Abstract

Let \( X \) be a scheme. Let \( \text{Gal}(X) \) be the topological category whose objects are geometric points of \( X \) and whose morphisms are specialisations. \( \text{Gal}(X) \) is thus a form of MacPherson's exit-path category for the étale topology. We construct an equivalence between representations of \( \text{Gal}(X) \) and constructible sheaves on \( X \), which we call the \textit{exodromy equivalence}. We show that this exodromy equivalence holds with nonabelian coefficients and with finite abelian coefficients. More generally, by using the pyknotic formalism, we extend this equivalence to coefficients in the category of modules over profinite rings and algebraic extensions of \( \mathbb{Q}_\ell \).

We also prove a higher categorical form of Hochster Duality, which reconstructs the entire étale topos of a quasicompact and quasiseparated scheme from the category of nonabelian constructible sheaves, and thus from \( \text{Gal}(X) \) itself.

If \( X \) is a scheme of finite type over a finitely generated field \( k \) of characteristic zero, then the category \( \text{Gal}(X \times_k \overline{k}) \) acquires a continuous action of the absolute Galois group \( G_k \) of \( k \). Voevodsky’s proof of a conjecture of Grothendieck now implies our main result: the resulting functor from normal schemes \( X \) of finite type over \( k \) to topological categories with an action of \( G_k \) and equivariant functors that preserve minimal objects is \textit{fully faithful}. 

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Introduction

Let $X$ be a scheme with underlying Zariski topological space $X^{\text{zar}}$. Consider the following category $\text{Gal}(X)$.

- An object is a geometric point $x \to X$, by which we mean a point whose residue field $\kappa(x)$ is a separable closure of the residue field $\kappa(x_0)$ of the image $x_0 \in X^{\text{zar}}$ of $x$.

- For two geometric points $x \to X$ and $y \to X$, a morphism $x \to y$ is a specialisation $x \prec y$ in the étale topology – that is, a geometric point $y \to X_{(x)}$ of the strict localisation $X_{(x)}$ lying over $y \to X$.

Specialisations $x \prec y$ and $y \prec z$ compose to give a specialisation $x \prec z$. Equivalently, $\text{Gal}(X)$ is the category of points of the étale topos of $X$.

The category $\text{Gal}(X)$ is a kind of categorification of the absolute Galois group. The assignment $x \mapsto x_0$ is a conservative functor from $\text{Gal}(X)$ to the specialisation poset of $X^{\text{zar}}$ – that is, the poset of points in which $x_0 \leq y_0$ if and only if $x_0$ lies in the closure of $y_0$. The fibre over a point $x_0$ is $\text{BD}_{x_0}$, where $\text{BD}_{x_0}$ is the absolute Galois group of $\kappa(x_0)$. If $X$ is normal, then the space of sections over a map $x_0 \leq y_0$ is the decomposition group of $x_0$ in the closure of $y_0$.

As with absolute Galois groups, there is a natural topology on the set of morphisms of $\text{Gal}(X)$, which is generated as follows. For any point $u \to X$ that is finite over its image $u_0 \in X^{\text{zar}}$, we form the unramified extension $A$ of the henselisation $O^{\text{h}}_{X,u_0}$ with residue field the separable closure of $\kappa(u_0)$ in $\kappa(u)$, and we write $X_{(u)} = \text{Spec} A$. If $v \to X$ is finite over its image $v_0 \in X^{\text{zar}}$, then a specialisation $u \prec v$ is a point $v \to X_{(u)}$ of $X_{(u)}$ lying over $v \to X$. For any specialisation $u \prec v$, we define the subset $U(u \prec v)$ of the set of morphisms of $\text{Gal}(X)$ consisting of those specialisations $x \prec y$ that lie over $u \prec v$.

We endow the morphisms of $\text{Gal}(X)$ with the topology generated by the sets $U(u \prec v)$. With this topology, $\text{Gal}(X)$ becomes a topological category.

A Theorem (see Theorem 14.4.7). Let $k$ be a finitely generated field of characteristic zero, and let $G_k$ be the absolute Galois group of $k$. Then the assignment $X \mapsto \text{Gal}(X)$ is fully faithful as a functor from normal $k$-varieties to topological categories over $\text{BG}_k$ and continuous functors over $\text{BG}_k$ that carry minimal objects to minimal objects.

Thus, for any normal $k$-varieties $X$ and $Y$, any continuous functor $\text{Gal}(X) \to \text{Gal}(Y)$ over $\text{BG}_k$ that preserves minimal objects is induced by a unique morphism of schemes $X \to Y$. In particular, the functor $X \mapsto \text{Gal}(X)$ is conservative for these schemes. Moreover, a $k$-morphism $f : X \to Y$ is an isomorphism if and only if $f$ induces an equivalence $\text{Gal}(X) \Rightarrow \text{Gal}(Y)$ on underlying ordinary categories.

This theorem can be regarded as a categorical version of the Anabelian Conjecture of Alexander Grothendieck: in effect, Theorem A states that Galois-theoretic information, when organised carefully, provides a complete invariant of normal varieties.

The category $\text{Gal}(X)$ is in effect an étale exit-path category. Bob MacPherson introduced the exit-path categories of stratified topological spaces to classify constructible sheaves in what we call the exodromy equivalence. Accordingly, our proof of Theorem A involves the development of a stratification of the étale homotopy type and the new theory of exodromy in the étale context.
**Monodromy for topological spaces**

It is a truth universally acknowledged, that a local system of \( \mathbb{C} \)-vector spaces on a connected topological manifold \( X \) is completely determined by its attached *monodromy representation*, so that the choice of a point \( x \in X \) specifies an equivalence of categories

\[
\text{Mon}_x : \text{LS}(X; \text{Vect}(\mathbb{C})) \Rightarrow \text{Rep}_{\mathbb{C}}(\pi_1(X, x)) .
\]

If one wants to avoid selecting a point, or if one wants to drop the connectivity hypothesis on \( X \), then one may combine the set of connected components and the various fundamental groups of \( X \) to form the *fundamental groupoid* \( \Pi_1(X) \). Then the monodromy equivalence becomes an equivalence

\[
\text{Mon} : \text{LS}(X; \text{Vect}(\mathbb{C})) \Rightarrow \text{Fun}(\Pi_1(X), \text{Vect}(\mathbb{C})) .
\]

An early insight of Dan Kan was that in a similar fashion, *all* the homotopy groups and all the \( k \) invariants of \( X \) could, in effect, be combined to form a single combinatorial gadget – a simplicial set \( \Pi_\infty(X) \) called the *singular simplicial set* or the *fundamental \( \infty \)-groupoid* of \( X \) – which knows everything about the homotopy type of \( X \).

Perhaps the clearest formulation of this insight was that of Dan Quillen, who showed that the category \( \text{TSp} \) of topological spaces and the category \( \text{sSet} \) of simplicial sets each admit model structures – each with the conventional choice of weak equivalence – relative to which the functor

\[
\Pi_\infty : \text{TSp} \rightarrow \text{sSet}
\]

is a right Quillen equivalence. Nowadays we go a step farther and think of \( \Pi_\infty \) as an equivalence \( S \Rightarrow \text{Gpd}_\infty \) between the underlying \( \infty \)-category of spaces and that of \( \infty \)-groupoids.

This fundamental \( \infty \)-groupoid of \( X \) appears in derived versions of the monodromy equivalence: for instance, the monodromy of a local system of *complexes* of \( \mathbb{C} \)-vector spaces is a functor from \( \Pi_\infty(X) \) to complexes, and this induces an equivalence of \( \infty \)-categories

\[
\text{Mon} : \text{LS}(X; \text{D}(\mathbb{C})) \Rightarrow \text{Fun}(\Pi_\infty(X), \text{D}(\mathbb{C})) .
\]

All of these equivalences follow from the ur-example of local systems of *spaces* on \( X \), which are known as *parametrised homotopy types* in the homotopy theory literature \([3; 79]\). These form an \( \infty \)-category \( \text{LS}(X) \), and there is a natural monodromy equivalence of \( \infty \)-categories

\[
\text{Mon} : \text{LS}(X) \Rightarrow \tilde{\Pi}_\infty(X) := \text{Fun}(\Pi_\infty(X), S)
\]

\([\text{HA}, \text{§A.9}]\).

**Monodromy for schemes**

To replace the manifold in this story with a scheme, Grothendieck identified étale local systems on a suitable connected scheme \( X \) with representations of its étale fundamental group. Here it is not the Zariski topological space of \( X \) that is germane but its *étale topos*, and one obtains not a group but a progroup: the *extended étale fundamental group*
\[ \pi^\#_1(X) \]. If preferred, one can consider the profinite completion of \( \pi^\#_1(X) \): the usual étale fundamental group \( \hat{\pi}^\#_1(X) \).

The étale fundamental group is an information-dense invariant, and Grothendieck’s Anabelian Conjectures are roughly an investigation of the extent to which the étale fundamental group is a complete invariant for certain classes of schemes. In dimension 0, the classical theorem of Jürgen Neukirch and Kōji Uchida [89; 90; 118] ensures that two number fields are isomorphic if and only if their absolute Galois groups are. In dimension 1, Akio Tamagawa [116] and Shinichi Mochizuki [83] show that dominant morphisms between smooth hyperbolic curves over suitable fields of characteristic zero can be detected at the level of fundamental groups. Work of Florian Pop [93, Theorem 1] shows that an isomorphism between two function fields over finitely generated fields can be detected at the level of Galois groups.

Eduardo Dubuc [30, §§5–6] generalised the étale fundamental group by extracting from a topos \( X \) a fundamental progroupoid \( \Pi^\#_1(X) \) and a monodromy equivalence

\[ X^{\text{locsys}} = \text{Fun}(\Pi^\#_1(X), \text{Set}) \]

between the local systems of sets on \( X \) and Set-valued functors on the \( \Pi^\#_1(X) \) (in the ‘pro’ sense). Following this, from an \( \infty \)-topos \( X \), Jacob Lurie extracted a fundamental \( \infty \)-groupoid \( \Pi^\infty_1(X) \) whose representations are monodromy representations. Again, the shape \( \Pi^\infty_\text{co}(X) \) has a profinite completion: the homotopy type \( \hat{\Pi}^\infty_\text{co}(X) \) of \( X \). Tom Bachmann and Marc Hoyois show [10, Proposition 10.1] that for any \( \infty \)-topos \( X \), one has a natural monodromy equivalence of \( \infty \)-categories

\[ X^{\text{lisse}} \cong \text{Fun}(\hat{\Pi}^\infty_\text{co}(X), S_\pi) \]

between the lisse sheaves on \( X \) – i.e., locally constant sheaves of \( \pi \)-finite spaces on \( X \) that can be trivialised on a finite cover – and functors on \( \Pi^\infty_\text{co}(X) \) valued in the \( \infty \)-category \( S_\pi \) of \( \pi \)-finite spaces (see also Proposition 4.4.18). This monodromy equivalence is a form of Galoisian duality. At the most abstract level, this duality arises from the fully faithful inclusion \( S_\pi \hookrightarrow \text{Top}^\infty_\text{co} \), given by \( \Pi \mapsto \hat{\Pi} = \text{Fun}(\Pi, S) \) and its proëxistent left adjoint. Hoyois showed that if \( X_\delta \) is the (1-localic) étale \( \infty \)-topos of a locally noetherian scheme \( X \), then the profinite space \( \hat{\Pi}^\infty_\text{co}(X_\delta) \) coincides with the étale homotopy type \( \hat{\Pi}^\infty_\text{et}(X) \) of Mike Artin and Barry Mazur [56, Corollary 5.6].

If the étale fundamental group \( \pi^\#_1 \) is information-dense, then the étale homotopy type \( \Pi^\#_\text{et} \) must be even more so. Indeed, Alexander Schmidt and Jacob Stix [105, Theorem 1.2] show that over a finitely generated field \( k \) of characteristic 0, if \( X \) and \( Y \) are smooth, geometrically connected varieties that can be embedded as locally closed subschemes of a product of hyperbolic curves, then the map

\[ \text{Isom}_k(X, Y) \to \text{Isom}_{BG_k}((\hat{\Pi}^\#_\text{et}(X), \hat{\Pi}^\#_\text{et}(Y))) \]

is a split injection with a natural retraction. Here \( \text{Isom}_{BG_k} \) denotes the set of homotopy classes of equivalences of profinite spaces over \( BG_k \).

**Exodromy for topological spaces**

A string of results has suggested the possibility that stratified spaces and constructible sheaves might be modeled in a similarly combinatorial fashion. Bob MacPherson proved
that constructible sheaves of sets on a (suitably nice) stratified topological space $X$ over a poset $P$ determine and are determined by a functor from the exit-path category $\Pi_{(1,1)}(X; P)$ of $X$, whose objects are points of $X$ and whose morphisms are stratified homotopy equivalence classes of exit paths – paths from a stratum $X_p$ to a stratum $X_q$ for $q \geq p$. We call this equivalence

$$ \text{Ex}^P : \text{Sh}(X; \text{Set})^{P\text{-cons}} \to \text{Fun}(\Pi_{(1,1)}(X; P), \text{Set}) $$

between $P$-constructible sheaves of sets on $X$ and functors $\Pi_{(1,1)}(X; P) \to \text{Set}$ the exodromy equivalence.\(^1\) One notes that $\Pi_{(1,1)}(X; P)$ is a category with a conservative functor to $P$. For each point $p \in P$, the fibre of the functor $\Pi_{(1,1)}(X; P) \to P$ over $p$ is the fundamental groupoid $\Pi_{1}(X_p)$ of the stratum $X_p$.

David Treumann \cite{117} then extended MacPherson’s result to give an exodromy equivalence between constructible stacks with functors from an exit-path 2-category of $X$ valued in groupoids. Lurie \cite[Appendix A]{HA} extended this further to give an exodromy equivalence

$$ \text{Ex}^P : \text{Sh}(X; S)^{P\text{-cons}} \to \Pi_{(\infty,1)}(X; P) = \text{Fun}(\Pi_{(\infty,1)}(X; P), S) $$

between $P$-constructible sheaves on $X$ with values in the $\infty$-category of spaces and functors from an exit-path $\infty$-category $\Pi_{(\infty,1)}(X; P)$ to $S$. The objects of $\Pi_{(\infty,1)}(X; P)$ are points of $X$, the morphisms are exit-paths, the 2-morphisms are stratified homotopies, the 3-morphisms are stratified homotopies of homotopies, etc., ad infinitum. One notes that $\Pi_{(\infty,1)}(X; P)$ is an $\infty$-category with a conservative functor to $P$ itself. Over each point $p \in P$, the fibre of this functor is the fundamental $\infty$-groupoid $\Pi_{\infty}(X_p)$ of the stratum $X_p$.

One is led to seek an analogue of the Kan–Quillen Theorem that states that the formation of the exit-path $\infty$-category is an equivalence of suitable homotopy theories between stratified spaces and suitable $\infty$-categories. A geometric form of this result was proved by David Ayala, John Francis, and Nick Rozenblyum \cite{9}, who showed that the exit-path $\infty$-category construction is fully faithful from a homotopy theory of conically smooth stratified spaces to $\infty$-categories.

A still closer stratified analogue of the Kan–Quillen equivalence has now been provided by the simultaneous, work of three authors:\(^2\) Sylvain Douteau \cite{28;29}, Stephen Nand-Lal and Jon Woolf \cite{87;88}, and the third-named author \cite{45}. These papers each take a slightly different point of view, but for our purposes here, the salient point (expressed in \cite{45}) is this: the functor $\Pi_{(\infty,1)}(\dash; P)$ is an equivalence between the following homotopy theories:

- topological spaces with a stratification over $P$ – in which a weak equivalence of such is a weak equivalence on strata and (homotopy) links – and
- $\infty$-categories with a conservative functor to $P$.

---

\(^1\) ἔξω: outer; δρόμος: avenue.

\(^2\) In his thesis \cite{48}, André Henriques conjectures that one should be able to define a model structure on $P$-stratified simplicial sets. In his later note \cite{49} he defines a model structure on $P$-stratified simplicial sets and relates it to a model structure on Fun$(sd(P)^O$, $\text{sSet})$. These model structures present a delocalisation of the $\infty$-category we’re interested in.
We are thus entitled to refer to ∞-categories with a conservative functor to a poset \( P \) as \( P \)-stratified spaces. This makes it possible to port some of the ideas of stratified homotopy theory to the study of schemes. Importantly, if \( S \) is a spectral topological space (i.e., the underlying Zariski topological space \( X^{\text{zar}} \) of a coherent\(^3\) scheme \( X \), or equivalently a profinite poset), then we are able to extend this description to define the homotopy theory of \( S \)-stratified spaces.

**Exodromy for schemes**

In the present paper, we define \( P \)-stratified \( \infty \)-topoi and more generally \( S \)-stratified \( \infty \)-topoi, and we study the constructible sheaves therein. For any \( S \)-stratified space \( \Pi \), the \( \infty \)-topos \( \Pi = \text{Fun}(\Pi, S) \) admits a natural \( S \)-stratification. This defines a functor

\[
\text{Str}_S \to \text{StrTop}^\wedge_S
\]

Restricting to profinite stratified spaces, we obtain a fully faithful functor

\[
\text{Str}_{\pi, S} \hookrightarrow \text{StrTop}^\wedge_S
\]

which admits a left adjoint \( \hat{\Pi}^S_{(\infty, 1)} \).

**B Theorem (Theorem 10.1.7).** Let \( S \) be a spectral topological space. For any \( S \)-stratified \( \infty \)-topos \( X \), the unit

\[
X \to \text{Fun}^{\text{str}}(\hat{\Pi}^S_{(\infty, 1)}(X), S)
\]

of the adjunction to profinite stratified spaces restricts to an equivalence

\[
\text{Fun}^{\text{str}}(\hat{\Pi}^S_{(\infty, 1)}(X), S_n) \simeq X^{\pi \text{-cons}}
\]

between the \( \infty \)-category of functors valued in \( \pi \)-finite spaces and \( S \)-constructible sheaves \( X \). We call this identification the exodromy equivalence for stratified \( \infty \)-topoi.

We call the profinite \( \infty \)-category \( \hat{\Pi}^S_{(\infty, 1)}(X) \) the \( S \)-stratified homotopy type of \( X \). This is a refinement of the usual homotopy type of \( X \): the classifying profinite space of \( \hat{\Pi}^S_{(\infty, 1)}(X) \) is precisely \( \hat{\Pi}^S_{\infty}(X) \) (Example 10.1.6).

Profinite stratified spaces admit Postnikov towers

\[
\Pi \to \cdots \to h_2 \Pi \to h_1 \Pi \to h_0 \Pi \;
\]

thus an \( S \)-stratified \( \infty \)-topos \( X \) has attached fundamental profinite \((n, 1)\)-categories

\[
\hat{\Pi}^S_{(n, 1)}(X) = h_n \hat{\Pi}^S_{(\infty, 1)}(X).
\]

Our interest in these refinements arose primarily due to the following example.

---

\(^3\)Following the Grothendieck school we use the term 'coherent scheme' synonymously with 'quasicompact quasiseparated scheme' (o.h.3).
C Example. If $X$ is a coherent scheme, then the 1-localic $\infty$-topos $X_{\text{et}}$ of $X$ admits a natural $X^{\text{zar}}$-stratification. We call the profinite $\infty$-category

$$\hat{\Pi}^{\text{et}}_{(\infty,1)}(X) = \hat{\Pi}^{X^{\text{zar}}}_{(\infty,1)}(X)$$

the stratified étale homotopy type of $X$. An important point is that the stratified étale homotopy type turns out to be 1-truncated, so that $\hat{\Pi}^{\text{et}}_{(\infty,1)}(X) = \hat{\Pi}^{(1,1)}_{(\infty,1)}(X)$. For stratified 1-types, we are able to identify them with 1-categories equipped with a suitable topology. Under this correspondence, the stratified étale homotopy type of $X$ agrees with the topological category $\text{Gal}(X)$ of points of $X$ that we introduced just before the statement of Theorem A.

For a finite ring $\Lambda$, the exodromy equivalence thus yields an exodromy equivalence

$$\text{Fun}(\text{Gal}(X), \text{Perf}(\Lambda)) \simeq D_{\text{cons}}(X; \Lambda).$$

Thus the datum of a constructible sheaf $E$ of $\Lambda$-complexes on $X$ is essentially the same information as that of a (continuous) exodromy representation

$$\rho_E : \text{Gal}(X) \to \text{Perf}(\Lambda).$$

Furthermore, by employing a small piece of the pyknotic (aka condensed) formalism, we show that the exodromy equivalence above can be extended to the more general class of coefficients that typically appear:

D Theorem (Theorem 13.0.2 & Theorem 13.0.3). Let $X$ be a topologically noetherian scheme, and let $E$ be an algebraic extension of $\mathbb{Q}_\ell$ with ring of integers $O_E$. Then we have exodromy equivalences

$$\text{Fun}(\text{Gal}(X), \text{Perf}(O_E)) \simeq D_{\text{cons}}(X; O_E) \quad \text{and} \quad \text{Fun}(\text{Gal}(X), \text{Perf}(E)) \simeq D_{\text{cons}}(X; E).$$

Hochster Duality for higher topoi

The main novel step in our proof of Theorem A is that the whole étale $\infty$-topos of any coherent scheme can be completely recovered from the stratified étale homotopy type. This is a generalisation of what we call Hochster Duality.

Melvin Hochster's thesis [51; 52] identifies the category of profinite posets with the category of spectral topological spaces — those topological spaces that underlie coherent schemes. This functions as a simultaneous generalisation of Alexandroff Duality (which identifies finite posets with finite $T_0$ topological spaces) and Stone Duality (which identifies profinite sets with quasicompact and totally separated topological spaces).

Lurie has already extended Stone Duality to the context of higher topoi: he proves that the functor that carries a profinite space $\Pi$ to the $\infty$-topos $\tilde{\Pi} = \text{Fun}(\Pi, S)$ is fully faithful, and its essential image consists of bounded coherent $\infty$-topoi in which the truncated coherent objects coincide with the lisse sheaves [SAG, §E.3]. We call these $\infty$-topoi Stone $\infty$-topoi. (Lurie calls them profinite $\infty$-topoi.)

In this paper, we prove the following:
E Theorem (∞-Categorical Hochster Duality; Theorem 9.3.1). The assignment that carries a profinite stratified space $\Pi$ to the $\infty$-topos $\tilde{\Pi} = \text{Fun}(\Pi, S)$ is fully faithful, and its essential image consists of bounded coherent $\infty$-topoi in which the truncated coherent objects coincide with the constructible sheaves.

We call these $\infty$-topoi spectral $\infty$-topoi (Definition 9.2.1). This is partially justified by the fact that they are the natural higher categorical extension of Hochster’s spectral topological spaces. Better still, we have the following.

F Example. Let $X$ be a coherent scheme. Then the étale $\infty$-topos $X_{\text{et}}$ of $X$ is spectral.

Since one may identify the constructible sheaves on $X$ with the truncated and coherent objects of $X_{\text{et}}$, we deduce that in fact $X_{\text{et}}$ is equivalent to the $\infty$-topos $\tilde{\text{Gal}}(X)$. In other words, the stratified étale homotopy type of $X$ recovers the entire étale $\infty$-topos attached to $X$.

Armed with this, Theorem A follows as soon as we know that our schemes can be recovered from their étale $\infty$-topoi. On this score, in his letter to Gerd Faltings, Grothendieck conjectured – and Vladimir Voevodsky proved [119] – that the assignment $X \mapsto X_{\text{et}}$ is a fully faithful functor from normal schemes of finite type over a finitely generated field $k$ of characteristic 0 to $\infty$-topoi with an action of the absolute Galois group $G_k$ and ‘admissible’ $G_k$-equivariant morphisms. Combined with our results on the profinite stratified shape, we obtain our Theorem A.

Whereas one can only hope that the étale homotopy type is a complete invariant for certain varieties constructed iteratively from hyperbolic curves, the addition of the natural stratification on the étale homotopy type makes the stratified étale homotopy type effective a complete invariant of all varieties.

In characteristic $p$ and for more general arithmetic schemes, the presence of inseparable extensions forces us to give a more careful formulation of Grothendieck’s conjecture (Conjecture 14.4.4), and both it and the analogue of Theorem A remain open.

Stratified Riemann Existence

If $X$ is a $\mathbf{C}$-scheme of finite type, then the Riemann Existence Theorem amounts to an equivalence between the étale homotopy type $\tilde{\Pi}^\text{et}(X)$ and the profinite completion $\tilde{\Pi}^\text{co}(X^{an})$ of the homotopy type of the topological space $X^{an}$ of complex points of $X$ with its analytic topology [7, Theorem 12.9; 18, Proposition 4.12]. In the same vein, the stratified Riemann Existence Theorem provides the following equivalence.

G Theorem (Stratified Riemann Existence; Proposition 12.6.3). Let $X$ be a $\mathbf{C}$-scheme of finite type, and $X \to P$ a finite constructible stratification. Then there is a natural equivalence

$$\tilde{\Pi}^\text{et}_{\text{co}}(X; P) \cong \tilde{\Pi}^\text{co}(X^{an}; P).$$

Combining Theorem G with Theorem A, we find that if $k$ is a finitely generated field of characteristic 0, then a normal $k$-variety can be reconstructed from the stratified homotopy type of the topological space

$$(X \times_{\text{Spec} \mathbf{k}} \text{Spec} \overline{k})^{an}$$
along with its action of $G_k$. In dimension 1, for example, a connected, smooth, and complete curve over $k$ is uniquely specified by a genus $g$ and a suitable action of $G_k$ on a diagram of free groups whose ranks depend on $g$ (see §14.5).

Technical overview

This book is broken into four parts. Parts I to III reflect the three ingredients necessary to construct the stratified étale homotopy type and to prove the central Hochster Duality Theorem for higher categories (Theorem E=Theorem 9.3.1). The last part, Part IV, is then focused applying this machinery to the étale $\infty$-topoi of schemes.

The first ingredient is a small (and quite elementary) piece of abstract homotopy theory in the study of stratified spaces and profinite stratified spaces. Most of this work is relatively formal, but one important notion is that of a spatial décollage, which is a presheaf on the subdivision of a poset satisfying a Segal condition. We prove that the $\infty$-category of stratified spaces is equivalent to that of spatial décollages via a nerve construction. The upshot is that a stratified space can be recovered from its ‘unglued’ form\(^1\) – a collection of strata and links, suitably organised.

On the toposic side, one wants to be able to perform the same ungluing procedure, so that one can recover an $\infty$-topos $X$ from the data of a closed subtopos $Z$, its open complement $U$, and the gluing information in the form of the deleted tubular neighbourhood $W$ of $Z$ in $U$. This is the second major ingredient – gluing squares of $\infty$-topoi, which are certain squares

\[
\begin{array}{ccc}
W & \rightarrow^q & U \\
\downarrow^p & \searrow^\sigma & \downarrow^j \\
Z & \leftarrow^i & X
\end{array}
\]

of geometric morphisms with a noninvertible natural transformation $\sigma$. In order to make sense of this, there are three nontrivial tasks:

\begin{itemize}
  \item We must work – systematically and \textit{ab initio} – with bounded coherent $\infty$-topoi. This involves some care, particularly as these conditions are not stable under the formation of recollements. See Chapter 3.
  \item We must develop the higher categorical analogue of Pierre Deligne’s oriented fibre product [61; 74; 92]. The tubular neighbourhood of $Z$ in $X$ is the evanescent co-topos $Z \times_X X$, and the deleted tubular neighbourhood $W$ is then the open subtopos $Z \times_X U \subseteq Z \times_X X$. See Chapter 5.
  \item Finally, and most crucially, we must prove a rather delicate Basechange Theorem for oriented fibre products (Theorem 7.1.7), which ensures that the two gluing functors $i^* j_*$ and $p_* q^*$ agree, at least on truncated objects. See Chapters 6 and 7.
\end{itemize}

\(^1\)Whence the term ‘décollage’.
of topoi décollages – i.e., presheaves of ∞-topoi on the subdivision of a poset that satisfy a kind of oriented Segal condition. This condition ensures that a string \( \{ p_0 \leq \cdots \leq p_n \} \) is carried to the iterated oriented fibre product \( X_{p_0} \times_X \cdots \times_X X_{p_n} \) of the strata. We may also pass to profinite objects in the base, which permits us to contemplate stratified ∞-topoi over spectral topological spaces.

Among the bounded coherent stratified ∞-topoi are those in which the strata are Stone ∞-topoi. These are the spectral ∞-topoi. They turn out to agree with those bounded coherent stratified ∞-topoi in which the truncated coherent objects are exactly the constructible sheaves – i.e., those sheaves that restrict to a lisse sheaf on any stratum. If \( \Pi \) is a profinite stratified space, then the stratified ∞-topos \( \tilde{\Pi} \) is spectral in this sense. As in Lurie’s ∞-Categorical Stone Duality, there is a left adjoint to the functor \( \Pi \mapsto \tilde{\Pi} \), which carries a stratified ∞-topos to its stratified homotopy type.

Now the ∞-Categorical Hochster Duality Theorem – which provides an equivalence between spectral ∞-topoi with profinite stratified spaces – follows from a sequence of three moves:

- We reduce to the case of a finite poset \( P \). This is formal.
- We then show that the stratified homotopy type of a spectral ∞-topos can be computed by ungluing to the topoi décollage, forming the homotopy type objectwise to get a spatial décollage, and then regluing to a profinite stratified space.
- We then appeal to Lurie’s ∞-Categorical Stone Duality Theorem.

Open problems

There are a number of questions we have not answered in this paper. Here are just a few.

**Question.** Our work here leaves Conjecture 14.4.4 frustratingly open. In effect, it predicts that a large class of absolute schemes \( X \) (see Definition 14.4.1) can be reconstructed from their Galois categories \( \text{Gal}(X) \).

**Question.** We may ask whether one can recover an absolute scheme \( X \) from the profinite stratified space at a finite stage. That is, is there a finite constructible stratification \( X \to P \) such that for any absolute scheme \( Y \), the map

\[
\text{Mor}_k(X, Y) = \text{Map}_{BG_k}(\tilde{\Pi}^{\text{ét}}_{(\infty,1)}(Y), \tilde{\Pi}^{\text{ét}}_{(\infty,1)}(X)) \to \pi_0 \text{Map}_{BG_k}(\tilde{\Pi}^{\text{ét}}_{(\infty,1)}(Y), \tilde{\Pi}^{\text{ét}}_{(\infty,1)}(X; P))
\]

is a bijection? (One might expect that it suffices to choose stratification in which the strata in \( X \) are strongly hyperbolic Artin neighbourhoods [SGA 4_{III}, Exposé XI, §§2 & 3]; at this point, we do not know.)

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The Université Montpellier has recently released a collection of notes of Alexander Grothendieck [40], including ‘Côte n° 151: Espaces stratifiés’, in which he develops some elements of stratified topos theory and some elements of an attached shape theory, to
which he referred in his Esquisse d’un Programme [42, p. 36]. It is not clear to us how much of the work here he anticipated.

We have used the framework and results in Jacob Lurie’s three big books [HTT], [HA], and [SAG] everywhere here. The impact of his ideas here is obvious and extensive. We are also grateful to him for his very helpful answers to a number of technical questions we pelted him with over the course of this project.

Much of our understanding of stratified spaces is directly the result of our conversations with David Ayala, who independently developed the ‘spatial décollage’ perspective on stratified spaces. We are exceedingly grateful to him for sharing with us a portion of these ideas. Ayala’s thinking has had a tremendous influence on us, and conversations with him in the first phase of this project have been vital to our work. Additionally, he has read some early editions of this monograph, spotted some mistakes, and helped us improve our writing.

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0 Terminology & notations

0.1 Set theoretic conventions

0.1.1. Recall that if $\delta$ is a strongly inaccessible cardinal (which we always assume to be uncountable), then the set $V_\delta$ of all sets of rank strictly less than $\delta$ is a Grothendieck universe of rank and cardinality $\delta$ \[SGA 4, Exposé I, Appendix\]. Conversely, if $V$ is a Grothendieck universe that contains an infinite cardinal, then $V = V_\delta$ for some strongly inaccessible cardinal $\delta$.

In order to deal precisely and simply with set-theoretic problems arising from the consideration of 'large' collections, we append to $\text{ZFC}$ the Axiom of Universes ($\text{AU}$). This asserts that any cardinal is dominated by a strongly inaccessible cardinal.

We write $\delta_0$ for the smallest strongly inaccessible cardinal. Now $\text{AU}$ implies the existence of a hierarchy of strongly inaccessible cardinals

$$\delta_0 < \delta_1 < \delta_2 < \cdots,$$

in which for each ordinal $\alpha$, the cardinal $\delta_\alpha$ is the smallest strongly inaccessible cardinal $\delta_\beta$ for any $\beta < \alpha$.\(^5\)

We certainly will not use the full strength of $\text{AU}$; the existence of only $\delta_0$ and $\delta_1$ suffices for our work here. At the cost of some circumlocutions, one could even get away with $\text{ZFC}$ alone.

0.1.2. We write $\mathbb{N}$ for the poset of nonnegative integers. We write $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, and $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$.

0.2 Higher categories

0.2.1. We use the language and tools of higher category theory, particularly in the model of quasicategories, as defined by Michael Boardman and Rainer Vogt and developed by André Joyal and Lurie. We will generally follow the terminological and notational conventions of Lurie’s trilogy [HTT; HA; SAG]. In particular:

- An $\infty$-category here will always mean quasicategory.

- A subcategory $C'$ of an $\infty$-category $C$ is a simplicial subset that is stable under composition in the strong sense, so that if $\sigma : \Delta^n \to C$ is an $n$-simplex of $C$, then $\sigma$ factors through $C' \subseteq C$ if and only if each of the edges $\sigma(\Delta^{[i+1]})$ does so.

- Let $\delta$ be a strongly inaccessible cardinal. We say that a set, group, simplicial set, $\infty$-category, ring, etc., is $\delta$-small\(^6\) if it is equivalent (in whatever appropriate sense) to one that lies in $V_\delta$. We abbreviate $\delta_0$-small to small.

- An $\infty$-category $C$ is locally $\delta$-small if and only if, for any objects $x, y \in C$, the mapping space $\text{Map}_C(x, y)$ is $\delta$-small. We abbreviate locally $\delta_0$-small to locally small.

---

\(^5\)Thus $V_{\delta_0}$ models $\text{ZFC}$ plus the axiom ‘the set of strongly inaccessible cardinals is order-isomorphic to $\alpha$’.

\(^6\)The adverb ‘essentially’ is often deployed in this situation.
Accessibility of ∞-categories and functors and presentability of ∞-categories will always refer to accessibility and presentability with respect to some δ₀-small cardinal. Please observe that an accessible ∞-category is always essentially δ₁-small and locally δ₀-small.

We will use the terms ∞-groupoid or space interchangeably for an ∞-category in which every morphism is invertible.

Let δ be a strongly inaccessible cardinal. Then we write S_δ for the ∞-category of δ-small spaces and Cat_{∞,δ} for the ∞-category of δ-small ∞-categories. In particular, we shall write S and Cat_{∞,δ₀} for S_δ₀ and Cat_{∞,δ₀}, respectively.

Let C be an ∞-category and W ⊆ C_1 a set of morphisms of C. Then we write W⁻¹C for the result of inverting the morphisms of W. If δ is an inaccessible cardinal for which C is δ-small, then W⁻¹C is δ-small as well. This ∞-category comes equipped with a functor C → W⁻¹C that, for any ∞-category D, induces a fully faithful functor

\[ \text{Fun}(W⁻¹C, D) \hookrightarrow \text{Fun}(C, D) \]

that identifies Fun(W⁻¹C, D) with the full subcategory spanned by those functors C → D that carry the morphisms of W to equivalences in D. One can (rather inexplicitly) describe W⁻¹C by forming the model category of (δ-small) marked simplicial sets (over Δ₀), and forming a fibrant replacement of the marked simplicial set (C, W).

0.2.2. For any n ∈ N⁺, write Cat_n ⊆ Cat_{∞} for the full subcategory spanned by the n-categories; that is, an ∞-category C lies in Cat_n if and only if for any x, y ∈ C, the ∞-groupoid Map_C(x, y) is equivalent to an (n−1)-groupoid. In particular, Cat_0 = Pos, the 1-category of partially ordered sets.

