Exodromy for stacks

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Abstract

In this short note we extend the Exodromy Theorem of [3] to a large class of stacks and higher stacks. We accomplish this by extending the Galois category construction to simplicial schemes. We also deduce that the nerve of the Galois category of a simplicial scheme is equivalent to its étale topological type in the sense of Friedlander.

Contents

0 Introduction

In [3], we identified a profinite category $\mathrm{Gal}(X)$ attached to any scheme[1] X [2; 3, Construction 13.5]. The profinite category $\mathrm{Gal}(X)$ classifies nonabelian constructible sheaves on X (our *Exodromy Equivalence* [3, Theorem 11.7]) and the protruncated classifying space of $\mathrm{Gal}(X)$ recovers the étale topological type of X in the sense of Friedlander [9]. A natural question, then, arises: what is the analogue of this construction for a simplicial scheme or stack? For example, what is the correct exodromy representation corresponding to an equivariant constructible sheaf on a scheme with an action of a group scheme?

Here, we answer this question by extending the Galois category construction and the Exodromy Theorem to a large class of stacks and higher stacks. Here is the basic construction.

0.1 Construction. Let Y_* be a simplicial scheme. Denote by $\mathrm{Gal}^{\Delta}(Y_*)$ the following 1-category. The objects are pairs (m, ν) consisting of an object $m \in \Delta$ and a geometric point $\nu \to Y_m$. A morphism $(m, \nu) \to (n, \xi)$ of $\mathrm{Gal}^{\Delta}(Y_*)$ is a morphism $\sigma \colon m \to n$ of Δ and a

[1] All our schemes and stacks in this paper will be assumed to be coherent.

specialisation $\nu \leftsquigarrow \sigma^*(\xi)$. This category has an obvious forgetful functor $\mathrm{Gal}^{\Delta}(Y_*) \to \Delta$, which is a cartesian fibration. A morphism $(m, \nu) \to (n, \xi)$ is cartesian over $\sigma : m \to n$ in Δ if and only if the specialisation $\nu \leftsquigarrow \sigma^*(\xi)$ is an isomorphism.

The fibre over $m \in \Delta$ is the category $\mathrm{Gal}(Y_m)$, which we regard as a profinite category. (See Definition 1.7 for the precise notion of categories fibred in profinite categories.)

Also attached to a simplicial scheme Y_* is the étale topological type of Y_* as constructed by Eric Friedlander [8, §4] and refined by David Cox [7], Ilan Barnea and Tomer Schlank [1], David Carchedi [5], and Chang-Yeon Cho [6]. The étale topological type of Y_* can be identified with the colimit in protruncated spaces of the simplicial object that carries $m \in \Delta$ to the protruncated étale homotopy type of Y_m (see [7, Theorem III.8]). Since the protruncated homotopy type of the fibres of the cartesian fibration $\mathrm{Gal}^{\Delta}(Y_*) \to \Delta$ agree with the étale homotopy type of the schemes Y_m, it follows that the protruncated homotopy type of the the total category $\mathrm{Gal}^{\Delta}(Y_*)$ is the colimit of this simplicial diagram. In other words:

0.2 Theorem. *The classifying protruncated space of* $\mathrm{Gal}^{\Delta}(Y_*)$ *recovers the protruncated étale topological type of* Y_*.

This is a consequence of Proposition 1.15 below. We will also show:

0.3 Theorem (Proposition 2.5). *If Y_* is a presentation of an Artin n-stack X, then the localisation of* $\mathrm{Gal}^{\Delta}(Y_*)$ *at the cartesian edges classifies constructible sheaves on X; in other words, a constructible sheaf on X is tantamount to a functor* $\mathrm{Gal}^{\Delta}(Y_*) \to \boldsymbol{S}_\pi$ *to π-finite spaces that carries all cartesian edges to equivalences and restricts to a continuous functor* $\mathrm{Gal}^{\Delta}(Y_m) \to \boldsymbol{S}_\pi$ *for all $m \in \Delta$.*

This theorem speaks only of Artin n-stacks, but it applies just as well to any coherent fpqc stack with a presentation as a simplicial scheme.

Additionally, this theorem speaks only about nonabelian constructible sheaves, but in fact the Galois categories we construct suffice to recover constructible $\overline{\mathbf{Q}}_\ell$ sheaves as well. The proof will appear in a forthcoming note [4].

