27 January 2022

GRT Seminar: Singular Support (Nadler)

Review of Symplectic/Contact geometry

Notation. X manifold — local coords $q_1, \dots, q_n$

T^*X cotan bundle — $q_1, \dots, q_n, p_1, \dots, p_n$

$$\begin{array}{c} T^*X \\ \pi \downarrow \\ X \end{array}$$

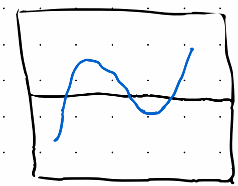
$$v \longmapsto \omega(v,-)$$
$$T(T^*X) \xrightarrow{\sim} T^*(T^*X)$$

> Exact Symplectic Structure $\omega = d\lambda$

symp — primitive / Louiville

$$\lambda = \sum_{i=1}^{n} p_i dq_i$$

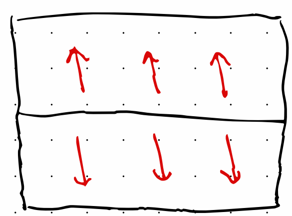
Exercise. λ is the unique 1-form on T^*X such that: if α is a 1-form on X, i.e., a section $\sigma_\alpha : X \to T^*X$, then $\sigma_\alpha^*(\lambda) = \alpha$



Exer. Define Louiville vector field $Z = \omega^{-1}(\lambda)$ on T^*X.
Show that

$$Z = \sum_{i=1}^{n} p_i \partial_{p_i}$$

generates $\mathbb{R}_{>0}$-dilation along fibers of T^*X.



Exer. Choose a Riemanian metric on X. Consider the length-squared function

$$h : T^*X \longrightarrow \mathbb{R}$$

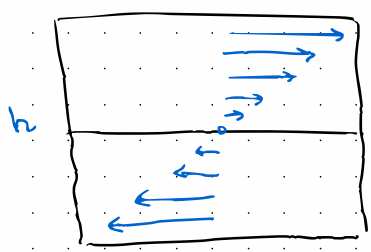
$$h(q_i, p_i) = \sum_{i=1}^{n} |p_i|^2$$

and also length $\sqrt{h}$.

not differentiable along 0-section

> Describe the Hamiltonian vector field associated to h and $\sqrt{h}$ (away from 0-section). Relate to geodesic flow of metric.

$\nu_h := \omega^{-1}(dh)$



Exer. $f : X \longrightarrow Y$ map of smooth manifolds. Construct a correspondence

$$
T^*X \xleftarrow{f^*} X \times_Y T^*Y \xrightarrow{pr_2} T^*Y
$$

and show:

(1) functoriality under composition of maps

$$
\left(
\begin{tikzcd}[column sep=small]
 & X_{01} \arrow[dl] \arrow[dr] & \\
X_0 & & X_1
\end{tikzcd}
\right)
\circ
\left(
\begin{tikzcd}[column sep=small]
 & X_{12} \arrow[dl] \arrow[dr] & \\
X_1 & & X_2
\end{tikzcd}
\right)
=
\begin{tikzcd}[column sep=small]
 & & X_{01} \times_{X_1} X_{12} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee"] \arrow[dddll, bend right] \arrow[dddrr, bend left] & & \\
 & X_{01} \arrow[ddl] \arrow[ddr] & & X_{12} \arrow[ddl] \arrow[ddr] & \\
 & & & & \\
X_0 & & X_1 & & X_2
\end{tikzcd}
$$

(2) Show $X \times_Y T^*Y \subset T^*X \times (T^*Y)^-$ is a Lagrangian correspondence

$\omega = \omega_X + (-\omega_Y)$

$\frac{1}{2}$ dim subman

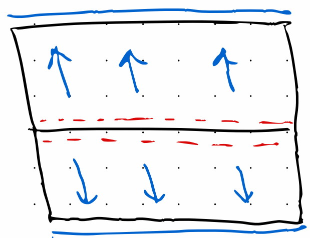
L such that

$\omega|_L = 0$.

2 Contact geometry

Notation. cosphere bundle $S^*X := (T^*X \setminus X)/\mathbb{R}_{>0}$

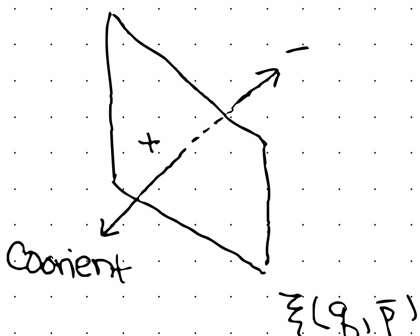
"ideal boundary of T^*X"



> Co-oriented contact structure $\xi \subset T(S^*X)$

2n-2 hyperplane field

2n-1 mfld



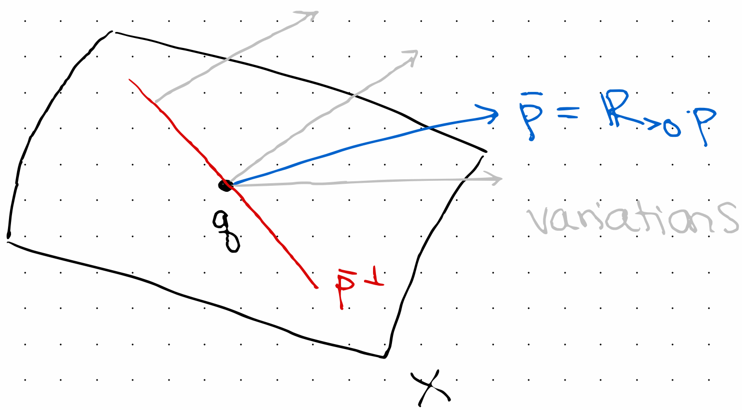
- Local coordinates $q_1, \dots, q_n, \bar{p}_1, \dots, \bar{p}_n$

