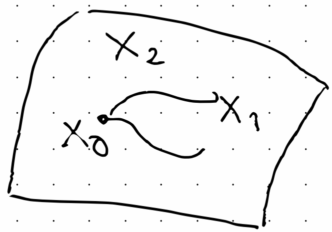
3 February 2022

GRT Seminar
Singular support (= microsupport)
(Nadler)

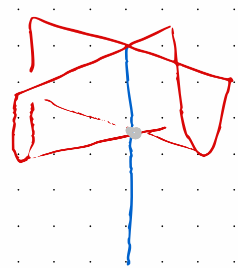
Context. X smooth manifold, $\mathcal{S} = \{X_\alpha\}_{\alpha \in A}$ stratification of X

$\dim X_\alpha = \alpha$



+ REGULARITY

to get rid of examples like Whitney's umbrella



> $Sh_{\mathcal{S}}(X) \subset Sh(X)$ weakly $\mathcal{S}$-constructible sheaves on X

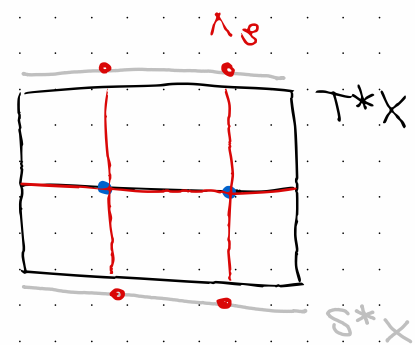
$F|_{X_\alpha}$ is locally constant

no restrictions on the size of stalks

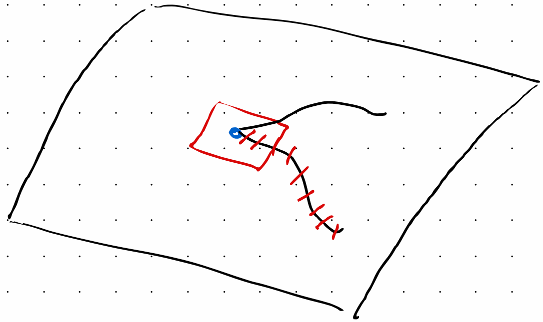
> $Sh_{wc}(X) = \bigcup_{\mathcal{S}} Sh_{\mathcal{S}}(X) \subset Sh(X)$

Recall. Inside of T^*X we have

$$T^*_{\mathcal{S}}X := \bigcup_{\alpha} \underbrace{T^*_{X_\alpha}X}_{\text{Conormal to } X_\alpha}$$



Exercise. $\mathcal{S}$ satisfies Whitney's Condition A if and only if $T^*_{\mathcal{S}}X \subset T^*X$ is closed



Question. What kind of object is $T^*_\xi X$?

> Closed, $\underbrace{\text{conic}}$, singular Lagrangian subvariety

=: $\mathbb{R}_{>0}$-conic. In fact, $\mathbb{R}^\times$-conic, but that isn't crucial

- Also contains the 0-section, but this isn't crucial

- Can remove zero-section

$$\Lambda_\xi = (T^*_\xi X \setminus X)/\mathbb{R}_{>0} \quad \text{inside } S^*X = T^*X \setminus X$$

is closed Legendrian (singular).

Point. Singular support is a subset with these adjectives.

$$\mathrm{Sh}_{\mathcal{S}}(X) \ni F \longmapsto \mu\mathrm{supp}(F) \subset T^*X$$

in fact, in $T^*_{\mathcal{S}}X$

Closed, $\mathbb{R}_{>0}$-conic, Lagrangian

$\mu\mathrm{supp}^\infty(F) \subset S^*X$

'story at ∞'