The inclusion Cat_n ⊆ Cat_{∞} admits a left adjoint h_n [103]. If C is an ∞-category, then the unit C → h_n(C) exhibits h_n(C) as the n-categorical truncation, so that the objects of h_n(C) are exactly those of C and whose mapping spaces are defined by the condition that the map

\[ \text{Map}_C(x, y) \rightarrow \text{Map}_{h_n(C)}(x, y) \]

exhibits Map_{h_n(C)}(x, y) as the (n−1)-truncation of Map_C(x, y). The 1-categorical truncation h₁(C) is also known as the homotopy category of C. The 0-categorical truncation is equivalent to the poset whose elements are the equivalence classes of objects of C in which x ≤ y if and only if there exists a morphism x → y.

0.3 Proöbjects in higher categories

0.3.1. We say that a δ₀-small ∞-category A is inverse if and only if its opposite ∞-category A^{op} is filtered. Hence an inverse system in an ∞-category C is a functor A → C from an inverse ∞-category A, and an inverse limit is a limit of an inverse system.
0.3.2. Let $C$ be an $\infty$-category. The $\infty$-category $\text{Pro}(C)$ of \textit{pro-objects in $C$} and the Yoneda embedding\(^7\)
\[ \chi : C \hookrightarrow \text{Pro}(C), \]
are defined by the following universal property

- The $\infty$-category $\text{Pro}(C)$ admits $\delta_0$-small inverse limits.

- For any $\infty$-category $D$ with $\delta_0$-small inverse limits, composition with $\chi$ induces an equivalence

\[ \text{Fun}^{\text{inv}}(\text{Pro}(C), D) \simeq \text{Fun}(C, D), \]
where $\text{Fun}^{\text{inv}}(\text{Pro}(C), D) \subset \text{Fun}(\text{Pro}(C), D)$ full subcategory of functors that preserve $\delta_0$-small inverse limits.

The existence of $\text{Pro}(C)$ is a special case of (the dual of) [HTT, Proposition 5.3.6.2].

0.3.3. The formation of pro-objects is formally dual to the formalization of ind-objects: for any $\infty$-category $C$, we have a natural identification $\text{Pro}(C)^{\text{op}} = \text{Ind}(C^{\text{op}})$.

0.3.4. For any accessible $\infty$-category $C$ that has finite limits, the $\infty$-category $\text{Pro}(C)$ admits an explicit description: $\text{Pro}(C)$ is equivalent to the full subcategory $\text{Fun}(C, S)^{\text{op}}$ spanned by the left exact accessible functors [SAG, Proposition A.8.1.6]. Under this identification, the Yoneda embedding $\chi : C \rightarrow \text{Pro}(C)$ is the restriction of the opposite of the Yoneda embedding $C^{\text{op}} \hookrightarrow \text{Fun}(C, S)$

If $X : A \rightarrow C$ is an inverse system, then its limit in $\text{Pro}(C)$ is the functor
\[ Y \mapsto \text{colim}_{\alpha \in A^\text{op}} \text{Map}_C(X_\alpha, Y). \]
We abuse notation and denote this pro-object by $X = \{X_\alpha\}_{\alpha \in A}$. Every pro-object of $C$ can be exhibited in this manner, and for pro-objects $X = \{X_\alpha\}_{\alpha \in A}$ and $Y = \{Y_\beta\}_{\beta \in B}$ we obtain the familiar formula
\[ \text{Map}_{\text{Pro}(C)}(X, Y) = \lim_{\beta \in B} \text{colim}_{\alpha \in A^\text{op}} \text{Map}_C(X_\alpha, Y_\beta). \]
We thus often speak of objects of $\text{Pro}(C)$ as if they were inverse systems. In particular, we call a pro-object $X$ \textit{constant} if and only if $X$ lies in the essential image of $\chi$; equivalently, $X$ is constant if and only if, as a functor $C \rightarrow S$, $X$ preserves inverse limits.

0.3.5. Let $\delta \geq \delta_0$ be an inaccessible cardinal, $C$ a locally $\delta$-small $\infty$-category that admits all $\delta_0$-small limits, $D$ an accessible $\infty$-category that admits finite limits, and $u : D \rightarrow C$ a left exact functor. The functor $u$ will not in general admit a left adjoint, but passage to pro-objects often repairs this. Indeed, one may extend $u$ to a (unique) functor $U : \text{Pro}(D) \rightarrow C$ that preserves inverse limits, and in the other direction, one may consider the composite
\[ F := u^* \circ \chi : C \rightarrow \text{Fun}(C, S_b)^{\text{op}} \rightarrow \text{Fun}(D, S_b)^{\text{op}} \]
of the Yoneda embedding $\chi$ with the restriction along $u$. The functor $F$ carries an object $c \in C$ to the assignment $d \mapsto \text{Map}_C(c, u(d))$. We have to make two set-theoretic assumptions:

---

\(^7\)The Hiragana character ‘\(\chi\)’ is pronounced ‘yo’.
Assume that for any object \( c \in C \) and any object \( d \in D \), the space \( \text{Map}_C(c, u(d)) \) is \( \delta_0 \)-small.

Assume that for any object \( c \in C \), there exists a regular cardinal \( \tau < \delta_0 \) such that for any \( \tau \)-filtered diagram \( d_\ast : A \to D \), the natural map

\[
\colim_{a \in A} \text{Map}_C(c, u(d_a)) \to \text{Map}_C(c, \colim_{a \in A} u(d_a))
\]

is an equivalence.

In this case, the functor \( F \) lands in \( \text{Pro}(D) \), and \( F \) is left adjoint to \( U \). We shall call \( F \) the proëxistent left adjoint to \( u \). If \( u \) already admits a left adjoint \( f \), then \( F \) lands in \( D \) and coincides with \( f \).

### 0.4 Recollements

**0.4.1.** Given functors \( F : X \to Z \) and \( G : Y \to Z \) between \( \infty \)-categories, we write

\[
X \downarrow_Z Y := X \left\uparrow_{\text{Fun}([0], Z)} \text{Fun}([1], Z) \right\downarrow_{\text{Fun}([1], Z)} Y
\]

defines the oriented fibre product of \( \infty \)-categories.

**0.4.2.** Let \( X \) and \( Y \) be essentially \( \delta_0 \)-small \( \infty \)-categories, let \( Z \) be a locally \( \delta_0 \)-small \( \infty \)-category, and let \( F : X \to Z \) and \( G : Y \to Z \) be functors. Write \( Z' \subset Z \) for the full subcategory spanned by those objects in the image of \( F \) or the image of \( G \). Then \( Z' \) is essentially \( \delta_0 \)-small and the oriented fibre product \( X \downarrow_Z Y \) is equivalent to \( X \downarrow_{Z'} Y \), whence \( X \downarrow_Z Y \) is essentially \( \delta_0 \)-small.

**0.4.3** (see [HA, §A.8]). Let \( C \) be an \( \infty \)-category that admits finite limits. Then two functors \( i_* : C_Z \to C \) and \( j_* : C_U \to C \) exhibit \( C \) as a recollement of \( C_Z \) and \( C_U \) if and only if the following conditions are satisfied.

- Both \( i_* \) and \( j_* \) are fully faithful.
- There are left exact left adjoints \( i^* \) and \( j^* \) to the functors \( i_* \) and \( j_* \).
- The functor \( j^* i_* \) is constant at the terminal object of \( C_U \).
- The functor \( (i^*, j^*) : C \to C_Z \times C_U \) is conservative.

We refer to the \( \infty \)-category \( C_Z \) as the closed subcategory, the \( \infty \)-category \( C_U \) as the open subcategory, and the functor \( i^* j_* : C_U \to C_Z \) as the gluing functor.

If \( C \) is the recollement of \( \infty \)-categories \( C_Z \) and \( C_U \), then \( C_Z \) is canonically equivalent to the kernel of \( j^* \) (i.e., the full subcategory spanned by those objects \( x \) such that \( j^* (x) = 1_{C_U} \)).

If \( C_Z \) and \( C_U \) are any \( \infty \)-categories with finite limits, and \( \delta : C_U \to C_Z \) is a left exact functor, then we write

\[
C_Z \triangledown_\delta C_U := C_Z \downarrow_{C_U} C_U
\].

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The projections
\[ i^* : C_Z \cup^\phi C_U \to C_Z \quad \text{and} \quad j^* : C_Z \cup^\phi C_U \to C_U \]
admit right adjoints
\[ i_* : C_Z \to C_Z \cup^\phi C_U \quad \text{and} \quad j_* : C_U \to C_Z \cup^\phi C_U \]
that together exhibit \( C_Z \cup^\phi C_U \) as a recollement of \( C_Z \) and \( C_U \). Furthermore, every recollement is of this form, where \( \phi \) is the gluing functor.

If \( C_Z \) contains an initial object, then \( j^* \) admits a further left adjoint \( j! \), so in this case we may also write \( j! = j_* \).

0.4.4. Let \( C \) be an \( \infty \)-category with finite limits and let \( i_* : C_Z \hookrightarrow C \) and \( j_* : C_U \hookrightarrow C \) be two functors which exhibit \( C \) as a recollement of \( C_Z \) and \( C_U \). Then for any integer \( n \geq -2 \), since the left exact functor \( (i^*, j^*) : C \to C_Z \times C_U \) is conservative, a morphism \( f \) of \( C \) is \( n \)-truncated if and only if \( i^*(f) \) and \( j^*(f) \) are both \( n \)-truncated.

0.5 Relative adjunctions

0.5.1. Given a commutative triangle of \( \infty \)-categories
\[
\begin{array}{ccc}
C & \xrightarrow{G} & D \\
\downarrow p & & \downarrow q \\
E & \xleftarrow{G} & D
\end{array}
\]
where \( p \) and \( q \) are isofibrations, we say that \( G \) admits a left adjoint relative to \( E \) if the following condition holds:

- There exists a functor \( F : C \to D \) and a natural transformation \( \eta : \text{id}_C \to GF \) which exhibits \( F \) as a left adjoint to \( G \) such that \( p\eta : p \to pGF = qF \) is an equivalence in \( \text{Fun}(C,E) \).

In this situation, given a functor \( E' \to E \), define \( C_{E'} = C \times_E E' \), \( D_{E'} = D \times_E E' \), and write \( G_{E'} : D_{E'} \to C_{E'} \) and \( F_{E'} : C_{E'} \to D_{E'} \) for the induced functors on pullbacks. Then the induced natural transformation \( \text{id}_{C_{E'}} \to G_{E'} F_{E'} \) exhibits \( F_{E'} \) as a left adjoint to \( G_{E'} \) relative to \( E' \). See [HA, Proposition 7.3.2.5].

If \( p \) and \( q \) are cartesian fibrations, \( G \) admits a left adjoint relative to \( E \) if and only if the following conditions hold:

- For every object \( e \in E \), the induced functor \( G_e : D_e \to C_e \) admits a left adjoint.

- The functor \( G \) carries \( p \)-cartesian morphisms in \( D \) to \( q \)-cartesian morphisms in \( C \).
See [HA, Proposition 7.3.2.6]. In this case, if \( f : a \to b \) is a morphism of \( E \), then one has a natural equivalence
\[
f^* G_b \simeq G_a f^* .
\]

Dually, if \( p \) and \( q \) are cocartesian fibrations, \( G \) admits a left adjoint relative to \( E \) if and only if the following (somewhat more complicated) conditions hold:

- For every object \( e \in E \), the induced functor \( G_e : D_e \to C_e \) admits a left adjoint \( F_e \).
- Let \( c \in C \) and \( \alpha : e \to e' \) be a morphism of \( e \) where \( e = p(c) \). Let \( \tilde{\alpha} : F_e(c) \to d \) be a \( q \)-cocartesian morphism in \( D \) lying over \( \alpha \), and let \( \beta : c \to G(d) \) be the composite \( \beta = G(\tilde{\alpha}) \circ \eta(e) \). Choose a factorisation of \( \beta \) as
  \[
  \beta : c \xrightarrow{\beta'} c' \xrightarrow{\beta''} G(d),
  \]
  where \( \beta' \) is a \( p \)-cocartesian morphism lifting \( \alpha \) and \( \beta'' \) is a morphism in \( C_{e'} \). Then \( \beta'' \) induces an equivalence \( F_{e'}(c') \to d \) in the \( \infty \)-category \( D_{e'} \).

See [HA, Proposition 7.3.2.11]. In this case, if \( f : a \to b \) is a morphism of \( E \), then one has a natural equivalence
\[
G_b f_! = f_! G_a .
\]

0.6 Schemes

0.6.1. Following the Grothendieck school [SGA 4_{ii}, Exposé VI, Exemples 1.22; SGA 4_{iii}, Exposé XVII, 0.12; 61; 92], we say that scheme \( X \) is coherent if and only if \( X \) is quasicompact and quasiseparated.

0.6.2. Following the Grothendieck school [SGA 4_{ii}, Exposé VI, Exemples 1.22; SGA 4_{iii}, Exposé XVII, 0.12; 61; 92], we say that scheme \( X \) is coherent if and only if \( X \) is quasicompact and quasiseparated.
Part I

Stratified spaces

In Chapter 1, we recall the Alexandroff topology on a poset. Just as the category of profinite sets can be identified with that of Stone topological spaces, the category of profinite posets be identified with that of spectral topological spaces. These topological spaces allow one to define stratifications of topological spaces over finite and profinite posets.

The homotopy theory of spaces stratified over a poset $P$ is introduced in Chapter 2. These can be described in two equivalent ways: as a $\infty$-category with a conservative functor to a poset and as a spatial décollage – a diagram of spaces indexed by the subdivision of $P$ that satisfies a Segal condition. These descriptions are equivalent, and they both permit one to study the Postnikov tower of stratified spaces and identify finiteness conditions on them.
1 Aide-mémoire on the topology of posets & profinite posets

In this short chapter we review the topologies on posets, and stratifications of topological spaces by posets. We also recall Hochster’s Theorem classifying spectral topological spaces in terms of pro-objects in finite posets (Theorem 1.3.5).

1.1 Alexandroff Duality

We start by reviewing the relationship between topological spaces and preorders. The first thing to note is that every topological space gives rise to a preorder.

1.1.1 Definition. Let $T$ be a topological space. The specialisation preorder on $T$ is the preorder on the underlying set of $T$ with order relation $x \leq y$ if and only if $x \in \{y\}$. We denote the specialization preorder on $T$ by $S(T)$.

Every preorder also gives rise to a topological space.

1.1.2 Definition. Let $P$ be a preorder.

(1.1.2.1) We say that a subset $U \subseteq P$ a cosieve if for any points $p, q \in P$ such that $p \leq q$, if $p \in U$ then $q \in U$.

(1.1.2.2) We say that a subset $Z \subseteq P$ is a sieve if for any points $p, q \in P$ such that $p \leq q$, if $q \in Z$ then $p \in Z$.

(1.1.2.3) We say that subset $W \subseteq P$ is an interval if for any points $p, q, r \in P$ such that $p \leq q \leq r$, if $p, r \in W$ then $q \in W$.

1.1.3 Definition. Let $P$ be a preorder. The Alexandroff topology on $P$ is the topology on the underlying set of $P$ in which a subset $U \subseteq P$ is open if and only if $U$ is a cosieve. We write $\text{Alex}(P)$ or simply $P$ for the set $P$ equipped with the Alexandroff topology.

Note that, a subset $Z \subseteq P$ is closed if and only if $Z$ is a sieve, and subset $W \subseteq P$ is locally closed if and only if $W$ is an interval.

Alexandroff topologies admit a well-known characterisation.

1.1.4 Proposition. The following are equivalent for a topological space $T$.

- The space $T$ is finitely generated; that is, a subset $U \subseteq T$ is open if for any finite topological space $F$ and continuous map $f : F \to T$, the inverse image $f^{-1}(U)$ is open.

- The union of any collection of closed subsets of $T$ is again closed.

- The topology on $T$ coincides with the Alexandroff topology attached to the specialisation preorder on $T$.

1.1.5 (Alexandroff Duality). The formation of the Alexandroff topology defines an equivalence of categories

$$\text{Alex} : \text{Pord} \to \text{TSpc}^{\text{fg}}$$
from the category of preorders to the category of finitely generated topological spaces. The inverse \( S : \text{TSp}c^\aleph \Rightarrow \text{Pord} \) given by taking the specialisation preorder. In particular, the functors \( \text{Alex} \) and \( S \) restrict to an equivalence between the category of finite preorders and the category of finite topological spaces.

The functors \( \text{Alex} \) and \( S \) also restrict to an equivalence between:

- the category of posets and the category of \( T_0 \) finitely generated topological spaces,
- the category of noetherian preorders (i.e., those for which every nonempty subset contains a maximal element) and the category of quasi-sober finitely generated topological spaces, and thus
- the category of noetherian posets and the category of sober finitely generated topological spaces.

1.1.6 Notation. Let \( P \) be a preorder. For any subset \( W \subseteq P \), we write \( P_{\geq W} \) for the cosieve generated by \( W \), which is the smallest open neighbourhood of \( W \). Dually, we write \( P_{\leq W} \) for the sieve generated by \( W \), which is the closure of \( W \).

We call the sets of the form

\[ P_{\geq p} = \{ q \in P \mid q \geq p \} \]

for \( p \in P \) the principal open sets. We call the sets of the form

\[ P_{\leq p} = \{ q \in P \mid q \leq p \} \]

the principal ideals.

Similarly, we write \( P_{\geq p} := P_{\geq p} \setminus \{ p \} \) and \( P_{\leq p} := P_{\leq p} \setminus \{ p \} \).

1.1.7. A poset is quasicompact in the Alexandroff topology if and only if its set of minimal elements is finite and limit-cofinal. A monotone map \( f : Q \to P \) of posets is quasicompact if and only if, for any \( p \in P \), the poset \( f^{-1}(P_{\geq p}) \) is quasicompact.

1.1.8 Notation. Let \( P \) be a preorder. We call a nonempty linearly ordered finite subset \( \Sigma \subset P \) a string in \( P \). We write \( \text{sd}(P) \) for the subdivision of \( P \), that is, \( \text{sd}(P) \) is the poset of strings \( \Sigma \subset P \) ordered by containment.

Note that there is a natural forgetful functor \( \text{sd}(P) \to \Delta \).

Let \( \Sigma \subset P \) be a string. Then every closed subset \( Z \subset \Sigma \) is again a string, and we denote the inclusion \( Z \subset \Sigma \) by \( i_{Z \subset \Sigma} \) (or simply \( i \) if \( Z \) and \( \Sigma \) are clear from the context). Dually, every open subset \( U \subset \Sigma \) is also a string, and we denote the inclusion \( U \subset \Sigma \) by \( j_{U \subset \Sigma} \) (or simply \( j \) if \( U \) and \( \Sigma \) are clear from the context).

In more general situations, we write \( \epsilon_{W \subset \Sigma} : W \to \Sigma \) for an inclusion \( W \subset \Sigma \) that is not known to be be either closed or open.

1.2 Stratifications of topological spaces

The theory of stratified topological spaces can now be neatly organized in terms of topological spaces equipped with a continuous map to a poset in the Alexandroff topology.
1.2.1 Definition. A stratification of a topological space $T$ by a poset $P$ is a continuous map $f : T \to P$. For any point $p \in P$, we write

$$T \geq p := f^{-1}(P \geq p),$$
$$T > p := f^{-1}(P > p),$$
$$T \leq p := f^{-1}(P \leq p),$$
$$T < p := f^{-1}(P < p),$$
$$T_p := T \geq p \cap T \leq p.$$

The subspaces $T_p$, $T_q$ are open in $T$, and $T_p$, $T_q$ are closed in $T$. The subspace $T_p \subset T$ is locally closed; we call $T_p$ the $p$th stratum of $f : T \to P$.

We say that the stratification $f : T \to P$ is nondegenerate if each stratum $T_p$ is nonempty, and for all $p, q \in P$ such that $p \leq q$, we have $T_p \subseteq T_q$.

We say that a stratification is finite if and only if its base poset is finite. We say that the stratification $f : T \to P$ is constructible if and only if, for all $p \in P$, the open subset $T_p \subset T$ is retrocompact – i.e., for every quasicompact open $V \subset T$, the intersection $V \cap T_p$ is quasicompact.

1.3 Hochster Duality

In this section we recall Hochster’s characterization of topological spaces that arise as the Zariski space of a coherent scheme in terms of pro-objects in the category of finite posets. This characterization provides a convenient way to study stratifications of schemes.

1.3.1 Notation. For any topological space $T$, we write $FC(T)$ for the 1-category of finite, nondegenerate, constructible stratifications $T \to P$. Please observe that $FC(T)$ is an inverse 1-category that is (equivalent to) a poset.

1.3.2 Definition. A topological space $S$ is spectral if and only if $S$ is the limit of its finite, nondegenerate, constructible stratifications; that is, if and only if

$$S = \lim_{P \in FC(S)} P$$

in the 1-category of topological spaces. We write $T\text{Sp}c^{pec} \subset T\text{Sp}c$ for the subcategory of spectral topological spaces and quasicompact continuous maps.

1.3.3 Notation. We write the $\text{Pos}$ for 1-category of posets, and $\text{Pos}^{fin}$ for the 1-category of finite posets. We regard the 1-category $\text{Pos}$ as a full subcategory of $\text{Cat}_{ec}$; indeed one has $\text{Pos} = \text{Cat}_0$.

Passing to pro-objects, we obtain the 1-category $\text{Pro}(\text{Pos})$ of proposets and the full subcategory $\text{Pro}(\text{Pos}^{fin})$ of pro-objects in the category of finite posets – which we call profinite posets.

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8Others call such topological spaces coherent; see for example [SAG, A.1; 67, Chapter III §3.4 & p. 78]. We use Hochster’s algebro-geometric terminology [51; 52].
1.3.4. The formation of the Alexandroff topology extends to an equivalence of 1-categories

\[
\text{Alex}: \text{Pro} (\text{Pos}^{\text{fin}}) \cong \text{TSp}^{\text{spec}}.
\]

We will therefore fail to distinguish between a spectral topological space and its corresponding profinite poset.

Hochster’s characterization of spectral topological spaces justifies their name:

1.3.5 Theorem (Hochster Duality [51; 52]). The following are equivalent for a topological space \( S \).

- The topological space \( S \) is spectral.
- The topological space \( S \) is sober, quasicompact, and quasiseparated; additionally, the set of quasicompact open subsets forms a basis for the topology of \( S \).
- The topological space \( S \) is homeomorphic to the underlying Zariski topological space of \( \text{Spec} R \) for some ring \( R \).
- The topological space \( S \) is homeomorphic to the underlying Zariski topological space of some coherent scheme \( Y \).

1.3.6. On one hand, Alexandroff Duality characterises posets as finitely generated topological spaces. On the other, Stone Duality characterises profinite sets as Stone spaces, i.e., totally separated quasicompact topological spaces. Hochster Duality provides a common extension of each of these forms of duality. The situation is summarised in the cube

\[
\begin{array}{ccc}
\text{Set}^{\text{fin}} & \xrightarrow{\sim} & \text{TSp}^{\text{fin, disc}} \\
\text{Pro} (\text{Set}^{\text{fin}}) & \xrightarrow{\sim} & \text{TSp}^{\text{fin}} \\
\text{Pos}^{\text{fin}} & \xrightarrow{\sim} & \text{TSp}^{\text{spec}} \\
\end{array}
\]

where \( \text{TSp}^{\text{fin}} \) denotes the 1-category of finite spectral topological spaces, and the horizontal functors marked ‘\( \sim \)’ are equivalences of 1-categories.

One of the main technical results of this book – the \( \infty \)-Categorical Hochster Duality Theorem (Theorem E=Theorem 9.3.1) – is an extension of this cube of dualities to one in which the 1-category of finite sets is replaced with the \( \infty \)-category of \( \pi \)-finite \( \infty \)-groupoids. Part of this extension is already established in the literature: Lurie proves an \( \infty \)-categorical form of Stone Duality [SAG, §E.3]. This \( \infty \)-Categorical Stone Duality Theorem identifies the \( \infty \)-category of profinite \( \infty \)-groupoids with the \( \infty \)-category of what we call \textit{Stone }\( \infty \)-topoi.\(^9\)

\(^9\)Lurie calls these \textit{profinite }\( \infty \)-topoi. In Chapter 9 we introduce a more general class of \( \infty \)-topoi that could also reasonably be called \textit{profinite }\( \infty \)-topoi, so we use the distinct term \textit{Stone }\( \infty \)-topoi to avoid confusion.
1.4 Profinite stratifications

The theory of stratifications also works well for profinite stratifications.

1.4.1 Definition. A profinite stratification of a topological space $T$ is a spectral topological space $S$ and a continuous map $f : T \to S$. We say that $f$ is constructible if and only if, for every quasicompact open subset $U \subseteq S$, the inverse image $f^{-1}(U) \subseteq T$ is retrocompact.

1.4.2. A profinite stratification with base $S$ is the same as a compatible family of stratifications with base $P$ for each nondegenerate, finite, constructible stratification $S \to P$.

1.4.3 Notation. Let $X$ be a scheme. We write $X^{\text{zar}}$ for the underlying Zariski topological space of $X$.

1.4.4 Example. Let $X$ be a scheme of finite type over the complex numbers. Write $X^{\text{an}}$ for the set $X(\mathbb{C})$ of complex points of $X$ equipped with the complex analytic topology. Then the natural continuous map $X^{\text{an}} \to X^{\text{zar}}$ is a profinite stratification of $X^{\text{an}}$ by $X^{\text{zar}}$. 

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2 The homotopy theory of stratified spaces

In this chapter we develop the homotopy theory of stratified spaces. To start, §2.1 explains how to think about the homotopy theory of stratified spaces in terms of co-categories with a conservative functor to a poset. Section 2.2 explains how the co-categories of stratified spaces relate as the poset varies. Section 2.3 explains the correct notions of connectedness and truncatedness for stratified spaces. Section 2.4 explains the analogue of π-finite spaces (i.e., truncated spaces with finite homotopy groups) in the stratified setting. These π-finite stratified spaces are crucial in our formulation of one of the main results of this text: ∞-Categorical Hochster Duality (Theorem E). Section 2.5 explains the theory of profinite stratified spaces. Sections 2.6 and 2.7 explain a complete Segal space style approach to stratified spaces that we’ll use again and again throughout the text; the power of this approach is that it reduces many questions about stratifications over a general poset P to questions about strata and links (and essentially to the poset P = [1]). Finally, §2.8 explains how the complete Segal space approach works in the profinite setting.

2.1 Stratified spaces as ∞-categories with a conservative functor to a poset

The equivalence between the homotopy theory of topological spaces and that of simplicial sets – supplied in the 1950s and 1960s by the work of Dan Kan and Dan Quillen [69; 70; 71; 72; 73; 96] – justifies the treatment of the co-category of Kan complexes as ‘the’ homotopy theory of spaces. Today, the theory of stratified spaces stands on similarly good footing. Work of David Ayala, John Francis, and Nick Rozenblyum [9], Stephen Nand-Lal and Jon Woolf [87; 88], Sylvain Douteau [29], and finally the third-named author [45] furnish an equivalence between the homotopy theory of stratified topological spaces and that of co-categories with a conservative functor to a poset.

In this section we briefly recall this result and begin to develop the theory of stratified spaces as co-categories with a conservative functor to a poset. To state the core result, we need to fix a convenient category of topological spaces.

2.1.1 Notation. We write 
\[ \text{TSp}^{\text{num}} \subset \text{TSp} \]
for the full subcategory spanned by the numerically generated topological spaces. The category 
\[ \text{TSp}^{\text{num}} \]
is a convenient category of topological spaces and is presentable; see [31; 33; 46; 47; §3; 114]. Moreover, every poset in the Alexandroff topology is a numerically generated topological space.

2.1.2 Theorem ([45, §3]). Let \( P \) be a poset. There is a class \( W \) of stratified weak equivalences in the category \( \text{TSp}_{/P}^{\text{num}} \) with the following properties.

\[ \text{TSp}_{/P}^{\text{num}}[W^{-1}] \Rightarrow \text{Cat}_{\text{cons}}^{\text{coo}/P}, \]

where the target is the co-category of co-categories with a conservative functor to \( P \).
Let $T$ be a $P$-stratified topological space whose stratification is conical in the sense of Lurie [HA, Definition A.5.5], e.g., $T$ is a topologically stratified in the sense of Goresky–MacPherson [39, §1.1]. Then the equivalence (2.1.3) sends $T$ to the exit path ∞-category of $T$.

If $T$ and $T'$ are conically $P$-stratified spaces, then a $P$-stratified map $f: T \to T'$ is a stratified weak equivalence if and only if $f$ induces a weak homotopy equivalence on strata and links.

2.1.4. An ∞-category $C$ admits a conservative functor to a poset if and only if every endomorphism of an object of $C$ is an equivalence. In this case, the homotopy 0-category $h_0(C)$ is a poset and the natural functor $C \rightarrow h_0(C)$ is conservative.

We therefore give the following definition.

2.1.5 Definition. We define the ∞-category $\text{Str}$ as the full subcategory of $\text{Fun}(\{1\}, \text{Cat}_{\infty})$ spanned by those functors $f: \Pi \rightarrow P$ where $P$ is a poset and the functor $f$ is conservative. We call an object of $\text{Str}$ a stratified space.

Let $P$ be a poset. We write $\text{Str}_P$ for the fiber the target functor $t: \text{Str} \rightarrow \text{Pos}$ over $P$. That is to say, $\text{Str}_P = \text{Cat}_{\text{cons}}^0_{P}$ is the ∞-category of ∞-categories with a conservative functor to $P$. We call an object of $\text{Str}_P$ a $P$-stratified space.

2.1.6. The ∞-category $\text{Str}_P$ can also be described as the underlying ∞-category of the third-named author's Joyal–Kan model category $s\text{Set}_P$ [45, Corollary 2.5.11].

2.1.7. Please observe that if $\Pi$ and $\Pi'$ are $P$-stratified spaces, then the ∞-category $\text{Fun}_p(\Pi, \Pi')$ of functors $\Pi \rightarrow \Pi'$ over $P$ is an ∞-groupoid. Moreover, $\text{Fun}_p(\Pi, \Pi')$ coincides with the mapping space $\text{Map}_{\text{Str}_P}(\Pi, \Pi')$.

2.1.8. Let $\Pi$ and $\Pi'$ be $P$-stratified spaces. To simplify notation we write

$$\text{Map}_p(\Pi, \Pi') := \text{Map}_{\text{Str}_P}(\Pi, \Pi') .$$

2.1.9 Definition. Let $f: \Pi \rightarrow P$ be a $P$-stratified space. For each point $p \in P$, we call the space

$$\Pi_p := \text{Map}_p(\{p\}, \Pi) = \{p\} \times_P \Pi$$

the $p$-th stratum of $\Pi$. For each pair of points $p, q \in P$ with $p \leq q$, we call the space

$$N_p(\Pi)\{p \leq q\} := \text{Map}_p(\{p \leq q\}, \Pi)$$

the link\(^{10}\) from the $p$-th stratum to the $q$-th stratum.

Please observe that the link comes equipped with source and target maps

$$(s, t): N_p(\Pi)\{p \leq q\} \rightarrow \Pi_p \times \Pi_q,$$

and the fibre of $(s, t)$ over a point $(x, y) \in \Pi_p \times \Pi_q$ is the space $\text{Map}_{\Pi}(x, y)$. When $p = q$, each of $s$ and $t$ is an equivalence, whence $(s, t)$ is equivalent to the diagonal

$$\Pi_p \rightarrow \Pi_p \times \Pi_p .$$

\(^{10}\)Our link corresponds to what Frank Quinn and others called the homotopy link or holink. The significance of our chosen notation will become clear in Construction 2.7.1.
2.1.10. A morphism \( f : \Pi' \to \Pi \) of \( \text{Str}_P \) is an equivalence if and only if, for every pair of points \( p, q \in P \) with \( p \leq q \), the map on links

\[
N_p(\Pi')(p \leq q) \to N_p(\Pi)(p \leq q)
\]

is an equivalence (in particular, when \( p = q \), the map on strata \( \Pi'_p \to \Pi_p \) is an equivalence). That is, \( f \) is an equivalence in \( \text{Str}_P \) if and only if \( f \) induces and equivalence on all strata and links.

2.2 Functoriality in the poset

In this section we explain how the \( \infty \)-categories of stratified spaces relate as the poset varies. Notice that if \( \phi : P' \to P \) is a morphism of posets, then the functor

\[
\text{Cat}_{\infty, /P'} \to \text{Cat}_{\infty, /P}
\]

given by postcomposition with \( \phi \) does not generally send \( P' \)-stratified spaces to \( P \)-stratified spaces. However, we can easily repair this by inverting all morphisms that lie over identities in \( P \). To explain this point, let us first explain the left and right adjoints to the inclusion \( \text{Str}_P \subset \text{Cat}_{\infty, /P} \).

2.2.1 Notation. We write \( \iota : \text{Cat}_{\infty} \to S \) for the right adjoint to the inclusion, given by sending an \( \infty \)-category \( C \) to the largest \( \infty \)-groupoid \( \iota C \subseteq C \) contained in \( C \). We call \( \iota C \) the interior of \( C \).

We write \( E : \text{Cat}_{\infty} \to S \) for the left adjoint to the inclusion. The functor \( E \) is given by sending an \( \infty \)-category \( C \) to the \( \infty \)-groupoid \( E(C) \) obtained by inverting every morphism of \( C \). We call \( E(C) \) the classifying space of \( C \).

The \( \infty \)-groupoid \( E(C) \) can be computed as the colimit \( E(C) \cong \text{colim}_C 1_S \) of the constant diagram \( C \to S \) at the terminal object.

2.2.2 Construction. Let \( P \) be a poset. Then inclusion \( \text{Str}_P \hookrightarrow \text{Cat}_{\infty, /P} \) admits a right adjoint \( t_p : \text{Str}_P \to \text{Cat}_{\infty, /P} \). Indeed, if \( C \) is an \( \infty \)-category, and \( f : C \to P \) is any functor, we write \( t_p(C) \subset C \) for the largest subcategory of \( C \) with the property that the composite

\[
t_p(C) \to C \xrightarrow{f} P
\]

is conservative. Concretely, \( t_p(C) \subset C \) is the subcategory containing all objects such that morphism \( e \) of \( C \) lies in \( t_p(C) \) if and only if \( e \) satisfies one of the following (disjoint) conditions:

- The morphism \( e \) is an equivalence.
- The morphism \( e \) is not sent to an identity morphism in \( P \).

\[\text{In simplicial sets the functor } E \text{ can be modeled as Kan’s } \text{Ex}^\infty \text{ functor. The notation } BC \text{ is often used for the classifying space of } C. \text{ We use the notation } E(C) \text{ for three reasons: to avoid conflict with the notation } BG \text{ for the 1-object groupoid with automorphism group } G, \text{ to pay homage to Kan’s } \text{Ex}^\infty \text{ functor, and because } ‘E’ \text{ stands for ‘invert everything.’}\]
Given a $P$-stratified space $\Pi$, every functor $\Pi \to C$ over $P$ factors through $\iota_P(C)$. Hence the assignment $C \mapsto \iota_P(C)$ defines a right adjoint to the inclusion $\text{Str}_P \subset \text{Cat}_{\infty,/}^P$.

### 2.2.3 Construction

Let $P$ be a poset. Then the inclusion $\text{Str}_P \hookrightarrow \text{Cat}_{\infty,/}^P$ admits a left adjoint $E_P : \text{Str}_P \to \text{Cat}_{\infty,/}^P$. Indeed, if $C$ is an $\infty$-category, and $f : C \to P$ is any functor, we can formally invert those morphisms of $C$ that are sent to identities in $P$ by forming the pullback

$$E_P(C) := E(C) \times_{E(P)} P$$

in $\text{Cat}_{\infty}$. Note that the second projection $E_P(C) \to P$ is conservative: for each $p \in P$ we have

$$E_P(C) \times_{P} \{p\} = E(C) \times_{E(P)} \{p\},$$

so that $E_P(C) \times_{P} \{p\}$ is the fiber of a map between $\infty$-groupoids. We regard $E_P(C)$ as a $P$-stratified space via the second projection $E_P(C) \to P$.

The functor $f : C \to P$ and the unit $C \to E_P(C)$ induce a natural functor $C \to E_P(C)$.

Now we are ready to describe the functionality of the construction $P \mapsto \text{Str}_P$.

### 2.2.4 Proposition

The target functor $t : \text{Str} \to \text{Pos}$ is a bicartesian fibration.

