10 February 2022

Singular Support II
(Nadler)

Recall Last Time. X smooth manifold, $\mathcal{S} = \{X_\alpha\}$ strat of X

> $Sh_{\mathcal{S}}(X)$ = dg derived cat of $\mathcal{S}$-weakly constructible sheaves

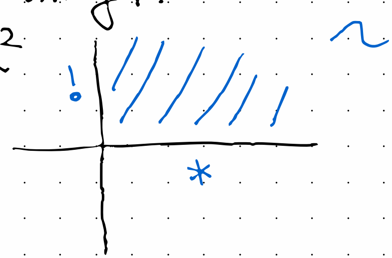
$$\begin{array}{l} \ \ \rotatebox{90}{$\in$} \\ F \rightsquigarrow \mu supp(F) \subset T^*(X). \end{array}$$

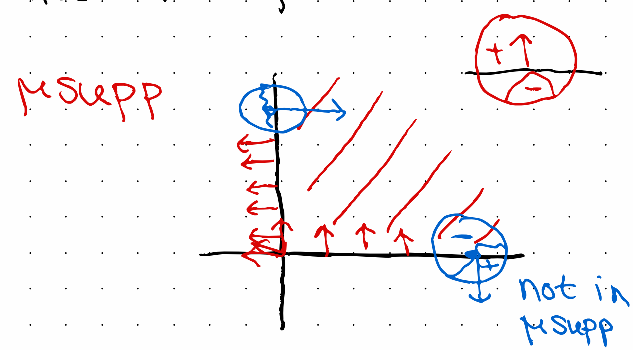
Proposition. $\mu supp(F) \subset T^*_{\mathcal{S}}(X) = \bigcup_\alpha T^*_{X_\alpha}(X)$ and is closed conic and Lagrangian, $\mu supp(F) \cap X = supp(F$

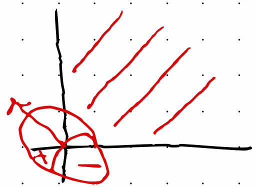
> In fact:

Thm (KS). $\mu supp(F)$ is involutive (aka coisotropic in this singular setting).

Ex. $X = \mathbb{R}^2$







Other Identities

(1) $\mu supp(\mathbb{D}(F)) = -\mu supp(F)$

(2) $\mu supp(F_1 \boxtimes F_2) = \overline{\mu supp(F_1) + \mu supp(F_2)}$

maybe closure is not necessary

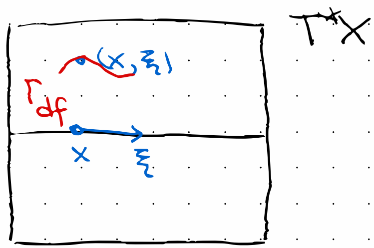
Microstalks

> Going to generalize stalks.

μstalk : stalk :: μsupp : supp

Input.

test brane



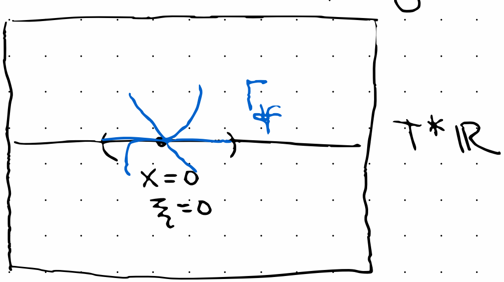
$B(x)$ small ball

$f : B(x) \longrightarrow \mathbb{R}$ $\quad f(x) = 0$

$df(x) = \xi$

Output. $\mathrm{Cone}\big(\Gamma(B(x);F) \longrightarrow \Gamma(f^{-1}(\mathbb{R}_{<0});F)\big) =: \phi_f(F)$.

Note. The function F plays a role: $f^{-1}(\mathbb{R}_{<0})$ is radically changing!

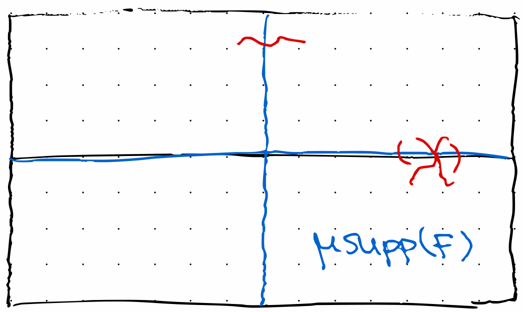


$f=0$: $f^{-1}(\mathbb{R}_{<0}) = \emptyset$ $\qquad \phi_f(k_{\mathbb{R}}) = k[1]$

$f=-x^2$: $f^{-1}(\mathbb{R}_{<0}) = B \setminus \{x\}$ $\qquad \phi_f(F) = \mathrm{Cone}(k \longrightarrow k^{\oplus 2}) \simeq k$

To try to make indepent of f as much as possible. Let's require that Γ_{df} intersects $\mu supp(F)$ transversley in a smooth point.

Call this μMorse



$T^*\mathbb{R}$

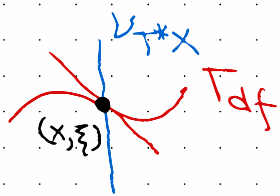
> This is the stratified/microlocal version of a Morse critical point! $\rightsquigarrow$ Morse test branes

Definition. Given a Morse test brane Γ_{df} through (x,ξ), write

$$\mu\mathrm{stalk}_{\Gamma_{df}}(F) := \phi_f(F).$$

Good future talk topic

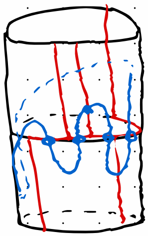
Thm. μstalk only depends on the quadratic part of f, i.e., $T_{(x,\xi)}\Gamma_{df}$. In fact, μstalk only depends on the triple of Lagrangians $(\nu_{T^*X}, T_{(x,\xi)}\Gamma_{df}, T_{(x,\xi)}\mu\mathrm{supp}(F))$.



Furthermore, all different quadratic parts give same μstalk up to $(-)\otimes \ell[i]$.

Some line

Theorem (Dubson-Kashiwara). X smooth compact, $F \in Sh_{\mathbb{S}}(X)$, $f: X \to \mathbb{R}$ that is μMorse.



Then

$\Gamma(X;F) =$ complex in $\mu stalks_{\Gamma_{df}}(F)$

$$\Rightarrow \chi(X;F) = \sum_{\Gamma_{df} \cap \mu supp(F)} \chi(\mu stalks(F)).$$

- Also a good thing to talk about.

Next time. Noncharacteristic propagation.

Example (μstalks)