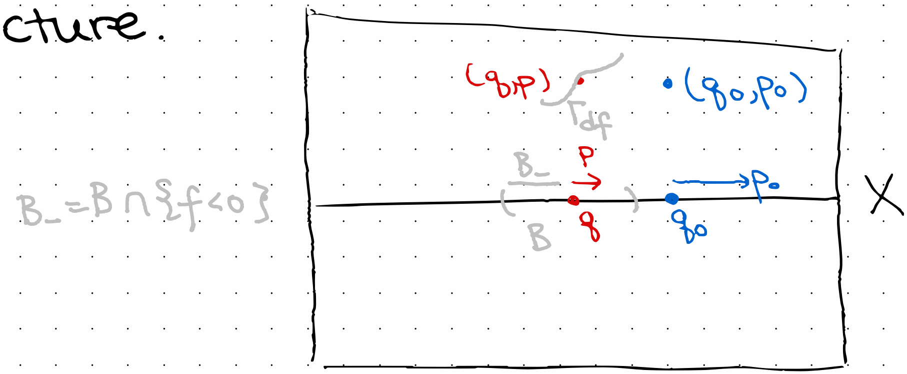
$\mu\mathrm{supp}(F) \cap X = \mathrm{supp}(X)$.

'traditional story'

Definition of $\mu supp$

- Easier to say when $(q,p) \in T^*X$ is not in $\mu supp(F)$.

Picture.



Def. $(q_0,p_0) \notin \mu supp(F)$ if there exists an open neighborhood U of (q_0,p_0) in T^*X such that for all germs of functions with $f(q)=0$ and $df_q = p$, there exists an open $B \subset X$ around q, the restriction map

$$\Gamma(B;F) \xrightarrow{r} \Gamma(B_-;F)$$

Exer. $\forall$ germs of graphical Lagr. through (q,p)

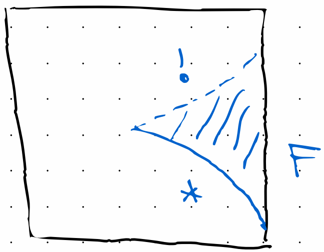
Γ = 'test Lag.'

is an equivalence. Equivalently, $\underbrace{\mathrm{Cone}(r)}_{\text{up to a shift} \simeq \phi_{f,q}(F)} \simeq 0$.

- This says that the 'the past and present are the same'.

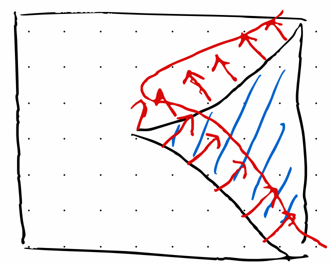
⚠ $\mu supp(F)$ is closed, but $(q,p) \in \mu supp(F)$ does not imply that $\phi_{f,q}(F) \neq 0$ for all f.

Example.



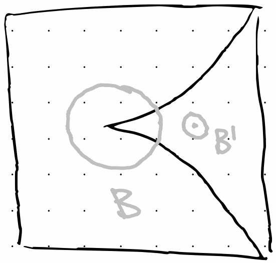
$y^2 = x^3$

$X = \mathbb{R}^2$



$\mu supp(F)$

$\mu supp^{\infty}(F)$ is a smooth Lagrangian



$q_0 = (0,0)$, $p_0 = 0$

$\Gamma(B;F) \simeq 0$

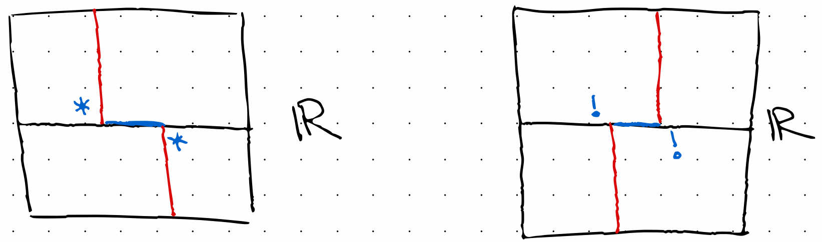
$\Gamma(B';F) = 0$

$q = (\varepsilon, 0)$

$p = 0$ $\quad f \equiv 0$

⚠ $f \equiv 0$, $\phi_{f,0}(F) = F|_0 = 0$.

Ex.



Future Talk A. Verdier duality $\mathbb{D}_X : Sh_{wc}(X) \to Sh_{wc}(X)^{op}$ acts antisymplectically!

$$\mu supp(\mathbb{D}_X F) = -\mu supp(F)$$

$-$ in fibers

$(q,p) \longmapsto (q,-p)$

$dq\,dp \longmapsto -dq\,dp$

Future Talk B. Functoriality of μsupp.

Future Talk C. Microlocal Sheaves

Future Talk D. General position.

Next time. Will explain what Talks B–D are about.