**Proof.** Let $\phi : P' \to P$ be a morphism of posets. Since the pullback of a conservative functor is conservative, the pullback functor

$$\phi^* := (-) \times_P P' : \text{Cat}_{\infty,/}^{P'} \to \text{Cat}_{\infty,/}^P$$

carries $P$-stratified spaces to $P'$-stratified spaces. The pullback functor $\phi^* : \text{Str}_P \to \text{Str}_{P'}$ admits a left adjoint $\phi_!$ given by the composite

$$\text{Str}_{P'} \to \text{Cat}_{\infty,/}^{P'} \xrightarrow{E_{P'}} \text{Str}_P$$

of postcomposition with $\phi$ followed by the left adjoint to the inclusion $\text{Str}_P \subset \text{Cat}_{\infty,/}^P$.

### 2.2.5 Proposition

The target functor $t : \text{Str} \to \text{Pos}$ is a bicartesian fibration.
is a t-cocartesian morphism of Str lying over φ.

We can use the pullbacks to describe how to compute limits in Str:

2.2.6. To compute the limit of a diagram \( \alpha \mapsto [\Pi_\alpha \to P_\alpha] \) in Str, we first form the limit \( P := \lim_\alpha P_\alpha \); then pulling back along the various projections \( p_\alpha : P \to P_\alpha \), we obtain the diagram \( \alpha \mapsto p_\alpha^* \Pi_\alpha \) of P-stratified spaces. We then form the limit \( \Pi := \lim_\alpha \Pi_\alpha \) in \( \text{Str}_P \). If the diagram is connected, then the limit \( \lim_\alpha \Pi_\alpha \) is computed in \( \text{Cat}_{\infty} \).

2.3 The stratified Postnikov tower

In this section we investigate a Postnikov tower for stratified spaces. Importantly, the correct notion of ‘\( n \)-truncatedness’ is not the notion of \( n \)-truncatedness internal to the \( \infty \)-category \( \text{Str}_P \) (in the sense of [HTT, §5.5.6]); rather it corresponds to the categorical level of the stratified space.\(^{12}\) For this, recall that we write \( h_n : \text{Cat}_\infty \to \text{Cat}_n \) for the left adjoint to the inclusion \( \text{Cat}_n \subset \text{Cat}_\infty \) (0.2.2).

2.3.1 Definition. Let \( P \) be a poset and \( \Pi \) a \( P \)-stratified space. We call the tower of \( P \)-stratified spaces

\[
\Pi \to \cdots \to h_3 \Pi \to h_2 \Pi \to h_1 \Pi \to h_0 \Pi \to P,
\]

the stratified Postnikov tower of \( \Pi \).

In particular, please observe that \( h_0 \Pi \to P \) is a morphism of posets.

2.3.2. If \( P = \{0\} \), then the stratified Postnikov tower coincides with the usual Postnikov tower of spaces.

2.3.3. The following are equivalent for a poset \( P \), a \( P \)-stratified space \( f : \Pi \to P \), and a nonnegative integer \( n \in \mathbb{N} \):

- the \( \infty \)-category \( \Pi \) is an \( n \)-category;
- the natural functor \( \Pi \to h_n \Pi \) is an equivalence;
- for all objects \( x, y \in \Pi \), the space \( \text{Map}_{\Pi}(x, y) \) is \((n-1)\)-truncated;
- for all points \( p, q \in P \) such that \( p \preceq q \), the source and target map

\[
(s, t) : N_p(\Pi) \{p \preceq q\} \to \Pi_p \times \Pi_q
\]

is \((n-1)\)-truncated (in particular, when \( p = q \), the stratum \( \Pi_p \) is \( n \)-truncated).

2.3.4 Definition. Let \( P \) be a poset and \( n \in \mathbb{N} \). We say that a \( P \)-stratified space \( \Pi \) is \( n \)-truncated if \( \Pi \) satisfies the equivalent conditions of (2.3.3). We write \( \text{Str}_{P,\leq n} \subset \text{Str}_P \) for the full subcategory spanned by the \( n \)-truncated \( P \)-stratified spaces.

We caution that an \( n \)-truncated \( P \)-stratified space is generally not the same thing as an \( n \)-truncated object of the \( \infty \)-category \( \text{Str}_P \) in the sense of [HTT, Definition 5.5.6.1]. Nor is it the same thing as a \( P \)-stratified space whose strata are \( n \)-truncated; truncatedness in our sense involves a condition on the links as well.

\(^{12}\) Recall that an \( n \)-category is an \( n \)-truncated object of \( \text{Cat}_\infty \), but the converse is false.

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2.3.5. Dually, the following are equivalent for a poset $P$, a $P$-stratified space $f : \Pi \to P$, and a nonnegative integer $n \in \mathbb{N}$:

- the natural functor $h_n \Pi \to P$ is an equivalence;
- for all objects $x, y \in \Pi$ such that $f(x) \leq f(y)$, the space $\text{Map}_\Pi(x, y)$ is $n$-connective;
- for all points $p, q \in P$ such that $p \leq q$, the map
  \[(s, t) : N_{h_0(\Pi)}[p \leq q] \to \Pi_p \times \Pi_q\]
  is $n$-connective (in particular, when $p = q$, the stratum $\Pi_p$ is $(n + 1)$-connective).

2.3.6 Definition. Let $P$ be a poset and $n \in \mathbb{N}$. We say that a $P$-stratified space $\Pi$ is $n$-connective if $\Pi$ satisfies the equivalent conditions of (2.3.5). We write $\text{Str}_P, \geq n \subset \text{Str}_P$ for the full subcategory spanned by the $n$-connective $P$-stratified spaces. We can easily identify the $0$-connective stratified spaces.

2.3.7 Definition. We say that a $1$-category $C$ is layered\textsuperscript{13} if and only if every endomorphism of an object of $C$ is an isomorphism. We say that an $\infty$-category $\Pi$ is layered if and only if its homotopy category $h_1(\Pi)$ is a layered $1$-category. This holds if and only if the natural functor $\Pi \to h_0(\Pi)$ is conservative. Thus a layered $\infty$-category $\Pi$ is naturally an $h_0(\Pi)$-stratified space.

We write $\text{Lay}_\infty$ for the full subcategory of $\text{Cat}_\infty$ spanned by the layered $\infty$-categories.

2.3.8. The assignment $[\Pi \to P] \mapsto \Pi$ defines a functor $\text{Str} \to \text{Lay}_\infty$ with a fully faithful left adjoint that carries $\Pi$ to the $h_0(\Pi)$-stratified space $\Pi$. Consequently, we obtain an identification

\[\text{Lay}_\infty = \text{Str}_{\geq 0}.\]

Here $\text{Str}_{\geq 0} \subset \text{Str}$ is the full subcategory spanned by the $0$-connective stratified spaces.

2.4 $\pi$-finite stratified spaces

In this section we introduce the key finiteness condition that we impose on almost all of the stratified spaces we consider in this book. This finiteness condition is the stratified version of $\pi$-finiteness for spaces.

2.4.1 Recollection ([SAG, Definition E.0.7.8]). An $\infty$-groupoid $K$ is $\pi$-finite if and only if the following conditions are satisfied.

- The set $\pi_0(K)$ is finite.
- For every point $x \in K$ and any $i \geq 1$, the group $\pi_i(K, x)$ is finite.
- The $\infty$-groupoid $K$ is $n$-truncated for some $n \in \mathbb{N}$.

We write $S_n \subset S$ for the full subcategory spanned by the $\pi$-finite $\infty$-groupoids.

\textsuperscript{13}Layered categories are often called EI categories.
2.4.2 Warning. We caution that a \( \pi \)-finite space is not the same thing as what is normally called a \textit{finite space} – one obtained via finite colimits from the point. In fact, the overlap between these two classes of spaces is essentially trivial: the spaces satisfying both of these conditions are exactly the discrete spaces with finitely many connected components.

We now define the analogous condition for a \textit{stratified space}.

2.4.3 Definition. We say that a stratified space \( \Pi \to P \) is \( \pi \)-finite if and only if the following conditions are satisfied.

\begin{itemize}
    \item The poset \( P \) is finite.
    \item For every point \( p \in P \), the set \( \pi_0(\Pi_p) \) is finite.
    \item For all \( x, y \in \Pi \), the mapping space \( \text{Map}_\Pi(x, y) \) is a \( \pi \)-finite space.
    \item The \( \infty \)-category \( \Pi \) is an \( n \)-category for some \( n \in \mathbb{N} \).
\end{itemize}

In particular, a nondegenerate stratified space \( \Pi \to P \) is \( \pi \)-finite if and only if \( \Pi \) has finitely many objects up to equivalence and is \textit{locally} \( \pi \)-finite in the sense that each mapping space \( \text{Map}_\Pi(x, y) \) is \( \pi \)-finite.

We write \( \text{Str}_\pi \subset \text{Str} \) for the full subcategory spanned by the \( \pi \)-finite stratified spaces. Given a finite poset \( P \), we write \( \text{Str}_{\pi, P} \subset \text{Str}_P \) for the full subcategory spanned by the \( \pi \)-finite \( P \)-stratified spaces.

2.4.4 The target functor \( t : \text{Str}_\pi \to \text{Pos}^\text{fin} \) is a cartesian fibration. However, it is not a cocartesian fibration because the pullback functor doesn’t admit a left adjoint when restricted to \( \pi \)-finite stratified spaces. To see this, note that the free pair of parallel arrows \( 0 \Rightarrow 1 \) is \( \pi \)-finite as a \([1]\)-stratified space, but its classifying space is equivalent to \( B\mathbb{Z} \), which is not \( \pi \)-finite.

However, the pullback does preserve finite limits, hence has a proëxistent left adjoint; we will discuss this in §2.5.

2.4.5 Lemma. The full subcategory \( \text{Str}_\pi \subset \text{Str} \) is an accessible subcategory that is closed under finite limits.

\textit{Proof.} Finite limits of finite posets are finite, pullbacks of \( \pi \)-finite stratified spaces along maps of finite posets are \( \pi \)-finite, and finite limits of locally \( \pi \)-finite \( \infty \)-categories are locally \( \pi \)-finite. Finally, \( \text{Str}_\pi \) is \( \delta_0 \)-small and idempotent complete. \( \square \)

2.5 Profinite stratified spaces

In light of \((0.3.5)\), Lemma 2.4.5 allows us to speak of profinite stratified spaces in terms of left exact accessible functors. We now turn to stratifications over profinite posets (i.e., spectral topological spaces).

2.5.1 Definition. We call objects of the \( \infty \)-category \( \text{Pro}(\text{Str}) \) \textit{stratified prospaces}; the target functor \( t : \text{Str} \to \text{Pos} \) from stratified spaces to posets extends to a target functor

\[ t : \text{Pro}(\text{Str}) \to \text{Pro}(\text{Pos}) . \]
Let \( P \) be a poset, regarded as a constant proposet. The fibre \( \text{Pro}(\text{Str})_P \) of \( t \) over \( P \) can be identified with the \( \infty \)-category \( \text{Pro}(\text{Str}_P) \) of proobjects in \( \text{Str}_P \). We refer to objects of \( \text{Pro}(\text{Str})_P \) as \( P \)-stratified prospace.

Similarly, if \( P \) is a proposet, then we refer to the fibre \( \text{Pro}(\text{Str})_P \) of \( t \) over \( P \) as the \( \infty \)-category of \( P \)-stratified prospace.

2.5.2. A stratified prospace can be exhibited as an inverse system \( \{P_a \to P_{a+1}\}_{a \in A} \) of stratified spaces. The target functor \( \text{t} \) carries this stratified prospace to the proposet \( \eta \). Note in particular that if \( \eta \) is constant. Regarding \( \text{t} \), it is occasionally necessary to deal with more general stratified prospace.

We now turn to the functoriality of the assignment \( P \mapsto \text{Pro}(\text{Str})_P \). The key point is that the adjunctions relating the \( \infty \)-categories \( \text{Str}_P \) as \( P \) varies extend to stratified prospace.

2.5.4 Construction (pullbacks). Please observe that the target functor

\[
\text{t} : \text{Pro}(\text{Str}) \to \text{Pro}(\text{Pos})
\]

is a cartesian fibration. Indeed, let \( \phi : P' \to P \) be a morphism of proposets, and exhibit \( P' \) and \( P \) as inverse systems of proposets \( \{P'_a\}_{a \in A} \) and \( \{P_a\}_{a \in A} \), respectively. Given a \( P \)-stratified prospace \( \Pi = \{P_a \to P_{a+1}\}_{a \in A} \), we define a \( P' \)-stratified prospace \( \phi^*(\Pi) \) as the inverse system

\[
\phi^*(\Pi) = \{P_a \times_{P_{a+1}} P'_b \to P'_b\}_{a \in A}.
\]

The morphism \( \phi^*(\Pi) \to \Pi \) is a \( t \)-cartesian morphism lying over \( \phi \) and the assignment \( \Pi \mapsto \phi^*(\Pi) \) defines a functor \( \phi^* : \text{Pro}(\text{Str})_P \to \text{Pro}(\text{Str})_{P'} \).

We now describe the left adjoint to this functor.

2.5.5 Construction (pushforwards). Let \( \eta : P' \to P \) a morphism of proposets where \( P \) is constant. Regarding \( P' \) as a left exact accessible functor \( \text{Pos} \to \text{Set} \), the morphism \( \eta \) defines an element \( \eta \in \text{Pos}^1(P) \). For a \( P \)-stratified prospace \( \Pi' \), there exists a \( t \)-cocartesian edge \( \Pi' \to \eta \Pi' \) covering \( \eta \); indeed, for any \( P \)-stratified space \( \Pi \), we have an equivalence

\[
(\eta_\Pi')(\Pi) \simeq \eta_\Pi' \times_{P^1(P)} [\eta].
\]

Equivalently, if we exhibit \( \Pi' \) as an inverse system \( \{\Pi'_a \to P'_{a+1}\}_{a \in A} \) in \( \text{Str} \), then the \( P \)-stratified prospace \( \eta_\Pi' \) can be exhibited as the inverse system \( A \times_{\text{Pos}_Q} \text{Pos}_{P'} \to \text{Str}_P \) given by

\[
(\alpha, P'_a \to P) \mapsto E_\Pi(\Pi'_a).
\]

Note in particular that if \( P' \) and \( \Pi' \) are constant, then so is \( \eta_\Pi' \).

In the \( \infty \)-category \( \text{Pro}(\text{Str}) \), the inverse system \( \text{Pos}_{P'} \to \text{Str} \) given by \( \eta \mapsto \eta_\Pi' \) is identified with \( \Pi' \) itself.

Given any morphism of proposets \( \phi : P' \to P \) and \( P' \)-stratified prospace \( \Pi' \), the inverse system \( \text{Pos}_{P'} \to \text{Str} \) given by the assignment

\[
\eta \mapsto (\eta \circ \phi)_\Pi'
\]
defines a $P$-stratified prospace $\phi^!\Pi'$. As this notation suggests, the morphism $\Pi' \to \phi^!\Pi'$ is a t-cartesian edge over $\phi$. Thus $t : \text{Pro}(\text{Str}) \to \text{Pro}(\text{Pos})$ is a cocartesian fibration.

We thus combine the previous two points:

2.5.6 Proposition. The target functor $t : \text{Pro}(\text{Str}) \to \text{Pro}(\text{Pos})$ is a bicartesian fibration.

We now turn to proobjects in $\pi$-finite stratified spaces.

2.5.7 Definition. A profinite stratified space is a proobject of the $\infty$-category $\text{Str}_{\pi}$. We write $\text{Str}^\wedge_{\pi} \coloneqq \text{Pro}(\text{Str}_{\pi})$ for the $\infty$-category of profinite stratified spaces.

The target functor $t : \text{Str}^\wedge_{\pi} \to \text{Pro}(\text{Pos})_{\text{fin}} \cong \text{TSp}(\text{spec})$ is a cartesian fibration. Given a spectral topological space $S$, we write $\text{Str}^\wedge_{\pi,S}$ for the fibre of $t$ over $S$. We call $\text{Str}^\wedge_{\pi,S}$ the profinite completion of $\Pi$.

2.5.8. The profinite completion functor $\Pi \mapsto \Pi^\wedge_{\pi}$ is not a relative left adjunction over $\text{Pro}(\text{Pos})$; however, the inclusion $\text{Str}_{\pi} \hookrightarrow \text{Str}$ extends to a fully faithful functor $\text{Str}^\wedge_{\pi} \hookrightarrow \text{Pro}(\text{Str}) \times_{\text{Pro}(\text{Pos})} \text{TSp}(\text{spec})$, and profinite completion does define a relative left adjunction over $\text{TSp}(\text{spec})$. In particular, if $S$ is a spectral topological space and $\Pi$ is an $S$-stratified space, then $\Pi^\wedge_{\pi}$ is a profinite $S$-stratified space, and the morphism $\Pi \to \Pi^\wedge_{\pi}$ lies over $S$.

2.5.9 Construction (pushforward). Let $\phi : S' \to S$ be a quasicompact continuous map of spectral topological spaces, and let $\Pi' \to S'$ be a profinite $S'$-stratified space. Then following Construction 2.5.5, we obtain an $S$-stratified prospace $\phi^!\Pi'$ over $S$. Forming the profinite completion $(\phi^!\Pi')^\wedge_{\pi} \to S$ of $\phi^!\Pi'$, we see that the map $\Pi' \to (\phi^!\Pi')^\wedge_{\pi}$ is a cocartesian edge over $\phi$ for the target functor $t : \text{Str}^\wedge_{\pi} \to \text{TSp}(\text{spec})$.

We thus obtain:

2.5.10 Proposition. The target functor $t : \text{Str}^\wedge_{\pi} \to \text{TSp}(\text{spec})$ is a bicartesian fibration.

2.5.11 Proposition. Let $S$ be a spectral topological space. Then the natural functor

$$\text{Str}^\wedge_{\pi,S} \to \lim_{P \in FC(S)} \text{Str}^\wedge_{\pi,P}$$

is an equivalence.

Proof. The formation of the limit in $\text{Str}^\wedge_{\pi,S}$ is an inverse. □
2.6 Complete Segal spaces & spatial décollages

2.6.1 Recollection. An ∞-category can be modeled as a simplicial space. In effect, if $C$ is an ∞-category, then one may extract a functor $N(C) : \Delta^\text{op} \to S$ in which $N(C)_m$ is the ∞-groupoid of functors $[m] \to C$ (the ‘moduli space of sequences of arrows in $C$’). The simplicial space $N(C)$ is what Charles Rezk \[98\] called a complete Segal space – i.e., a functor $D : \Delta^\text{op} \to S$ satisfying the following conditions.

- For all $m \in \mathbb{N}$, the natural map
  
  \[ D_m \to D[0 < 1] \times_{D[1]} D[1 < 2] \times_{D[2]} \cdots \times_{D[m-1]} D[m-1 < m] \]

  is an equivalence.

- Let $I$ denote the unique contractible 1-groupoid with two objects. Then the map
  \[ D_0 \to \text{Map}_{\text{Fun}(\Delta^\text{op}, S)}(N(I), D) \]

  induced by the projection $I \to \{0\}$ is an equivalence.

(Joyal and Tierney \[68\] showed that the assignment $C \mapsto N(C)$ defines an equivalence from the ∞-category $\text{Cat}_{\infty}$ of ∞-categories to the ∞-category CSS of complete Segal spaces.

We can isolate the ∞-groupoids in CSS: an ∞-category $C$ is an ∞-groupoid if and only if $N(C) : \Delta^\text{op} \to S$ is left Kan-extended from $\{0\} \subset \Delta^\text{op}$.

In the remainder of this section and the next, we shall demonstrate that the homotopy theory of stratified spaces admits an analogous description.

2.6.2 Notation. For a poset $P$, we write $\text{sd}^\text{op}(P) := \text{sd}(P)^{\text{op}}$.

2.6.3 Definition. Let $P$ be a poset. A functor $D : \text{sd}^\text{op}(P) \to S$ is a spatial décollage (over $P$) if and only if, for every string $\{p_0 < \cdots < p_m\} \subseteq P$, the map

\[ D[p_0 < \cdots < p_m] \to D[p_0 < p_1] \times_{D[p_1]} D[p_1 < p_2] \times_{D[p_2]} \cdots \times_{D[p_{m-1}]} D[p_{m-1} < p_m] \]

is an equivalence. We write

\[ \text{Déc}_P(S) \subseteq \text{Fun}(\text{sd}^\text{op}(P), S) \]

for the full subcategory spanned by the spatial décollages.

2.6.4 Example. Let $P$ be a poset of rank $\leq 1$. Then every functor $\text{sd}^\text{op}(P) \to S$ automatically satisfies the décollage condition. So in this case,

\[ \text{Déc}_P(S) = \text{Fun}(\text{sd}^\text{op}(P), S) \]

Now we turn to the functoriality of the assignment $P \mapsto \text{Déc}_P(S)$.
2.6.5 Construction. We write $\int \mathrm{sd}^{\text{op}}$ for the 1-category given by the Grothendieck construction of the functor

$$\mathrm{sd}^{\text{op}} : \text{Pos} \to \text{Pos} \subset \text{Cat}_1$$

The 1-category $\int \mathrm{sd}^{\text{op}}$ has objects pairs $(P, \Sigma)$ consisting of a poset $P$ and a string $\Sigma \subseteq P$. A morphism $(P, \Sigma) \to (Q, T)$ in $\int \mathrm{sd}^{\text{op}}$ is a morphism of posets $f : P \to Q$ such that $T \subseteq f(\Sigma)$. The assignment $(P, \Sigma) \mapsto P$ is a cocartesian fibration

$$\int \mathrm{sd}^{\text{op}} \to \text{Pos}$$

whose fibre over a poset $P$ is the poset $\mathrm{sd}^{\text{op}}(P)$.

We write $\text{Pair}_{\text{Pos}}(\int \mathrm{sd}^{\text{op}}, S)$ for the simplicial set over Pos defined by the following universal property: for any simplicial set $K$ over Pos, there is a bijection

$$\text{Mor}_{\text{Set}}(K, \text{Pair}_{\text{Pos}}(\int \mathrm{sd}^{\text{op}}, S)) \cong \text{Mor}_{\text{Set}}(K \times_{\text{Pos}} \int \mathrm{sd}^{\text{op}}, S)$$

natural in $K$. By [HTT, Corollary 3.2.2.13], the functor

$$\text{Pair}_{\text{Pos}}(\int \mathrm{sd}^{\text{op}}, S) \to \text{Pos}$$

is a cartesian fibration whose fibre over a poset $P$ is the $\infty$-category $\text{Fun}(\mathrm{sd}^{\text{op}}(P), S)$. Now let

$$\text{Déc}(S) \subset \text{Pair}_{\text{Pos}}(\int \mathrm{sd}^{\text{op}}, S)$$

denote the full subcategory spanned by the pairs $(P, D)$ in which $D$ is a spatial décollage. Since $\text{Déc}(S)$ contains all the cartesian edges, the functor $\text{Déc}(S) \to \text{Pos}$ is a cartesian fibration.

2.7 The nerve of a stratified space

We now show that the $\infty$-category $\text{Str}$ of stratified spaces and the $\infty$-category $\text{Déc}(S)$ of décollages are equivalent over $\text{Pos}$.

2.7.1 Construction (nerve of a stratified space). Let $P$ be a poset. Any string contained in $P$ can be regarded as a $P$-stratified space via the inclusion map. This assignment defines a functor $\mathrm{sd}(P) \to \text{Str}_P$. For any $P$-stratified space $\Pi$, we define the nerve of $\Pi$ to be the functor

$$N_p(\Pi) : \mathrm{sd}^{\text{op}}(P) \to S$$

given by the assignment $\Sigma \mapsto \text{Map}_P(\Sigma, \Pi)$. (This is the moduli space of sections over $\Sigma$.) An equivalence of $P$-stratified spaces is carried to an objectwise equivalence of functors; hence the nerve defines a functor

$$N : \text{Str}_P \to \text{Fun}(\mathrm{sd}^{\text{op}}(P), S)$$

Furthermore, the assignment $[\Pi \to P] \mapsto (P, N_p(\Pi))$ defines a functor

$$N : \text{Str} \to \text{Pair}_{\text{Pos}}(\int \mathrm{sd}^{\text{op}}, S)$$
2.7.2 Example. For any poset $P$, $P$-stratified space $\mathcal{P}$, and points $p, q \in P$ such that $p \leq q$, the space
\[ N_p(\mathcal{P})\{p \leq q\} = \text{Map}_P(\{p \leq q\}, \mathcal{P}) \]
is the link between the $p$-th and $q$-th strata of $\mathcal{P}$.

Let us demonstrate that the functor $N$ lands in the full subcategory
\[ \text{Déc}(S) \subset \text{Pair}_{\text{Pos}}(\int \text{sd}^{op}, S). \]

2.7.3 Lemma. For any poset $P$ and $P$-stratified space $\mathcal{P}$, the functor $N$ lands in the full subcategory $\text{Déc}(S) \subset \text{Pair}_{\text{Pos}}(\int \text{sd}^{op}, S)$.

Proof. In $\text{Cat}_{\infty, P}$, for any string $\{p_0 < \cdots < p_n\} \subset P$, there is an equivalence
\[ \{p_0 < p_1\} \cup \cdots \cup \{p_{n-1} < p_n\} \Rightarrow \{p_0 < \cdots < p_n\}, \]
which induces an equivalence
\[ \text{Map}_P(\{p_0 < \cdots < p_n\}, \mathcal{P}) \Rightarrow \text{Map}_P(\{p_0 < p_1\}, \mathcal{P}) \times \cdots \times \text{Map}_P(\{p_{n-1} < p_n\}, \mathcal{P}), \]
as desired. \qed

2.7.4 Theorem. The functor $N : \text{Str} \to \text{Déc}(S)$ is an equivalence of $\infty$-categories over $\text{Pos}$.

Proof. Let $P$ be a poset and write $\Delta_{/P}$ for the category of simplices of $P$. The Joyal–Tierney theorem \[68\] implies that the functor
\[ N : \text{Cat}_{\infty, P} \to \text{Fun}(\Delta_{/P}, S)_{N(P)} = \text{Fun}(\Delta_{/P}^{op}, S), \]
\[ C \mapsto [\Sigma \mapsto \text{Fun}_P(\Sigma, C)] \]
is fully faithful, and the essential image $\text{CSS}_{N(P)}$ consists of those functors $\Delta_{/P}^{op} \to S$ that satisfy both the Segal condition and the completeness condition. Now notice that left Kan extension along the inclusion $\text{sd}^{op}(P) \hookrightarrow \Delta_{/P}^{op}$ defines a fully faithful functor $\text{Déc}_P(S) \hookrightarrow \text{CSS}_{N(P)}$ whose essential image consists of those complete Segal spaces $C \to N(P)$ such that for any $p \in P$, the complete Segal space $C_p$ is an $\infty$-groupoid. \qed

2.7.5. Write $\text{Déc}(S) \subset \text{Dec}(S)$ for the full subcategory spanned by those pairs $(P, D)$ where $P$ is a finite poset and $D$ is a spatial décollage over $P$ whose values are all $\pi$-finite. Then nerve restricts to an equivalence of $\infty$-categories $N : \text{Str}_\pi \Rightarrow \text{Déc}(S)$.

2.8 Profinite spatial décollages

In this section we extend the theory of décollages to proobjects and explain how to understand profinite $P$-stratified spaces in terms of décollages valued in the $\infty$-category of profinite spaces (Construction 2.8.8).
2.8.1. We extend the nerve $N : \text{Str} \to \text{Déc}(S)$ to pro-objects to obtain an equivalence of $\infty$-categories

$$N : \text{Pro}(\text{Str}) \Rightarrow \text{Pro}(\text{Déc}(S))$$

over $\text{Pro}(\text{Pos})$.

In order to understand profinite décollages, we recall some basic facts about profinite spaces and the product in the $\infty$-category of profinite spaces.

2.8.2 Recollection. We write $S^\wedge_\pi \coloneqq \text{Pro}(S^\pi)$ for the $\infty$-category of profinite spaces. We regard the $S^\wedge_\pi$ as a full subcategory of the $\infty$-category $\text{Pro}(S)$. Precomposition with the inclusion $S^\pi \hookrightarrow S$ defines a functor $(-)^\wedge_\pi : \text{Pro}(S) \to S^\wedge_\pi$ that exhibits $S^\wedge_\pi$ as a localization of $\text{Pro}(S)$. Given a prospace $X$, we call $X^\wedge_\pi$ the profinite completion of $X$.

2.8.3 Recollection ([SAG, Remark E.2.1.2]). In addition to the cartesian symmetric monoidal structure on $\text{Pro}(S)$, there is a related ‘composition’ monoidal structure: since the composition of two left exact accessible functors $S \to S$ is again left exact and accessible, the composition monoidal structure on $\text{Fun}(S,S)^{\text{op}}$ restricts to a monoidal structure $(X,Y) \mapsto X \circ Y$ on $\text{Pro}(S)$. The identity functor is both the unit for $\circ$ and terminal object of $\text{Pro}(S)$. Hence the universal property of the product provides is a comparison morphism

$$c_{X,Y} : X \circ Y \to X \times Y$$

that is natural in $X$ and $Y$. However, this morphism is not an equivalence in general.

Since the subcategory $S^\wedge_\pi \subset \text{Pro}(S)$ is closed under products, if $X, Y \in S^\wedge_\pi$, then the morphism $X \circ Y \to X \times Y$ induces a morphism

$$(2.8.4) \quad (X \circ Y)^\wedge_\pi \to X \times Y.$$ 

We claim that the morphism (2.8.4) is an equivalence. To see this, we show that the comparison morphism $X \circ Y \to X \times Y$ becomes an equivalence after evaluation at any truncated space.\textsuperscript{14}

2.8.5 Lemma. Let $X$ be a profinite space and $Y$ a prospace. Then for every truncated space $K$, the morphism

$$c_{X,Y} : (X \circ Y)(K) \to (X \times Y)(K)$$

is an equivalence in $S^{\text{op}}$.

Proof. Exhibit $X$ an inverse system $\{X_\alpha\}_{\alpha \in A}$ of $\pi$-finite spaces and $Y$ an inverse system $\{Y_\beta\}_{\beta \in B}$ of spaces. For each $\alpha \in A$, the fact that the space $X_\alpha$ is $\pi$-finite for implies that the functor corepresented by $X_\alpha$ preserves colimits of filtered diagrams of uniformly truncated spaces [SAG, Corollary A.2.3.2]. Since the space $K$ is truncated, the filtered diagram

$$\beta \mapsto \text{Map}_S(Y_\beta, K)$$

\textsuperscript{14}We are grateful to Jacob Lurie for this observation.
is uniformly truncated. Hence we have the following equivalences in $S$:

$$(X \times Y)(K) = \colim_{(\alpha, \beta) \in A^p \times B^p} \Map_S(X_\alpha \times Y_\beta, K)$$

$$= \colim_{(\alpha, \beta) \in A^p \times B^p} \Map_S(X_\alpha, \Map_S(Y_\beta, K))$$

$$\Rightarrow \colim_{a \in A^p} \Map_S(X_a, \colim_{\beta \in B^p} \Map_S(Y_\beta, K))$$

$$= (X \circ Y)(K) .$$

2.8.6 Corollary. Let $X$ and $Y$ be profinite spaces. Then the natural morphism

$$(X \circ Y)_n^\wedge \to X \times Y$$

in is an equivalence in $S_n^\wedge$.

2.8.7. Corollary 2.8.6 is helpful for describing fibre products in $S_n^\wedge$ as well: given morphisms of profinite spaces $p : X \to Z$ and $q : Y \to Z$, the pullback $X \times_Z Y$ of $p$ along $q$ is given by a cobar construction:

$$X \times_Z Y = \lim_{m \in \Delta} (X \circ Z^{\leq m} \circ Y)_n^\wedge .$$

We are now ready to describe $\Str_{n, p}$ in terms of profinite décollages.

2.8.8 Construction (profinite décollages). For any finite poset $P$, write $D\text{éc}_P(S_n^\wedge)$ for the full subcategory of $\Fun(sd^{op}(P), S_n^\wedge)$ spanned by those functors

$$D : sd^{op}(P) \to S_n^\wedge$$

such that for any string $\{p_0 < \cdots < p_n\} \subseteq P$, the natural map

$$D[p_0 < \cdots < p_n] \to \lim_{m \in \Delta} (D[p_0 < p_1] \circ D[p_1]^{\leq m} \cdots \circ D[p_{n-1}]^{\leq m} \circ D[p_{n-1} < p_n])_n^\wedge$$

is an equivalence of profinite spaces. We call objects of $D\text{éc}_P(S_n^\wedge)$ profinite décollages over $P$.

Combining the equivalence

$$\text{Pro}(\Fun(sd^{op}(P), S_n^\wedge)) \Rightarrow \Fun(sd^{op}(P), S_n^\wedge)$$

furnished by [HTT, Proposition 5.3.5.15] with the equivalences (2.7.5) and (2.8.1), we obtain equivalences of $\infty$-categories

$$(2.8.9) \quad \Str_{n, p} \Rightarrow \text{Pro}(D\text{éc}_P(S_n)) \Rightarrow D\text{éc}_P(S_n^\wedge) .$$

Specifically, the composite equivalence (2.8.9) is given by extending the nerve

$$N_p : \Str_{n, p} \Rightarrow D\text{éc}_P(S_n) \subseteq D\text{éc}_P(S_n^\wedge)$$

along inverse limits.
Part II

Elements of higher topos theory

In this part we develop the higher-toposic tools that we'll need to state and prove our ∞-Categorical Hochster Duality Theorem (Theorem E=Theorem 9.3.1). In Chapter 3 we recall a number of important results from higher topos theory and develop the basic frameworks of (bounded) coherent ∞-topoi and (bounded) ∞-pretopoi. The theories of coherent ∞-topoi and bounded ∞-pretopoi will be used heavily in the remainder of the text. In Chapter 4, we describe the basic theorems of shape theory for ∞-topoi; the generalization of shape theory to stratified ∞-topoi forms the foundation of our work. Chapter 5 develops the basics of Deligne’s oriented fibre product; this plays a fundamental role in our approach to stratified higher topos theory in Part III. In Chapter 6 we introduce the analogue of local rings in the context of higher topos theory. In Chapter 7 we generalize work of Moerdijk–Vermeulen and Illusie–Gabber by proving a key base-change theorem for oriented fiber products of bounded coherent ∞-topoi. As with the proof of the proper basechange theorem in algebraic geometry, reduction to the local case plays a key role in our proof.
3 Aide-mémoire on higher topoi

In this chapter we recall a number of important results from higher topos theory (mostly from Jacob Lurie’s [SAG, Appendices A & E]), and develop some basic results that we’ll use throughout the rest of the text. This chapter is here mostly for ease of reference; much of the material is expository, and the original material mostly serves to fill small gaps in the existing foundations.

Section 3.1 sets our notational conventions for $\infty$-topoi. Section 3.2 introduces the first of two finiteness conditions that we impose on almost all of the $\infty$-topoi we consider in this text: boundedness. Section 3.3 introduces the second finiteness condition: coherence. Section 3.4 studies the relationship between coherence and $n$-topoi. Section 3.5 provides a convenient reformulation of coherence for $n$-localic $\infty$-topoi. Section 3.6 shows that a morphism of finitary $\infty$-sites induces a coherent geometric morphism on corresponding $\infty$-topoi; along with the material from §3.5, this implies that the theory of 1-localic coherent $\infty$-topoi recovers Grothendieck’s theory of coherent ordinary topoi [SGA 4, Exposé VI, Definition 2.3]. Section 3.7 uses the material from §3.6 to provide examples of coherent $\infty$-topoi and geometric morphisms coming from algebraic geometry. Section 3.8 explains how bounded coherent $\infty$-topoi are classified by their truncated coherent objects. This perspective will be used extensively throughout our work. Section 3.9 recalls the fact that inverse limits of bounded coherent $\infty$-topoi are again bounded coherent. Section 3.10 shows that the pushforward in a coherent geometric morphism commutes with filtered colimits of uniformly truncated diagrams. Section 3.11 discusses points of $\infty$-topoi and hypercompletness, Deligne Completeness, and Lurie’s Conceptual Completeness Theorem for bounded coherent $\infty$-topoi. Section 3.12 explains how bases for Grothendieck topologies work for $\infty$-topoi (which is more subtle than for ordinary topos); the key example that we need is that the $\infty$-topos of sheaves on a finite poset $P$ is the functor $\infty$-category $\text{Fun}(P, S)$.

3.1 Higher topoi

We begin by setting our basic notational conventions for higher topoi.