0.4 Example. Let G be an affine group scheme over a ring k, and let X be a k-scheme with an action of G. Then we have the usual simplicial k-scheme $B_{k,*}(X, G, k)$ whose n-simplices are $X \times_k G^n$; this presents the quotient stack X/G.

Thus the category of G-equivariant (nonabelian) constructible sheaves on X is equivalent to the category of continuous functors

$$\mathrm{Gal}^{\Delta}(B_{k,*}(X, G, k)) \to \boldsymbol{S}_\pi$$

that carry the cartesian edges to equivalences. If Λ is a ring, then the derived category of G-equivariant constructible sheaves of Λ-modules on X is equivalent to the category of continuous functors

$$\mathrm{Gal}^{\Delta}(B_{k,*}(X, G, k)) \to \mathbf{Perf}(\Lambda)$$

that carry cartesian edges to equivalences.

The objects of the category $\mathrm{Gal}^{\Delta}(B_{k,*}(X,G,k))$ can be thought of as tuples

$$(m, \Omega, x_0, g_1, \dots, g_m)$$

in which $m \in \Delta$ is an object, Ω is a separably closed field, and $x_0 \colon \operatorname{Spec}\Omega \to X$ and $g_1, \dots, g_m \colon \operatorname{Spec}\Omega \to G$ are points with the property that $(x_0, g_1, \dots, g_m)$ is a geometric point of $X \times_k G^m$, so that Ω is the separable closure of the residue field of the image of the $(x_0, g_1, \dots, g_m)$ in the Zariski space of $X \times_k G^m$.

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1 Fibred Galois categories

1.1. We use the language and tools of higher category theory, particularly in the model of *quasicategories*, as defined by Michael Boardman and Rainer Vogt and developed by André Joyal and Jacob Lurie. We will generally follow the terminological and notational conventions of Lurie's trilogy [HTT; HA; SAG], but we will simplify matters by *systematically using words to mean their good homotopical counterparts*. So 'category' here means '∞-category', 'topos' means '∞-topos', & c.

We write $\boldsymbol{S}$ for the category of spaces and $\boldsymbol{S}_\pi \subset \boldsymbol{S}$ for the full subcategory spanned by the π-finite spaces.

We use [HTT, Corollary 3.2.2.13] systematically to construct cartesian fibrations; we leave the details of this by now standard construction implicit in what follows.

1.2 Notation. If $\boldsymbol{X} \to S$ is a topos fibration [HTT, Definition 6.3.1.6], then for any morphism $f \colon s \to t$ of S, there is a corresponding geometric morphism $f_* \colon \boldsymbol{X}_t \to \boldsymbol{X}_s$ of topoi; its left exact left adjoint will be denoted f^*.

1.3 Definition. Let S be a category. A *bounded coherent topos fibration* $\boldsymbol{X} \to S$ is a topos fibration in which each fibre $\boldsymbol{X}_s$ is bounded coherent, and for any morphism $f \colon t \to s$ of S, the induced geometric morphism $f_* \colon \boldsymbol{X}_s \to \boldsymbol{X}_t$ is coherent [SAG, Definitions A.2.0.12 & A.7.1.2; 3, Definition 5.28]. A *spectral topos fibration* $\boldsymbol{X} \to S$ is a bounded coherent topos fibration in which each fibre $\boldsymbol{X}_s$ is a spectral topos (for the canonical profinite stratification [3, Lemma 9.40 & Definition 10.3]).

1.4. The usual straightening/unstraightening equivalence restricts to an equivalence between the category of bounded coherent (respectively, spectral) topos fibrations $\boldsymbol{X} \to S$ and the category of functors from S^{op} to the category of bounded coherent (resp., spectral) topoi (cf. [HTT, Proposition 6.3.1.7]).

For a bounded coherent topos fibration $\boldsymbol{X} \to S$ we write $\boldsymbol{X}^{coh}_{<\infty} \subseteq \boldsymbol{X}$ for the full subcategory spanned by the objects that are truncated and coherent in their fibre [3, Definition 5.18]. Then $\boldsymbol{X}^{coh}_{<\infty} \to S$ is a cocartesian fibration that is classified by a functor from S to the category of bounded pretopoi [SAG, Definition A.7.4.1 & Theorem A.7.5.3].

