17 February 2022

Singular Support IV
(Nadler)

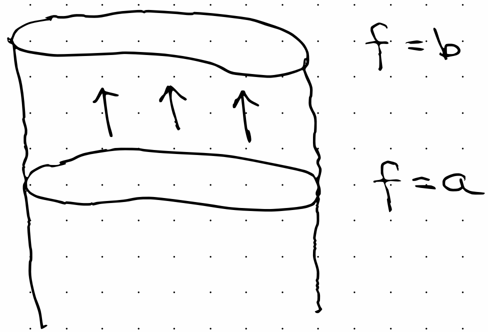
Theorem (Dubson-Kashiwara). X compact manifold, $F \in Sh(X)$ with $\mu supp(F) \subset T^*X$, $f: X \to \mathbb{R}$ function such that $\Gamma_{df} \pitchfork \mu supp(F) =: S$.

Then $\Gamma(X;F)$ is a complex in the microstalks at S.

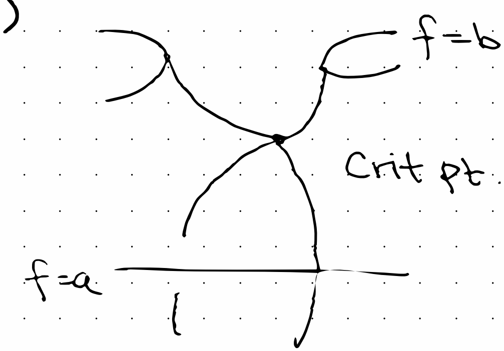
Remark. Recall that F is locally constant if and only if $\mu supp(F) \subset X$. In this case, $\Gamma_{df} \cap \mu supp(F)$ consists of Morse critical points and this recovers Morse theory.

Recall. In the Morse theory setting, there are two things that happen.

(1) No critical points in $f^{-1}([a,b])$. Then $f_{\leq a}$ and $f_{\leq b}$ have the same cohomology.



(2)



$\Longrightarrow$ Cohomology fits into a triangle

$$\begin{tikzcd}
 & H^*(f_{\leq a}) \arrow[dl] & \\
\text{Morse} & & H^*(f_{\leq b}) \arrow[ul] \arrow[ll]
\end{tikzcd}$$

Point. (1) is a special case of noncharacteristic propagation.

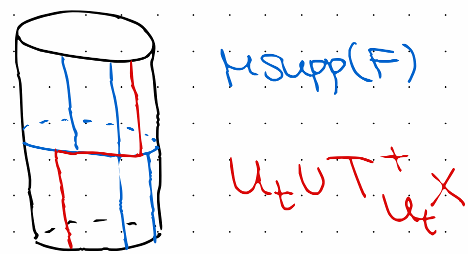
Noncharacteristic Propagation

> Good future talk topic.

Setup. X manifold, $F \in Sh(X)$, $U_t \subset X$ family of open submanifolds with smooth boundary ∂U_t. Assume that $\overline{U}_t$ is compact.

> Really, the U_t define a submanifold in $X \times \mathbb{R}$ with no critical points.

Assume that the positive conormal $T^+_{\partial U_t}X$ intersects $\mu supp(F)$ in the 0-section.



Theorem. Under these assumptions $\Gamma(U_t; F)$ is locally constant in t.

Exercise. Suppose that F is weakly $\mathcal{S}$-constructible. Then

$$\partial U_t \pitchfork X_\alpha \text{ for all } \alpha \Rightarrow T^+_{\partial U_t} X \cap \mu\mathrm{supp}(F) \subset X.$$

Functoriality of microsupport

Setup. $f : X \longrightarrow Y$ map of manifolds. We get a correspondence

$$\begin{tikzcd}
 & L_f = T^*Y \times_Y X \arrow[dl, "f^*"'] \arrow[dr, "\text{`}f\text{'} = \text{projection}"] & \\
T^*X & & T^*Y
\end{tikzcd}$$

Exercise. $L_f \subset (T^*X)^- \times T^*Y$ is a Lagrangian submanifold.

Def. Given $S \subset T^*X$, write

$$f_\#(S) := f\left((f^*)^{-1}(S)\right) \subset T^*Y$$

For $T \subset T^*Y$, write

$$f^\#(T) := f^*(f^{-1}(T)) \subset T^*X.$$

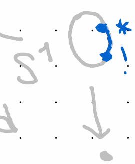
Theorem.

(1) f submersion (i.e., f^* injective), then

$$\mu\mathrm{supp}(f^*(F)) = f^\#(\mu\mathrm{supp}(F)) \qquad f^! \simeq f^* \otimes \omega_f$$

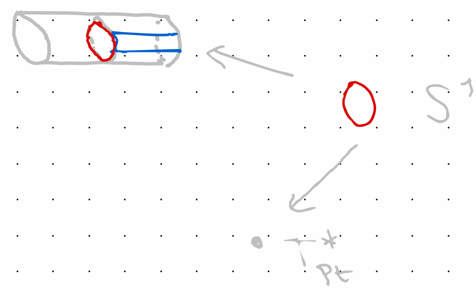
(2) If f is proper, then

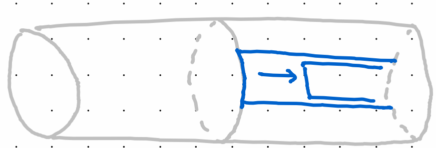
$$\mu\mathrm{supp}(f_! F) \subset f_\#(\mu\mathrm{supp}(F)) \qquad f_! = f_*.$$



pushforward to 0

$\Rightarrow$ empty μsupp





Can be displaced

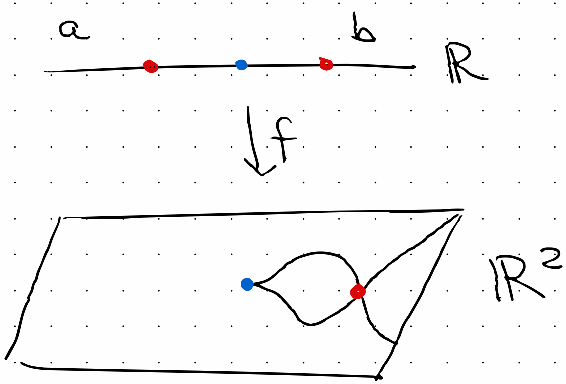
Def. Let $f: X \to Y$ be a proper map of manifolds. Write

$$\Lambda_f := f_\#(\text{0-Section}) \subset T^*Y.$$

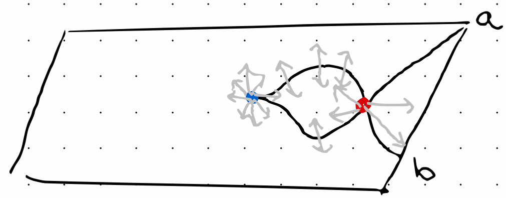
'remembers why points are critical'

Exercise. $\Lambda_f \cap Y = f(X)$

Example.



Λ_f



Next Time. Define microlocal sheaves!