3.1.1 Notation. We use here the theory of $n$-topoi for $n \in \mathbb{N}$; see [HTT, Chapter 6]. We write $\text{Top}_n \subset \text{Cat}_{\text{st}, \delta}$ for the subcategory of $\delta$-small $n$-topoi and geometric morphisms. All of the examples in this paper will have $n \in \{0, 1, \infty\}$.

For any $\delta$-small $\infty$-category $C$, we write $\text{PSh}(C) = \text{Fun}(C^{op}, S)$ for the $\infty$-topos of presheaves of spaces on $C$.

3.1.2 Example. Recall that 0-topoi are (essentially $\delta$-small) locales [HTT, Proposition 6.4.2.5], and 1-topoi are topoi in the classical sense of Grothendieck [HTT, Remark 6.4.1.3].

3.1.3 Example. Let $m, n \in \mathbb{N}$ with $m \leq n$. An $m$-site is a $\delta_0$-small $m$-category $X$ equipped with a Grothendieck topology $\tau$. Given an $m$-site $(X, \tau)$, we write $\text{Sh}_{\tau}(X)_{\leq (n-1)}$ for the $n$-topos of sheaves $\delta_0$-small $(n-1)$-groupoids on $X$.

It is not expected that all $\infty$-topoi are of the form $\text{Sh}_\tau(X)$ for an $\infty$-site $(X, \tau)$; however, if $n \in \mathbb{N}$, then every $n$-topos is of the form $\text{Sh}_\tau(X)_{\leq (n-1)}$ for some $n$-site $(X, \tau)$ [HTT, Theorem 6.4.1.5(1)].
3.1.4 Example. For any topological space \( W \), we write \( \hat{W} = \text{Sh}(W) \) for the 0-localic \( \infty \)-topos of sheaves of spaces on \( W \).

3.1.5 Notation. Let \( n \in \mathbb{N}^* \) and let \( X \) and \( Y \) be \( n \)-topoi. We write
\[
\text{Fun}_n(X, Y) \subseteq \text{Fun}(X, Y)
\]
for the full subcategory spanned by the geometric morphisms. We note that \( \text{Fun}_n(X, Y) \) is accessible [HTT, Proposition 6.3.1.13]. We write
\[
\text{Fun}^*(Y, X) \subseteq \text{Fun}(Y, X)
\]
for the full subcategory spanned by those functors that are left exact left adjoints, so that \( \text{Fun}^*(Y, X) \cong \text{Fun}^*(X, Y) \text{op} \).

3.1.6. The \( \infty \)-topos \( S \) of spaces is terminal in \( \text{Top}_\infty \).

3.1.7 Notation. Let \( X \) be an \( \infty \)-topos. We write \( \Gamma_X^* \) or \( \Gamma_X^* \) for the essentially unique geometric morphism \( X \to S \); the functor \( \Gamma_X^* \) is corepresented by the terminal object \( 1_X \in X \). The geometric morphism \( \Gamma_X^* \) is called the global sections geometric morphism.

3.1.8 Definition. Let \( X \) be an \( \infty \)-topos. A point of \( X \) is a geometric morphism \( x^* : S \to X \). We often write \( \hat{x} \) for this copy of \( S \), regarded as lying over \( X \) via the geometric morphism \( x^* \).

3.1.9 Recollection (étale geometric morphisms). Let \( X \) and \( Y \) be \( \infty \)-topoi. A geometric morphism \( p^* : X \to Y \) is étale if \( p^* \) admits a further left adjoint \( p_! : X \to Y \) that exhibits \( X \) as the slice \( \infty \)-topos \( Y/p_!(1_X) \). In this case \( p_! \) is identified with the forgetful functor \( Y/p_!(1_Y) \to Y \).

By [HTT, Corollary 6.3.5.6], the functor
\[
\text{Fun}_n(Z, X) \to \text{Fun}_n(Z, Y)
\]
is a right fibration whose fibre over a geometric morphism \( f^* : Z \to Y \) is the \((\delta_0, \text{small})\) \( \infty \)-groupoid \( \text{Map}_X(1_X, f^* p_!(1_X)) \).

3.1.10. If \( X \) and \( Y \) are \( \infty \)-topoi, the product \( X \times Y \) in \( \text{Top}_\infty \) is not the product of \( \infty \)-categories; rather, the \( \infty \)-topos \( X \times Y \) can be identified with the tensor product of presentable \( \infty \)-categories.\(^{15}\)

Similarly, if \( f^* : X \to Z \) and \( g^* : Y \to Z \) are geometric morphisms, then the pullback \( X \times_Z Y \) in \( \text{Top}_\infty \) exists [HTT, Proposition 6.3.4.6], but the \( \infty \)-topos \( X \times_Z Y \) is not the pullback of \( \infty \)-categories.

In Chapter 5 we also study an oriented fibre product of \( \infty \)-topoi. Again, this oriented fiber product does not coincide with the oriented fibre product of \( \infty \)-categories (0.4.1). We therefore endeavour to indicate clearly when a product, pullback, or oriented fibre product is formed in \( \text{Top}_\infty \) or \( \text{Cat}_{\infty, \delta_1} \).

We repeatedly make use of the fact that inverse limits in \( \text{Top}_\infty \) are computed in \( \text{Cat}_{\infty, \delta_1} \).

3.1.11 Theorem ([HTT, Theorem 6.3.3.1]). The forgetful functor \( \text{Top}_\infty \to \text{Cat}_{\infty, \delta_1} \) preserves inverse limits.

\(^{15}\)For this reason, Lurie writes \( X \otimes Y \) for the product in \( \text{Top}_\infty \).
3.2 Boundedness

We now turn to the first of two finiteness conditions that we impose on almost all of the \( \infty \)-topoi we consider in this book.

3.2.1 Notation. Let \( C \) be a presentable \( \infty \)-category. For each integer \( n \geq -2 \), write \( C_{\leq n} \subset C \) for the full subcategory spanned by the \( n \)-truncated objects, and \( \tau_{\leq n} : C \to C_{\leq n} \) for the \( n \)-truncation functor, which is left adjoint to the inclusion \( C_{\leq n} \subset C \) [HTT, Proposition 5.5.6.18]. Write \( C_{\leq n} \subset C \) for the full subcategory spanned by those objects which are \( n \)-truncated for some integer \( n \geq -2 \).

3.2.2 Notation. If \( m, n \in \mathbb{N} \) with \( m < n \), then passage to \((m-1)\)-truncated objects defines a functor

\[ (-)_{\leq m-1} : \text{Top}_n \to \text{Top}_m. \]

We call a \((1)\)-truncated object of an \( n \)-topos \( X \) an open in \( X \) and write \( \text{Open}(X) \equiv X_{\leq -1} \).

3.2.3 Definition. If \( m, n \in \mathbb{N} \) with \( m < n \), then the functor \((-)_{\leq m-1} : \text{Top}_n \to \text{Top}_m \) admits a fully faithful right adjoint. Write \( \text{Top}_m^{\leq n} \subset \text{Top}_n \) for the essential image of this functor; this consists of those \( n \)-topoi \( X \) such that, for every \( n \)-topos \( Y \), the functor

\[ \text{Fun}^* (Y, X) \to \text{Fun}^* (Y_{\leq m-1}, X_{\leq m-1}) \]

is an equivalence. We call such \( n \)-topoi \( m \)-localic [HTT, §6.4.5].

The inclusion \( \text{Top}_{\infty}^{n} \subset \text{Top}_{\infty} \) of the full subcategory of \( n \)-localic \( \infty \)-topoi admits a left adjoint

\[ L_n : \text{Top}_{\infty} \to \text{Top}_{\infty}^{n} \]

called \( n \)-localic reflection.

3.2.4. If \( n \in \mathbb{N} \), then the proof of [HTT, Proposition 6.4.5.9] demonstrates that an \( \infty \)-topos \( X \) is \( n \)-localic if and only if \( X = \text{Sh}_{\tau}(X) \), where \( (X, \tau) \) is a \( \delta_0 \)-small \( n \)-site with all finite limits.

3.2.5 Example. If \( W \) is a topological space, then the \( \infty \)-topos \( \mathcal{W} \) of sheaves on \( W \) is 0-localic.

3.2.6 Warning. If \( (X, \tau) \) is an \( n \)-site and the \( n \)-category \( X \) does not have finite limits, then the \( \infty \)-topos \( \text{Sh}_{\tau}(X) \) is not generally \( N \)-localic for any \( N \geq 0 \). See [SAG, Counterexample 20.4.0.1] for a basis \( B \) for the topology on the Hilbert cube \( \prod_{i \in \mathbb{Z}} [0, 1] \) for which the \( \infty \)-topos of sheaves on \( B \) is not \( N \)-localic for any \( N \geq 0 \).

3.2.7 Example. If \( X \) is a scheme, then the \( \infty \)-topos \( X_{\text{et}} \) of étale sheaves on the 1-site of étale \( X \)-schemes is 1-localic.

3.2.8 Example. Let \( n \in \mathbb{N} \) and let \( X \) be an \( n \)-localic \( \infty \)-topos. Then [SAG, Lemma 1.4.7.7] demonstrates that for an object \( U \in X \), the over \( \infty \)-topos \( X_{/U} \) is \( n \)-localic if and only if \( U \) is \( n \)-truncated.
3.2.9 Definition. We write \( \text{Top}_\infty^\wedge \) the inverse limit of \( \infty \)-categories

\[
\text{Top}_\infty^\wedge = \lim_{n \in \mathbb{N}^\omega} \text{Top}_n
\]

along the various truncation functors \( (-)^{\leq m-1} \). This is the \( \infty \)-category of sequences \( \{X_n\}_{n \in \mathbb{N}} \) in which each \( X_n \) is an \( n \)-topos, along with identifications \( (X_n)^{\leq m-1} \Rightarrow X_m \) whenever \( m \leq n \). The truncation functors provide a functor

\[
\tau : \text{Top}_\infty \to \text{Top}_\infty^\wedge,
\]

which carries an \( \infty \)-topos \( X \) to the sequence \( \{X^{\leq n-1}\}_{n \in \mathbb{N}} \).

3.2.10 Construction. The functor \( \tau : \text{Top}_\infty \to \text{Top}_\infty^\wedge \) admits a fully faithful right adjoint, which identifies \( \text{Top}_\infty^\wedge \) with the full subcategory of \( \text{Top}_\infty \) spanned by the \( \text{bounded} \) \( \infty \)-topoi [SAG, Proposition A.7.1.5]. These are the \( \infty \)-topoi that can be exhibited as inverse limits in \( \text{Top}_\infty \) of a diagram of localic \( \infty \)-topoi. Equivalently, an \( \infty \)-topos \( X \) is bounded if and only if the natural geometric morphism

\[
X \to \lim_{n \in \mathbb{N}^\omega} L_n(X)
\]

is an equivalence.

Surprisingly, the functor the functor \( \tau : \text{Top}_\infty \to \text{Top}_\infty^\wedge \) also admits a left adjoint; to state what the essential image of this left adjoint is, we first recall a bit about Postnikov completeness.

3.2.11 Definition. Let \( C \) be a presentable \( \infty \)-category. We say that:

(3.2.11.1) *Postnikov towers converge* in \( C \), if for every object \( X \in C \), the natural morphism \( X \to \lim_{n \in \mathbb{N}^\omega} \tau_{\leq n} X \) is an equivalence in \( C \).

(3.2.11.2) The \( \infty \)-category \( C \) is *Postnikov complete* if the natural functor

\[
C \to \lim \left( \cdots \to C_{S_{n+1}} \xrightarrow{\tau_{S_{n+1}}} C_{S_n} \xrightarrow{\tau_{S_n}} \cdots \xrightarrow{\tau_{S_0}} C_{S_0} \right),
\]

is an equivalence of \( \infty \)-categories. (Here the limit is formed in \( \text{Cat}_{\infty, \delta} \).)

3.2.12. Note that Postnikov completeness implies the convergence of Postnikov towers, but not conversely.

3.2.13 Construction. The functor \( \tau : \text{Top}_\infty \to \text{Top}_\infty^\wedge \) also admits a left adjoint, which is necessarily fully faithful. This identifies \( \text{Top}_\infty^\wedge \) with the full subcategory of \( \text{Top}_\infty \) spanned by the Postnikov complete \( \infty \)-topoi [SAG, Corollary A.7.2.8].

We write \( (-)^{\text{post}} \) for the right adjoint to the inclusion of the full subcategory of \( \text{Top}_\infty \) spanned by the Postnikov complete \( \infty \)-topoi, and write \( (-)^{b} \) for the left adjoint to the inclusion of the full subcategory of \( \text{Top}_\infty \) spanned by the bounded \( \infty \)-topoi, so that

\[
X^{\text{post}} = \lim_{n \in \mathbb{N}^\omega} X_{S_n} \quad \text{and} \quad X^{b} = \lim_{n \in \mathbb{N}^\omega} L_n(X).
\]

For an \( \infty \)-topos \( X \), we call \( X^{\text{post}} \) the *Postnikov completion* of \( X \) and call \( X^{b} \) the *bounded reflection* of \( X \).
The relationship between bounded $\infty$-topoi and Postnikov complete $\infty$-topoi is formally analogous to the relationship between $p$-nilpotent and $p$-complete abelian groups. Of course $p$-nilpotent and $p$-complete abelian groups form equivalent categories, but their embeddings into the category of all abelian groups differ.

### 3.3 Coherence

The second finiteness conditions that we impose on almost all of the $\infty$-topoi we consider is coherence. Coherence is the topos-theoretic analogue of being quasicompact and quasiseparated in the world of schemes. Moreover, the étale $\infty$-topos of a scheme $X$ is coherent if and only if $X$ is quasicompact and quasiseparated (see Proposition 3.7.3).

#### 3.3.1 Definition (coherence)

Let $0 \leq r \leq \infty$, and let $\mathcal{X}$ be an $r$-topos. We say that $\mathcal{X}$ is $0$-coherent if and only if the $0$-topos (=locale) $\text{Open}(\mathcal{X})$ is quasicompact. Let $n \in \mathbb{N}^*$, and define $n$-coherence of $r$-topoi and their objects recursively as follows.

- An object $U \in \mathcal{X}$ is $n$-coherent if and only if the $r$-topos $\mathcal{X}_{/U}$ is $n$-coherent.

- The $r$-topos $\mathcal{X}$ is locally $n$-coherent if and only if every object $U \in \mathcal{X}$ admits a cover $\{V_i \to U\}_{i \in I}$ in which each $V_i$ is $n$-coherent.

- The $r$-topos $\mathcal{X}$ is $(n+1)$-coherent if and only if $\mathcal{X}$ is locally $n$-coherent, and the $n$-coherent objects of $\mathcal{X}$ are closed under finite products.

An $r$-topos $\mathcal{X}$ is coherent if and only if $\mathcal{X}$ is $n$-coherent for every $n \in \mathbb{N}$, and an object $U$ of an $r$-topos $\mathcal{X}$ is coherent if and only if $\mathcal{X}_{/U}$ is a coherent $r$-topos. Finally, an $r$-topos $\mathcal{X}$ is locally coherent if and only if every object $U \in \mathcal{X}$ admits a cover $\{V_i \to U\}_{i \in I}$ in which each $V_i$ is coherent.

#### 3.3.2

In particular, if $\mathcal{X}$ is locally $n$-coherent, then $U \in \mathcal{X}$ is $(n+1)$-coherent if and only if $U$ is $n$-coherent and for any pair $U', V \in \mathcal{X}_{/U}$ of $n$-coherent objects, the fibre product $U' \times_U V$ is $n$-coherent.

#### 3.3.3

We are mostly interested in coherence for $\infty$-topoi, however we have introduced the notion for $r$-topoi in general because an $\infty$-topos $\mathcal{X}$ is $n$-coherent if and only if its underlying $n$-topos $\mathcal{X}_{\leq n-1}$ is $n$-coherent (this is the content of §3.4).

#### 3.3.4 Notation

Let $0 \leq r \leq \infty$, and let $\mathcal{X}$ be an $r$-topos. Write $\mathcal{X}^{\text{coh}} \subset \mathcal{X}$ for the full subcategory of $\mathcal{X}$ spanned by the coherent objects and $\mathcal{X}^{\text{coh}}_{\leq n} \subset \mathcal{X}$ for the full subcategory of $\mathcal{X}$ spanned by the truncated coherent objects. For each integer $n \geq 0$, write $\mathcal{X}^{n, \text{coh}} \subset \mathcal{X}$ for the full subcategory spanned by the $n$-coherent objects.

#### 3.3.5 Example

A space $K \in S$ is truncated coherent if and only if $K$ is $\pi$-finite. That is to say,

$$S^{\text{coh}}_{\leq n} = S_{\pi}.$$  

#### 3.3.6 Example

By [SAG, Proposition A.7.5.1], if $\mathcal{X}$ is a bounded coherent $\infty$-topos, then $\mathcal{X}$ is also locally coherent.
3.3.7. Let $0 \leq r \leq \infty$, let $X$ be an $r$-topos, and let $U \in X$. Then for any integer $n \geq 0$, an object $U^r_\to U$ of $X_{U^r}$ is $n$-coherent if and only if $U^r$ is $n$-coherent when viewed as an object of $X$. Thus we have canonical identifications

$$(X^{n\text{-coh}})_U = (X_U)^{n\text{-coh}}$$

and

$$(X^{\text{coh}})_U = (X_U)^{\text{coh}}$$

as full subcategories of $X_U$. If $U \in X_{\infty\text{co}}$ is a truncated object, then we have a canonical identification

$$(X^{\text{coh}})_U = (X_U)^{\text{coh}}$$

as full subcategories of $X_U$.

3.3.8 Definition. Let $X$ and $Y$ be $\infty$-topoi. We say that a geometric morphism

$$f^* : X \to Y$$

is coherent if and only if, for any coherent object $F \in Y^{\text{coh}}$, the object $f^*(F) \in X$ is coherent as well. We write $\text{Top}^{\text{coh}}$ for the subcategory of $\text{Top}^{\infty}$ whose objects are coherent $\infty$-topoi and whose morphisms are coherent geometric morphisms.

We defer examples of coherent $\infty$-topoi to §3.7; we do this in order to put all of our examples from algebraic geometry on the same footing after developing the basic calculus of finitary sites in this section and in §3.6.

3.3.9 Definition. An $\infty$-site $(X, \tau)$ is finitary if and only if it admits all fibre products, and, for every object $U \in X$ and every covering sieve $S \subset X_U$, there is a finite subset $\{U_i\}_{i \in I} \subset S$ that generates a covering sieve.

Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be finitary $\infty$-sites. A morphism of $\infty$-sites

$$f^* : (Y, \tau_Y) \to (X, \tau_X)$$

is a morphism of finitary $\infty$-sites if $f^*$ preserves fibre products.

3.3.10 Proposition ([SAG, Proposition A.3.1.3]). Let $(X, \tau)$ be a finitary $\infty$-site and write $\mathcal{V}_\tau : X \to \text{Sh}_\tau(X)$ for the sheafified Yoneda embedding. Then:

(3.3.10.1) The $\infty$-topos $\text{Sh}_\tau(X)$ locally coherent.

(3.3.10.2) For every object $x \in X$, the sheaf $\mathcal{V}_\tau(x)$ is a coherent object of $\text{Sh}_\tau(X)$.

(3.3.10.3) If, in addition, $X$ admits a terminal object, then $\text{Sh}_\tau(X)$ is coherent.

An elementary way to construct a finitary $\infty$-site is to make use of an $\infty$-categorical analogue of the notion of pretopology on a 1-category.

3.3.11 Definition. An $\infty$-pre-site is a pair $(X, E)$ consisting of an $\infty$-category $X$ along with a subcategory $E \subset X$ satisfying the following conditions:

- The subcategory $E$ contains all equivalences of $X$.
- The $\infty$-category $X$ admits finite limits, and $E$ is stable under base change.
- The $\infty$-category $X$ admits finite coproducts, and $E$ is closed under finite coproducts.

3.3.12 Construction ([SAG, Proposition A.3.2.1]). Let $(X, E)$ be an $\infty$-pre-site. Then there exists a topology $\tau_E$, in which the $\tau_E$-covering sieves are generated by finite families $\{y_i \to x\}_{i \in I}$ such that $\bigsqcup_{i \in I} y_i \to x$ lies in $E$. The $\infty$-site $(X, \tau_E)$ is finitary.
3.4 Coherence & $n$-topoi

In this section we prove that the property that an $\infty$-topos $X$ be $n$-coherent only depends on its underlying $n$-topos $X_{\leq n-1}$ of $(n-1)$-truncated objects (Corollary 3.4.10). We begin with some preliminaries on the relationship between coherence and connectivity.

3.4.1 Proposition ([SAG, Proposition A.2.4.1]). Let $X$ be an $\infty$-topos, let $f : U \to V$ be a morphism in $X$, and let $n \in \mathbb{N}$. Then:

(3.4.1.1) If $U$ is $n$-coherent and $f$ is $n$-connective, then $V$ is $n$-coherent.

(3.4.1.2) If $V$ is $n$-coherent and $f$ is $(n+1)$-connective, then $U$ is $n$-coherent.

Since the natural morphism from an object in an $\infty$-topos to its $n$-truncation is $(n+1)$-connective, we deduce:

3.4.2 Corollary. Let $X$ be an $\infty$-topos and $n \in \mathbb{N}$. An object $U \in X$ is $n$-coherent if and only if $\tau_{\leq n-1}(U)$ is an $n$-coherent object of $X$.

It is also easy to deduce the following.

3.4.3 Corollary ([SAG, Corollary A.2.4.4]). Let $X$ be a coherent $\infty$-topos and $n \in \mathbb{N}$. Then for any $n$-coherent object $U \in X$, the $(n-1)$-truncation $\tau_{\leq n-1}(U)$ of $U$ is a coherent object of $X$.

3.4.4 Corollary. Let $f_* : X \to Y$ be a geometric morphism between coherent $\infty$-topoi. Then $f_*$ is coherent if and only if $f^*$ carries $Y^{coh}_{\leq n} \to X^{coh}_{\leq n}$.

We also deduce that coherence of a geometric morphism between coherent $\infty$-topoi is equivalent to the a priori stronger condition that the pullback functor preserve $n$-coherent objects for all $n \geq 0$.

3.4.5 Corollary. Let $f_* : X \to Y$ be a geometric morphism between coherent $\infty$-topoi. Then $f_*$ is coherent if and only if $f^*$ carries $n$-coherent objects of $Y$ to $n$-coherent objects of $X$ for all $n \in \mathbb{N}$.

**Proof.** It is immediate from the definition that if $f^*$ preserves $n$-coherence for all $n \geq 0$, then $f_*$ is coherent. Conversely, assume that $f_*$ is coherent, and let $U \in Y$ be an $n$-coherent object. Since $Y$ is coherent, Corollary 3.4.3=[SAG, Corollary A.2.4.4] shows that $\tau_{\leq n-1}(U)$ is an $n$-coherent object of $Y$. Since $f_*$ is coherent, we see that

$$f^* \tau_{\leq n-1}(U) = \tau_{\leq n-1}(f^*(U))$$

is a coherent object of $X$. Corollary 3.4.2 then shows that $f^*(U)$ is an $n$-coherent object of $X$. 

---

16 We are grateful to Jacob Lurie for conveying this observation.

17 This second notion is how Grothendieck and Verdier originally defined coherence for geometric morphisms between ordinary topoi [SGA 4_ii, Exposé VI, Définition 3.1].
Before showing that $n$-coherence only depends on the underlying $n$-topos, we need two preliminary facts on $m$-connective morphisms in an $\infty$-topos.

3.4.7 Lemma. Let $X$ be an $\infty$-topos and $m \geq 0$ an integer. Let $W \in X$ and let $u : U' \to U$ and $v : V' \to V$ be morphisms in $X_{/W}$. If $u$ and $v$ are $m$-connective morphisms of $X$, then the induced morphism $u \times_W v : U' \times_W V' \to U \times_W V$ is $m$-connective.

Proof. First we treat the case where $W = 1_X$ is the terminal object of $X$. In this case, since $\tau_{\leq m-1} : X \to X$ preserves finite products [HTT, Lemma 6.5.1.2] and $\tau_{\leq m-1}(u)$ and $\tau_{\leq m-1}(v)$ are equivalences by assumption, we see that $\tau_{\leq m-1}(u \times v) \simeq \tau_{\leq m-1}(u) \times \tau_{\leq m-1}(v)$ is an equivalence.

Now we treat the general case. In the diagram

$$
\begin{array}{ccc}
U' \times_W V' & \xrightarrow{u \times_W v} & U \times_W V \\
\downarrow & & \downarrow \\
U' \times V' & \xrightarrow{u \times v} & U \times V \\
\downarrow & & \downarrow \\
U' \times V & \xrightarrow{u \times v} & U \times V
\end{array}
$$

both squares are pullbacks and $u \times v$ is $m$-connective (by the preceding paragraph). This completes the proof since the class of $m$-connective morphisms in an $\infty$-topos is stable under pullback [HTT, Proposition 6.5.1.16].

The following is a useful strengthening of the fact that $n$-truncation commutes with basechange along a morphism between $n$-truncated objects [36, Lemma 1.8]:

3.4.8 Lemma. Let $X$ be an $\infty$-topos and $n \in \mathbb{N}$. Let $W \in X$ and let $U \to W$ and $V \to W$ be morphisms in $X$. If $W$ is $n$-truncated, then the natural morphism

$$
\tau_{\leq n}(U \times_W V) \to \tau_{\leq n}(U) \times_W \tau_{\leq n}(V)
$$

is an equivalence.

Proof. Since the natural morphisms $U \to \tau_{\leq n}(U)$ and $V \to \tau_{\leq n}(V)$ are $(n+1)$-connective, by Lemma 3.4.7 the natural morphism

$$
\phi : U \times_W V \to \tau_{\leq n}(U) \times_W \tau_{\leq n}(V)
$$

is $(n+1)$-connective. Since $W$ is $n$-truncated and the $n$-truncated objects of an $\infty$-topos are closed under limits, the object $\tau_{\leq n}(U) \times_W \tau_{\leq n}(V)$ is $n$-truncated. By the uniqueness of the factorisation of a morphism in an $\infty$-topos into an $(n+1)$-connective morphism followed by a $n$-truncated morphism, we see that $\phi$ exhibits $\tau_{\leq n}(U) \times_W \tau_{\leq n}(V)$ as the $n$-truncation of $U \times_W V$.

3.4.9 Proposition. Let $X$ be an $\infty$-topos and $n \in \mathbb{N}$. The following are equivalent for an $(n-1)$-truncated object $W \in X_{\leq n-1}$:
(3.4.9.1) As an object of the ∞-topos $X$, the object $W$ is $n$-coherent.

(3.4.9.2) As an object of the $n$-topos $X_{S^{n-1}}$, the object $W$ is $n$-coherent.

Proof. Clearly (3.4.9.1) implies (3.4.9.2). We prove that (3.4.9.2) implies (3.4.9.1) by induction on $n$. The base case $n = 0$ is immediate from the definition of 0-coherence.

For the induction step assume we have shown that an $(n - 1)$-truncated object of $X$ is $n$-coherent if it is $n$-coherent as an object of the $n$-topos $X_{S^{n-1}}$. Let $W$ be an $n$-truncated object of $X$ that is $(n + 1)$-coherent as an object of the $(n + 1)$-topos $X_{S^n}$; we prove that $W$ is $(n + 1)$-coherent as an object of the ∞-topos $X$. First we show that $X_{jW}$ is locally $n$-coherent. Let $f : U \to W$ be a morphism in $X$. Since $W$ is $n$-truncated, $f$ factors as a composite

$$U \to \tau_{S^n}(U) \to W.$$ 

Since $X_{S^n/X}$ is locally $n$-coherent by the inductive hypothesis, there exists a cover

$$\{U_i \to \tau_{S^n}(U)\}_{i \in I}$$

of $\tau_{S^n}(U)$ such that for each $i \in I$, the object $U_i \in X_{S^n/X}$ is an $n$-coherent object of $X_{S^n/X}$. Equivalently, each $U_i$ is an $n$-coherent object of $X_{S^n}$ (3.3.7). Since the morphism $U \to \tau_{S^n}(U)$ is $(n+1)$-connective, Proposition 3.4.1=[SAG, Proposition A.2.4.1] and the fact that $(n + 1)$-connective morphisms in an ∞-topos are stable under pullback [HTT, Proposition 6.5.1.16] show that the family

$$\{U_i \times_{\tau_{S^n}(U)} U \to U\}_{i \in I}$$

is a cover of $U$ in $X_{jW}$ by $n$-coherent objects. That is, $X_{jW}$ is locally $n$-coherent.

Now let us show that the $n$-coherent objects of $X_{jW}$ are stable under finite products. Let $f : U \to W$ and $g : V \to W$ be morphisms in $X_{jW}$, where $U$ and $V$ are $n$-coherent. Then since the $n$-coherent objects of $X_{S^n/X}$ are stable under finite products by the inductive hypothesis, we see that $\tau_{S^n}(U) \times_W \tau_{S^n}(V)$ is an $n$-coherent object of $X_{S^n/W}$. By the inductive hypothesis and Corollary 3.4.2, the object $\tau_{S^n}(U) \times_W \tau_{S^n}(V)$ is an $n$-coherent object of $X_{jW}$. The claim now follows from the fact that the natural morphism

$$U \times_W V \to \tau_{S^n}(U) \times_W \tau_{S^n}(V)$$

is $(n + 1)$-connective (Lemma 3.4.8) and Proposition 3.4.1=[SAG, Proposition A.2.4.1].

Setting $W = 1_X$ in Proposition 3.4.9 we deduce:

3.4.10 Corollary. Let $n \in \mathbb{N}$. The following are equivalent for an ∞-topos $X$:

(3.4.10.1) The ∞-topos $X$ is $n$-coherent.

(3.4.10.2) The $n$-topos $X_{S^{n-1}}$ is $n$-coherent.

For the next few results, please recall the notations of Construction 3.2.10.
3.4.11 Corollary. Let \( n \in \mathbb{N} \) and let \( f_* : X \to Y \) be a geometric morphism of \( \infty \)-topoi. If \( f_* \) induces an equivalence \( X_{\leq n-1} \Rightarrow Y_{\leq n-1} \), then \( X \) is \( n \)-coherent if and only if \( Y \) is \( n \)-coherent. Equivalently, if \( f_* \) induces an equivalence \( L_n(X) \Rightarrow L_n(Y) \) on \( n \)-localic reflections, then \( X \) is \( n \)-coherent if and only if \( Y \) is \( n \)-coherent.

Corollary 3.4.11 shows that there are many different ways to check the \( n \)-coherence of an \( \infty \)-topos.

3.4.12 Lemma. Let \( n \in \mathbb{N} \). The following are equivalent for an \( \infty \)-topos \( X \):

(3.4.12.1) The \( \infty \)-topos \( X \) is \( n \)-coherent.

(3.4.12.2) The \( n \)-localic reflection \( L_n(X) \) of \( X \) is \( n \)-coherent.

(3.4.12.3) The hypercompletion \( X^{hyp} \) of \( X \) is \( n \)-coherent (see Definition 3.11.5).

(3.4.12.4) The Postnikov completion \( X^{post} \) of \( X \) is \( n \)-coherent.

(3.4.12.5) The bounded reflection \( X^b \) of \( X \) is \( n \)-coherent.

Proof. The equivalence of these statements follows from repeated application of Corollary 3.4.11. The equivalence of (3.4.12.1) and (3.4.12.2) follows immediately from Corollary 3.4.11.

To see that (3.4.12.1)\( \Leftrightarrow \) (3.4.12.3), note that since truncated objects are hypercomplete, the natural fully faithful geometric morphism \( X^{hyp} \hookrightarrow X \) induces an equivalence on \( (n-1) \)-truncated objects.

To see that (3.4.12.1)\( \Leftrightarrow \) (3.4.12.4), note that by [SAG, Proposition A.7.3.7] the natural geometric morphism \( X^{post} \to X \) is an equivalence when restricted to \( (n-1) \)-truncated objects.

To see that (3.4.12.1)\( \Leftrightarrow \) (3.4.12.5), note that since the \( n \)-localic reflection functor \( L_n : \text{Top}_\infty \to \text{Top}_\infty \) preserves inverse limits [SAG, Lemma A.7.1.4], the natural geometric morphism

\[
X \to X^b = \lim_{k \in \mathbb{N}^{op}} L_k(X)
\]

induces an equivalence on \( n \)-localic reflections.

3.4.13 (Equivalent conditions for coherence of geometric morphisms). Let

\[
\begin{array}{ccc}
X' & \xrightarrow{x_*} & X \\
\downarrow f'_* & & \downarrow f_* \\
Y' & \xrightarrow{y_*} & Y
\end{array}
\]

be a commutative square of coherent \( \infty \)-topoi in \( \text{Top}_\infty \). Lemma 3.4.12 and Corollary 3.4.5 show that if \( x_* \) and \( y_* \) induce equivalences

\[
X'_{\leq \infty} \Rightarrow X_{\leq \infty} \quad \text{and} \quad Y'_{\leq \infty} \Rightarrow Y_{\leq \infty}
\]
on truncated objects, then \( f_* \) is coherent if and only if \( f'_* \) is coherent.

In particular, the following are equivalent for a geometric morphism \( f_* : X \to Y \) between coherent \( \infty \)-topoi:

\[52\]
(3.4.13.1) The geometric morphism $f_* : X \to Y$ is coherent.

(3.4.13.2) The induced geometric morphism $f_*^{hyp} : X^{hyp} \to Y^{hyp}$ on hypercompletions is coherent.

(3.4.13.3) The induced geometric morphism $f_*^{post} : X^{post} \to Y^{post}$ on Postnikov completions is coherent.

(3.4.13.4) The induced geometric morphism $f_*^{b} : X^{b} \to Y^{b}$ on bounded reflections is coherent.

Thus the equivalence between Postnikov complete $\infty$-topoi and bounded $\infty$-topoi (Construction 3.2.10) restricts to an equivalence between the subcategory of Postnikov complete coherent $\infty$-topoi and coherent geometric morphisms and the subcategory of bounded coherent $\infty$-topoi and coherent geometric morphisms.

3.5 Coherence of morphisms & $n$-localic $\infty$-topoi

In this section we prove that coherence for an $n$-localic $\infty$-topos is equivalent to $(n+1)$-coherence, and may be checked on its underlying $n$-topos (Proposition 3.5.6). First we’ll need $\infty$-toposic versions of a number of points from [SGA 4ii, Exposé VI, §§1–3]; these follow easily from [SAG, §A.2.1].

3.5.1 Definition. Let $n \in \mathbb{N}$ and let $X$ be a locally $n$-coherent $\infty$-topos. A morphism $U \to V$ in $X$ is called relatively $n$-coherent if for every $n$-coherent object $V' \in X$ and every morphism $V' \to V$, the fibre product $U \times_V V'$ is also $n$-coherent.

3.5.2 Example ([SAG, Example A.2.1.2]). Let $X$ be a locally $n$-coherent $\infty$-topos and $f : U \to V$ a morphism in $X$. If $U$ is $n$-coherent and $V$ is $(n+1)$-coherent, then $f$ is relatively $n$-coherent.

3.5.3 Example. As a consequence of Proposition 3.4.1=[SAG, Proposition A.2.4.1] and the fact that the class of $(n+1)$-connective morphisms in an $\infty$-topos is stable under pullback [HTT, Proposition 6.5.1.16], the $(n+1)$-connective morphism of an $\infty$-topos are ‘relatively $n$-coherent’ in a very strong sense: they satisfy the condition of relative $n$-coherence without the need of local $n$-coherence assumptions on the $\infty$-topos.

3.5.4 Lemma. Let $n \in \mathbb{N}$ and let $X$ be a locally $n$-coherent $\infty$-topos. Let $u : U' \to U$ and $v : V' \to V$ be relatively $n$-coherent morphisms in $X$, $W \in X$ an object, and $U \to W$ and $V \to W$ are any morphisms. Then the induced morphism $U' \times_W V' \to U \times_W V$ is relatively $n$-coherent.