1.5 Example. If X_* is a simplicial (coherent!) scheme, then the fibred topos $X_{*,\acute{e}t} \to \Delta$ is a spectral topos fibration.

1.6. Hochster duality [3, Theorem 10.10] expresses an equivalence between the category of profinite layered categories[2] and the category of spectral topoi, which carries a profinite layered category $\Pi = \{\Pi_\alpha\}_{\alpha\in A}$ to the spectral topos $\widetilde{\Pi}$ of sheaves in the effective epimorphism topology [SAG, §A.6.2] on the bounded pretopos

$$\mathrm{Fun}^{cts}(\Pi, \boldsymbol{S}_\pi) \coloneqq \operatorname*{colim}_{\alpha\in A^{op}} \mathrm{Fun}(\Pi_\alpha, \boldsymbol{S}_\pi)$$

of *continuous functors* $\Pi \to \boldsymbol{S}_\pi$. Under Hochster duality, the category of spectral topos fibrations $\boldsymbol{X} \to S$ is equivalent to the category of functors from S^{op} to the category of profinite layered categories.

A fibred form of Hochster duality is what allows us to construct fibred Galois categories. To define it, we need to make sense categories fibred in profinite stratified spaces.

1.7 Definition. Let S be a category. A functor $\Pi \to S$ will be said to be a *category over S fibred in layered categories* if it is a catesian fibration whose fibres are layered categories. We write $\mathbf{Lay}^{cart}_{/S}$ for the category of categories over S fibred in layered categories.

1.8 Construction. There is a monad T on the category $\mathbf{Lay}$ of small layered categories given by sending a layered category Π to the limit over the π-finite layered categories to which it maps.[3] The category of T-algebras is equivalent to the category of profinite layered categories. If S is a category, this monad can be applied fibrewise to give a monad T_S on the category $\mathbf{Lay}^{cart}_{/S}$ of categories fibred in layered categories.

Under the straightening/unstraightening identification

$$\mathbf{Lay}^{cart}_{/S} \simeq \mathrm{Fun}(S^{op}, \mathbf{Lay}) ,$$

the monad T_S corresponds to the monad on $\mathrm{Fun}(S^{op}, \mathbf{Lay})$ given by applying T objectwise. Consequently, the category of T_S-algebras is equivalent to the category of functors from S^{op} to the category of profinite layered categories.

1.9 Definition. Let S be a category. A *category over S fibred in profinite layered categories* is a T_S-algebra. If $\Pi \to S$ is a category fibred in layered categories, then a *fibrewise profinite structure* on $\Pi \to S$ is a T_S-algebra structure on $\Pi \to S$. We write $\mathbf{Lay}^{cart,\wedge}_{\pi,/S}$ for the category of T_S-algebras.

1.10 Warning. One might also contemplate the category $\mathrm{Pro}(\mathbf{Lay}^{cart}_{\pi,/S})$ of proöbjects in the full subcategory

$$\mathbf{Lay}^{cart}_{\pi,/S} \subseteq \mathbf{Lay}^{cart}_{/S}$$

spanned by those cartesian fibrations whose fibres are π-finite layered categories. This is generally *not* equivalent to the category of categories over S fibred in profinite layered categories. Under straightening/unstraightening, the category $\mathbf{Lay}^{cart,\wedge}_{\pi,/S}$ is equivalent to the category $\mathrm{Fun}(S^{op}, \mathbf{Lay}^{\wedge}_{\pi})$, whereas $\mathrm{Pro}(\mathbf{Lay}^{cart}_{\pi,/S})$ is equivalent to the category $\mathrm{Pro}(\mathrm{Fun}(S^{op}, \mathbf{Lay}_\pi))$. These coincide when S is a finite poset [HTT, Proposition 5.3.5.15], but otherwise typically do not coincide.

[2] A category C is *layered* if every endomorphism in C is an equivalence.

[3] That is, T is the right Kan extension of the inclusion $\mathbf{Lay}_\pi \hookrightarrow \mathbf{Lay}$ of π-finite layered categories along itself.

