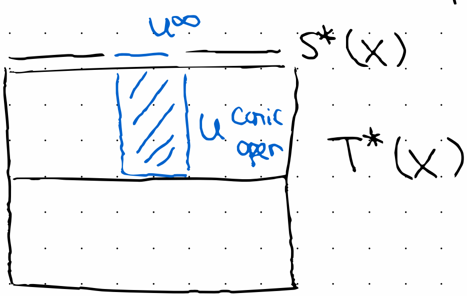
24 February 2022

Microlocal Sheaves (Nadler)

Setup. X manifold, F weakly constructible sheaf on X

$\rightsquigarrow \mu supp(F) \subset T^*(X)$ closed conic Lagrangian

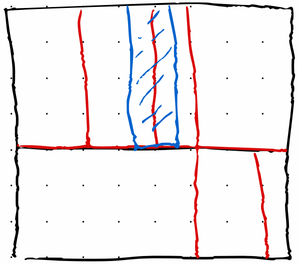
Idea. Focus on conic opens $U \subset T^*(X)$



Important. A microlocal sheaf is **not** a sheaf.

Idea. "Throw out (quotient by) sheaves F with $\mu supp(F) \cap U = \emptyset$."

For concreteness. Fix $\Lambda \subset T^*(X)$ closed conic Lagrangian



- We'll define $\mu Sh_\Lambda(T^*X)$
- Recall $Sh_\Lambda(X)$ = Sheaves on X with $\mu supp \subset \Lambda$.

Def.

(1) μSh_Λ^{pre} : presheaf of (dg derived) categories on T^*X supported on Λ.

$$\mu Sh_\Lambda^{pre}(U) = \underbrace{Sh_\Lambda(X;U)}_{\text{Sheaves } F \text{ on } X \text{ st. } \mu supp(F)\cap U \subset \Lambda \cap U} \Big/ \overbrace{Null(X;U)}^{\text{Shvs } F \text{ on } X \text{ st } \mu supp(F)\cap U = \emptyset}$$

Only depend on the cone on U

(2) $\mu Sh_\Lambda := $ Sheafification of μSh_Λ^{pre}

(3) $\mu Sh_\Lambda(T^*X) := \Gamma(T^*X; \mu Sh_\Lambda)$.

(4) $L \subset S^*(X)$ closed legandrian.

$$\mu Sh_L^\infty \text{ ``}:=\text{''} \mu Sh_{Cone(L)}\big|_{T^*X \setminus X} \overset{\sim}{\underset{\text{conic}}{\uparrow}} S^*X$$

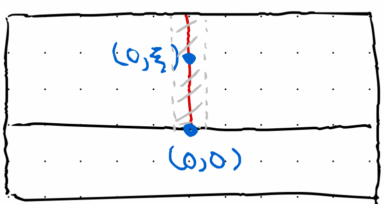
good to give a talk

Prop./Exer. $\mu Sh_\Lambda(T^*X) = Sh_\Lambda(X)$

> Hint: if $U \subset T^*X$ is given by T^*B for $B \subset X$ open, then $Null(X,U)$ = Sheaves supported away from B.

Example

(0) $\Lambda = T^*_{\{0\}}\mathbb{R}$



$\mu Sh_\Lambda(T^*\mathbb{R}) = 0$!

Let's calculate $\mu Sh_\Lambda|_{(0,\xi)}$ and $\mu Sh_\Lambda|_{(0,0)}$

$\mu Sh_\Lambda|_{(0,\xi)} = \mu Sh_\Lambda(\mathcal{U})$

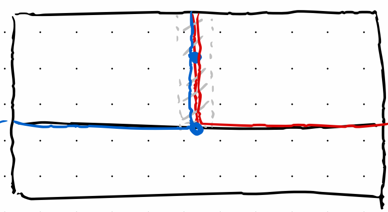
small conic open

$\simeq \mu Sh_\Lambda^{pre}(\mathcal{U})$

$\overset{def}{=} Sh_\Lambda(X;\mathcal{U}) / Null(X;\mathcal{U})$

$\simeq Mod_k$ with F a generator

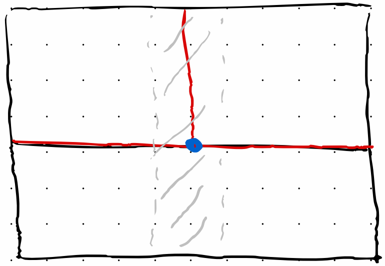
non-canon



$\mu supp(j_{>0,!}k_{\mathbb{R}_{>0}})$

$\mu supp(j_{<0,!}k_{\mathbb{R}_{<0}})$

$j_{>0,!}k_{\mathbb{R}_{>0}}$ and $F := j_{<0,!}k_{\mathbb{R}_{<0}}$ rep. the same microsheaf (up to a shift).