Proof. Let $f : X \to U \times_W V$ be a morphism in $X$ where $X$ is $n$-coherent. Note that we have equivalences of iterated fibre products

$X \times_{U \times_W V} (U' \times_W V') \cong (X \times_X U') \times_X (X \times_X V')$

$\cong (X \times_X U') \times_V V'.$
First, since \( X \times_U U' \) is the pullback of \( \text{pr}_1 \circ f : X \to U \) along the relatively \( n \)-coherent morphism \( u \), the object \( X \times_U U' \) is \( n \)-coherent. Second, \((X \times_U U') \times_U V'\) is the pullback of the morphism \( X \times_U U' \to V \) induced by \( \text{pr}_2 \circ f : X \to V \) along the relatively \( n \)-coherent morphism \( v \). Hence \((X \times_U U') \times_U V'\) is an \( n \)-coherent object of \( X \), as desired. \( \square \)

#### 3.5.5 Lemma
Let \( X \) be an \( \infty \)-topos and \( m \in \mathbb{N} \). Let \( X_0 \subset X \) be a full subcategory satisfying the following conditions:

1. The full subcategory \( X_0 \subset X \) is closed under finite products.
2. Every object of \( X_0 \) is \( m \)-coherent.
3. For every object \( U \in X \), there exists an effective epimorphism \( \bigsqcup_{i \in I} U_i \to U \) where \( U_i \in X_0 \) for each \( i \in I \).

Then the \( m \)-coherent objects of \( X \) are closed under finite products.

**Proof.** Let \( X^I_0 \subset X \) denote the closure of \( X_0 \) under finite coproducts; then every object of \( X^I_0 \) is \( m \)-coherent. Since colimits in \( X \) are universal and \( X_0 \) is closed under finite products, \( X^I_0 \subset X \) is closed under finite products.

Let \( U, V \in X \) be \( m \)-coherent objects; we show that \( U \times V \) is \( m \)-coherent. Since \( U \) and \( V \) are quasicompact, there exist effective epimorphisms \( u : U' \to U \) and \( v : V' \to V \) where \( U', V' \in X^I_0 \). By Example 3.5.2=[SAG, Example A.2.1.2] both \( u \) and \( v \) are relatively \((m - 1)\)-coherent. Lemma 3.5.4 shows that

\[
u \times v : U' \times V' \to U \times V
\]

is a relatively \((m-1)\)-coherent effective epimorphism. Since \( U' \times V' \in X^I_0 \) is \( m \)-coherent and \( X \) is locally \( m \)-coherent, [SAG, Proposition A.2.1.3] shows that \( U \times V \) is \( m \)-coherent, as desired. \( \square \)

#### 3.5.6 Proposition
Let \( n \in \mathbb{N} \). The following are equivalent for an \( n \)-localic \( \infty \)-topos \( X \):

1. The \( n \)-topos \( X_{\leq n-1} \) is \( (n + 1) \)-coherent.
2. The \( \infty \)-topos \( X \) is \( (n + 1) \)-coherent.
3. The \( \infty \)-topos \( X \) is coherent.
4. The \( n \)-topos \( X_{\leq n-1} \) is coherent.

**Proof.** Clearly (3.5.6.1)\(\Rightarrow\) (3.5.6.4) and (3.5.6.4)\(\Rightarrow\) (3.5.6.1).

First we show that (3.5.6.1)\(\Rightarrow\) (3.5.6.2). Corollary 3.4.10 shows that \( X \) is \( n \)-coherent. First notice that every object of \( X \) admits a cover by \((n - 1)\)-truncated \( n \)-coherent objects (so, in particular, \( X \) is locally \( n \)-coherent). This follows from the following observations:

- Since \( n \)-cohesive \( X \) is \( n \)-localic, every object of \( X \) admits a cover by \((n - 1)\)-truncated objects.
- Since the \( n \)-topos \( X_{\leq n-1} \) is locally \( n \)-coherent, Proposition 3.4.9 shows that every \((n - 1)\)-truncated object of \( X \) admits a cover by \((n - 1)\)-truncated \( n \)-coherent objects.
Moreover, since the \((n-1)\)-truncated objects of an \(\infty\)-topos are closed under limits and \(X_{m-1}\) is \((n+1)\)-coherent, Proposition 3.4.9 shows that the \((n-1)\)-truncated \(n\)-coherent objects of \(X\) are closed under finite products. Lemma 3.5.5 applied to the full subcategory \(X_0 \subset X\) spanned by the \((n-1)\)-truncated \(n\)-coherent objects (so that \(m = n\) in the notation of Lemma 3.5.5) now shows that the \(n\)-coherent objects of \(X\) are closed under finite products.

Since an \(n\)-localic \(\infty\)-topos is \(N\)-localic for all \(N \geq n\), to prove the implication \((3.5.6.2) \Rightarrow (3.5.6.3)\), it suffices to prove that if \(X\) is \((n+1)\)-coherent, then \(X\) is \((n+2)\)-coherent. First we show that \(X\) is locally \((n+1)\)-coherent. We have already seen that every object of \(X\) admits a cover by a \((n-1)\)-truncated \(n\)-coherent objects, and that the subcategory \(X_0\) of \((n-1)\)-truncated \(n\)-coherent objects is closed under finite products. Since \(X\) is \((n+1)\)-coherent, [SAG, Corollary A.2.4.3] shows that \((n-1)\)-truncated \(n\)-coherent objects of \(X\) are automatically \((n+1)\)-coherent, immediately implying that \(X\) is locally \((n+1)\)-coherent. Lemma 3.5.5 applied to the subcategory \(X_0\) of \((n-1)\)-truncated \((n+1)\)-coherent objects (so that \(m = n + 1\) in the notation of Lemma 3.5.5) shows that the \((n+1)\)-coherent objects of \(X\) are closed under finite products.

3.6 Coherent geometric morphisms via sites & coherent ordinary topos

In this section we explain the relationship between coherent ordinary topos in the sense of Grothendieck [SGA 4_{1\text{II}}, Exposé VI] and their corresponding 1-localic \(\infty\)-topoi. \(^{18}\) (See [76, 77, Appendix C, §§5–6] for an excellent accounts of coherent ordinary topos.) We show that the \(\infty\)-category of coherent 1-localic \(\infty\)-topoi is equivalent to the 2-category of coherent ordinary topos. In fact, the results of §3.4 allow us to show that the \(\infty\)-category of coherent \(n\)-localic \(\infty\)-topoi is equivalent to the \((n+1)\)-category of coherent \(n\)-topoi (Proposition 3.6.11).

3.6.1 Recollection. A 1-topos \(X\) is coherent in the sense of [SGA 4_{1\text{II}}, Exposé VI, Definition 2.3] if and only if \(X\) is 2-coherent in the sense of Definition 3.3.1. This is true if and only if \(X\) is equivalent to the 1-topos of sheaves of sets on a finitary 1-site \((X, \tau)\) with a terminal object. Proposition 3.5.6 shows that \(X\) is coherent if and only if its corresponding 1-localic \(\infty\)-topos is coherent.

A geometric morphism \(f_* : X \rightarrow Y\) is coherent [SGA 4_{1\text{II}}, Exposé VI, Definition 3.1] if and only if \(f_*\) is induced by a morphism of finitary 1-sites \(f^* : (Y, \tau_Y) \rightarrow (X, \tau_X)\).

The content of the equivalence between coherent \(n\)-topoi and coherent \(n\)-localic \(\infty\)-topoi reduces to showing that a coherent morphism of coherent \(n\)-topoi induces a coherent morphism of corresponding \(n\)-localic \(\infty\)-topoi. This follows from the fact that coherence of a geometric morphism between locally coherent \(\infty\)-topoi can be checked on a generating set of coherent objects (Corollary 3.6.6). A particularly useful consequence is that morphisms of finitary \(\infty\)-sites induce coherent geometric morphisms (Corollary 3.6.8).

First we need a few preliminary results. For this, please recall the notion of relative \(n\)-coherence (Definition 3.5.1) introduced in §3.5.

\(^{18}\)The contents of this section originally appeared in a (partially expository) preprint of the third-named author [44].
3.6.2 Lemma. Let $X$ be an $\infty$-topos. If $e: U \to V$ is an effective epimorphism in $X$ and $U$ is quasicompact, then $V$ is quasicompact.

Proof. This is a special case of [SAG, Proposition A.2.1.3], or, alternatively, Proposition 3.4.1=[SAG, Proposition A.2.4.1].

3.6.3 Lemma. Let $n \geq 1$ be an integer and $X$ a locally $(n-1)$-coherent $\infty$-topos. Let $U \in X$ and let $e: \bigsqcup_{i \in I} U_i \to U$ be a cover of $U$ where $I$ is finite and $U_i$ is $n$-coherent for each $i \in I$. The following are equivalent:

(3.6.3.1) The effective epimorphism $e$ is relatively $(n-1)$-coherent.

(3.6.3.2) For all $i, j \in I$, the object $U_i \times_U U_j$ is $(n-1)$-coherent.

(3.6.3.3) The object $U$ is $n$-coherent.

Proof. If $e$ is relatively $(n-1)$-coherent, then since coproducts in $X$ are universal, the fibre product

$\left( \bigsqcup_{i \in I} U_i \right) \times \left( \bigsqcup_{j \in I} U_j \right) = \bigsqcup_{i,j \in I} U_i \times_U U_j$

is $(n-1)$-coherent. Thus $U_i \times_U U_j$ is $(n-1)$-coherent for all $i, j \in I$ [SAG, Remark A.2.0.16].

If each $U_i \times_U U_j$ is $(n-1)$-coherent, then since each $U_i$ is $n$-coherent we see that the pullback of $e$ along itself

$\bigsqcup_{i,j \in I} U_i \times_U U_j \to \bigsqcup_i U_i$

is relatively $(n-1)$-coherent (Example 3.5.2=[SAG, Example A.2.1.2]). Applying [SAG, Corollary A.2.1.5] we deduce that $e: \bigsqcup_{i \in I} U_i \to U$ is relatively $(n-1)$-coherent.

To conclude, note that if $e: \bigsqcup_{i \in I} U_i \to U$ is relatively $(n-1)$-coherent, then [SAG, Proposition A.2.1.3] shows that $U$ is $n$-coherent. On the other hand, if $U$ is $n$-coherent, then $e$ is $(n-1)$-coherent by Example 3.5.2=[SAG, Example A.2.1.2].

3.6.4 Proposition. Let $f_*: X \to Y$ be a geometric morphism of $\infty$-topoi and $n \in \mathbb{N}$. Assume that:

(3.6.4.1) There exists a collection of $n$-coherent objects $Y_0 \subset \text{Obj}(Y)$ of $Y$ such that for every $n$-coherent object $U \in Y$ there exists a cover $\bigsqcup_{i \in I} U_i \to U$ where $U_i \in Y_0$ for each $i \in I$.

(3.6.4.2) The pullback functor $f^*: Y \to X$ takes objects of $Y_0$ to $n$-coherent objects of $X$.

(3.6.4.3) If $n \geq 1$, the $\infty$-topoi $X$ and $Y$ are locally $(n-1)$-coherent and $f^*: Y \to X$ takes $(n-1)$-coherent objects of $Y$ to $(n-1)$-coherent objects of $X$.

Then $f^*$ takes $n$-coherent objects of $Y$ to $n$-coherent objects of $X$. 

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Proof. Let $U \in Y$ be an $n$-coherent object; we show that $f^* (U)$ is $n$-coherent. Since $U$ is 0-coherent, by (3.6.4.1) there exists a cover

$$e : \bigsqcup_{i \in I} U_i \twoheadrightarrow U$$

where $U_i \in Y_0$ for each $i \in I$ and $I$ is finite. By (3.6.4.2), for all $i \in I$ the object $f^* (U_i)$ is $n$-coherent, so since $n$-coherent objects are closed under finite coproducts [SAG, Remark A.2.0.16], the object

$$f^* \left( \bigsqcup_{i \in I} U_i \right) = \bigsqcup_{i \in I} f^* (U_i)$$

is $n$-coherent.

Note that

$$f^* (e) : \bigsqcup_{i \in I} f^* (U_i) \twoheadrightarrow f^* (U)$$

is an effective epimorphism in $X$. If $n = 0$, this proves the claim (Lemma 3.6.2). If $n \geq 1$, then Lemma 3.6.3 shows that it suffices to show that for all $i, j \in I$, the object

$$f^* (U_i) \times f^* (U_j) = f^* (U_i \times_U U_j)$$

is $(n-1)$-coherent. This follows from the fact that $U_i \times_U U_j$ is $(n-1)$-coherent (by Lemma 3.6.3) and the assumption that $f^*$ sends $(n-1)$-coherent objects of $Y$ to $(n-1)$-coherent objects of $X$. \qed

Proposition 3.6.4 shows that coherence of a geometric morphism between \textit{locally} coherent \textit{∞}-topoi is equivalent to the \textit{a priori} stronger condition that the pullback functor preserve $n$-coherent objects for all $n \geq 0$; see also Corollary 3.4.6.

3.6.5 Corollary. Let $f_* : X \to Y$ be a geometric morphism between locally coherent \textit{∞}-topoi. Then $f_*$ is coherent if and only if $f^*$ takes $n$-coherent objects of $Y$ to $n$-coherent objects of $X$ for all $n \geq 0$.

Proposition 3.6.4 also shows that coherence of a geometric morphism can be checked on a generating set of coherent objects.

3.6.6 Corollary. Let $f_* : X \to Y$ be a geometric morphism between locally coherent \textit{∞}-topoi. Let $Y_0 \subset \text{Obj}(Y^\text{coh})$ be a collection of coherent objects such that for every object $U \in Y$ there exists a cover $\bigsqcup_{i \in I} U_i \to U$ where $U_i \in Y_0$ for each $i \in I$. If for all $U \in Y_0$ the object $f^* (U)$ is coherent, the geometric morphism $f_* : X \to Y$ is coherent.

For the next result, we need the following lemma.

3.6.7 Lemma. Let

$$f^* : (Y, \tau_Y) \to (X, \tau_X)$$

be a morphism of \textit{∞}-sites, and write $\xi_{\tau_Y} : Y \to \text{Sh}_{\tau_Y} (Y)$ for the sheafified Yoneda embedding. If the topology $\tau_X$ is finitary, then

$$f^* \xi_{\tau_Y} : Y \to \text{Sh}_{\tau_X} (X)$$

factors through $\text{Sh}_{\tau_X} (X)^\text{coh} \subset \text{Sh}_{\tau_X} (X)$.
Proof. We have a commutative square

\[
\begin{array}{ccc}
Y & \xrightarrow{f^*} & X \\
\downarrow{\Phi_{\tau_Y}} & \xrightarrow{\Phi_{\tau_X}} & \downarrow{\Phi_{\tau_X}} \\
\text{Sh}_{\tau_Y}(Y) & \xrightarrow{f} & \text{Sh}_{\tau_X}(X)
\end{array}
\]

where the vertical functors are sheafified Yoneda embeddings. Since the topology $\tau_X$ is finitary, the sheafified Yoneda embedding $\Phi_{\tau_X} : X \to \text{Sh}_{\tau_X}(X)_{\text{coh}}$ (Proposition 3.3.10=[SAG, Proposition A.3.1.3]).

3.6.8 Corollary. Let $f^* : (Y, \tau_Y) \to (X, \tau_X)$ be a morphism of finitary $\infty$-sites. Then the induced geometric morphism

\[ f_* : \text{Sh}_{\tau_X}(X) \to \text{Sh}_{\tau_Y}(Y) \]

is coherent.

Proof. By Proposition 3.3.10, both $\text{Sh}_{\tau_X}(X)$ and $\text{Sh}_{\tau_Y}(Y)$ are locally coherent. The image $\Phi_{\tau_Y}(Y)$ of $Y$ under the sheafified Yoneda embedding generates $\text{Sh}_{\tau_Y}(Y)$ under colimits, so by Corollary 3.6.6 it suffices to check that $f^*$ carries objects in $\Phi_{\tau_Y}(Y)$ to coherent objects of $X$; this the content of Lemma 3.6.7.

3.6.9. Proposition 3.5.6 and Corollaries 3.6.6 and 3.6.8 together show that a geometric morphism of coherent $1$-topoi is coherent in the sense of [SGA 4\text{ii}, Exposé VI, Definition 3.1] if and only if the geometric morphism of corresponding of $1$-localic $\infty$-topoi is coherent if and only if the geometric morphism of coherent $1$-topoi is coherent in the sense of Definition 3.3.8.

We now turn to the equivalence between coherent $n$-topoi and coherent $n$-localic $\infty$-topoi.

3.6.10 Notation. Let $n \in \mathbb{N}$. Write

\[ \text{Top}_{\infty}^{n,\text{coh}} \subset \text{Top}_{\infty}^{\text{coh}} \]

for the full subcategory spanned by the $n$-localic coherent $\infty$-topoi. Write $\text{Top}_{n}^{\text{coh}} \subset \text{Top}_{n}$ for the subcategory of the $(n + 1)$-category of $n$-topoi with objects coherent $n$-topoi and morphisms coherent geometric morphisms. When $n = 1$, the $2$-category $\text{Top}_{1}^{\text{coh}}$ is the $2$-category of ordinary coherent topos and coherent geometric morphisms (both in the sense of [SGA 4\text{ii}, Exposé VI]).

Proposition 3.5.6 and Corollary 3.6.6 immediately imply the following:

3.6.11 Proposition. Let $n \in \mathbb{N}$. The equivalence of $\infty$-categories $(-)_{\text{tr} - 1} : \text{Top}_{\infty}^n \Rightarrow \text{Top}_n$ restricts to an equivalence

\[ (-)_{\text{tr} - 1} : \text{Top}_{\infty}^{n,\text{coh}} \Rightarrow \text{Top}_n^{\text{coh}} \]
3.6.12 Corollary. Let $n \in \mathbb{N}$. The following are equivalent for a geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ between $n$-localic coherent $\infty$-topoi:

3.6.12.1) The geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ is coherent.

3.6.12.2) The pullback functor $f^* : \mathcal{Y} \to \mathcal{X}$ carries $(n-1)$-truncated $n$-coherent objects of $\mathcal{Y}$ to $n$-coherent objects of $\mathcal{X}$.

3.7 Examples of coherent $\infty$-topoi from algebraic geometry

In this section we use Corollary 3.6.8 to provide a few examples of coherent $\infty$-topoi arising from algebraic geometry.

3.7.1 Example. For a spectral topological space $S$, write $\text{Open}^{qc}(S) \subset \text{Open}(S)$ for the locale of quasicompact opens in $S$. Since the quasicompact opens of $S$ form a basis for the topology on $S$ that is closed under finite intersections, the $\infty$-topos $\text{Sh}(\text{Open}^{qc}(S))$ is $0$-localic (3.2.4). Applying [77, Proposition B.6.4] we see that the inclusion

$$\text{Open}^{qc}(S) \subset \text{Open}(S)$$

induces an equivalence of $0$-localic $\infty$-topoi

$$\tilde{S} = \text{Sh}(\text{Open}^{qc}(S))$$

(see also Corollary 3.12.14). Since Grothendieck topology on $\text{Open}^{qc}(S)$ is finitary, the $\infty$-topos $\tilde{S}$ of sheaves on $S$ is a coherent $\infty$-topos. (Cf. [SAG, Lemma 2.3.4.1]).

If $f : S \to T$ is a quasicompact continuous map of spectral topological spaces, the inverse image map $f^{-1} : \text{Open}(T) \to \text{Open}(S)$ restricts to a map

$$f^{-1} : \text{Open}^{qc}(T) \to \text{Open}^{qc}(S).$$

Corollary 3.6.8 shows that the induced geometric morphism $f_* : \tilde{S} \to \tilde{T}$ is coherent. Since spectral topological spaces are sober, a continuous map $f : S \to T$ of spectral topological spaces induces a coherent geometric morphism on the level of $\infty$-topoi if and only if $f$ is quasicompact.

3.7.2. Note that if $X$ is a coherent $\infty$-topos, then the underlying topological space of $X$ is spectral (Corollary 3.4.10).

Combining the fact that the Zariski, Nisnevich, étale, and proétale topoi of a scheme all have the same underlying topological space with the fact that if a scheme $X$ is quasicompact and quasiseparated, then the $1$-topoi of sheaves on $X$ in each of these topologies is coherent [SAG, Proposition 2.3.4.2 & Remark 3.7.4.2; 10, Appendix A; 77, Example 7.1.7], we deduce the following:

3.7.3 Proposition. The following are equivalent for a scheme $X$:

3.7.3.1) The scheme $X$ is coherent (i.e., quasicompact and quasiseparated).

\footnote{For background on the Nisnevich topology, see [SAG, §3.7; 59; 55; 91].}

\footnote{For background on the proétale topology, see [STK, Tags 0988 & 099R; 15].}
(3.7.3.2) The Zariski ∞-topos $X_{\text{zar}}$ of $X$ is a coherent ∞-topos.

(3.7.3.3) The Nisnevich ∞-topos $X_{\text{nis}}$ of $X$ is a coherent ∞-topos.

(3.7.3.4) The étale ∞-topos $X_{\text{ét}}$ of $X$ is a coherent ∞-topos.

(3.7.3.5) The proétale ∞-topos $X_{\text{proét}}$ of $X$ is a coherent ∞-topos.

3.7.4. In the case of the étale topology, see also [SAG, Proposition 2.3.4.2].

3.7.5 Example. Let $f : X \to Y$ be a morphism of coherent schemes and let

$$\tau \in \{\text{zar}, \text{nis}, \text{ét}, \text{proét}\}.$$  

Then the induced geometric morphism $f_* : X_\tau \to Y_\tau$ on ∞-topoi of $\tau$-sheaves is a coherent geometric morphism of coherent ∞-topoi. (Cf. [SAG, Proposition 2.3.5.1])

3.7.6 Example. Let $X$ be a coherent scheme. Then the natural geometric morphisms

$$X_{\text{proét}} \to X_{\text{ét}}, \quad X_{\text{ét}} \to X_{\text{nis}}, \quad \text{and} \quad X_{\text{nis}} \to X_{\text{zar}}$$

are all coherent geometric morphisms of coherent ∞-topoi.

3.8 Classification of bounded coherent ∞-topoi via ∞-pretopoi

In this section we explain how an ∞-topos that is both bounded and coherent is determined by its truncated coherent objects.

3.8.1 Notation. Write $\text{Top}_{\text{bc}}^\infty \subset \text{Top}_{\text{coh}}^\infty$ for the full subcategory spanned by those coherent ∞-topoi that are also bounded, that is, the bounded coherent ∞-topoi.

To a large extent, bounded coherent ∞-topoi function in much the same way as coherent 1-topoi. In particular, any bounded coherent ∞-topos is, in a canonical fashion, the ∞-category of sheaves on an ∞-site with excellent formal properties.

3.8.2 Definition. An ∞-category $X$ called an ∞-pretopos if and only if the following conditions are satisfied.

- The ∞-category $X$ admits finite limits.
- The ∞-category $X$ admits finite coproducts, which are universal and disjoint.
- Groupoid objects in $X$ are effective, and their geometric realisations are universal.

If $X$ and $Y$ are ∞-pretopoi, then a functor $f^* : Y \to X$ is a morphism of ∞-pretopoi if $f^*$ preserves finite limits, finite coproducts, and effective epimorphisms. We write

$$\text{preTop}_{\text{bc}}^\infty \subset \text{Cat}_{\text{bc}}^\infty$$

for the subcategory consisting of ∞-pretopoi and morphisms of ∞-pretopoi.

3.8.3 Example. If $X$ is a coherent ∞-topos, then the full subcategory $X_{\text{coh}}^\infty \subset X$ spanned by the coherent objects is an ∞-pretopos [SAG, Corollary A.6.1.7].
The following two useful facts are immediate from the definitions.

**3.8.4 Lemma.** Let \( \{ X_i \}_{i \in I} \) be a collection of \( \infty \)-pretopoi. Then the product \( \prod_{i \in I} X_i \) in \( \text{Cat}_{\infty, \delta_1} \) is an \( \infty \)-pretopos and for each \( j \in I \) the projection

\[
\text{pr}_j : \prod_{i \in I} X_i \to X_j
\]

is a morphism of \( \infty \)-pretopoi.

**3.8.5 Lemma.** Given morphisms of \( \infty \)-pretopoi \( X \to Z \) and \( Y \to Z \), the pullback \( X \times_Z Y \) in \( \text{Cat}_{\infty, \delta_1} \) is an \( \infty \)-pretopos, and the projections

\[
\text{pr}_1 : X \times_Z Y \to X \quad \text{and} \quad \text{pr}_2 : X \times_Z Y \to Y
\]

are morphisms of \( \infty \)-pretopoi.

**3.8.6 Notation.** Let \( X \) be an \( \infty \)-pretopos, and write \( E \subseteq X \) for the collection of effective epimorphisms in \( X \). Then \( (X, E) \) is an \( \infty \)-presite, and we write \( \text{eff} = \tau_E \) for the resulting finitary topology on \( X \). We call this topology the effective epimorphism topology on \( X \) [SAG, §A.6.2].

**3.8.7.** The effective epimorphism topology on an \( \infty \)-pretopos is a subcanonical topology [SAG, Corollary A.6.2.6].

**3.8.8 Definition.** An \( \infty \)-pretopos \( X \) is bounded if and only if \( X \) is \( \delta_0 \)-small and every object of \( X \) is truncated. We write

\[
\text{preTop}_{\infty}^b \subset \text{preTop}_{\infty}
\]

for the full subcategory spanned by the bounded \( \infty \)-pretopoi.

**3.8.9 Theorem ([SAG, Theorem A.7.5.3]).** The constructions \( X \mapsto X_{\text{coh}}^{<\infty} \) and \( X \mapsto \text{Sh}_{\text{eff}}(X) \) are mutually inverse equivalences of \( \infty \)-categories

\[
\text{Top}_{\infty}^{bc} \simeq \text{preTop}_{\infty}^b, \text{op}.
\]

In light of (3.4.13) we deduce the following variant for Postnikov complete coherent \( \infty \)-topoi:

**3.8.10 Corollary.** Write \( \text{Top}_{\infty}^{\text{post,coh}} \subset \text{Top}_{\infty}^{\text{coh}} \) for the full subcategory spanned by the Postnikov complete coherent \( \infty \)-topoi. The constructions \( X \mapsto X_{\text{coh}}^{<\infty} \) and \( X \mapsto \text{Sh}_{\text{eff}}(X)_{\text{post}} \) are mutually inverse equivalences of \( \infty \)-categories

\[
\text{Top}_{\infty}^{\text{post,coh}} \simeq \text{preTop}_{\infty}^b, \text{op}.
\]

We finish this section by recording the following bounded analogue of Lemma 3.8.4 that we use later.

**3.8.11 Lemma.** Let \( \{ X_i \}_{i \in I} \) be a finite collection of bounded \( \infty \)-pretopoi. Then the \( \infty \)-pretopos given by the product \( \prod_{i \in I} X_i \) in \( \text{Cat}_{\infty, \delta_1} \) is a bounded \( \infty \)-pretopos.

**Proof.** For each \( i \in I \) the \( \infty \)-category \( X_i \) is \( \delta_0 \)-small, so the product \( \prod_{i \in I} X_i \) is also \( \delta_0 \)-small. For any integer \( n \geq -2 \), an object \( F \in \prod_{i \in I} X_i \) is \( n \)-truncated if and only if \( \text{pr}_i(F) \in X_i \) is \( n \)-truncated for all \( i \in I \). Since \( I \) is finite and every object of each of the \( \infty \)-categories \( \{ X_i \}_{i \in I} \) is truncated by assumption, every object of the product \( \prod_{i \in I} X_i \) is truncated. \( \square \)
3.9 Coherence of inverse limits

We now recall that bounded coherent $\infty$-topoi and coherent geometric morphisms are stable under inverse limits in $\text{Top}_\infty$.

3.9.1 Proposition ([SAG, Proposition A.8.3.1]). The $\infty$-category $\text{preTop}_\infty^b$ admits filtered colimits and the forgetful functor $\text{preTop}_\infty^b \to \text{Cat}_{\omega, \delta}$ preserves filtered colimits.

3.9.2 Proposition ([SAG, Proposition A.8.3.2]). Let $X : A \to \text{preTop}_\infty^b$ be a filtered diagram of bounded $\infty$-pretopoi. Then the natural geometric morphism

$$\text{Sh}_\text{eff}(\text{colim}_{\alpha \in A} X_\alpha) \to \lim_{\alpha \in A^{\text{op}}} \text{Sh}_\text{eff}(X_\alpha)$$

is an equivalence in $\text{Top}_\infty$.

3.9.3. See [24, Lemma 3.3] for a more general statement about filtered colimits of finitary $\omega$-sites.

The following is immediate from the previous two propositions and Theorem 3.1.11 = [HTT, Theorem 6.3.3.1].

3.9.4 Corollary ([SAG, Corollary A.8.3.3]). The $\infty$-category $\text{Top}_\infty^{bc}$ admits inverse limits and the inclusion $\text{Top}_\infty^{bc} \to \text{Top}_\infty$ and forgetful functor $\text{Top}_\infty^{bc} \to \text{Cat}_{\omega, \delta}$ both preserve inverse limits.

3.10 Coherence & preservation of filtered colimits

The goal of this section is to prove the appropriate co-toposic generalisation of the fact that a coherent geometric morphism of 1-topoi preserves filtered colimits (see Corollary 3.10.5).\(^{21}\)

We begin by recalling a basic fact about filtered colimits of truncated objects and introducing some convenient terminology.

3.10.1 Recollection. Since filtered colimits commute with finite limits in an $\infty$-topos, for any $\infty$-topos $X$ and integer $n \geq -2$, the inclusion $X_{\leq n} \hookrightarrow X$ preserves filtered colimits. Thus $X_{\leq n}$ is an $\omega$-accessible localisation of $X$.

3.10.2 Definition. Let $C$ be a presentable $\infty$-category. We say that an object $X \in C$ is almost compact if $\tau_{\leq n}(C)$ is a compact object of the $n$-category $C_{\leq n}$.

We say that functor $F : C \to D$ between presentable $\omega$-categories almost preserves filtered colimits if for each integer $n \geq -2$, the functor $F : C_{\leq n} \to D$ preserves filtered colimits.

3.10.3 Lemma. Let $(X, \tau)$ be a finitary $\omega$-site, write $X = \text{Sh}_\tau(X)$, and write $\kappa_\tau : X \to X$ for the sheafified Yoneda embedding. Then $x \in X$, the object $\kappa_\tau(x)$ is almost compact.

\(^{21}\)We learned how to simplify and generalise the material in this section from its original form through a preprint of Chang-Yeon Chough [22, Theorem 3.4].
Proposition 3.10.4
Let \( \text{Proposition A.3.1.3} \)
Theorem A.7.5.3
Let \( \text{Proposition A.2.3.1} \)

Almost preserves filtered colimits \([\text{sections functor morphism. Let} \text{coherent}\]

3.10.5 Corollary.

Proof.

In light of Theorem 3.8.9=[SAG, Theorem A.7.5.3], Proposition 3.10.4 specialises to the following.

3.10.5 Corollary.

Let \( f_*: X \to Y \) be a coherent geometric morphism between bounded coherent co-topoi. Then the functor \( f_* \) almost preserves filtered colimits.
3.11 Points, Conceptual Completeness, & Deligne Completeness

In this section we discuss points of ∞-topoi as well as the ∞-toposic generalisations of the Conceptual Completeness Theorem of Makkai–Reyes and Deligne’s Completeness Theorem.

3.11.1 Notation. For an ∞-topos $\mathcal{X}$, we write $\text{Pt}(\mathcal{X}) = \text{Fun}^\ast_S(\mathcal{X})$ for the ∞-category of points of $\mathcal{X}$.

Note that a morphism $x'_* \to x'_*$ of $\text{Pt}(\mathcal{X})$ is a natural transformation $x'_* \to x'_*$. (The morphisms are the 'geometric transformations' usually preferred in 1-topos theory.) This choice syncs well with the direction of posets: for instance, when $P$ is a noetherian poset, one has $\text{Pt}(\tilde{P}) = P$.

In general, the passage from an ∞-topos to its ∞-category of points loses quite a bit of information. However, the ∞-toposic version of the Conceptual Completeness Theorem of Makkai–Reyes [78, Theorem 9.2] tells us that bounded coherent ∞-topoi are determined by their ∞-categories of points.

3.11.2 Theorem (Conceptual Completeness; [SAG, Theorem A.9.0.6]). A geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ between bounded coherent ∞-topoi is an equivalence if and only if $f_*$ is coherent and the induced functor $\text{Pt}(f_*): \text{Pt}(\mathcal{X}) \to \text{Pt}(\mathcal{Y})$ is an equivalence of ∞-categories.

3.11.3 Definition. An ∞-topos $\mathcal{X}$ has enough points if a morphism $\phi$ in $\mathcal{X}$ is an equivalence if and only if for every point $x_* \in \text{Pt}(\mathcal{X})$ the stalk $x_*\phi$ is an equivalence.

In classical topos theory, the Deligne Completeness Theorem [SGA 4ii, Exposé VI, Proposition 9.0] states that a locally coherent 1-topos has enough points. This is no longer true in the setting of ∞-topoi, the main obstruction being that ∞-connective morphisms in an ∞-topos need not be equivalences. For this reason the ∞-categorical version of Deligne’s theorem takes place in the setting of ∞-topoi where ∞-connective morphisms are equivalences, i.e., ∞-topoi in which Whitehead’s Theorem is valid.

3.11.4 Recollection. A morphism $\phi : U \to V$ in an ∞-topos $\mathcal{X}$ is ∞-connective if $\phi$ is $n$-connective for each integer $n \geq -1$.

Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. Since the left adjoint $f^*$ is left exact and preserves effective epimorphisms, if $\phi$ is an ∞-connective morphism of $\mathcal{Y}$, then $f^*(\phi)$ is an ∞-connective morphism of $\mathcal{X}$.

3.11.5 Definition. Let $\mathcal{X}$ be an ∞-topos. An object $U \in \mathcal{X}$ is hypercomplete if $U$ is local with respect to the class of ∞-connective morphisms in $\mathcal{X}$. We write $\mathcal{X}^{\text{hyp}} \subset \mathcal{X}$ for the full subcategory spanned by the hypercomplete objects of $\mathcal{X}$. We say that $\mathcal{X}$ is hypercomplete if $\mathcal{X}^{\text{hyp}} = \mathcal{X}$.

3.11.6. The ∞-category $\mathcal{X}^{\text{hyp}} \subset \mathcal{X}$ is a left exact localisation of $\mathcal{X}$, hence an ∞-topos [HTT, p. 699]. Moreover, the ∞-topos $\mathcal{X}^{\text{hyp}}$ is hypercomplete [HTT, Lemma 6.5.2.12].
3.11.7. Let \( f_* : X \to Y \) be a geometric morphism of \( \infty \)-topoi. Since \( f^* \) preserves \( \infty \)-connective morphisms, the pushforward \( f_* : X \to Y \) preserves hypercomplete objects, hence restricts to a functor \( f_* : X^{hyp} \to Y^{hyp} \). The functor \( f_* : X^{hyp} \to Y^{hyp} \) is the right adjoint in a geometric morphism with left exact left adjoint given by the composite

\[
Y^{hyp} \xrightarrow{f_*} X \xrightarrow{f^{hyp}} X^{hyp}
\]

of \( f^* \) with the left adjoint to the inclusion \( X^{hyp} \subset X \). We denote this geometric morphism by \( f^{hyp} \).

3.11.8 Warning. Let \( f_* : X \to Y \) be a geometric morphism of \( \infty \)-topoi. The pullback functor \( f^* : Y \to X \) generally does not preserve hypercomplete objects; since every \( \infty \)-topos is a left exact localization of a presheaf \( \infty \)-topos, if this were true then every \( \infty \)-topos would be hypercomplete.

3.11.9 Proposition ([HTT, Proposition 6.5.2.13]). Let \( X \) be an \( \infty \)-topos. Then for every hypercomplete \( \infty \)-topos \( H \), composition with the inclusion \( X^{hyp} \subset X \) induces an equivalence

\[
\text{Fun}_*(H, X^{hyp}) \simeq \text{Fun}_*(H, X).
\]

Consequently, the assignment \( X \mapsto X^{hyp} \) defines a functor right adjoint to the inclusion of hypercomplete \( \infty \)-topoi into all \( \infty \)-topoi. For this reason we call \( X^{hyp} \) the hypercompletion of \( X \).

3.11.10 Example. An \( \infty \)-topos with enough points is hypercomplete.

3.11.11 Example. Let \( X \) be a 1-topos with corresponding 1-localic \( \infty \)-topos \( X' \). Then \( X \) has enough points (in the sense of [SGA 4, Exposé IV, Définition 6.4.1]) if and only if the hypercomplete \( \infty \)-topos \( (X')^{hyp} \) has enough points.