1.11. Let S be a category. Then the category of spectral topos fibrations over S is equivalent to the category $\mathbf{Lay}^{cart,\wedge}_{\pi,/S}$. Let us make the equivalence explicit. If $\boldsymbol{X} \to S$ is a spectral topos fibration, then we define a category over S fibred in layered categories

$$\Pi^{S,\wedge}_{(\infty,1)}(\boldsymbol{X}) \to S$$

as follows. An object of $\Pi^{S,\wedge}_{(\infty,1)}(\boldsymbol{X})$ is a pair (s,ν), where $s \in S$ and $\nu_* : \boldsymbol{S} \to \boldsymbol{X}_s$ is a point. A morphism $(s,\nu) \to (t,\xi)$ is a morphism $f : s \to t$ of S and a natural transformation $\nu_* \to f_*\xi_*$. The category $\Pi^{S,\wedge}_{(\infty,1)}(\boldsymbol{X})$ fibred in layered categories admits a canonical fibrewise profinite structure; the fibre $\Pi^{S,\wedge}_{(\infty,1)}(\boldsymbol{X})_s$ over an object $s \in S$ is the profinite stratified shape $\Pi^{\wedge}_{(\infty,1)}(\boldsymbol{X}_s)$ of [3, Construction 11.1].

In the other direction, if $\Pi \to S$ is a category over S fibred in profinite layered categories, then let $X_0 \to S$ denote the cocartesian fibration in which the objects are pairs (s,F) consisting of an object $s \in S$ and a functor $F \colon \Pi_s \to \boldsymbol{S}_\pi$, and a morphism $(f,\phi) \colon (s,F) \to (t,G)$ consists of a morphism $f : s \to t$ of S and a natural transformation $\phi \colon f_! F \to G$. Then $(\widetilde{\Pi})^{coh}_{<\infty}$ is equivalent to the subcategory of X_0 whose objects are those pairs (s,F) in which F is continuous and whose morphisms are those pairs (f,ϕ) in which ϕ is continuous (1.6).

1.12 Construction. If S is a category and $\boldsymbol{Y}$ is a bounded coherent topos, then the projection $\boldsymbol{Y}\times S \to S$ is a bounded coherent topos fibration. The assignment $\boldsymbol{Y} \mapsto \boldsymbol{Y} \times S$ defines a functor from the category of bounded coherent topoi to the category of bounded coherent topos fibrations over S. This functor admits a left adjoint, which we denote by $|\cdot|_S$. At the level of pretopoi, $(|\boldsymbol{X}|_S)^{coh}_{<\infty}$ is equivalent to the category of cocartesian sections of $\boldsymbol{X}^{coh}_{<\infty} \to S$, i.e., the limit of the corresponding functor from S to bounded pretopoi.

Now we arrive at the main topos-theoretic result.

1.13 Proposition. *Let S be a category, and let $\boldsymbol{X} \to S$ be a spectral topos fibration. Then the pretopos $(|\boldsymbol{X}|_S)^{coh}_{<\infty}$ is equivalent to the category of functors $F \colon \Pi^{S,\wedge}_{(\infty,1)}(\boldsymbol{X}) \to \boldsymbol{S}_\pi$ with the following properties.*

- *F carries any cartesian edge to an equivalence.*
- *For any object $s \in S$, the restriction $F|_{\Pi^{\wedge}_{(\infty,1)}(\boldsymbol{X}_s)}$ is continuous.*
- *F is uniformly truncated in the sense that there exists an $N \in \boldsymbol{N}$ such that for any object $(s,\nu) \in \Pi^{S,\wedge}_{(\infty,1)}(\boldsymbol{X})$, the space $F(s,\nu)$ is N-truncated.*

Proof. The pretopos $(|\boldsymbol{X}|_S)^{coh}_{<\infty}$ can be identified with the category of cocartesian sections of $\boldsymbol{X}^{coh}_{<\infty} \to S$. The description of (1.11) completes the proof. □

Please note that the last condition of Proposition 1.13 is automatic if S has only finitely many connected components (e.g., $S = \Delta$).

1.14 Example. If X_* is a simplicial scheme, then the category over Δ fibred in profinite layered categories $\Pi^{\Delta,\wedge}_{(\infty,1)}(X_{*,ét})$ associated to the spectral topos fibration $X_{*,ét} \to \Delta$ is the category $\mathrm{Gal}^{\Delta}(X_*)$ of Construction 0.1. In this case, Proposition 1.13 implies that $(|X_{*,ét}|_{\Delta})^{coh}_{<\infty}$ is equivalent to the category of functors $\mathrm{Gal}^{\Delta}(X_*) \to \boldsymbol{S}_\pi$ that carry cartesian edges to equivalences and restrict to continuous functors $\mathrm{Gal}^{\Delta}(X_m) \to \boldsymbol{S}_\pi$ for all $m \in \Delta$.

