17 March 2022

Microlocal Sheaves as an Invariant
(Nadler)

Main question we haven't addressed. What are microlocal sheaves an invariant of?

Notation. X manifold, $\Lambda \subset T^*X$ closed conic Lag

$$\Lambda^\infty \subset S^*X$$

$$\rightsquigarrow \mu Sh_\Lambda \quad , \quad \mu Sh_{\Lambda^\infty} .$$

Should have Symp invariance

Should have contact invariance

... maybe more, deformation invariance

Remark. If $f : X \xrightarrow{\sim} X$ is a diffeomorphism, then

$$f_* \mu Sh_\Lambda \simeq \mu Sh_{f_*(\Lambda)} .$$

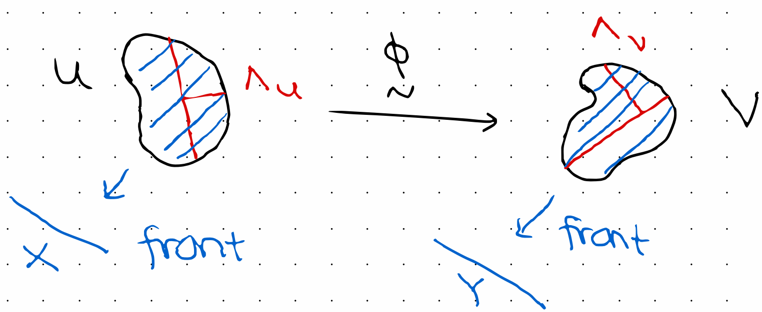
Today. Contact invariance of μSh_{Λ^∞}.

Theorem. Let X and Y be manifolds,

$$\Lambda_X^\infty \subset S^*X \quad \text{and} \quad \Lambda_Y^\infty \subset S^*Y$$

Legendrians. Let $U \subset S^*X$ and $V \subset S^*Y$ be opens. Assume that we have a commutative square

$$\begin{tikzcd}
U \arrow[r, "\overset{\phi}{\sim}", "\text{cooriented contacto.}"'] & V \\
\Lambda_U := U \cap \Lambda_X^\infty \arrow[u, hook] \arrow[r, "\sim"] & V \cap \Lambda_Y^\infty =: \Lambda_V \arrow[u, hook]
\end{tikzcd}$$

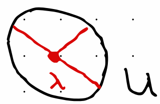


Then

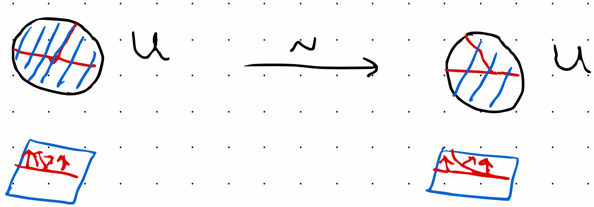
$$\boxed{\phi_* \mu Sh_{\Lambda_U} \simeq \mu Sh_{\Lambda_V}}$$

Point. Taking $X = Y$, $U = V$, $\Lambda_X^\infty = \Lambda_Y^\infty$ tells us that μSh_Λ is 'independent' of the polarization on T^*X.

