the Dold-Kan theorem, contained in his book "Higher Algebra".

Theorem 16. *Let $M_\bullet \in \mathrm{Fun}(\Delta^{op}, Mod_R)$ be a simplicial R-module. By composition with $Mod_R \simeq D(R)_0 \subset D(R)$, view it as a simplicial object in the ∞-category $D(R)$. Define a filtered object of $D(R)$ by defining its n^{th} filtered piece to be*

$$|M_\bullet|_{\leq n} := \mathrm{colim}_{\Delta^{op}_{\leq n}} M_\bullet$$

and note that the underlying object $|M_\bullet| = \varinjlim |M_\bullet|_{\leq n}$ identifies with $\varinjlim_{\Delta^{op}} M_\bullet$.

Then this filtered object lies in $Fun(\mathbb{Z}_\leq, D(R))_{cx}$, and under the equivalence with Ch_R it corresponds to the chain complex of R-modules given by the image of $M_\bullet$ under the Dold-Kan correspondence, namely the normalized chain complex associated to $M_\bullet$.

Thus the most natural filtration present on the colimit (geometric realization) of a simplicial R-module, namely the filtration by dimension, actually does correspond to a complex. This is a fundamental property of the simplex category; the analog fails, for example, for the category of (unordered) nonempty finite sets, even though that category model homotopy types, derived categories, etc. just like the simplex category does. It is essentially for this reason that we use simplices to define homology of topological spaces, for example.

Also, it shows the consistency of our notation for $|\cdot| : Ch_R \to D(R)$ with the standard notation for geometric realization (colimit over Δ^{op}) under the Dold-Kan correspondence.

Remark 17. *In relation to this discussion of filtered objects, note that one can define a spectral sequence on homology associated to an arbitrary filtered object of $D(R)$, exactly as one does for a filtered complex classically. The E^1 page is made up of the homology of the associated graded terms. If the filtered object is convergent, then the spectral sequence is conditionally convergent in the sense of Boardman (in practice it will be honestly convergent and this will be apparent without much effort) with abutment the homology of the underlying object. The case where the E^1 page is concentrated on the horizontal axis exactly corresponds to the case of a filtration in $Fun(\mathbb{Z}_\leq, D(R))_{cx}$, and the E^1 differential on the horizontal axis is the corresponding complex. This follows from the definitions.*

Exercise. This is a four-part exercise. It's enough for you to do 2 out of the 4 parts, and if you're doing part n you're allowed to use the statements in parts $< n$ as a black box.

1. For $M, N \in Mod_R$, view them inside $D(R)_0 \subset D(R)$. Produce a natural isomorphism $[M,N]_d = Ext_R^{-d}(M,N)$.

2. Show that if $I \in Mod_R$ is injective and $X \in D(R)$, then $[X, I] = Hom_R(H_0 X, I)$.

3. Show that every object in $D(R)_{\leq 0}$ can be represented by a (non-positively homologically graded) chain complex of injective R-modules (meaning, for any $X \in D(R)_{\leq 0}$ there is such a chain complex $I_\bullet$ and an isomoprhism $|I_\bullet| \simeq X$.

4. Show that every object in $D(R)$ can be represented by a chain complex of R-modules.