Finally, since the profinite stratified shape is a delocalisation of the protruncated shape [9, Theorem 2.5] we deduce the following:

1.15 Proposition. *Let S be a category, and let $\mathbf{X} \to S$ be a spectral topos fibration. Then the protruncated shape of $|\mathbf{X}|_S$ is equivalent to the protruncated homotopy type of $\Pi^{S,\wedge}_{(\infty,1)}(\mathbf{X})$.*

1.16 Example. If X_* is a simplicial scheme, then the protruncated homotopy type of the fibrewise profinite category $\mathrm{Gal}^{\Delta}(X_*)$ is equivalent to the Friedlander étale topological type of X_* [9, Theorem A].

2 Sheaves on stacks

2.1 Construction. Write **Aff** for the 1-category of affine schemes. We employ [HTT, Corollary 3.2.2.13] to construct a category $\mathbf{PSh}_{\acute{e}t}$ and a cocartesian fibration

$$\mathbf{PSh}_{\acute{e}t} \to \mathbf{Aff}^{op}$$

in which the objects of $\mathbf{PSh}_{\acute{e}t}$ are pairs (S, F) consisting of an affine scheme S and a presheaf (of spaces) on the small étale site of S, and a morphism $(S, F) \to (T, G)$ is a pair (f, ϕ) consisting of a morphism $f\colon T \to S$ and a morphism of presheaves $\phi\colon f^{-1}F \to G$ on the small étale site of T. Define $\mathbf{Sh}_{\acute{e}t} \subset \mathbf{PSh}_{\acute{e}t}$ to be the full subcategory spanned by those pairs (S, F) in which F is a sheaf; then $\mathbf{Sh}_{\acute{e}t} \to \mathbf{Aff}^{op}$ is a topos fibration. Define $\mathbf{Constr}_{\acute{e}t} \subset \mathbf{Sh}_{\acute{e}t}$ to be the further full subcategory spanned by those pairs (S, F) in which F is a (nonabelian) constructible sheaf [3, Definition 10.11]; then $\mathbf{Constr}_{\acute{e}t} \to \mathbf{Aff}^{op}$ is a cocartesian fibration.

2.2 Definition. Let $X \to \mathbf{Aff}$ be a stack, i.e., a right fibration that is classified by an accessible fpqc sheaf $\mathbf{Aff}^{op} \to \mathbf{S}$. A *(nonabelian) constructible sheaf* on X is a cocartesian section

$$F\colon X^{op} \to \mathbf{Constr}_{\acute{e}t}$$

over $\mathbf{Aff}^{op}$. We write $\mathbf{Constr}_{\acute{e}t}(X)$ for the category of constructible sheaves on X.

2.3 Warning. This can only be expected to be a reasonable definition for coherent stacks.

2.4. Informally, a constructible sheaf F on X assigns to every affine scheme S over X a constructible sheaf F_S and to every morphism $f\colon S \to T$ of affine schemes an equivalence $F_S \simeq f^*F_T$. In other words, the category of constructible sheaves on X is the limit of the diagram $X^{op} \to \mathbf{Cat}$ given by the assignment $S \mapsto \mathbf{Constr}_{\acute{e}t}(S)$.

Of course, since X is not a small category, it is not obvious that this limit exists in **Cat**. However, if X contains a small limit-cofinal full subcategory Y, then the desired limit exists.

Now we conclude:

2.5 Proposition. *If $p\colon X \to \mathbf{Aff}$ is a stack, and if X is presented by a simplicial scheme Y_*, then we obtain an equivalence between the category $\mathbf{Constr}_{\acute{e}t}(X)$ and the category of functors*

$$\mathrm{Gal}^{\Delta}(Y_*) \to \mathbf{S}_\pi$$

that carry cartesian edges to equivalences and for all $m \in \Delta$ restrict to a continuous functor $\mathrm{Gal}(Y_m) \to \mathbf{S}_\pi$.

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