3.11.12. Please observe that for an \( \infty \)-topos \( X \), the hypercompletion \( X^{hyp} \) has enough points if and only if \( \infty \)-connectiveness of morphisms in \( X \) can be checked on stalks, i.e., a morphism \( \phi \) in \( X \) is \( \infty \)-connective if and only if for every point \( x_* \) of \( X \), the stalk \( x^* \phi \) is an equivalence in \( S \). The Deligne Completeness Theorem (Theorem 3.11.15=[SAG, Proposition A.4.0.5]) and Corollary 3.11.17 show that \( \infty \)-connectiveness in a locally coherent \( \infty \)-topos can be checked on stalks.

3.11.13 Example. An \( \infty \)-topos \( X \) is hypercomplete if and only if the pullback functor \( p^* : X \to X^{post} \) is conservative. In particular, if Postnikov towers converge in \( X \) (Definition 3.2.11), then \( X \) is hypercomplete. However, the converse is false:

3.11.14 Warning. Postnikov towers need not converge in a hypercomplete \( \infty \)-topos. Morel and Voevodsky provide the following counterexample [86, §2.1, Example 1.30]. Let \( G \) denote the profinite group \( \prod_{[12]} \mathbb{Z}/2 \). Write \( X \) for the \( \infty \)-topos of hypersheaves on the site of finite continuous \( G \)-sets with respect to the topology in which a family of
maps is a covering if and only if it is jointly surjective. Since $\Gamma^*_X : S \to X$ preserves connected objects, the constant sheaf $U = \Gamma^*_X(\prod_{i \geq 1} K(\mathbb{Z}/2, i))$ at the product of Eilenberg–MacLane spaces $K(\mathbb{Z}/2, i)$ is connected. On the other hand, one can show that the limit of the Postnikov tower of $U$ is the product

$$\lim_{n \geq 0} \tau_{5n}U = \prod_{i \geq 1} \Gamma^*_X(K(\mathbb{Z}/2, i)),$$

and, moreover, that this object is not connected. In particular, the map $U \to \lim_{n \geq 0} \tau_{5n}U$ is not an equivalence.

In light of Example 3.11.10, the following is the correct $\infty$-toposic generalisation of Deligne's Completeness Theorem.

**3.11.15 Theorem (\(\infty\)-Categorical Deligne Completeness; \([\text{SAG}, \text{Proposition A.4.0.5}]\)).** An $\infty$-topos that is locally coherent and hypercomplete has enough points.

We have already seen that the coherence of an $\infty$-topos only depends on its hypercompletion (Lemma 3.4.12). The following proposition gives a more refined assertion about the relationship between the coherent objects of an $\infty$-topos and its hypercompletion.

**3.11.16 Proposition ([SAG, Proposition A.2.2.2]).** Let $X$ be an $\infty$-topos, and write $L^\text{hyp} : X \to X^\text{hyp}$ for the left adjoint to the inclusion $X^\text{hyp} \hookrightarrow X$. If $X$ is locally $n$-coherent for all $n \geq 0$, then:

1. The $\infty$-topos $X^\text{hyp}$ is locally $n$-coherent for all $n \geq 0$.
2. An object $U \in X^\text{hyp}$ is coherent if and only if $U$ is coherent when viewed as an object of $X$.
3. An object $U \in X$ is coherent if and only if $L^\text{hyp}(U) \in X^\text{hyp}$ is coherent.

**3.11.17 Corollary.** Let $X$ be an $\infty$-topos. If $X$ is (locally) coherent, then the hypercompletion $X^\text{hyp}$ of $X$ is (locally) coherent.

**3.11.18 Example.** Let $X$ be a bounded coherent $\infty$-topos. Then since $X$ is also locally coherent (Example 3.3.6), the hypercompletion $X^\text{hyp}$ of $X$ is coherent and locally coherent.

### 3.12 Bases for $\infty$-topoi

Let $W$ be a topological space and $B \subset \text{Open}(W)$ a basis for $W$. Upon passing to sheaves of sets, right Kan extension defines an equivalence of 1-topoi

$$\text{Sh}(B; \text{Set}) \simeq \text{Sh}(W; \text{Set})$$

with inverse given by restriction of presheaves \([77, \text{Proposition B.6.4}]\). The analogous statement for sheaves of spaces is false; open subsets of the Hilbert cube $\prod_{i \in \mathbb{Z}} [0, 1]$ homeomorphic to a product $[0, 1] \times \prod_{i \in \mathbb{Z}} [0, 1]$ form a basis $B$ for the topology on the Hilbert
cube, but sheaves of spaces on $B$ do not coincide with sheaves on the Hilbert cube [SAG, Counterexample 20.4.0.1]. The goal of this section is to show that although right Kan extension need not define an equivalence

\[(3.12.1) \quad \text{Sh}(B) \to \text{Sh}(W),\]

the failure of (3.12.1) to be an equivalence is fundamentally infinitary in nature and (3.12.1) is an equivalence when we restrict to hypercomplete objects.

We begin by recalling the basics of bases for sites and ∞-sites.

3.12.2 Definition. Let $(C, \tau)$ be an ∞-site. A basis for the topology $\tau$ on $C$ is a full subcategory $B \subset C$ satisfying the following property: for every object $c \in C$, there exists a set of morphisms $\{f_i : b_i \to c\}_{i \in I}$ such that $b_i \in B$ for each $i \in I$ and the set $\{f_i\}_{i \in I}$ generates a $\tau$-covering sieve on $c \in C$.

3.12.3. Let $(C, \tau)$ be an ∞-site and $B \subset C$ a basis for $\tau$. Then there is a unique topology $\tau|_B$ on $B$ satisfying the following property: for each object $b \in B$, a sieve $S \subset B/_{b}$ is a $\tau|_B$-covering sieve if and only if the image of $S$ under the embedding $B/_{b} \hookrightarrow C/_{b}$ generates a $\tau$-covering sieve on $b \in C$.

We always regard a basis $B \subset C$ as an ∞-site equipped the topology $\tau|_B$. To simplify notation, we often write $\tau$ instead of $\tau|_B$.

3.12.4. Let $(C, \tau)$ be an ∞-site and $B \subset C$ a basis for $\tau$. Then for every object $c \in C$, the full subcategory $B_c = B \times_C C_c \subset C_c$ is a basis for the topology on $C_c$ induced by $\tau$.

3.12.5 Example. Let $W$ be a topological space. A full subposet $B \subset \text{Open}(W)$ is a basis for the standard topology on $\text{Open}(W)$ (in the sense of Definition 3.12.2) if and only if $B$ defines a basis for the topological space $W$ in the usual sense: every open set of $W$ can be written as a union of opens belonging to $B$.

3.12.6 Example. Let $P$ be a poset. Then the functor $P^{op} \to \text{Open}(P)$ defined by $p \mapsto P_{\geq p}$ is fully faithful and defines a basis for the topology on the Alexandroff topological space $P$. The induced topology on $P^{op} \subset \text{Open}(P)$ is the trivial topology.

The first property of bases for ∞-sites is that a presheaf on $B$ is a sheaf if and only if its right Kan extension along the inclusion $B \hookrightarrow C$ is a sheaf on $C$.

3.12.7 Lemma ([5, Proposition A.5; 77, Proposition B.6.6]). Let $(C, \tau)$ be an ∞-site and $i : B \hookrightarrow C$ a basis for the topology $\tau$ on $C$. Then:

(3.12.7.1) A presheaf $F : B^{op} \to S$ on $B$ is a $\tau|_B$-sheaf if and only if the right Kan extension of $F$ along $i : B^{op} \hookrightarrow C^{op}$ is a $\tau$-sheaf on $C$.

(3.12.7.2) Right Kan extension along $i$ defines a fully faithful right adjoint

\[i_* : \text{Sh}_\tau(B) \hookrightarrow \text{Sh}_\tau(C)\]
with left exact left adjoint given by the composite
\[ \text{Sh}_\tau(C) \xrightarrow{\iota^*} \text{PSh}(B) \xrightarrow{\tau_*} \text{Sh}_\tau(B) \]
of presheaf restriction followed by \(\tau|_B\)-sheafification.

**Proof.** Note that (3.12.7.2) is an immediate consequence of (3.12.7.1). Write \(i_* F\) for the right Kan extension of \(F\) along \(i : B^{op} \hookrightarrow C^{op}\).

To prove (3.12.7.1), we first show that if \(i_* F\) is a \(\tau\)-sheaf on \(C\), then \(F\) is a \(\tau|_B\)-sheaf on \(B\). Let \(b \in B\) and let \(S_B \subset B\) be a covering sieve on \(b\); we need to show that the natural map
\[ \rho_b^C : F(b) \to \lim_{b' \in S_B^C} F'(b') \]
is an equivalence. Let \(S_C \subset C\) denote the sieve generated by \(S_B \subset C\). Since \(S_B\) is a covering sieve for the topology \(\tau|_B\) on \(B\), the sieve \(S_C\) is a \(\tau\)-covering sieve. Now notice that the map \(\rho_b^C\) factors as a composite
\[ F(b) = i_* F(b) \xrightarrow{\rho_b^C} \lim_{c' \in S_C^C} i_* F(c') \xrightarrow{\rho_b^C} \lim_{b' \in S_B^C} i_* F(b') = \lim_{b' \in S_B^C} F'(b') . \]
The morphism \(\rho_b^C\) is an equivalence because \(i_* F\) is a \(\tau\)-sheaf on \(C\) and \(S_C\) is a \(\tau\)-covering sieve. The morphism \(\rho_b^C\) is an equivalence because \(i_* F\) is the right Kan extension of \(F\).

Now we show that if \(F\) is a \(\tau|_B\)-sheaf on \(B\), then \(i_* F\) is a \(\tau\)-sheaf on \(C\). Let \(c \in C\) and let \(S_C \subset C\) be a \(\tau\)-covering sieve of \(c\). Define
\[ S_B := S \times_C C_{lec} \subset B_{lec} . \]
We need to show that the top horizontal map in the commutative square
\[ \begin{array}{ccc}
i_* F(c) & \longrightarrow & \lim_{c' \in S_C^C} i_* F(c') \\
\downarrow & & \downarrow \\
\lim_{b \in S_B^C} i_* F(b) & \longrightarrow & \lim_{b \in S_B^C} i_* F(b) \\
\end{array} \]
is an equivalence. The vertical maps are equivalences because \(i_* F\) is the right Kan extension of its restriction to \(B\); hence it suffices to show that the lower horizontal map is an equivalence. To see this, first note that the lower horizontal map can be rewritten as a limit of maps
\[ (3.12.8) \lim_{b \in S_B^C} \lim_{b' \in S_B^C} i_* F(b') \to \lim_{\{f : b \to c\} \in S_B^C} \lim_{b' \in S_B^C} i_* F(b') . \]
To conclude, note that since \(i^* i_* F = F\) is a \(\tau|_B\)-sheaf on \(B\) and \(B\) is a basis for \((C, \tau)\), the morphism (3.12.8) is an equivalence. \(\square\)
For sheaves of sets, Lemma 3.12.7 admits a converse: a presheaf of sets $F$ on $C$ is a sheaf if and only if the restriction of $F$ to $B$ is a sheaf and $F$ is the right Kan extension of its restriction. This fails for sheaves of spaces in general. The goal of the remainder of this chapter is to show that this is true if we restrict too hyper sheaves. The technique is to squeeze $\text{Sh}^{\text{hyp}}_c(C)$ between $\text{Sh}^{\text{hyp}}_c(B)$ and $\text{Sh}_c(B)$ and apply the following observation.

3.12.9 Lemma. Let $X$ and $Y$ be $\infty$-topoi, and assume that there are fully faithful geometric morphisms

$$Y^{\text{hyp}} \xrightarrow{f_*} X \xleftarrow{g_*} Y$$

and the composite $g_* f_* : Y^{\text{hyp}} \to Y^{\text{hyp}}$ is the identity. Then $f_* : Y^{\text{hyp}} \to X^{\text{hyp}}$ and $g_* : X^{\text{hyp}} \to Y^{\text{hyp}}$ are mutually inverse equivalences.

Proof. Immediate from the fact that $f_*$ and $g_*$ preserve hypercomplete objects (3.11.7) and the assumption that $g_* f_* \simeq \text{id}_{Y^{\text{hyp}}}$. □

The key technical lemma we need to show that a hypersheaf on $C$ is the right Kan extension of its restriction to $B$ is a relative version of [SAG, Lemma 20.4.5.4]. The following lemma was first noticed by Aoki [5, Lemma A.10].

3.12.10 Lemma. Let $f_* : X \to Y$ be a geometric morphism of $\infty$-topoi and $B \subset Y$ a small full subcategory. Assume that for each object $Y \in Y$, there exists a morphism $e : \coprod_{i \in I} U_i \to Y$ such that $U_i \in B$ for each $i \in I$ and $f^*(e)$ is an effective epimorphism in $X$. Then $f^*(\text{colim}_{U \in B} U)$ is an $\infty$-connective object of $X$.

Proof. Write $X = f^*(\text{colim}_{U \in B} U)$. We prove that $X$ is $n$-connective for each $n \geq 0$ by induction on $n$. For the base case, note that since $f^*$ is a left exact left adjoint, the unique morphism $e : \coprod_{U \in B} f^*(U) \to 1_X$ is an effective epimorphism. The effective epimorphism $e$ factors as a composite

$$\coprod_{U \in B} f^*(U) \longrightarrow \text{colim}_{U \in B} f^*(U) \longrightarrow 1_X,$$

hence the unique morphism $X \to 1_X$ is an effective epimorphism (i.e., $X$ is 0-connective).

For the inductive step, we assume that $X$ is $(n - 1)$-connective and prove that $X$ is $(n - 1)$-connective. That is, we need to show that the diagonal $\Delta_X : X \to X \times X$ is $(n - 1)$-connective. Since $f^*$ is a left exact left adjoint and colimits in an $\infty$-topos are universal, we can rewrite $X \times X$ as the colimit

$$X \times X = \text{colim} \ f^*(U) \times f^*(U').$$

Rewriting $X$ as an iterated colimit

$$X = \text{colim} \ \text{colim} \ f^*(V),$$

we see that we can rewrite the diagonal $\Delta_X$ as a colimit of maps

$$\delta_{U,U'} : \text{colim} \ f^*(V) \to f^*(U) \times f^*(U').$$
Thus it suffices to show that each of the maps $\delta_{U,U'}$ is $(n-1)$-connective. This follows from the inductive hypothesis applied to the geometric morphism

$$X_{f(f^*(U)\times f^*(U'))} \to Y_{f(f^*(U')\times f^*(U'))}$$

whose left exact left adjoint is given by

$$f^* : Y_{f(f^*(U')\times f^*(U'))} \to X_{f(f^*(U)\times f^*(U'))}.$$  

We are finally ready to prove the main result of this section. The following result has appeared in work of Porta–Yue Yu under the additional assumption that representable presheaves are already hypersheaves [94, Proposition 2.22]. We learned of the present proof from Aoki [5, Appendix A].

3.12.11 Proposition. Let $(C, \tau)$ be an $\infty$-site and $i : B \hookrightarrow C$ a basis for the topology $\tau$. Then:

(3.12.11.1) If $F$ is a $\tau$-hypersheaf on $C$, then $F$ is the right Kan extension of its restriction $i_* F$ to $B$.

(3.12.11.2) Right Kan extension defines an equivalence of hypercomplete co-topoi

$$i_* : \text{Sh}_{\tau}^{\text{hyp}}(B) \Rightarrow \text{Sh}_{\tau}^{\text{hyp}}(C)$$

with inverse given by presheaf restriction $i^*$.  

(3.12.11.3) A presheaf $F : C^{\text{op}} \to S$ is a $\tau$-hypersheaf if and only if $i^* F$ is a $\tau|_B$-hypersheaf on $B$ and $F$ is the right Kan extension of $i^* F$.

Proof. Write $L_{\tau} : \text{PSh}(C) \to \text{Sh}_{\tau}(C)$ for the $\tau$-sheafification functor and $\varkappa : C \hookrightarrow \text{PSh}(C)$ for the Yoneda embedding.

First we prove (3.12.11.1). Let $F$ be a $\tau$-hypersheaf on $C$; we prove that the unit $F \to i_* i^* F$ is an equivalence. By the formula for right Kan extension, for each object $c \in C$, the unit $F(c) \to i_* i^* F(c)$ is given by applying the functor $\text{Map}_{\text{PSh}(C)}(-, F)$ to the natural morphism

$$e_c : \text{colim}_{b \in B^\varkappa} \varkappa(b) \to \varkappa(c)$$

in $\text{PSh}(C)$. Since $F$ is a hypercomplete object of $\text{Sh}_{\tau}(C)$, to prove the claim it suffices to show that the morphism $L_{\tau}(e_c)$ is $\infty$-connective for every object $c \in C$. Since $B \subset C$ is a basis for $\tau$, this follows from Lemma 3.12.10 applied to the geometric morphism with left adjoint

$$\text{PSh}(C_{/c}) = \text{PSh}(C)/\varkappa(c) \xrightarrow{1} \text{Sh}_{\tau}(C)/L_{\tau}(\varkappa(c))$$

given by $\tau$-sheafification.

Now we prove (3.12.11.2). By (3.12.11.1), Lemma 3.12.7, and the fact that the right adjoint in a geometric morphism preserves hypercomplete objects (3.11.7), we have fully faithful functors

$$\text{Sh}_{\tau}^{\text{hyp}}(B) \xhookrightarrow{i_*} \text{Sh}_{\tau}^{\text{hyp}}(C) \xhookrightarrow{i^*} \text{Sh}_{\tau}(B),$$
where the functor $i_* : \mathbf{Sh}_\tau^\text{hyp}(B) \hookrightarrow \mathbf{Sh}_\tau^\text{hyp}(C)$ is the right adjoint in a geometric morphism. Thus by Lemma 3.12.9 it suffices to show that the restriction functor

\[(\ref{3.12.12})\]

\[i^* : \mathbf{Sh}_\tau^\text{hyp}(C) \hookrightarrow \mathbf{Sh}_\tau(B)\]

admits a left exact left adjoint. Write $(-)^{\text{hyp}} : \mathbf{Sh}_\tau(C) \to \mathbf{Sh}_\tau^\text{hyp}(C)$ for the left adjoint to the inclusion $\mathbf{Sh}_\tau^\text{hyp}(C) \to \mathbf{Sh}_\tau(C)$. We claim that the composite

\[\mathbf{Sh}_\tau^\text{hyp}(B) \xleftarrow{i_*} \mathbf{Sh}_\tau(C) \xrightarrow{(-)^{\text{hyp}}} \mathbf{Sh}_\tau^\text{hyp}(C)\]

is left adjoint to the restriction (\ref{3.12.12}). To see this, let $G \in \mathbf{Sh}_\tau(B)$ and $F \in \mathbf{Sh}_\tau^\text{hyp}(C)$ and note that since $i_*$ is fully faithful, by (\ref{3.12.11.1}) and the hypercompleteness of $F$ we have natural equivalences

\[
\text{Map}_{\mathbf{Sh}_\tau(B)}(G, i^* F) = \text{Map}_{\mathbf{Sh}_\tau(C)}(i_* G, i_* i^* F) \\
= \text{Map}_{\mathbf{Sh}_\tau(C)}(i_* G, F) \\
= \text{Map}_{\mathbf{Sh}_\tau^\text{hyp}(C)}(i_* G^{\text{hyp}}, F).
\]

Finally, (\ref{3.12.11.3}) is immediate from (\ref{3.12.11.2}) and Lemma 3.12.7. □

3.12.13 Corollary. Let $(C, \tau)$ be an $\infty$-site and $i : B \hookrightarrow C$ a basis for the topology $\tau$. If $\mathbf{Sh}_\tau(C)$ is hypercomplete, then $\mathbf{Sh}_\tau(B)$ is hypercomplete and the geometric morphism $i_* : \mathbf{Sh}_\tau(B) \hookrightarrow \mathbf{Sh}_\tau(C)$ is an equivalence.

3.12.14 Corollary. Let $(C, \tau)$ be an $\infty$-site, $i : B \hookrightarrow C$ a basis for the topology $\tau$, and $n \geq 0$ be an integer. If $\mathbf{Sh}_\tau(C)$ and $\mathbf{Sh}_\tau(B)$ are both $n$-localic, then the geometric morphism $i_* : \mathbf{Sh}_\tau(B) \hookrightarrow \mathbf{Sh}_\tau(C)$ is an equivalence.

3.12.15 Example. Let $P$ be a poset. From Example 3.12.6 we see that right Kan extension along the inclusion $P \subset \text{Open}(P)^\text{op}$ defined by $p \mapsto P_{\geq p}$ defines a fully faithful geometric morphism

\[\text{Fun}(P, S) \hookrightarrow \bar{P}\]

that identifies $\text{Fun}(P, S)$ with the hypercompletion of $\bar{P}$.

In particular, if $\bar{P}$ is a finite poset, then $\bar{P}$ is already hypercomplete [HTT, Remark 7.2.4.18; 24, Lemma 3.13], so right Kan extension defines an equivalence

\[\text{Fun}(P, S) \simeq \bar{P}.
\]

See [8, Corollary 2.4] for a direct proof of this fact.

3.12.16 Warning. If $P$ is an infinite poset, then $\bar{P}$ need not be hypercomplete; see [5, Example A.13] for a counterexample.

3.12.17 Remark. See [54, Lemma C.3] for another very useful criterion for checking that the inclusion of a basis induces an equivalence after passage to sheaves of spaces.
4 Shape theory

This chapter is dedicated to shape theory for ∞-topoi. Section 4.1 establishes the basic material we'll need on protruncated objects. Section 4.2 recalls the definition of the shape and explains a few basic properties of the shape; one of the most important of these is that the protruncated shape of an ∞-topos and its hypercompletion agree. Section 4.3 is devoted to proving that the protruncated shape commutes with inverse limits of bounded coherent ∞-topoi (Corollary 4.3.7). This result provides a computational tool for shapes and will be used repeatedly throughout the text. Section 4.4 explains how to regard profinite spaces as ∞-topoi following [SAG, Appendix E].

4.1 Protruncated objects

In this section, we recall some facts about protruncated objects that we'll need throughout the text. We also record an interesting observation which does not seem to be in the literature (Lemma 4.1.6).

4.1.1 Notation. Let $C$ be a presentable ∞-category. For each integer $n \geq -2$, the protruncated functor $\tau_{\leq n} : \text{Pro}(C) \to \text{Pro}(C \leq n)$ is the unique extension of the $n$-truncation functor $\tau_{\leq n} : C \to C \leq n$ to pro-objects that preserves inverse limits.

4.1.2. Let $C$ be a presentable ∞-category. Then the extension to pro-objects of the functor $C \to \text{Pro}(C_{<\infty})$ given by sending an object $X \in C$ to the inverse system given by its Postnikov tower $\{\tau_{\leq n}(X)\}_{n\geq -2}$ is left adjoint to the inclusion $\text{Pro}(C_{<\infty}) \hookrightarrow \text{Pro}(C)$. We call this left adjoint $\tau_{<\infty} : \text{Pro}(C) \to \text{Pro}(C_{<\infty})$ protruncation. A morphism of pro-objects $f : X \to Y$, regarded as left exact accessible functors $C \to S$, becomes an equivalence after protruncation if and only if for every truncated object $K \in C_{<\infty}$, the induced morphism $f(K) : X(K) \to Y(K)$ is an equivalence.

Since the truncation functors in an ∞-topos preserve finite product [HTT, Lemma 6.5.1.2], if $C$ is an ∞-topos, then the protruncation functor $\tau_{<\infty}$ also preserves finite products.

4.1.3. In the terminology of Mike Artin and Barry Mazur [7, Definition 4.2], Morphisms in the ∞-category $\text{Pro}(S)$ of prospaces that induce equivalences after protruncation are precisely those morphisms that become $\sharp$-isomorphisms in the category $\text{Pro}(h_1 S)$.

4.1.4. Isaksen's strict model structure on pro-simplicial sets [64] presents the ∞-category $\text{Pro}(S)$ of prospaces [57, Lemma 3.1]. The model structure that Isaksen defines in [62] is the left Bousfield localisation of the strict model structure at the $\tau_{<\infty}$-equivalences, hence presents the ∞-category $\text{Pro}(S_{<\infty})$ of protruncated spaces [57, Remark 3.2]. The latter model structure is what is almost always used étale homotopy theory, for example in the recent work of Schmidt–Stix [105] on the étale homotopy type and anabelian geometry.

4.1.5. Let $C$ be a presentable ∞-category. The essentially unique functor

$$\text{mat} : \text{Pro}(C) \to C$$
that preserves inverse limits and restricts to the identity $C \to C$ is right adjoint to the Yoneda embedding $\varepsilon : C \leftarrow \text{Pro}(C)$ [SAG, Example A.8.1.7]. We call mat the materialisation functor. Hence we have adjunctions

$$C \xleftrightarrow{\varepsilon} \text{Pro}(C) \xrightarrow{\tau_{\leq n}} \text{Pro}(C_{\leq n}).$$

If Postnikov towers converge in $C$ (Definition 3.2.11), then the composite left adjoint is also fully faithful:

**4.1.6 Lemma.** Let $C$ be presentable $\infty$-category. If Postnikov towers converge in $C$, the protruncation functor $\tau_{\leq n} : C \to \text{Pro}(C_{\leq n})$ is fully faithful. Moreover, the essential image of $\tau_{\leq n} : C \to \text{Pro}(C_{\leq n})$ is the full subcategory spanned by those protruncated objects $X$ such that for each integer $n \geq -2$, the pro-$n$-truncation $\tau_{\leq n}(X) \in \text{Pro}(C_{\leq n})$ is a constant pro-object.

**Proof.** It suffices to show that for any object $X \in C$, the unit morphism $X \to \text{mat}\tau_{\leq n}(X)$ is an equivalence. This follows from the equivalence

$$\text{mat}\tau_{\leq n}(X) \cong \lim_{n \geq -2} \tau_{\leq n}(X)$$

and the assumption that Postnikov towers converge in $C$. \qed

**4.1.7.** Composing the fully faithful functor $\tau_{\leq n} : S \hookrightarrow \text{Pro}(S_{\leq n})$ with the inclusion $\text{Pro}(S_{\leq n}) \hookrightarrow \text{Pro}(S)$ gives another embedding of spaces into pro-spaces: for a space $K$, the natural morphism of pro-spaces $\varepsilon(K) \to \tau_{\leq n}(K)$ is an equivalence if and only if $K$ is truncated. Unlike the Yoneda embedding, the functor $\tau_{\leq n} : S \hookrightarrow \text{Pro}(S)$ is neither a left nor a right adjoint.

### 4.2 Shape theory

We now recall the basics of shape theory for $\infty$-topoi. The shape is crucial to the study of Stone $\infty$-topoi presented in §4.4. Both shape theory and Stone $\infty$-topoi are key to our development of the stratified shape in Part III and stratified étale homotopy type in Part IV.

**4.2.1 Definition.** The shape $\Pi_{\infty} : \text{Top}_{\infty} \to \text{Pro}(S)$ is the left adjoint to the extension to pro-objects of the fully faithful functor $S \hookrightarrow \text{Top}_{\infty}$ given by

$$\Pi \mapsto S_{ht} = \text{Fun}(\Pi, S)$$

[SAG, §E.2.2]. The shape admits two other very useful descriptions:

- Let $X$ be an $\infty$-topos, and write $\Gamma_{X,1} : X \to \text{Pro}(S)$ for the proexistential left adjoint of $\Gamma_{X}^* : S \to X$. The shape of $X$ is equivalent to the pro-space $\Gamma_{X,1}(1_X)$ [HA, Remark A.1.10; 56, §2].
As a left exact accessible functor $S \to S$, the prospace $\Pi_\infty(X)$ is the composite $\Gamma_\ast, \Gamma_\ast^\ast$ [HTT, §7.1.6; 56, §2]. Under this identification, the shape assigns to a geometric morphism $f_\ast: X \to Y$ with unit $\eta: id_Y \to f_\ast f^\ast$ the morphism of prospaces corresponding to

$$\Gamma_Y, \ast \eta \Gamma_Y^\ast : \Gamma_Y, \ast f_\ast f^\ast \Gamma_Y^\ast = \Gamma_X, \ast \Gamma_X^\ast$$

in $\text{Pro}(S)^{op} \subset \text{Fun}(S, S)$.

4.2.2. The functor $\lambda: \text{Pro}(S) \to \text{Top}_\infty$ given by extending the fully faithful functor $S \to \text{Top}_\infty$ to proöbjects is not itself fully faithful.

For our first example shape computation, please recall that we write $E: \text{Cat}_\infty \to S$ for the left adjoint to the inclusion (Notation 2.2.1).

4.2.3 Example. If $C$ is a small $\infty$-category, then $\Gamma^\ast: S \to \text{Fun}(C, S)$ admits a genuine left adjoint $\Gamma_\ast: \text{Fun}(C, S) \to S$ given by taking the colimit of a diagram $C \to S$. The shape of the $\infty$-topos $\text{Fun}(C, S)$ is thus given by the colimit of the constant diagram at the terminal object of $S$:

$$\Pi_\infty(\text{Fun}(C, S)) = \Gamma_\ast(1_{\text{Fun}(C, S)}) = \text{colim}_C 1_S = E(C).$$

Moreover, the functor $E: \text{Cat}_\infty \to S$ is equivalent to the composite

$$\text{Cat}_\infty \xrightarrow{\text{Fun}(\cdot, S)} \text{Top}_\infty \xrightarrow{\Pi_\infty} S.$$ 

4.2.4 Definition. A geometric morphism $f_\ast: X \to Y$ of $\infty$-topoi is a shape equivalence if the induced morphism

$$\Pi_\infty(f_\ast): \Pi_\infty(X) \to \Pi_\infty(Y)$$

is an equivalence in $\text{Pro}(S)$. We say that an $\infty$-topos $X$ has trivial shape if $\Pi_\infty(X)$ is a terminal object of $\text{Pro}(S)$.

4.2.5. Work of Hoyois [56, Proposition 2.6] shows that a geometric morphism $f_\ast$ is a shape equivalence if and only if $f_\ast$ induces an equivalence of $\infty$-categories of space-valued torsors.

4.2.6 Warning. The pullback (in $\text{Top}_\infty$) of a shape equivalence is not generally a shape equivalence, even when both morphisms are shape equivalences. As an example, consider the space $X = [0, 1]$, and its closed subspace $Z = \{0\}$ and open complement $U = (0, 1]$. Then the $\infty$-topoi $\overline{X}, \overline{U}$, and $\overline{Z}$ all have trivial shape and the natural inclusions

$$\overline{Z} \hookrightarrow \overline{X} \quad \text{and} \quad \overline{U} \hookrightarrow \overline{X}$$

are both shape equivalences [HA, Example A.4.5], however the pullback $\overline{Z} \times_{\overline{X}} \overline{U}$ is the initial $\infty$-topos $\emptyset$, which has empty shape.

That is to say, presheaf $\infty$-topoi are locally of constant shape [HA, Definition A.1.5 & Proposition A.1.8].
4.2.7 Notation. Let \( n \geq -2 \) be an integer. We write
\[
\Pi_n \coloneqq \tau_{\leq n} \circ \Pi : \mathsf{Top} \to \mathsf{Pro}(S_{\leq n})
\]
for the pro-\( n \)-truncated shape (Notation 4.1.1). We write
\[
\Pi_{\leq n} \coloneqq \tau_{\leq n} \circ \Pi : \mathsf{Top} \to \mathsf{Pro}(S_{\leq n})
\]
for the protruncated shape (4.1.2).

4.2.8 Example. Since truncated objects of an \( \infty \)-topos are hypercomplete, for any \( \infty \)-topos \( \mathcal{X} \), the natural geometric morphism \( \mathcal{X} \to \mathcal{X} \) induces an equivalence
\[
\Pi_{\leq n}(\mathcal{X}) \cong \Pi_{\leq n}(\mathcal{X})
\]
on protruncated shapes.

4.3 Shapes of inverse limits

This section is dedicated to proving that the protruncated shape preserves limits of inverse systems of bounded coherent \( \infty \)-topoi and coherent geometric morphisms (Corollary 4.3.7). This follows from the more general fact that the protruncated shape preserves limits of inverse systems of \( \infty \)-topoi and geometric morphisms in which the push-forward preserve filtered colimits of uniformly truncated objects. We learned this from Chough [22, §3]; though Chough’s paper only states this for the profinite shape, his proof works for the protruncated shape.

We first fix some useful notation for the next few results. Please also recall Definition 3.10.2.

4.3.1 Notation. Let \( \mathcal{X} : I \to \mathsf{Top} \) be an inverse diagram of \( \infty \)-topoi. For each morphism \( \alpha : j \to i \) in \( I \), we write
\[
f_{\alpha, *} : \mathcal{X}_j \to \mathcal{X}_i
\]
for the transition morphism. For each \( i \in I \), we write
\[
\pi_{i, * : \lim_{i \in I} \mathcal{X}_i} \to \mathcal{X}_i
\]
for the projection. In addition, assume for each morphism \( \alpha : j \to i \) the functor \( f_{\alpha, * : \lim_{i \in I} \mathcal{X}_i} \to \mathcal{X}_i \)
almost preserves filtered colimits.

4.3.2 Proposition. Under the assumptions of Notation 4.3.1, for each \( i \in I \) and truncated object \( U \in \mathcal{X}_{\leq i} \) we have
\[
(4.3.3) \quad \pi^*_i(U) = \left\{ \operatorname{colim}_{(\alpha, \beta) \in \langle I \rangle \times \langle I \rangle, j \in I} f_{\beta, * : f_{\alpha, * : \lim_{i \in I} \mathcal{X}_i} \to \mathcal{X}_i} (U) \right\}_{j \in I}. \]

A proof of this can be found in work of the third-named author [43, Proposition 2.2], but we present a better proof here.
Proof. Since inverse limits in \( \text{Top}_{\text{coo}} \) are computed in \( \text{Cat}_{\text{coo}} \) (Theorem 3.1.11 = [HTT, Theorem 6.3.1.1]), the assumption that each \( f_{\alpha,*} \) almost preserves filtered colimits guarantees that the right-hand side of (4.3.3) is a well-defined object of \( \lim_{i \in I} X_i \).

For each \( i \in I \), the forgetful functor \( I_{ji} \to I \) is limit-cofinal [HTT, Example 5.4.5.9 & Lemma 5.4.5.12], so we may without loss of generality assume that \( i \in I \) is a terminal object. For each \( k \in I \), write \( f_{k,*} : X_k \to X_i \) for the geometric morphism induced by the unique morphism \( k \to i \). In this case, a simple cofinality argument shows that

\[
\colim_{(\alpha, \beta) \in (I_{ji})_0} f_{\beta,*} f_{\alpha,*} (U) = \colim_{[\beta : k \to j] \in (I_{ji})_0} f_{\beta,*} f_{k,*} (U).
\]

By definition, for all \( V \in X \) we have

\[
\Map_X \left( \left\{ \colim_{[\beta : k \to j] \in (I_{ji})_0} f_{\beta,*} f_{k,*} (U) \right\}, V \right) = \lim_{j \in I} \Map_X \left( \colim_{[\beta \in (I_{ji})_0]} f_{\beta,*} f_{k,*} (U), \pi_{j,*} (V) \right)
\]

\[
= \lim_{j \in I} \Map_X (f_{\beta,*} f_{k,*} (U), \pi_{j,*} (V))
\]

Rewriting the limit as a limit over \( \beta \in \text{Fun}([1], I) \) and using the fact that the constant functor \( I \to \text{Fun}([1], I) \) is limit-cofinal (since it is a left adjoint), we see that

\[
\Map_X \left( \left\{ \colim_{[\beta : k \to j] \in (I_{ji})_0} f_{\beta,*} f_{k,*} (U) \right\}, V \right) = \lim_{\beta \in \text{Fun}([1], I)} \Map_X (f_{\beta,*} f_{k,*} (U), f_{\beta,*} \pi_{k,*} (V))
\]

\[
= \lim_{k \in I} \Map_X (f_{k,*} (U), \pi_{k,*} (V))
\]

\[
= \lim_{k \in I} \Map_X (\pi_{k,*} f_k (U), V)
\]

\[
= \lim_{k \in I} \Map_X (\pi_{k,*} (U), V)
\]

\[
= \Map_X (\pi_{i,*} (U), V).
\]

\[\square\]

4.3.4 Corollary. Keep the assumptions of Proposition 4.3.2. Then for each \( i \in I \) and truncated object \( U \in X_{i, \text{coo}} \), we have an equivalence

\[
\pi_{i,*} \pi_i (U) = \colim_{a \in (I_{ji})_0} f_{a,*} f_{\alpha,*} (U)
\]

of objects of \( X_i \).

