3 March 2022

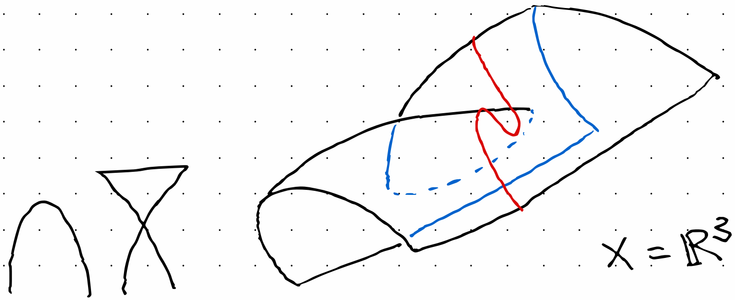
Examples of microlocal Sheaves (Nadler)

Recall. X manifold, $\Lambda \subset T^*X$ closed conic Lagrangian. We associated a conic sheaf of categories μSh_Λ on T^*X supported on Λ.

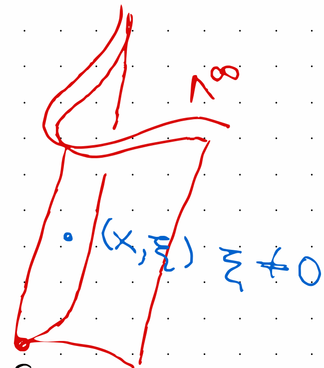
> $\pi : T^*X \longrightarrow X$ projection

Theorem (*). Let $\lambda \in \Lambda$, $x := \pi(x)$, $\lambda^\infty :=$ image of λ in Λ^∞, $\pi^\infty : \Lambda^\infty \longrightarrow X$ projection.
If Λ is in generic position at λ and $(\pi^\infty)^{-1}(x) \cap \Lambda^\infty = \{\lambda^\infty\}$
Then there is a small ball $B(x)$ around x and an equivalence

$$\mu Sh_\Lambda|_\lambda \simeq Sh_\Lambda(B(x))/Loc(B(x))$$
$$\simeq Sh_\Lambda(B(x))^{\Gamma=0} \leftarrow \text{vanishing global sections}$$



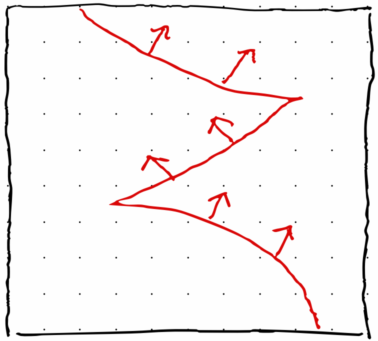
'Swallow tail'

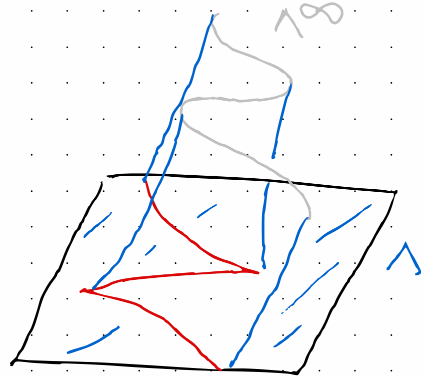


Important comment. μSh_Λ doesn't really depend on the vertical polarization of T^*X. It is really a symplectic invariant.

> Point: we can put things in generic position without changing μSh_Λ.

Zorro's Front.

