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**Microlocal Sheaves**

(Wodkier)

Setup. $X$ manifold, $F$ weakly constructible sheaf on $X$

$\Rightarrow \mu \text{supp}(F) \subseteq T^*(X)$ closed conic Lagrangian

Idea. Focus on conic opens $U \subset T^*(X)$

Important. A microlocal sheaf is *not* a sheaf.

Idea. "Throw out (quotient by) sheaves $F$ with $\mu \text{supp}(F) \cap U = \emptyset"
For concreteness. Fix $\Lambda \subset T^*(X)$ closed conic Lagrangian.

- We'll define $\muSh_{\Lambda}(T^*X)$.

- Recall $\text{Sh}_{\Lambda}(X) =$ sheaves on $X$ with $\mu\text{supp} \subset \Lambda$.

**Def.**

1. $\muSh_{\Lambda}^\text{pre}$: presheaf of (dg derived) categories on $T^*X$ supported on $\Lambda$.

   $\muSh_{\Lambda}^\text{pre}(U) = \text{Sh}_{\Lambda}(X, U) / \overline{\text{Null}(X, U)}$

   Only depend on the cone on $U$.

   Sheaves $\text{F}$ on $X$ st. $\mu\text{supp}(F) \cap U = \emptyset$

2. $\muSh_{\Lambda} = \text{sheafification of } \muSh_{\Lambda}^\text{pre}$
(3) $\mu Sh_{\Lambda}(T^*X) := \Gamma(T^*X; \mu Sh_{\Lambda})$.

(4) $L \subset S^*(X)$ closed Legendrian.

$\mu Sh_{\Lambda}^\infty := \mu Sh_{\text{cone}(L)} |_{T^*X \setminus X}$

$\downarrow$ good to give a talk

Prop./Exer. $\mu Sh_{\Lambda}(T^*X) = Sh_{\Lambda}(X)$

> Hint: if $U \subset T^*X$ is given by $T^*B$ for $B \subset X$ open, then $\text{Null}(X, U) =$ Sheaves supported away from $B$.

Example

(0) $\Lambda = T^*_{\mathbb{R}} \mathbb{R}$

$\mu Sh_{\Lambda}(T^*\mathbb{R}) = 0$!

Let's calculate $\mu Sh_{\Lambda}|_{(0, z)}$ and $\mu Sh_{\Lambda}|_{(0, 0)}$. 

\[ (0, 0) \quad (0, z) \quad (0, 0) \]
\[ \mu_{\text{Sh}}^*(\mathbb{C}_{(0,\infty)}) = \mu_{\text{Sh}}^*(\mathbb{C}) \uparrow \text{small conic open} \]

\[ = \mu_{\text{Sh}}^{\text{pre}}(\mathbb{C}) \]

\[ \text{def} \quad \text{Sh}^*(X;\mathbb{C}) / \text{Null}(X;\mathbb{C}) \]

\[ \cong \text{Mod}_K \text{ with } F \text{ a generator} \]

\[ \text{non-canon} \]

\[ \mu_{\text{supp}}(j_{>0,!*}K_{R>0}) \]

\[ \text{Supp}^*(j_{<0,!*}K_{R<0}) \]

\[ J_{>0,!*}K_{R>0} \text{ and } F = J_{<0,!*}K_{R<0} \text{ rep. the same micro sheaf up to a shift} \]
Theorem. If \((x, \xi) \in \Lambda\) is a smooth point, then \(\mu \text{Sh} \lvert_{(x, \xi)}\) is non-canonically equivalent to \(\text{Mod}_K\). Moreover, \(\mu \text{Sh}_\Lambda \lvert_{\Lambda_{\text{sm}}}\) is locally constant.

Example.
(1) \(X = \mathbb{R}^2\)

\[
\mu \text{Sh}_\Lambda \lvert_{(x, \xi)} \sim \text{Mod}_K
\]
Definition. We’ll say \( \Lambda \subset T^*X \) is in generic position if the projection \( \pi: \Lambda^\infty \rightarrow X \) is finite.

\[ \text{The image } \pi(\Lambda^\infty) \text{ is a front = hypersurface + coorientation} \]

Example of a Front.
Nonexample.

\[ \Lambda = T^*_{x_0} \mathbb{R}^2 \]

\[ \Lambda^\infty = \circ \]

\[ \begin{array}{c}
\text{Good to talk about}
\end{array} \]

\textbf{Theorem.} Assume that \( \Lambda \) is in generic position near \( (x, \xi) \in \Lambda \) and \( \pi^{-1}(x) = \mathbb{R}_{>0} \cdot (x, \xi) \).

\[ \mu \text{Sh}_\Lambda \big|_{(x, \xi)} = \frac{\text{Sh}_{\Lambda \cup B}(B)}{\text{Sh}_B(B)} \]

\[ \cong \text{Loc}(B)^+ \cong \text{Sh}_{\Lambda \cup B}(B) \]

\[ \Gamma = 0 \]

\text{vanishing global sections!}
Example (1) continued.

Indeed, the sheaf we considered has vanishing global sections!

Exercise. Show that $\mathcal{S}_\wedge(R^2) = \text{Mod}(\bullet \leftarrow \bullet \rightarrow \bullet)$

$\mathcal{S}_\wedge(1_{R^2}) / \text{Loc}(R^2) = \text{Mod}(\bullet \rightarrow \bullet)$