Proof. For each \( i \in I \), the forgetful functor \( I_{ji} \to I \) is limit-cofinal [HTT, Example 5.4.5.9 & Lemma 5.4.5.12], so we may without loss of generality assume that \( i \in I \) is a terminal object. Then the claim is clear from Proposition 4.3.2 and the definition of \( \pi_{i,*} \).

\[\square\]

4.3.5 Proposition. Keep the assumptions of Proposition 4.3.2, and in addition assume that for each \( i \in I \) the global sections functor \( \Gamma_{X_i,*} : X_i \to S \) almost preserves filtered colimits. Then the natural morphism

\[\Pi_{\text{coo}} (X) \to \lim_{i \in I} \Pi_{\text{coo}} (X_i)\]

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becomes an equivalence after protruncation.

Proof. For each \( i \in I \), the forgetful functor \( I_i \to I \) is limit-cofinal [HTT, Example 5.4.5.9 & Lemma 5.4.5.12], so we may without loss of generality assume that \( I \) admits a terminal object \( 1 \). Write \( \Gamma_i = \Gamma_{X_i, \ast}, f_i : X_i \to X_1 \) for the geometric morphism induced by the essentially unique morphism \( i \to 1 \) in \( I \), and \( \Gamma : \lim_{j \in I} X_j \to S \) for the global sections geometric morphism.

We want to show that the natural morphism

\[
\colim_{i \in I} \Gamma_i \ast \Gamma_i^\ast \to \Gamma \ast \Gamma^\ast
\]

in \( \text{Fun}(S, S) \) is an equivalence when restricted to truncated spaces (4.1.2). For any truncated space \( K \), we see that we have equivalences

\[
\colim_{i \in I} \Gamma_i \ast \Gamma_i^\ast (K) = \colim_{i \in I} \Gamma_i \ast f_i \ast f_i^\ast \Gamma_i^\ast (K)
\]

\[
= \Gamma_i \ast \left( \colim_{i \in I} f_i \ast f_i^\ast \Gamma_i^\ast (K) \right)
\]

\[
= \Gamma_i \ast \left( \colim_{i \in I} f_i \ast f_i^\ast \right) \ast \Gamma_i^\ast (K)
\]

\[
= \Gamma_i \ast \left( \pi_{1, \ast} \pi_1^\ast \Gamma_i^\ast (K) \right) \quad \text{(assumption on } \Gamma_i \ast \text{)}
\]

\[
= \Gamma_i \ast \pi_1^\ast \Gamma_i^\ast (K) \quad \text{(Proposition 4.3.2)}
\]

\[
= \Gamma_i \ast (\colim_{i \in I} f_i \ast f_i^\ast \Gamma_i^\ast (K)) \quad \text{(Proposition 4.3.2)}
\]

\[
= \Gamma_i \ast \Gamma_i^\ast (K)
\]

4.3.6. In particular, the assumptions of Proposition 4.3.5 are satisfied for inverse systems of coherent \( \infty \)-topoi where the transition morphisms almost preserve filtered colimits [SAG, Theorem A.2.3.1].

From Corollary 3.10.5 and Proposition 4.3.5 we deduce:

4.3.7 Corollary. The protruncated shape

\[
\Pi_{\infty} : \text{Top}_{\infty} \to \text{Pro}(S_{\infty})
\]

preserves inverse limits.

4.4 Profinite spaces & Stone \( \infty \)-topoi

In this section we discuss profinite spaces and their relation to \( \infty \)-topoi, as developed in [SAG, Appendix E]. Recall that we write \( S^\wedge_n \) for the \( \infty \)-category \( \text{Pro}(S_n) \) of profinite spaces (Recollection 2.8.2).

4.4.1. The restriction of the materialization functor \( \text{mat} : \text{Pro}(S) \to S \) to \( S^\wedge_n \) is right adjoint to the composite

\[
S \xrightarrow{\text{Y}} \text{Pro}(S) \xrightarrow{(-)^{\wedge}_n} S^\wedge_n
\]

of the Yoneda embedding \( S \subset \text{Pro}(S) \) followed by profinite completion.
4.4.2 Definition. The profinite shape functor is the composite

\[ \hat{\Pi}_\infty = (\neg)^{\lambda} \circ \Pi_\infty : S^\Lambda \rightarrow S^\Lambda \]

of the shape functor \( \Pi_\infty \) with the profinite completion functor \( (\neg)^{\lambda} : \text{Pro}(S) \rightarrow S^\Lambda \).

4.4.3 Theorem ([SAG, Theorem E.2.4.1]). The composite

\[ \lambda : S^\Lambda \hookrightarrow \text{Pro}(S) \xrightarrow{\lambda} \text{Top}_\infty \]

of the inclusion \( S^\Lambda \subset \text{Pro}(S) \) with the functor \( \lambda \) of (4.2.2) is fully faithful and right adjoint to the profinite shape functor \( \hat{\Pi}_\infty \).

4.4.4 Definition. An \( \infty \)-topos \( \mathcal{X} \) is Stone\( ^{24} \) if \( \mathcal{X} \) lies in the essential image of the fully faithful functor \( \lambda : S^\Lambda \hookrightarrow \text{Top}_\infty \). We write \( \text{Top}^{\text{Stn}}_\infty \subset \text{Top}_\infty \) for the full subcategory spanned by the Stone \( \infty \)-topoi.

Consequently, the inclusion \( \text{Top}^{\text{Stn}}_\infty \hookrightarrow \text{Top}_\infty \) admits a left adjoint

\[ (\neg)^{\text{Stn}} : \text{Top}_\infty \rightarrow \text{Top}^{\text{Stn}}_\infty \]

which we refer to as the Stone reflection.

4.4.5. Since bounded coherent \( \infty \)-topoi are closed under inverse limits in \( \text{Top}_\infty \) (Corollary 3.9.4=[SAG, Corollary A.8.3.3]), Example 3.3.5 shows that Stone \( \infty \)-topoi are bounded coherent.

4.4.6 Proposition ([SAG, Proposition E.3.1.4]). Let \( X \) and \( Y \) be \( \infty \)-topoi. If \( Y \) is Stone, then the \( \infty \)-category \( \text{Fun}_{\infty}(X, Y) \) is a (small) \( \infty \)-groupoid.

4.4.7. If \( Y \) is a Stone \( \infty \)-topos, then since \( S \) is Stone and \( \lambda \) is fully faithful with left adjoint given by the profinite shape, we see that

\[ \text{Pt}(Y) = \text{Map}_{\text{Top}_\infty}(S, Y) = \text{mat} \hat{\Pi}_\infty(Y) \]

Since Stone \( \infty \)-topoi are bounded coherent, (4.4.7) combined with Conceptual Completeness ([Theorem 3.11.2]=SAG, Theorem A.9.0.6)) imply the following 'Whitehead Theorem' for profinite spaces.

4.4.8 Theorem (Whitehead’s Theorem for profinite spaces; [SAG, Theorem E.3.1.6]). The materialisation functor mat: \( S^\Lambda \rightarrow S \) is conservative.

4.4.9 Proposition ([SAG, Proposition E.4.6.1]). Let \( n \in \mathbb{N} \). A morphism \( f \) in \( S^\Lambda \) is \( n \)-truncated if and only if mat(\( f \)) is an \( n \)-truncated morphism of \( S \).

Stone \( \infty \)-topoi have a number of useful alternative characterisations. The first is that, under the assumption of bounded coherence, the conclusion of Proposition 4.4.6=[SAG, Proposition E.3.1.4] actually characterises Stone \( \infty \)-topoi.

\[ ^{24} \text{Lurie calls these } \infty \text{-topoi profinite.} \]
4.4.10 Theorem ([SAG, Theorem E.3.4.1]). Let $X$ be an $\infty$-topos. Then $X$ is Stone if and only if the following conditions are satisfied:

- The $\infty$-topos $X$ is bounded and coherent.
- The $\infty$-category of points $\text{Pt}(X)$ of $X$ is an $\infty$-groupoid.

The next characterisation is that bounded coherent objects are in fact lisse. First we recall the definition of lisse sheaves as well as some material on lisse sheaves that we’ll utilize later on.

4.4.11 Recollection. Let $X$ be an $\infty$-topos. An object $F \in X$ is called a local system if and only if there exists a cover $\{U_i\}_{i \in I}$ of the terminal object of $X$, a corresponding family $\{K_i\}_{i \in I}$ of spaces, and an equivalence

$$F \times U_i \cong \Gamma_X^*(K_i) \times U_i$$

in $X_{/U_i}$ for each $i \in I$.

We say that a local system $F$ as above is a lisse sheaf or lisse object\textsuperscript{25} if, in addition, the set $I$ can be chosen to be finite, and the spaces $K_i$ can be chosen to be $\pi$-finite.

We write $X_{\text{loc sys}} \subseteq X$ and $X_{\text{lisse}} \subseteq X$ for the full subcategories spanned by the local systems and lisse sheaves, respectively. Please note that for any geometric morphism of $\infty$-topoi $f^* : Y \to X$, the pullback $f^* : X \to Y$ preserves lisse objects.

Later we’ll find the following simple characterisation of lisse sheaves as a single pullback very useful:

4.4.12 Lemma ([SAG, Proposition E.2.7.7]). Let $X$ be an $\infty$-topos. Then an object $F$ of $X$ is lisse if and only if there exist: a full subcategory $G \subseteq S_{\pi}$ spanned by finitely many objects, an unique geometric morphism $g_* : X \to S_{G}$, and an unique equivalence $F \cong g^*(I)$, where $I$ classifies the inclusion functor $G \to S$.

The following useful fact is equivalent to the fact that the profinite shape $\breve{\Pi}_{\infty} : \text{Top}_{bc}^{\infty} \to S_{\pi}^\wedge$ preserves inverse limits (see Corollary 4.3.7).

4.4.13 Lemma. For any $\pi$-finite space $G$, the $\infty$-topos $S_{G}$ is cocomplete in $\text{Top}_{bc}^{\infty}$. That is, for any inverse system $\{X_a\}_{a \in A}$ of bounded coherent $\infty$-topoi with limit $X$, the natural functor

$$\text{Fun}_a(X, S_{G}) \to \lim_{a \in A} \text{Fun}_a(X_a, S_{G})$$

is an equivalence.

Now we turn to the important characterization of Stone $\infty$-topoi in terms of lisse sheaves and its consequences.

4.4.14 Proposition ([SAG, Proposition E.3.1.1]). Let $X$ be $\infty$-topos. Then $X$ is Stone if and only if both of the following conditions are satisfied.

\textsuperscript{25}Lurie uses the phrase locally constant constructible.
4.4.15 Corollary ([SAG, Corollary E.3.1.2]). Let \( f_* : X \to Y \) be a geometric morphism between coherent \( \infty \)-topoi. If \( Y \) is Stone, then \( f_* \) is coherent.

By the characterization of Stone \( \infty \)-topoi in terms of lisse sheaves, it is not surprising that the Stone reflection of an \( \infty \)-topos \( X \) can be written as sheaves on \( X^{\text{lisse}} \) with respect to the effective epimorphism topology:

4.4.16 Theorem ([SAG, Theorem E.2.3.2]). Let \( X \) be an \( \infty \)-topos. Then:

- The \( \infty \)-category \( X^{\text{lisse}} \) is a bounded \( \infty \)-pretopos and the inclusion \( X^{\text{lisse}} \to X \) is a morphism of \( \infty \)-pretopoi.
- The inclusion \( X^{\text{lisse}} \to X \) induces a geometric morphism \( X \to \text{Sh}_{\text{eff}}(X^{\text{lisse}}) \) which exhibits \( \text{Sh}_{\text{eff}}(X^{\text{lisse}}) \) as the Stone reflection of \( X \).

Now we assemble all of the ways we can check that a geometric morphism induces an equivalence on Stone reflections.

4.4.17 Corollary ([SAG, Corollary E.2.3.3]). Let \( f_* : X \to Y \) be a geometric morphism of \( \infty \)-topoi. The following are equivalent:

- The induced geometric morphism \( f_*^{\text{Stn}} : X^{\text{Stn}} \to Y^{\text{Stn}} \) is an equivalence of \( \infty \)-topoi.
- The geometric morphism \( f_* \) is a profinite shape equivalence.
- The morphism \( \text{Pt}(f_*^{\text{Stn}}) \) is an equivalence of \( \infty \)-groupoids.
- The pullback functor \( f^* \) restricts to an equivalence of \( \infty \)-categories \( Y^{\text{lisse}} \cong X^{\text{lisse}} \).

Putting together the basics about Stone \( \infty \)-topoi gives an alternative proof of the monodromy equivalence for lisse local systems proven by Bachmann and Hoyois [10, Proposition 10.1].

4.4.18 Proposition. Let \( X \) be an \( \infty \)-topos the unit \( X \to X^{\text{Stn}} \) of the adjunction to Stone \( \infty \)-topoi restricts to an equivalence

\[
\text{Fun}(\Pi_\infty(X), S) = X^{\text{lisse}}.
\]

Proof. Represent the profinite shape \( \Pi_\infty(X) \) by an inverse system \( \{\Pi_\infty S\}_{\alpha \in A^\partial} \) of \( \pi \)-finite spaces so that

\[
\text{Fun}(\Pi_\infty(X), S) = \colim_{\alpha \in A^\partial} \text{Fun}(\Pi_\infty S, S).
\]

By Example 3.3.5 and (3.3.7), for any \( \pi \)-finite space \( \Pi \) we have \( \text{Fun}(\Pi S, S)^{\text{coh}}_{\infty} = \text{Fun}(\Pi, S) \), so

\[
\text{Fun}(\Pi_\infty(X), S) = \colim_{\alpha \in A^\partial} \text{Fun}(\Pi_\infty S)^{\text{coh}}_{\infty}
\]

\[
\cong (\lim_{\alpha \in A^\partial} \text{Fun}(\Pi_\infty S))^{\text{coh}}_{\infty}
\]

(Proposition 3.9.2)

\[
= X^{\text{Stn}}^{\text{coh}}
\]

(Definition 4.4.4)

\[
\cong X^{\text{lisse}}
\]

(Theorem 4.4.16).

\( \Box \)
4.4.19 Example. Let $X$ be a coherent scheme, and write $\mathcal{X}^\text{ét}$ for the finite étale site of $X$: the full subcategory of the étale site $X^\text{ét}$ spanned by the finite étale $X$-schemes, with the induced topology (see [1, §VI.9]). Since the finite étale site is a finitary site, the 1-localic finite étale $\infty$-topos $\mathcal{X}^\text{ét} := \text{Sh}(\mathcal{X}^\text{ét})$ is coherent (Proposition 3.3.10 = [SAG, Proposition A.3.1.3]). The finite étale $\infty$-topos $\mathcal{X}^\text{ét}$ is the classifying $\infty$-topos of the profinite étale fundamental groupoid of $X$ (cf. [SGA 1, Exposé V, Proposition 5.8; 1, Lemma VI.9.11]). In particular, the finite étale $\infty$-topos $\mathcal{X}^\text{ét}$ is Stone.
5 Oriented pushouts & oriented fibre products

Deligne [SGA 7, Exposé XIII; 74] (the latter text written by Gérard Laumon) constructed a 1-topos, called the evanescent or vanishing topos, which he identified as the natural target for the nearby cycles functor. To do so, he identified, in terms of generating sites, the oriented fibre product in a double category of 1-topoi (whose existence was proved first by Giraud [37]). In the ∞-categorical setting, we shall perform an analogous construction in order to describe the link between two strata in a stratified ∞-topos that satisfies suitable finiteness hypotheses.

In §5.1 we review recollements of ∞-topoi. The recollement of bounded coherent ∞-topoi generally is neither bounded nor coherent; Construction 5.1.12 explains how to fix this. Section 5.2 discusses squares of ∞-topoi that commute up to a natural transformation and the definition of oriented pushouts. Section 5.3 discusses internal homs in Top∞ and path ∞-topoi. In Section 5.5 we show that in the setting of smooth manifolds, the oriented fibre product is essentially a tubular neighborhood. In Section 5.6 we give a site-theoretic description of the oriented fibre product and use it to prove that the oriented fibre product of bounded coherent ∞-topoi is again bounded coherent (Lemma 5.6.6). Section 5.7 proves a compatibility between étale geometric morphisms and oriented fibre products (Proposition 5.7.5) that we’ll need to prove a basechange theorem for oriented fibre products in Chapter 7.

5.1 Recollements of higher topoi

We begin with open and closed subtopoi.

5.1.1. Let X be an ∞-topos and U ∈ X. Recall that the overcategory X_U is an ∞-topos, and the forgetful functor j_U: X_U → X admits a right adjoint j_U*, which itself admits a right adjoint j_U. (Recollection 3.1.9). If U is an open of X, the functor j_U is fully faithful.

In this case, we write X_U for the full subcategory of X spanned by those objects F such that F × U = U. The inclusion X_U ⊆ X is accessible and admits a left exact left adjoint, so that X_U is an ∞-topos [HTT, Proposition 7.3.2.3]. We call the ∞-topos X_U the closed complement of X_U, and i_U: X_U → X for the inclusion.

In this case, X is a recollement (0.4.3) of X_U and X_U with gluing functor i_U j^*= j_U*, viz.,

X = X_U U j^* i_U X_U.

5.1.2. Let X be an ∞-topos, and let

i: Z → X \quad \text{and} \quad j: U → X

be geometric morphisms of ∞-topoi that exhibit X as the recollement Z U j^* i: U. Then since i^* and j^* are left exact left adjoints, the natural conservative functor

(i^*, j^*): X → Z U

preserves and reflects colimits and finite limits. (Here Z U denotes the coproduct of Z and U in Top∞, which is the product of Z and U in Cat_{co?},.) In particular, a morphism f in X is:
an effective epimorphism if and only if both $i^*(f)$ and $j^*(f)$ are effective epimorphisms.

$n$-truncated for some integer $n \geq -2$ if and only if both $i^*(f)$ and $j^*(f)$ are $n$-truncated.

See (0.4.4).

5.1.3. A recollement of $\infty$-topoi is tantamount to a geometric morphism of $\infty$-topoi $X \to [1]$. Indeed, if $Z$ and $U$ are $\infty$-topoi, and $\phi : U \to Z$ is a left exact accessible functor, then the recollement $X = Z \cup^\phi U$ is an $\infty$-topos [HA, Proposition A.8.15], and the essentially unique geometric morphisms $Z \to S$ and $U \to S$ now induce a geometric morphism

$$X \to S \cup_{S_0} S = [1].$$

In the other direction, given a geometric morphism $X \to [1]$, the closed topos $X_0 = [0] \times [1] X$ and open topos $X_1 = [1] \times [1] X$ of $X$ form a recollement of $X$.

In a strong sense, the entire theory of stratified $\infty$-topoi (Definition 8.2.1) is a generalisation of this observation.

Since $n$-localic and bounded $\infty$-topoi (Definition 3.2.3 & Construction 3.2.10) are each closed under limits in $\text{Top}_{\text{co}}$, we deduce the following.

5.1.4 Lemma. Let $X$ be an $\infty$-topos, and let $i_* : Z \to X$ and $j_* : U \to X$ be geometric morphisms of $\infty$-topoi that exhibit $X$ as the recollement $Z \cup^{i_* J_*} U$. For any $n \in \mathbb{N}$, if $X$ is $n$-localic or bounded, then both $Z$ and $U$ are each $n$-localic or bounded, respectively.

5.1.5 Warning. We caution, however, that there isn’t a simple converse to Lemma 5.1.4: it is not the case that the recollement of two bounded $\infty$-topoi is necessarily bounded. To ensure this, we need a condition on the gluing functor.

5.1.6 Definition. Let $Z$ and $U$ be two bounded $\infty$-topoi, and let $\phi : U \to Z$ be an left exact accessible functor $\phi : U \to Z$. We say that $\phi$ is a bounded gluing functor if and only if the recollement $X = Z \cup^\phi U$ is bounded.

5.1.7 Question. Do bounded gluing functors admit a simple or useful intrinsic characterisation?

So much for the boundedness of recollements. Let us now turn to coherence (Definition 3.3.1). We can easily characterise the coherent objects of a coherent recollement.

5.1.8 Proposition ([DAG XIII, Proposition 2.3.22]). Let $n \in \mathbb{N}$, let $X$ be an $(n + 1)$-coherent $\infty$-topos, and let $i_* : Z \to X$ and $j_* : U \to X$ be geometric morphisms of $\infty$-topoi that exhibit $X$ as the recollement $Z \cup^{i_* J_*} U$. If $U$ is $0$-coherent, then an object $F \in X$ is $n$-coherent if and only if both $i^*(F)$ and $j^*(F)$ are $n$-coherent. In particular, the $\infty$-topoi $Z$ and $U$ are $n$-coherent.

5.1.9 Warning. We caution again that there isn’t a simple converse to Proposition 5.1.8: as with boundedness, it is not the case that the recollement of two coherent $\infty$-topoi is necessarily coherent.
5.1.10 Definition. Let \( Z \) and \( U \) be two coherent \( \infty \)-topoi, and let \( \phi : U \to Z \) be an left exact accessible functor. We say that \( \phi \) is a coherent gluing functor if and only if the recollement \( X = Z \cup_{\phi} U \) is coherent.

5.1.11. Let \( Z \) and \( U \) be two coherent \( \infty \)-topoi, and let \( \phi : U \to Z \) be an left exact accessible functor. Write \( i_* : Z \hookrightarrow X \) and \( j_* : U \hookrightarrow X \) for the fully faithful functors defining the recollement. Then one can show that the gluing functor \( \phi \) is coherent if the following conditions are satisfied.

- The functor \( j_* \) is quasicompact in the sense that for any quasicompact object \( F \) in \( X \), the object \( j_* F \) in \( U \) is also quasicompact.
- For every \( n \in \mathbb{N} \), every object \( F \) in \( U \) admits a family \( \{ G_{\alpha} \to F \}_{\alpha \in A} \) in which each \( G_{\alpha} \) is \( n \)-coherent, and the family \( \{ \phi(G_{\alpha}) \to \phi(F) \}_{\alpha \in A} \) is a covering in \( Z \).

5.1.12 Construction. Let \( Z \) and \( U \) be bounded coherent \( \infty \)-topoi, and let \( \phi : U \to Z \) be an left exact accessible functor. Form the recollement \( X' = Z \cup_{\phi} U \), and write \( i_* : Z \hookrightarrow X' \) and \( j_* : U \hookrightarrow X' \) for the induced closed and open embeddings. Consider the full subcategory \( X_0 \subseteq X' \) spanned by those objects \( F \) such that \( i_* F \) and \( j_* F \) are each truncated coherent, so that \( X_0 \) is the oriented fibre product (0.4.1) in \( \text{Cat}_{\infty,\delta} \):

\[
X_0 = Z_{\text{coh}}^\text{tr} \downarrow_Z U_{\text{coh}}^\text{tr}.
\]

Then since \( X_0 \subseteq X \) is closed under finite limits, finite coproducts, and the formation of geometric realisations of groupoid objects, the \( \infty \)-category \( X_0 \) is an \( \infty \)-pretopos and the inclusion \( X_0 \hookrightarrow X \) is a morphism of \( \infty \)-pretopoi (Definition 3.8.2). Moreover, by (5.1.2) every object of \( X_0 \) is truncated and by (0.4.2) the \( \infty \)-category \( X_0 \) is \( \delta_0 \)-small, hence \( X_0 \) is a bounded \( \infty \)-pretopos (Definition 3.8.8). Consequently, we may form the bounded coherent \( \infty \)-topos (Notation 3.8.6)

\[
X = \text{Sh}_{\text{eff}}(X_0).
\]

By [SAG, Proposition A.6.4.4], the inclusion \( X_0 \hookrightarrow X' \) extends (essentially uniquely) to a comparison geometric morphism \( r_* : X' \to X \), which is not in general an equivalence, but restricts to an equivalence \( r_* : X_0^\text{tr} \to X_0 \). The geometric morphisms \( r_* i_* \) and \( r_* j_* \) are each coherent by construction. We therefore call \( X \) the bounded coherent recollement, and we write

\[
Z \cup_{\text{be}}^\phi U = X.
\]

5.1.13 Lemma. Let \( Z \) and \( U \) be bounded coherent \( \infty \)-topoi, and let \( \phi : U \to Z \) be an left exact accessible functor. Then the natural geometric morphism

\[
Z \cup_{\text{be}}^\phi r_* i_* U \to Z \cup_{\text{be}}^\phi U
\]

is an equivalence.
Proof. Write $X = Z \cup_{bc} U$. The object $j_1_U \in Z \cup_{bc} U$, is the object 

$$(\emptyset_Z, 1_U, \emptyset_Z \to \phi(1_U)),$$

which is an open in $X$ as well as an object of the co-pretopos $X_0$ of Construction 5.1.12. Thus $j^* r^*$ restricts to an equivalence

$$(X_{j_1 U})^\text{coh}_{<\infty} \xrightarrow{\text{ coh}} U^\text{coh}_{<\infty},$$

whence the functor $r_\ast j^* : U \to X_{j_1 U}$ is an equivalence. The truncated coherent objects of the closed subtopos $X_{j_1 U}$ are precisely those of the form $(F_Z, 1_U, F_Z \to \phi(1_U))$ for some truncated coherent object $F_Z$ of $Z$. Hence $i^* r^*$ restricts to an equivalence

$$(X_{j_1 U})^\text{coh}_{<\infty} \xrightarrow{\text{ coh}} Z^\text{coh}_{<\infty},$$

whence the functor $i_\ast r_\ast : Z \to X_{j_1 U}$ is an equivalence. 

5.1.14 Lemma. Let $Z$, and $U$ be bounded coherent co-topoi, and let $\phi : U \to Z$ be a bounded coherent gluing functor. Then $Z \cup_{bc} U$ is the bounded coherent recollement.

Proof. This follows from Proposition 5.1.8=[DAG XIII, Proposition 2.3.22] combined with Theorem 3.8.9=[SAG, Theorem A.7.5.3].

The critical point that we use repeatedly in the sequel is the observation that the bounded coherent recollement depends only upon the restriction of the gluing functor to truncated coherent objects. More precisely, let $Z$ and $U$ be bounded coherent co-topoi, and let $\phi : U \to Z$ and $\phi' : U \to Z$ be two accessible, left exact functors. Let $\eta : \phi \to \phi'$ be a natural transformation. Now $\eta$ induces a functor

$$\eta^* : Z \cup_{bc} U \to Z \cup_{bc} U$$

given by the assignment

$$(z, u, \alpha : z \to \phi(u)) \mapsto (z, u, \eta_u \alpha : z \to \phi'(u)).$$

The functor $\eta^*$ preserves colimits and finite limits; consequently, $\eta^*$ is the left adjoint of a geometric morphism $\eta_*$. Then since $\eta^*$ restricts to a functor

$$\eta^* : Z^\text{coh}_{<\infty} \downarrow_{Z, \phi} U^\text{coh}_{<\infty} = X^\text{coh}_{<\infty} \to (X')^\text{coh}_{<\infty} = Z^\text{coh}_{<\infty} \downarrow_{Z, \phi'} U^\text{coh}_{<\infty},$$

i.e., $\eta^*$ preserves truncated coherent objects, $\eta^*$ induces a geometric morphism

$$\eta_* : Z \cup_{bc} U \to Z \cup_{bc} U$$

on bounded coherent recollements.

5.1.15 Proposition. Let $Z$ and $U$ be bounded coherent co-topoi, and let $\phi : U \to Z$ and $\phi' : U \to Z$ be two accessible, left exact functors. Let $\eta : \phi \to \phi'$ be a natural transformation. If $\eta_U^\text{coh}$ is an equivalence, then $\eta$ induces an equivalence

$$Z \cup_{bc} U \xrightarrow{\text{ coh}} Z \cup_{bc} U.$$
5.2 Oriented squares & oriented pushouts

To speak of oriented pullbacks of ∞-topoi without finding ourselves buried under a mass of pernicious details (or unproved claims) about double ∞-categories or (∞,2)-categories, we express the universal property of the oriented pullback in simple terms. The key kind of square we will have to contemplate is the following.

5.2.1 Notation. We exhibit data of geometric morphisms $f_* : X \to Z$, $g_* : Y \to Z$, $p_* : W \to X$, and $q_* : W \to Y$, along with a (not necessarily invertible) natural transformation $\tau : g_* q_* \to f_* p_*$ by the single square

$$
\begin{array}{ccc}
W & \xrightarrow{g_*} & Y \\
\downarrow{p_*} & \nearrow{\tau} & \downarrow{g_*} \\
X & \xrightarrow{f_*} & Z \\
\end{array}
$$

5.2.2 Warning. It seems that this convention for writing 2-cells is the opposite of what's written in some of the 1-topos theory literature [66; 84; 85], but it agrees with of the algebro-geometric literature [SGA 7II, Exposé XIII; 74]. We therefore textitisise that our 2-morphisms are natural transformations between the right adjoints.

The oriented fibre product in $\text{Cat}_{\infty,\delta}^1$ of a diagram of ∞-topoi recovers not the oriented fibre product in $\text{Top}_{\infty}$, but rather the oriented pushout in $\text{Top}_{\infty}$. We shall also have to contemplate the oriented pushout in $\text{Top}^{bc}_{\infty}$.

5.2.4 Construction. The ∞-category $\text{Top}_{\infty}$ is tensored over the ∞-category $\text{Cat}_{\infty,\delta}$. Indeed, if $W$ is an ∞-topos and $C$ is a $\delta_0$-small ∞-category, then the ∞-category $\text{Fun}(C, W)$ is an ∞-topos. Moreover, the functor $C \to \text{Fun}_s(W, \text{Fun}(C, W))$ that carries an object $c \in C$ to the right adjoint of the functor $\text{Fun}(C, W) \to W$ given by evaluation at $c$ induces an equivalence of ∞-categories

$$\text{Fun}_s(\text{Fun}(C, W), Z) \simeq \text{Fun}(C, \text{Fun}_s(W, Z))$$

for any ∞-topos $Z$.

Let $W$, $Z$, and $U$ be ∞-topoi, and let $p_* : W \to Z$ and $q_* : W \to U$ be two geometric morphisms. The recollement $Z \downarrow_{p_*, q_*} U$ can be identified with the oriented fibre product

$$Z \downarrow_{w} U$$

formed in $\text{Cat}_{\infty,\delta}$, with respect to the left adjoints $p^*$ and $q^*$. We note that $Z \uparrow_{p^*, q^*} U$ is an ∞-topos. This ∞-topos enjoys the following universal property: a geometric morphism

$$\omega(f, g, \tau)_* : Z \uparrow_{p^*, q^*} U \to X$$

determines and is determined by an oriented square

$$
\begin{array}{ccc}
W & \xrightarrow{g_*} & U \\
\downarrow{p_*} & \nearrow{\tau} & \downarrow{g_*} \\
Z & \xrightarrow{f_*} & X \\
\end{array}
$$

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This universal property specifies the $\infty$-topos $Z \sqcup^{p,q} U$ essentially uniquely. We write

$$Z \sqcup^W U := Z \sqcup^{p,q} U,$$

and we call this $\infty$-topos the oriented pushout of $p_*$ and $q_*$. In this case, we write $i_* : Z \hookrightarrow Z \sqcup^W U$ for the closed embedding and $j_* : U \hookrightarrow Z \sqcup^W U$ for its open complement.

**5.2.5 Warning.** If $Z$, $U$, and $W$ are all bounded coherent, and if $p_*$ and $q_*$ are both coherent geometric morphisms, Warning 5.1.5 & Warning 5.1.9 still apply: we cannot ensure that the oriented pushout $Z \sqcup^\emptyset U$ is either bounded or coherent (cf. [SGA 4_1, Exposé VI, §4]).

**5.2.6 Construction.** Consider an oriented square

$$
\begin{array}{ccc}
W & \xrightarrow{q_*} & U \\
\downarrow{p_*} & & \downarrow{g_*} \\
Z & \xrightarrow{j_*} & X
\end{array}
$$

where all $\infty$-topoi are bounded coherent and all geometric morphisms are coherent. For any truncated coherent object $G \in X$, the object $\omega(f, g, \tau)^* G$ is truncated, and the objects

$$i^* \omega(f, g, \tau)^* G = f^* G \quad \text{and} \quad j^* \omega(f, g, \tau)^* G = g^* G$$

are each truncated coherent, whence $\omega(f, g, \tau)_*$ factors through the bounded coherent recollement $Z \sqcup^{p,q} U$ (Construction 5.1.12) in an essentially unique manner. Consequently, we write

$$Z \sqcup^{W}_{bc} U := Z \sqcup^{p,q} Y_{bc} U,$$

and call this $\infty$-topos the bounded coherent oriented pushout. This is the oriented pushout that is correct in $\mathbf{Top}_{bc}^{\infty}$. Accordingly, one has an equivalence of $\infty$-pretopoi

$$(Z \sqcup^{W}_{bc} U)^{coh} \cong Z^{coh}_{bc} \downarrow U^{coh}_{bc}.$$

Please observe that by construction, in the square

$$
\begin{array}{ccc}
W & \xrightarrow{q_*} & U \\
\downarrow{p_*} & & \downarrow{j_*} \\
Z & \xrightarrow{j_*} & Z \sqcup^{W}_{bc} U
\end{array}
$$

the natural basechange morphism

$$\text{BC}_{\tau} : i^* j_* \rightarrow p_* q_*$$

becomes an equivalence after restriction to $U^{coh}_{bc}$. A thorough study of basechange morphisms will occupy Chapter 7.
5.3 Internal homs & path ∞-topoi

Oriented fibre products have the universal property that is dual to that of oriented push-outs. In order to define them, we must identify the cotensor of $\text{Top}_{\infty}$ over $\text{Cat}_{\infty,\Delta^0}$, or at least over $\text{Pos}$. Partly in order to define oriented fibre products of $\infty$-topoi now and partly to define the nerve construction for stratified $\infty$-topoi later (Construction 8.5.1), we recall some facts about the internal hom in $\infty$-topoi. The first point to be made about the internal hom is that it doesn’t always exist.

5.3.1 Recollection. Recall [SAG, Theorem 21.1.6.11] that an $\infty$-topos $\mathcal{W}$ is exponentiable if and only if the functor $-\times \mathcal{W} : \text{Top}_{\infty} \to \text{Top}_{\infty}$ admits a right adjoint $\text{Mor}(\mathcal{W}, -)$. If $\mathcal{W}$ is exponentiable, then for any $\infty$-topos $\mathcal{X}$, we have a natural equivalence

$$\text{Pt}(\text{Mor}(\mathcal{W}, \mathcal{X}))^{op} = \text{Fun}_*(S, \text{Mor}(\mathcal{W}, \mathcal{X})) \Rightarrow \text{Fun}_*(\mathcal{W}, \mathcal{X}).$$

We thus call $\text{Mor}(\mathcal{W}, \mathcal{X})$ the mapping $\infty$-topos. Any compactly generated $\infty$-topos is exponentiable, and exponentiable $\infty$-topoi admit several useful characterizations: see [SAG, Theorem 21.1.6.12].

5.3.2 Example. Let $S$ be spectral topological space $S$. Then $\tilde{S}$ is compactly generated [HTT, Proposition 6.5.4.4; SAG, Proposition 21.1.7.8], so for any $\infty$-topos $\mathcal{X}$, there exists a mapping $\infty$-topos $\text{Mor}(\tilde{S}, \mathcal{X})$. A point $s \in S$ induces a geometric morphism

$$\text{Mor}(\tilde{S}, \mathcal{X}) \to \text{Mor}(\tilde{s}, \mathcal{X}) \cong \mathcal{X},$$

and the geometric morphism $\tilde{S} \to S$ induces a diagonal geometric morphism

$$\Delta_* : Z = \text{Mor}(S, \mathcal{X}) \to \text{Mor}(\tilde{S}, \mathcal{X}).$$

5.3.3 Example. If $P$ is a finite poset, then the functor $\text{Mor}(\tilde{P}, -) : \text{Top}_{\infty} \to \text{Top}_{\infty}$ can be identified with the unique limit-preserving endofunctor of $\text{Top}_{\infty}$ such that, for any small $\infty$-category $C$, one has an equivalence

$$\text{Mor}(\tilde{P}, \text{Fun}(C, S)) = \text{Fun}(\text{Fun}(P, C), S)$$

via the natural functor. In particular, if $P$ and $Q$ are finite posets, then

$$\text{Mor}(\tilde{P}, \tilde{Q}) = \text{Fun}(P, Q).$$

5.3.4 Definition. For any $\infty$-topos $X$, the $\infty$-topos $\text{Mor}([1], X)$ is called the path $\infty$-topos of $X$ [SAG, Definition 21.3.2.3]. We write

$$\text{Path}(X) = \text{Mor}([1], X).$$

5.3.5 Example. As a special case of Example 5.3.3, for any small $\infty$-category $C$, there is a natural equivalence

$$\text{Path}(\text{Fun}(C, S)) = \text{Fun}(\text{Fun}([1], C), S).$$

5.3.6 Lemma. Let $n \in \mathbb{N}$, and let $Z$ be an $n$-localic $\infty$-topos. Then the path $\infty$-topos $\text{Path}(Z)$ is $n$-localic.