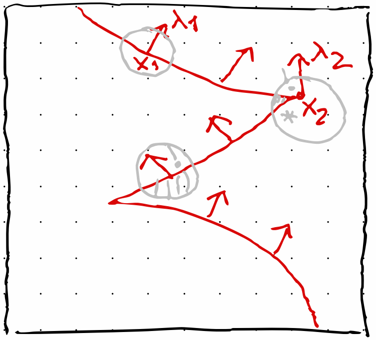


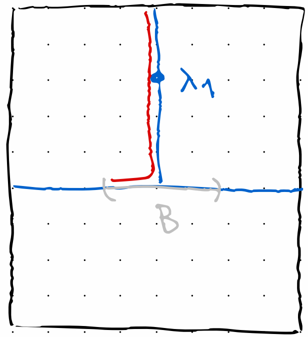


$$\mu Sh_\Lambda(T^*X - X) \overset{\text{def}}{=} \mu Sh^\infty_{\Lambda^\infty}(S^*X)$$

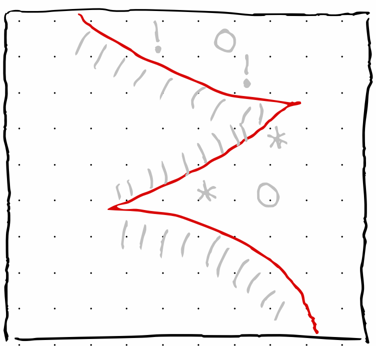
$$\simeq Loc(\Lambda^\infty)$$

Since Λ^∞ is smooth





μsupp of ! extension on $B_{<0}$



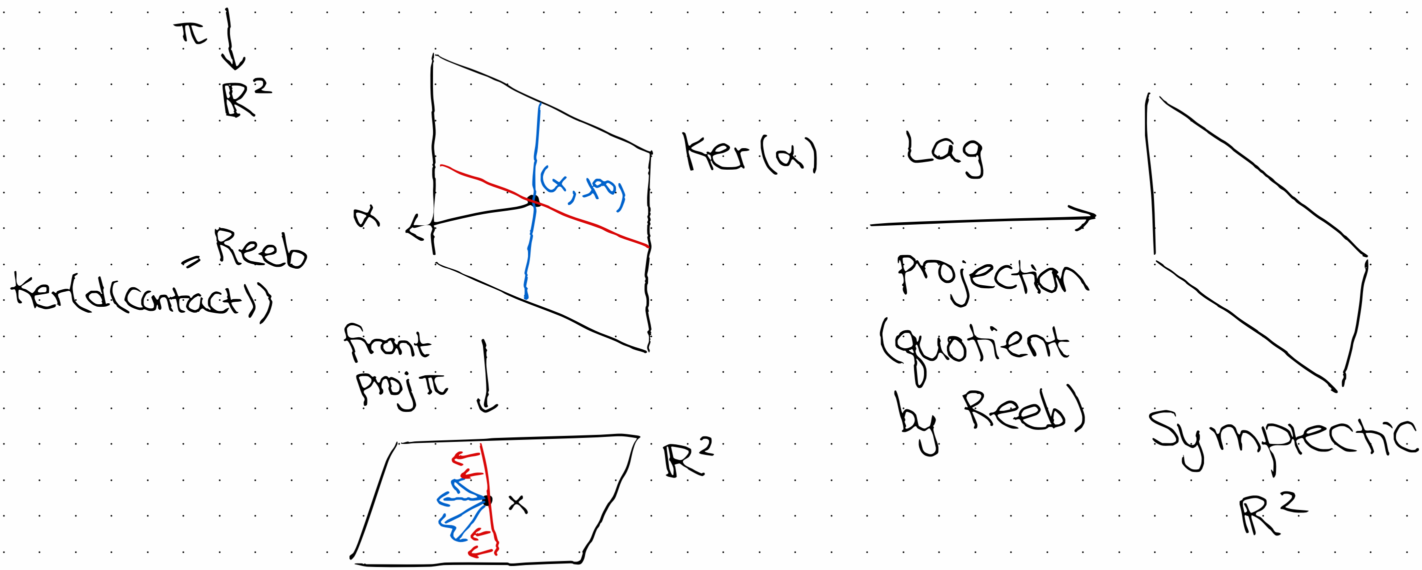
Exer 1. Impossible to have $\mu\mathrm{supp} \subset \Lambda$

Exer 2. $Sh_\Lambda(\mathbb{R}^2) \longrightarrow \mu Sh_{\Lambda\infty}(S^*\mathbb{R}^2) \simeq Loc(\Lambda^\infty)$