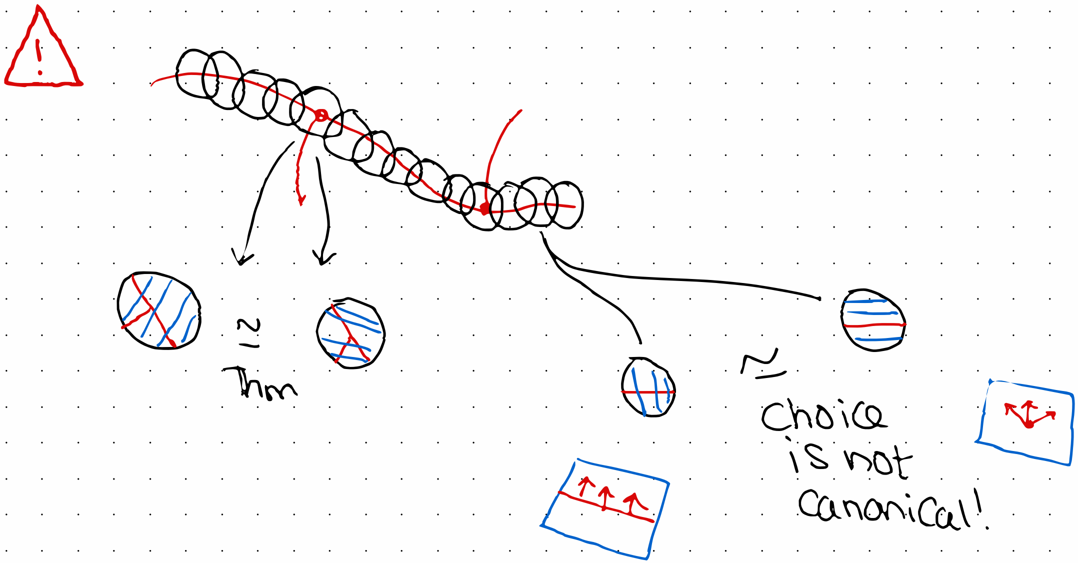
Application 1. U = small ball around $\lambda \in \Lambda_X^\infty$.



Vague Proposition. If Λ_X^∞ is 'reasonable' near λ, then there exists a coorient contactomorphism $\phi : U \xrightarrow{\sim} U$ such that $\phi(\Lambda_X^\infty)$ is in generic position at λ.



Application 2. Given any cooriented contact manifold N and Legendrian $\Lambda \subset N$, 'can construct' μSh_Λ by defining μSh_Λ on a basis of small balls.



Point. In order to construct μSh_Λ with additional choice of 'Maslov data'.

Example. Consider possible μSh_Λ for $\Lambda = S^1 \subset T^*S^1$

locally const sheaves of cats on S^1 with stalk $\mathrm{Mod}(k)$ $\simeq$ $\dfrac{\text{autoequiv of } \mathrm{Mod}(k)}{\text{`conjugation'}}$

Example of ϕ. This is the most important example.
$X = \mathbb{P}(V)$ projective space (over $\mathbb{R}$ or $\mathbb{C}$).

$\dim(V) = n$

$\mathbb{P}(V)$ $\qquad\qquad$ $\mathbb{P}(V^*)$ $\qquad$ dim $n-1$

line $\ell \subset V$ $\qquad\qquad$ hyperplane $H \subset V$

$T^*\mathbb{P}(V)$ $\qquad\qquad$ $T^*\mathbb{P}(V^*)$

$T_\ell \mathbb{P}(V) = \mathrm{Hom}(\ell, V/\ell)$ $\qquad$ $T^*_\ell \mathbb{P}(V) = \mathrm{Hom}(V/\ell, \ell)$

$$
\begin{array}{ccc}
S^*\mathbb{P}(V) & & S^*\mathbb{P}(V^*) \\
\| & & \| \\
Z = \{(\ell, H) \mid \ell \subset H\} & \cong & \{(H, \ell) \mid \ell \subset H\}
\end{array}
$$

dim $2n-3$

Exercise. Check this identification is a cooriented contactomorphism.

$$
\begin{tikzcd}
 & (\ell, H) \arrow[dl, mapsto, bend right] \arrow[drr, mapsto, bend left] & & \\
\ell & Z \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & & H \\
\mathbb{P}(V) = X & & \mathbb{P}(V^*) = Y &
\end{tikzcd}
$$

Point. Microsheaves on these are trivially the same from this perspective, but this doesn't extend to cotangent bundles!

Equivalence given by. $F \longmapsto p_{Y,!}\, p_X^*(F)$

> Here it's important that we've quotiented by constant sheaves.