Suppose $p_1 \neq 0$, scale $p_n = 1 \rightsquigarrow q_1, \dots, q_n, p_2, \dots, p_n$

$$\lambda|_{\{p_1=1\}} = dq_1 + \sum_{i=2}^{n} q_i \, dp_i$$

$$\xi = \ker(\lambda|_{\{p_1=1\}}).$$

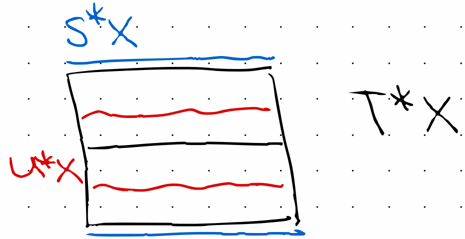
Geometric meaning of ξ from perspective of X.



Can vary $\bar{p}$ as long as the variation stays in $\bar{p}^{\perp}$

'Infintesimal variations of $(q,\bar{p})$ constrained to $q \in \bar{p}^{\perp}$'

Exer. Choose Riem. metric on $X \rightsquigarrow U^*X$ = unit cosphere bundle



(1) Show $U^*X \xrightarrow{\sim} S^*X$

(2) Show $\lambda|_{U^*X}$ is a contact form

(3) Show $\xi = \ker(\lambda|_{U^*X})$

under iso of (1)

Note. There isn't a canonical contact form on S^*X, but there is a canonical one valued in a nontrivial line bundle.

Exer. Formulate functoriality for cosphere bundles S^*X.

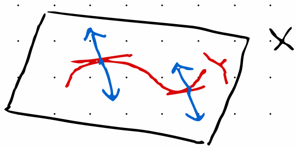
Exer. Show that $T^*X - X$ = Symplectization of S^*X.

> Extends to a dictionary

$\mathbb{R}_{>0}$-conic symplectic geom. of $T^*X - X$ $\longleftrightarrow$ contact geom. of S^*X.

Examples.

	Conic Lag.	Legendrian
$Y \subset X$	$T^*_Y X \subset T^*X$	$S^*_Y X \subset S^*X$
Submfld	conormal	



3 Back to Sheaves

Notation. $Sh(X)$ = dg derived category of complexes of sheaves of $\mathbb{C}$-modules on X

> Sh_X = Sheaf of dg derived cats. on X
> $U \mapsto Sh(U)$

Exer. Given $f: X \to Y$, define $f^* Sh_Y$, $f_* Sh_X$ and basic properties...

Recall. Given $F \in Sh(X) \rightsquigarrow supp(F) \underset{\text{closed}}{\subset} X$

$x \in X \setminus supp(F)$ means $\exists$ open $x \in U \subset X$ such that $F|_U \simeq 0$.

Ex. $X = \mathbb{R} \xhookleftarrow{j} X_+ = \mathbb{R}_{>0}$

$F = j_! \mathbb{C}_{X_+}$

$F|_{\mathbb{R}_{>0}} \simeq \mathbb{C}_{X_+}$, $F|_{\mathbb{R}_{\le 0}} = 0$

$\mathrm{Supp}(F) = \mathbb{R}_{\ge 0}$

Exer. Let $F \in Sh(X)$, $Y \subset X$ closed subset, consider $F \otimes \mathbb{C}_Y$. Construct a natural map $F \longrightarrow F \otimes \mathbb{C}_Y$ and formulate a universal property.

> Show that $\mathrm{supp}(F \otimes \mathbb{C}_Y) \subset Y$.

Remark. We can thus arbitrarly "cut off" the support of sheaves. I.e., arbitrararily prescribe support in position space $q_1, \ldots, q_n$.

4 Singular Support

$\mathcal{F}|_{X_\alpha}$ locally const

> At least for sheaves that are $\underbrace{\text{weakly constructible}}$ with respect to some $\underbrace{\text{stratification}}$ $\mathcal{S} = \{X_\alpha\}_{\alpha \in A}$

X_α = smooth subman. satisfying some regularity (Whitney Conds.)

Q. What kind of object will the singular support = microsupport be?

$\mu supp(F) \subset T^*X$ closed conic Lagrangian of pure dim = dim(X) and strat. by isotropics wrt ω.

> Moreover,

$$\mu\mathrm{supp}(F) \subset T^*_{\mathcal{S}}X = \bigcup_{\alpha} T^*_{X_\alpha}X$$

Conormal of Stratification

> Also: $\mu\mathrm{supp}(F) \cap X = \mathrm{supp}(F)$

zero section

Two parts.

$$\begin{tikzcd} \mu\mathrm{supp}(F) \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \mathrm{supp}(F) \subset X \\ \mu\mathrm{supp}^\infty(F) \subset S^*X & \end{tikzcd}$$

$$\mu\mathrm{supp}^\infty(F) = \big(\mu\mathrm{supp}(F) \setminus \mathrm{supp}(F)\big)/\mathbb{R}_{>0}$$