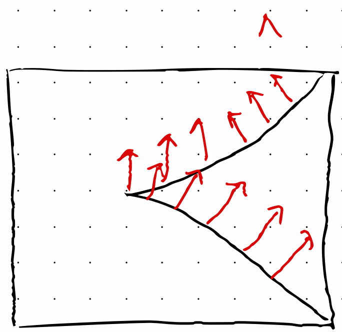
$$\mu Sh_\Lambda|_{(0,0)} = 0$$

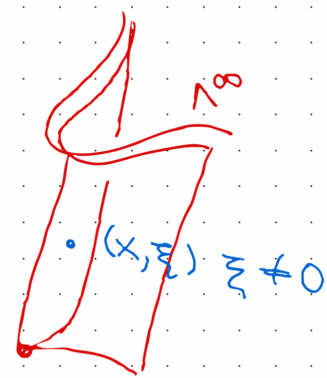
good to present on

Theorem. If $(x,\xi)\in\Lambda$ is a smooth point, then $\mu Sh_\Lambda|_{(x,\xi)}$ is noncanonically equivalent to Mod_k. Moreover, $\mu Sh_\Lambda|_{\Lambda^{sm}}$ is locally constant.

Example.

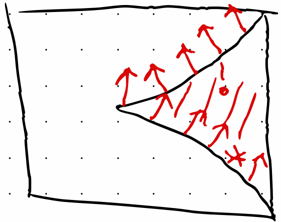
(1) $X = \mathbb{R}^2$





$$\mu Sh_\Lambda|_{(x,\xi)} \simeq Mod_k$$

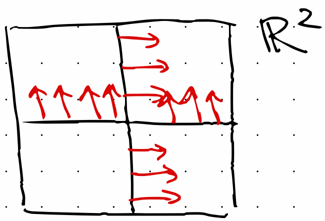
> Generator F



Definition. We'll say $\Lambda \subset T^*X$ is in generic position if the projection $\pi: \Lambda^\infty \longrightarrow X$ is finite.

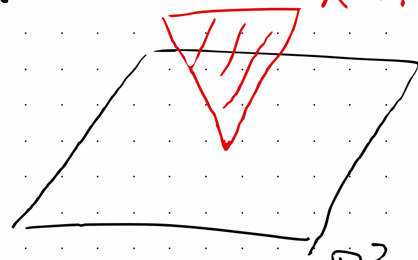
> The image $\pi(\Lambda^\infty)$ is a front = hypersurface + coorientation

Example of a Front.

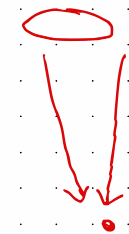


Nonexample.

$\Lambda = T^*_{\{0\}}\mathbb{R}^2$



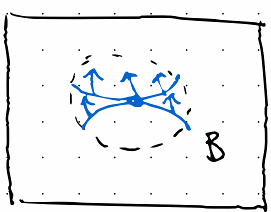
$\mathbb{R}^2$

$\Lambda^\infty =$ 

Good to talk about

Theorem. Assume that Λ is in generic position near $(x,\xi) \in \Lambda$ and $\pi^{-1}(x) = \mathbb{R}_{>0} \cdot (x,\xi)$.

$$\mu Sh_\Lambda|_{(x,\xi)} \simeq Sh_{\Lambda \cup B}(B) \,/\, \underbrace{Sh_B(B)}_{Loc(B)!}.$$

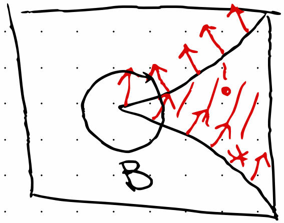


B

$$\simeq Loc(B)^\perp \simeq Sh_{\Lambda \cup B}(B)^{\Gamma = 0}$$

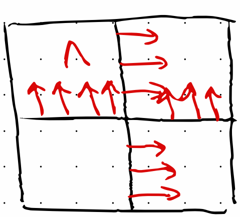
vanishing global sections!

Example (1) continued.



indeed, the sheaf we considered has vanishing global sections!

Exercise. Show that $Sh_\Lambda(\mathbb{R}^2) = Mod(\bullet \leftarrow \circ \rightarrow \circ)$



$Sh_\Lambda(\mathbb{R}^2) \,|\, Loc(\mathbb{R}^2) = Mod(\bullet \quad \bullet)$