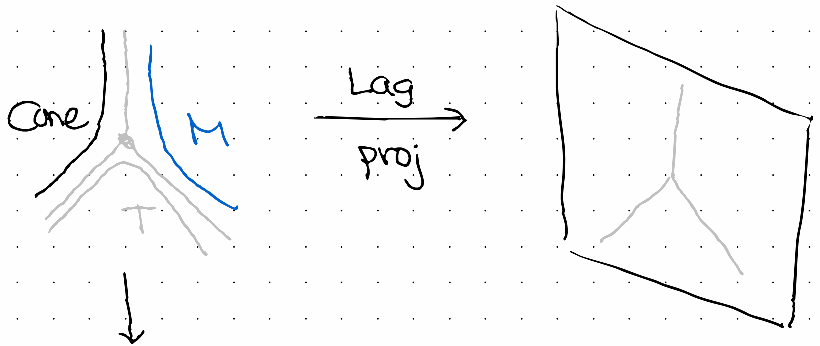
is zero!

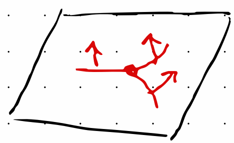
Some Singular 1d Legendrians Λ^∞

Recall. $S^*\mathbb{R}^2$ cooriented contact 3-manifold



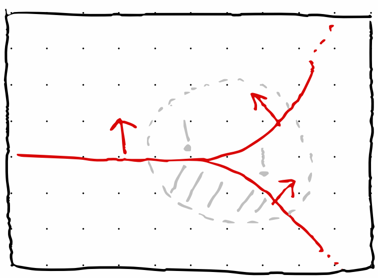
> Singular Lagrangian



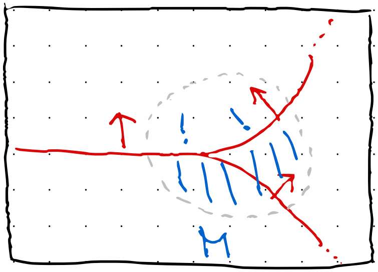


at • derivatives coincide

$\Lambda = \mathrm{Cone}(\Lambda^{\infty}) \cup$ 0-section



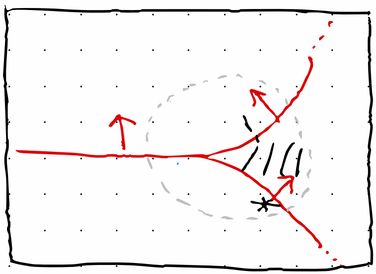
Call this Sheaf T



$\mathrm{Hom}(T, M) \simeq K$ (up to shift)

$\Downarrow$

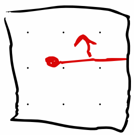
$c : T \longrightarrow M$

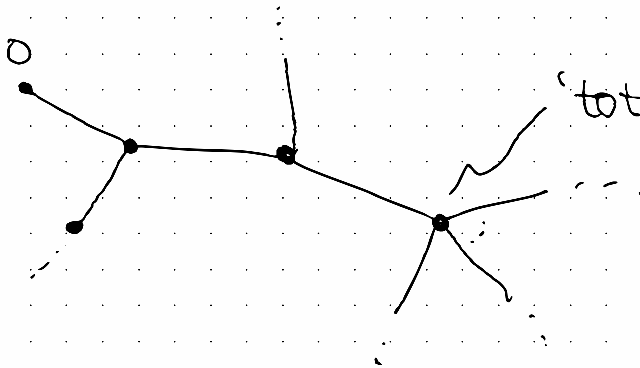


$\mathrm{Cone}(c)$

$$\mu Sh_\Lambda|_x \simeq \left(\begin{array}{c} \text{exact triangles} \\ \begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, "{[1]}"] & \end{tikzcd} \end{array} \right)$$

More generally. Λ^∞ = Legendrian graph





'totally acyclic n-cycles'