Proof. This is a special case of [SAG, Lemma 21.1.7.3]. \qed
5.4 Oriented fibre products

We are now ready to construct the oriented fibre product of ∞-topoi and to relate it to the classical oriented fibre product of 1-topoi (Lemma 5.4.13).

5.4.1 Definition. If $f_* : X \to Z$ and $g_* : Y \to Z$ are two geometric morphisms of ∞-topoi, then the oriented fibre product is the pullback

$$X \times_Z Y := X \times_{\text{Mor}(\{0\}, Z)} \text{Mor}(\{1\}, Z) \times_{\text{Mor}(\{1\}, Z)} Y$$

in $\text{Top}_\infty$. We write $\text{pr}_{1,*} : X \times_Z Y \to X$ and $\text{pr}_{2,*} : X \times_Z Y \to Y$ for the natural geometric morphisms.

Thus a geometric morphism $\psi(p, q, r)_* : W \to X \times_Z Y$ determines and is determined by a square (5.2.2). This universal property specifies the ∞-topos $X \times_Z Y$ essentially uniquely.

5.4.2 Warning. Please note that the oriented fibre product in $\text{Top}_\infty$ is not the oriented/lax pullback in $\text{Cat}_{\infty, \delta}$; we will therefore take pains to express clearly where the oriented fibre product is taking place.

Additionally, in this paper, the symbol '$\times$' is only ever used for the oriented fibre product in $\text{Top}_\infty$; we only use the notation $X \downarrow_\delta Y$ for the oriented fibre product in some $\text{Cat}_{\infty, \delta}$ (see (0.4.1)).

5.4.3. Please observe that since the exponential functor $\text{Path}(-) : \text{Top}_\infty \to \text{Top}_\infty$ is a right adjoint and limits in $\text{Fun}(\Delta^2, \text{Top}_\infty)$ are computed pointwise, the functor $\text{Fun}(\Delta^2, \text{Top}_\infty) \to \text{Top}_\infty$ given by the formation of the oriented fibre product preserves limits.

5.4.4 Example. When $Z = S$, the oriented fibre product reduces to the product in $\text{Top}_\infty$:

$$X \times_S Y = X \times Y.$$

5.4.5. Let $f_* : X \to Z$ and $g_* : Y \to Z$ be geometric morphisms of ∞-topoi. Then under the identifications $X = X \times_S S$ and $Y = S \times_S Y$, the projections $\text{pr}_{1,*} : X \times_Z Y \to X$ and $\text{pr}_{2,*} : X \times_Z Y \to Y$ are equivalent to $\text{id}_X \times_{\text{Id}_Z} \Gamma_{Y,*}$ and $\Gamma_{X,*} \times_{\text{Id}_Z} \text{id}_Y$, respectively (Notation 3.1.7).

5.4.6 Example. For any ∞-topos $X$, the oriented fibre product $X \times_X X$ is canonically identified with the path ∞-topos $\text{Path}(X)$.

The next thing to notice is that the points of an oriented fibre product of ∞-topoi are the oriented fibre product of the corresponding ∞-categories of points.

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5.4.7. For any $\infty$-topos $E$, the functor

$$\text{Fun}_* (E, -)^{op} : \text{Top}_\infty \to \text{Cat}_{\infty, \delta}$$

commutes with cotensors with $\text{Cat}_{\infty, \delta}^1$ (in particular, cotensoring with $[1]$) and pullbacks of $\infty$-topoi, hence $\text{Fun}_* (E, -)^{op}$ carries oriented fibre products in $\text{Top}_\infty$ to oriented fibre products in $\text{Cat}_{\infty, \delta}^1$.

Specialising to the case $E = S$, we deduce the following.

5.4.8 Lemma. The functor $\text{Pt} : \text{Top}_\infty \to \text{Cat}_{\infty, \delta}^1$ carries oriented fibre products in $\text{Top}_\infty$ to oriented fibre products in $\text{Cat}_{\infty, \delta}^1$. That is, if $f_* : X \to Z$ and $g_* : Y \to Z$ are geometric morphisms of $\infty$-topoi, then the natural functor

$$\text{Pt}(X \times_Z Y) \to \text{Pt}(X) \downarrow_{\text{Pt}(Z)} \text{Pt}(Y)$$

is an equivalence.

5.4.9 Example. There is a canonical geometric morphism

$$\psi(\text{pr}_1, \text{pr}_2, \text{id})_* : X \times_Z Y \to X \times_Z Y.$$ 

5.4.10 Example. The $\infty$-topos $X \times_Z Y$ is called the evanescent (or vanishing) $\infty$-topos of $f_*$, and the natural functor

$$\Psi_{f_*} := \psi(\text{id}_X, f, \text{id})_* : X \to X \times_Z Y$$

is called the nearby cycles functor. Dually, the $\infty$-topos $Z \times_Y Y$ is called the coëvanescent (or covanishing) $\infty$-topos of $g_*$, and the natural functor

$$\Psi_{g*} := \psi(g, \text{id}_Y, \text{id})_* : Y \to Z \times_Y Y$$

is called the conearby cycles functor.

The oriented fibre product can be decomposed into fibre products in $\text{Top}_\infty$ involving the evanescent and coëvanescent $\infty$-topoi as follows: we have equivalences

$$X \times_Z Y = (X \times_Z Z) \times_Z Y \quad \text{and} \quad X \times_Z Y = X \times_Z (Z \times_Z Y),$$

and, more symmetrically,

$$X \times_Z Y \cong (X \times_Z Z) \times_{\text{Path}(Z)} (Z \times_Z Y).$$

5.4.11 Example. Keep the notations of Definition 5.4.1, and let $p_* : Z \to Z'$ be a fully faithful geometric morphism. Then $p_*$ induces an equivalence of $\infty$-topoi

$$X \times_Z Y \Rightarrow X \times_{Z'} Y.$$

To see this, simply note that $X \times_{Z} Y$ and $X \times_{Z'} Y$ have the same universal property since $p_*$ is fully faithful. Hence for the purpose of computing oriented fibre products, we may assume that $Z$ is a presheaf $\infty$-topos.
We're mostly interested in working with 1-localic ∞-topoi or more generally bounded ∞-topoi. The following lemma says that taking oriented fibre products doesn’t take us out of these subcategories of all ∞-topoi.

5.4.12 Lemma. Let \( f_* : X \to Z \) and \( g_* : Y \to Z \) be geometric morphisms of ∞-topoi. If \( X, Y, \) and \( Z \) are \( n \)-localic (Definition 3.2.3), so is the oriented fibre product \( X \times_Z Y \).

Moreover, if \( X, Y, \) and \( Z \) are bounded (Construction 3.2.10), so is the oriented fibre product \( X \times_Z Y \).

Proof. For the first assertion, by Lemma 5.3.6 the oriented fibre product is a limit of \( n \)-localic ∞-topoi, hence \( n \)-localic. The second claim follows from the fact that formation of the oriented fibre product preserves limits (5.4.3).

The 1-toposic oriented fibre product \([60; 61; 74; 84; 92]\) is related to the oriented fibre product of corresponding 1-localic ∞-topoi via the following easy result.

5.4.13 Lemma. Let \( f_* : X \to Z \) and \( g_* : Y \to Z \) be geometric morphisms of 1-topoi, and write \( X', Y', \) and \( Z' \) for the corresponding 1-localic ∞-topoi associated to \( X, Y, \) and \( Z, \) respectively. Then the oriented fibre product of 1-topoi \( X \times_Z Y \) is canonically equivalent to the 1-topos of \( 0 \)-truncated objects of \( X' \times_Z Y' \).

Proof. As a consequence of Lemma 5.4.12, the equivalence of \( \infty \)-categories \( \tau_{\leq 0} : \text{Top}_1^\infty \Rightarrow \text{Top}_1 \) from 1-localic ∞-topoi to 1-topoi (Definition 3.2.3) respects cotensors by the 1-category \([1]\). In light of the equivalence \( \tau_{\leq 0} : \text{Top}_1^\infty \Rightarrow \text{Top}_1 \), the claim now follows from the definitions of the oriented fibre product in the setting of ∞-topoi and 1-topoi.

5.5 Oriented fibre products & tubular neighbourhoods of manifolds

Let us now relate oriented fibre products and tubular neighborhoods in the setting of smooth manifolds. This will serve as a partial justification for our use of the language of ‘deleted tubular neighbourhoods’ of closed subtopoi, and in addition it makes it possible to compute these deleted tubular neighbourhoods in the case of smooth complex varieties with smooth stratifications.

5.5.1 Notation. Let \( X \) be a smooth manifold and \( i : Z \hookrightarrow X \) the inclusion of a closed submanifold. Write \( U = X \setminus Z \) and \( j : U \hookrightarrow X \) for the inclusion of the open complement of \( Z \) in \( X \). Let \( p : N_i \to Z \) denote the normal bundle of \( Z \), and \( z : Z \hookrightarrow N_i \) its zero section. Let \( t : N_i \to X \) be a choice of tubular neighbourhood of \( Z \) in \( X \), so that \( tz = i \).

5.5.2. Keep Notation 5.5.1. Since \( pz = \text{id}_Z \), the geometric morphism

\[
p_* : N_i \to Z
\]

exhibits the ∞-topos \( N_i \) as local over \( Z \) with center \( z_* \). In particular, \( p_* \) is a shape equivalence.

5.5.3. Keep Notation 5.5.1, and write \( \eta : \text{id}_Z \to z_* z^* \) for the unit of the adjunction \( z^* \dashv z_* \). Then since \( tz = i \) and \( z^* = p_* \), we see that \( t_* z_* z^* = i_* p_* \). Thus \( \eta \) induces a natural transformation

\[
t_* \eta : t_* \to t_* z_* z^* = i_* z^*.
\]
The natural transformation $t_*\eta$ thus provides an oriented square

\[
\begin{array}{ccc}
\mathbb{N}_j & \xrightarrow{t_*} & X \\
\downarrow p_* & \downarrow \downarrow i_*\eta & \downarrow \\
\bar{Z} & \xleftarrow{i_*} & \bar{X}.
\end{array}
\] (5.5.4)

The square (5.5.4) induces an essentially unique geometric morphism

\[ f_t^*: \mathbb{N}_j \to \bar{Z} \times_X \bar{X} \]

such that $\text{pr}_{1*} f_t^* = p_*$, $\text{pr}_{2*} f_t^* = t_*$, and $t_*\eta = f_t^*\tau$ where $\tau: \text{pr}_{2*} \to i_* \text{pr}_{1*}$ is the defining natural transformation. Since $p_*$ and $\text{pr}_{1*}$ are shape equivalences, $f_t^*$ is also a shape equivalence. Moreover, note that since the geometric morphism $t_*$ is fully faithful, the geometric morphism $f_t^*$ is also fully faithful.

5.6 Generating $\infty$-sites for oriented fibre products

We now describe a generating $\infty$-site for the oriented fibre product in the setting of sheaf $\infty$-topoi. This description is adapted from Deligne's site-theoretic description \[61, \text{Exposé XI, §1; 74, 3.1.3}]. We employ it to deduce that the oriented fibre product of bounded coherent $\infty$-topoi and coherent geometric morphisms is bounded coherent (Lemma 5.6.6). We begin with oriented fibre products of presheaf $\infty$-topoi.

5.6.1 Construction. Let $X, Y,$ and $Z$ be $\delta_0$-small $\infty$-categories, each of which admit finite limits. Let $f^*: Z \to X$ and $g^*: Z \to Y$ be left exact functors that induce, via precomposition, geometric morphisms

\[ f_*: \text{PSh}(X) \to \text{PSh}(Z) \quad \text{and} \quad g_*: \text{PSh}(Y) \to \text{PSh}(Z) \]
on $\infty$-categories of presheaves of spaces.

Represent $f^*$ and $g^*$ as a cartesian fibration $m: M \to \Lambda^2_2$, so that the fibres over the vertices 0, 1, and 2 are $X, Y,$ and $Z$, respectively, and $m$ is classified by the diagram $X \leftarrow Z \to Y$. Now form the $\infty$-category

\[ \widetilde{W}(f, g) = \text{Fun}_{\Lambda^2_2}(\Lambda^2_2, M) = \text{Fun}(\Delta^1, X) \times_{\text{Fun}(\Delta^1, X)} Z \times_{\text{Fun}(\Delta^1, Y)} \text{Fun}(\Delta^1, Y) \]
of sections of $m$. Let us write $K_Y$ for the class of morphisms $\phi: \Delta^1 \times \Lambda^2_2 \to M$ in $\text{Fun}_{\Lambda^2_2}(\Lambda^2_2, M)$ of the form

\[
\begin{array}{ccc}
u_X & \xrightarrow{u_Z} & \xleftarrow{\phi_Y} \nu_Y \\
\downarrow \phi_X & \downarrow & \downarrow \phi_Y \\
u_X & \xrightarrow{u_Z} & \xleftarrow{u_Y}
\end{array}
\]
in which $\phi_X$ is an equivalence, and the diagram above exhibits $\phi_Y$ as the pullback of $g^*\phi_Z$. Dually, let us write $K_X$ for those morphisms $\phi$ in which $\phi_Y$ is an equivalence, and the diagram above exhibits $\phi_X$ as the pullback of $f^*\phi_Z$.
We now define two new ∞-categories by inverting these morphisms in the ∞-categorical sense (0.2.1):

\[ \overset{\sim}{W}(f, g) := K_Y^{-1} W(f, g) \quad \text{and} \quad W(f, g) = K_X^{-1} \overset{\sim}{W}(f, g). \]

5.6.2. The ∞-category \(\overset{\sim}{W}(f, g)\) admits finite limits, which are computed pointwise. The sets \(K_Y\) and \(K_X\) are stable under composition and pullback. It follows that the classes \(K_Y\) and \(K_X\) each give rise to right calculi of fractions on \(\overset{\sim}{W}(f, g)\) in the sense of Cisinski’s book [23, Theorem 7.2.16].

Consequently, the mapping spaces in \(\overset{\sim}{W}(f, g)\) admit a very simple description: for any objects \(u, v \in \overset{\sim}{W}(f, g)\), write

\[ A(u, v) \subseteq \overset{\sim}{W}(f, g)_u \times_{\overset{\sim}{W}(f, g)} \overset{\sim}{W}(f, g)_v \]

for the full subcategory spanned by those diagrams \(u \leftarrow w \rightarrow v\) in which the morphism \(u \leftarrow w\) lies in \(K_Y\). Then one has a natural equivalence of ∞-groupoids

\[ \text{Map}_{\overset{\sim}{W}(f, g)}(u, v) \simeq E A(u, v). \]

Furthermore, the ∞-categories \(\overset{\sim}{W}(f, g)\) and \(W(f, g)\) admit finite limits, and the localisations

\[ \overset{\sim}{W}(f, g) \to W(f, g) \quad \text{and} \quad W(f, g) \to W(f, g) \]

each preserve finite limits [23, Corollary 7.1.16 & Theorem 7.2.25].

5.6.3 Construction. Keep the notations of Construction 5.6.1. We also have left exact functors \(p^* : X \to \overset{\sim}{W}(f, g)\) and \(q^* : Y \to \overset{\sim}{W}(f, g)\) defined by the assignments

\[ x \mapsto [x \to 1 \leftarrow 1] \quad \text{and} \quad y \mapsto [1 \to 1 \leftarrow y]. \]

We also regard the left exact functors \(p^*\) and \(q^*\) as landing in \(\overset{\sim}{W}(f, g)\) and \(W(f, g)\) by composing with the relevant localisations.

There exists a section \(\sigma : Z \to \overset{\sim}{W}(f, g)\) of the natural projection \(\overset{\sim}{W}(f, g) \to Z\) that carries \(z\) to the cartesian section \(f^*(z) \to z \leftarrow g^*(z)\). We thus have natural transformations

\[ p^* f^* \xleftarrow{\theta} \sigma \xrightarrow{\xi} q^* g^* \]

where for any \(z \in Z\), the components \(\theta_z\) and \(\xi_z\) are given by the diagram

\[
\begin{array}{ccccccc}
1 & \xrightarrow{1} & 1 \\
\downarrow & & \downarrow \\
1 & \xrightarrow{1} & 1 & \xleftarrow{g^*(z)}
\end{array}
\]

\[
\begin{array}{ccccccc}
f^*(z) & \xrightarrow{1} & 1 & \xleftarrow{1} \\
\downarrow & & \downarrow \\
\uparrow & & \uparrow \\
\uparrow & & \uparrow \\
f^*(z) & \xrightarrow{z} & g^*(z) & \xleftarrow{g^*(z)}
\end{array}
\]

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In particular, note that $\theta_z \in K_Z$ and $\xi_z \in K_Y$. Consequently, when we pass to the localization $W(f, g)$ at $K_Y$, we obtain a natural transformation $\theta_{\xi^{-1}}: q^*g^* \to p^*f^*$, and this becomes an equivalence upon passage to $W(f, g)$.

The functors $p^*: X \to \overline{W}(f, g)$ and $q^*: Y \to \overline{W}(f, g)$, along with the natural transformation $\tau = \theta_{\xi^{-1}}$, give rise to a square

$$
\begin{array}{ccc}
PSh(\overline{W}(f, g)) & \xrightarrow{q^*} & PSh(Y) \\
\downarrow{p^*} & & \downarrow{g^*} \\
PSh(X) & \xleftarrow{\tau} & PSh(Z),
\end{array}
$$

which in turn gives rise to an identification of the oriented fibre product of presheaf $\infty$-topoi, viz.

$$PSh(X) \times_{PSh(Z)} PSh(Y) = PSh(\overline{W}(f, g)).$$

In the same manner, we obtain an identification of the fibre product of presheaf $\infty$-topoi, viz.

$$PSh(X) \times_{PSh(Z)} PSh(Y) = PSh(W(f, g)).$$

Now we explain how to inject a topology into Construction 5.6.3 to give a generating $\infty$-site for the oriented fibre product of sheaf $\infty$-topoi.

5.6.5 Construction. Let $(X, \tau_X), (Y, \tau_Y)$, and $(Z, \tau_Z)$ be $\delta_0$-small $\infty$-sites (Definition 3.3.9) with finite limits. Let $f^*: Z \to X$ and $g^*: Z \to Y$ be left exact morphisms of sites, so that we $f^*$ and $g^*$ induce geometric morphisms

$$f_*: X := Sh_{\tau_X}(X) \to Sh_{\tau_Z}(Z) = Z \quad \text{and} \quad g_*: Y := Sh_{\tau_Y}(Y) \to Sh_{\tau_Z}(Z) = Z.$$

Define the $\infty$-category $\overline{W}(f, g)$ as in Construction 5.6.1. Let $\overline{\tau}$ denote the topology on $\overline{W}(f, g)$ generated by the families $\{\phi_i : v_i \to u_i\}_{i \in I}$, in which for each $i \in I$, the morphism $\phi_i$ is the image of a morphism of $\overline{W}(f, g)$ of the form

$$
\begin{array}{cccc}
u_{i,X} & \longrightarrow & \nu_{i,Z} \leftarrow & \nu_{i,Y} \\
\phi_{i,X} & \downarrow & \phi_{i,Z} & \downarrow \phi_{i,Y} \\
u_{X} & \longrightarrow & \nu_{Z} \leftarrow & \nu_{Y}
\end{array}
$$

in which one of the following holds:

- the family $\{\phi_{i,X} : v_{i,X} \to u_{i,X}\}_{i \in I}$ generates a $\tau_X$-covering sieve, and for any $i \in I$, the morphisms $\phi_{i,Z}$ and $\phi_{i,Y}$ are equivalences;
- the family $\{\phi_{i,Y} : v_{i,Y} \to u_{i,Y}\}_{i \in I}$ generates a $\tau_Y$-covering sieve, and for any $i \in I$, the morphisms $\phi_{i,Z}$ and $\phi_{i,X}$ are equivalences.

Then there is a natural equivalence of $\infty$-topoi

$$X \times_{\overline{\tau}} Y \simeq Sh_{\overline{\tau}}(\overline{W}(f, g)).$$
Observe that the topology $\tau_Z$ is irrelevant here, as we should expect, since

$$X \times_Z Y = X \times_{\operatorname{PSH}(Z)} Y$$

(Example 5.4.11).

Please observe that the topology $\tau$ on $W(f, g)$ generated by these same families produces the usual (unoriented) fibre product of co-topoi, viz.,

$$X \times Y = \operatorname{Sh}_\tau(W(f, g)) .$$

If all of the topologies $\tau_X, \tau_Y,$ and $\tau_Z$ are finitary, then $(\tilde{W}(f, g), \tilde{\tau})$ and $(W(f, g), \tau)$ are finitary co-sites.

The topology $\tilde{\tau}$ lets us deduce that the oriented fibre product of bounded coherent co-topoi is again bounded coherent.

5.6.6 Lemma. Keep the notations of Construction 5.6.5. If the topologies $\tau_X, \tau_Y,$ and $\tau_Z$ are all finitary, then:

(5.6.6.1) The oriented fibre product $X \times_Z Y$ is coherent and locally coherent, and the projections $\operatorname{pr}_{1,*}$ and $\operatorname{pr}_{2,*}$ are coherent.

(5.6.6.2) The pullback $X \times Y$ is coherent and locally coherent, and the projections $\operatorname{pr}_{1,*}$ and $\operatorname{pr}_{2,*}$ are coherent.

Proof. Since the topology $\tilde{\tau}$ is finitary, Proposition 3.3.10=$[\text{SAG, Proposition A.3.1.3}]$ ensures that the co-topoi $X \times_Z Y$ and $X \times_Y Y$ are coherent and locally coherent. Since $\operatorname{pr}_{1,*}$ and $\operatorname{pr}_{2,*}$ are induced by the morphisms of finitary co-sites

$$(X, \tau_X) \to (\tilde{W}(f, g), \tilde{\tau}) \quad \text{and} \quad (Y, \tau_Y) \to (W(f, g), \tau) ,$$

the remainder of (5.6.6.1) follows from Corollary 3.6.8. The proof of (5.6.6.2) is the same as the proof of (5.6.6.1), replacing the finitary co-site $(\tilde{W}(f, g), \tilde{\tau})$ by $(W(f, g), \tau) . \Box$

Conceptual Completeness now implies that the oriented fibre product of bounded coherent co-topoi is determined by its co-category of points in the following sense.

5.6.7 Proposition. An oriented square

$$
\begin{array}{ccc}
W & \xrightarrow{q_*} & Y \\
\downarrow_{p_*} & & \downarrow_{g_*} \\
X & \xrightarrow{f_*} & Z ,
\end{array}
$$

of bounded coherent co-topoi and coherent geometric morphisms is an oriented fibre product square if and only if the induced oriented square

$$
\begin{array}{ccc}
\operatorname{Pt}(W) & \xrightarrow{q_*} & \operatorname{Pt}(Y) \\
\downarrow_{p_*} & & \downarrow_{g_*} \\
\operatorname{Pt}(X) & \xrightarrow{f_*} & \operatorname{Pt}(Z) ,
\end{array}
$$

in $\operatorname{Cat}_{\infty, \delta_1}$, exhibits $\operatorname{Pt}(W)$ as the oriented fibre product $\operatorname{Pt}(X) \downarrow_{\operatorname{Pt}(Z)} \operatorname{Pt}(Y)$ (0.4.1).
Proof. This follows from Conceptual Completeness (Theorem 3.11.2=[SAG, Theorem 3.9.0.6]), along with the fact that the functor $Pt: \text{Top}_{\infty} \to \text{Cat}_{\infty, \delta}$ preserves oriented fibre product squares (Lemma 5.4.8).

5.7 Compatibility of oriented fibre products & étale geometric morphisms

We turn to the compatibility of oriented fibre products with étale geometric morphisms. Our treatment is inspired by Illusie’s discussion [61, Exposé XI, 1.10(b)]. First we prove what must be a standard fact about the compatibility of ordinary pullbacks and étale geometric morphisms (Lemma 5.7.2) which we could not locate in the literature.

5.7.1 Notation. Let $f_*: X \to Z$ and $g_*: Y \to Z$ be geometric morphisms of $\infty$-topoi, and suppose we are given objects $X \in X$, $Y \in Y$, and $Z \in Z$, along with morphisms $\phi: X \to f^*(Z)$ and $\psi: Y \to g^*(Z)$. We write

$$X \times_Z Y = \{pr_1^*(X) \times pr_2^*(Y) \in X \times_Z Y$$

for the pullback of $pr_1^*(X)$ and $pr_2^*(Y)$ over $pr_1^* f^*(Z) \cong pr_2^* g^*(Z)$ formed in the (un-oriented) pullback $\infty$-topos $X \times_Z Y$.

5.7.2 Lemma. Keep the notations of Notation 5.7.1. Then the natural geometric morphism $p_*: X/_{X \times_Z Y} \to X \times_Z Y$ is étale and $p(1) = X \times_Z Y$.

Proof. First note that the commutative square

$$\begin{array}{ccc}
(X \times_Z Y)/_{X \times_Z Y} & \longrightarrow & (X \times_Z Y)/_{pr_2^*(Y)} \longrightarrow Y/_{Y} \\
\downarrow & & \downarrow \\
(X \times_Z Y)/_{pr_1^*(X)} & \longrightarrow & Y/_{Z/_{Z}}
\end{array}$$

defines a geometric morphism $e_*: (X \times_Z Y)/_{X \times_Z Y} \to X/_{X \times Z} Y/_{Y}$. We claim that $e_*$ is an equivalence of $\infty$-topoi. Indeed, for any $\infty$-topos $E$, consider the commutative square

$$\begin{array}{ccc}
\text{Fun}^*(X/_{X \times Z} Y/_{Y}, E) & \longrightarrow & \text{Fun}^*(X/_{X \times Z} Y/_{Y}, E) \times_{\text{Fun}^*(Z/_{Z}, E)} \text{Fun}^*(Y/_{Y}, E) \\
\downarrow & & \downarrow \\
\text{Fun}^*(X \times_Z Y, E) & \longrightarrow & \text{Fun}^*(X, E) \times_{\text{Fun}^*(Z, E)} \text{Fun}^*(Y, E).
\end{array}$$

Now it follows from Recollection 3.1.9=[HTT, Corollary 6.3.5.6] that the functor

$$\text{Fun}^*(X/_{X \times Z} Y/_{Y}, E) \to \text{Fun}^*(X \times_Z Y, E)$$

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is a left fibration whose fibre over a left exact left adjoint \( h^* : X \times_Z Y \to E \) is the space

\[
\text{Map}_E(1, h^* \text{pr}_1^*(X)) \times \text{Map}_E(1, h^* \text{pr}_2^*(Y)) = \text{Map}_E(1, h^*(X \times_Z Y)) .
\]

On the other hand, again by Recollection 3.1.9=[HTT, Corollaries 6.3.5.6], the natural geometric morphism \( (X \times_Z Y)_{/(X \times_Z Y)} \to X \times_Z Y \) induces a left fibration

\[
\text{Fun}^*(((X \times_Z Y)_{/(X \times_Z Y)}, E) \to \text{Fun}^*(X \times_Z Y, E)
\]

whose fibre over \( h^* \) is the space \( \text{Map}_E(1, h^*(X \times_Z Y)) \). Thus the geometric morphism \( e_* \) induces a fibrewise equivalence

\[
\text{Fun}^*((X \times_Z Y)_{/(X \times_Z Y)}, E) \to \text{Fun}^*(X/\times_Z Y, E)
\]
of left fibrations over \( \text{Fun}^*(X \times_Z Y, E) \), hence an equivalence \( \square \).

Now we turn to the compatibility of oriented fibre products and étale geometric morphisms. We treat the path ∞-topos first, and then apply Lemma 5.7.2 to deduce the general result by expressing the oriented fibre product as an iterated pullback.

**5.7.3 Lemma.** Let \( Z \) be an ∞-topos, and let \( Z \in Z \) be an object. Then the natural geometric morphism \( p_* : \text{Path}(Z/_{Z}) \to \text{Path}(Z) \) is étale and \( p_!(1) = \text{pr}_1^*(Z) \).

**Proof.** We have two geometric morphisms

\[
p_* : \text{Path}(Z/_{Z, pr_1(Z)}) \to Z/_{Z} \quad \text{and} \quad q_* : \text{Path}(Z/_{pr_2(Z)}) \to \text{Path}(Z/_{pr_2(Z)}) \to Z/_{Z}
\]

along with a natural transformation \( \sigma : q_* \to p_* \). These define a geometric morphism

\[
e_* : \text{Path}(Z/_{pr_2(Z)}) \to \text{Path}(Z/_{Z})
\]

over \( \text{Path}(Z) \). We claim that \( e_* \) is an equivalence of ∞-topoi.

First, for any ∞-topos \( E \), consider the commutative square

\[
\begin{align*}
\text{Fun}^*(\text{Path}(Z/_{Z}), E) & \xrightarrow{\sim} \text{Fun}(\Delta^1, \text{Fun}^*(Z/_{Z}, E)) \\
\text{Fun}^*(\text{Path}(Z), E) & \xrightarrow{\sim} \text{Fun}(\Delta^1, \text{Fun}^*(Z, E)) .
\end{align*}
\]

It follows from [HTT, Corollaries 2.1.2.9 & 6.3.5.6] that the functor

\[
\text{Fun}^*(\text{Path}(Z/_{Z}), E) \to \text{Fun}^*(\text{Path}(Z), E)
\]
is a left fibration whose fibre over \( h^* \) is the space

\[
\text{Map}_E(1, h^* \text{pr}_1^*(Z)) \times \text{Map}_E(1, h^* \text{pr}_2^*(Z)) = \text{Map}_E(1, h^*(Z)) .
\]

Here the map \( \text{Map}_E(1, h^* \text{pr}_1^*(Z)) \to \text{Map}_E(1, h^* \text{pr}_2^*(Z)) \) is induced by the natural transformation \( \tilde{r} : \text{pr}_1^* \to \text{pr}_2^* \) adjoint to the defining natural transformation \( r : \text{pr}_2^* \to \text{pr}_1^* \) of the path ∞-topos \( \text{Path}(Z) \).
On the other hand, by Recollection 3.1.9=[HTT, Corollary 6.3.5.6] for any ∞-topos $E$, the natural geometric morphism $\text{Path}(Z)_{/pr^*_1(\mathbb{Z})} \to \text{Path}(\mathbb{Z})$ induces a left fibration

$$\text{Fun}^*(\text{Path}(Z)_{/pr^*_1(\mathbb{Z})}, E) \to \text{Fun}^*(\text{Path}(Z), E)$$

whose fibre over $h^*$ is the space $\text{Map}_E(1, h^* pr^*_1(\mathbb{Z}))$. Thus for any ∞-topos $E$, the geometric morphism $e_*$ induces a fibrewise equivalence

$$\text{Fun}^*(\text{Path}(Z)_{/pr^*_1(\mathbb{Z})}, E) \to \text{Fun}^*(\text{Path}(Z_{/\mathbb{Z}}), E)$$

of left fibrations over $\text{Fun}^*(\text{Path}(Z), E)$. $\square$

Now we define an object $X \tilde{\times}_Z Y \in X \tilde{\times}_Z Y$ and prove that

$$X_{/X} \tilde{\times}_{Z_1} Y_{/Y} = (X \tilde{\times}_Z Y)_{X \tilde{\times}_Z Y}.$$ 

5.7.4 Construction. Let $f_* : X \to Z$ and $g_* : Y \to Z$ be geometric morphisms of ∞-topoi, and let $X \in X$, $Y \in Y$, and $Z \in Z$ be objects, and let $\phi : X \to f^*(Z)$ and $\psi : Y \to g^*(Z)$. Form the oriented fibre product

\[
\begin{array}{ccc}
X \tilde{\times}_Z Y & \xrightarrow{pr^*_2} & Y \\
\downarrow^{pr^*_1} & \searrow^{\tau} & \downarrow^{g_*} \\
X & \xrightarrow{f_*} & Z.
\end{array}
\]

Define an object $X \tilde{\times}_Z Y$ of $X \tilde{\times}_Z Y$ by the pullback square

\[
\begin{array}{ccc}
X \tilde{\times}_Z Y & \xrightarrow{pr^*_2} & \text{pr}_2^*(Y) \\
\downarrow^j & & \downarrow^{\text{pr}_2^*(\psi)} \\
\text{pr}_1^*(X) & \xrightarrow{\tilde{\tau}(Z) \circ \text{pr}_1^*(\phi)} & \text{pr}_2^* g^*(Z),
\end{array}
\]

where

$$\tilde{\tau} : \text{pr}_1^* f^* \to \text{pr}_2^* g^*$$

is the natural transformation adjoint to $\tau : g_* \text{pr}_{2,*} \to f_* \text{pr}_{1,*}$.

5.7.5 Proposition. Keep the notations of Construction 5.7.4. Then the natural geometric morphism $p_* : X_{/X} \tilde{\times}_{Z_{/\mathbb{Z}}} Y_{/Y} \to X \tilde{\times}_Z Y$ is étale and $p_!(1) = \text{pr}_1^*(X \tilde{\times}_Z Y)$.

Proof. The claim follows from Lemma 5.7.3 along with Lemma 5.7.2 applied to the top
right, top left, and bottom left cubes in the diagram

\[
\begin{array}{cccc}
X_{/X} \times_{Z_{/Z}} Y_{/Y} & \rightarrow & Z_{/Z} \times_{Z_{/Z}} Y_{/Y} & \rightarrow & Y_{/Y} \\
X \times_{Z} Y & \rightarrow & Z \times_{Z} Y & \rightarrow & Y \\
X_{/X} \times_{Z_{/Z}} Z_{/Z} & \rightarrow & \text{Path}(Z_{/Z}) & \rightarrow & Z_{/Z} \\
X \times_{Z} Z & \rightarrow & \text{Path}(Z) & \rightarrow & Z \\
X_{/X} & \rightarrow & Z_{/Z} & \rightarrow & Z_{/Z}
\end{array}
\]

where the front and back faces of the bottom right cube are oriented fibre product squares, all other squares are commutative, and the front and back faces of each of the top right, top left, and bottom left cubes are pullback squares.

\[\square\]

5.7.6 Corollary. Keep the notations of Construction 5.7.4. If the morphism

\[\text{pr}^*_2(\psi) : \text{pr}^*_2(Y) \rightarrow \text{pr}^*_2(g^*(Z))\]

is an equivalence, then we have a natural equivalence

\[(X \times_{Z} Y)_{/X \times_{Z} Y} = (X \times_{Z} Y)_{/\text{pr}^*_2(X)} .\]

5.7.7. Keep the notations of Construction 5.7.4 and assume, in addition, that \(X, Y\) and \(Z\) are bounded coherent, the geometric morphisms \(f_*\) and \(g_*\) are coherent, and the objects \(X, Y,\) and \(Z\) are all truncated coherent. Then the object \(X \times_{Z} Y \in X \times_{Z} Y\) is the image under the Yoneda embedding \(\mathcal{A} : \mathcal{W}(f, g) \hookrightarrow X \times_{Z} Y\) of the object of \(\mathcal{W}(f, g)\) given by \(X \rightarrow Z \leftrightarrow Y\) (Construction 5.6.5).
6 Local ∞-topoi & localisations

In this chapter we generalise the basic theory of what are usually called local geometric morphisms and local topoi to the setting of ∞-topoi [SGA 4\textsubscript{II}, Exposé IV, §8; 65, §C.3.6; 66]. Local ∞-topoi play the role of local rings in topos theory: one can localize an ∞-topos \( \mathcal{X} \) at a point \( x^* : S \to \mathcal{X} \) and this construction has the property that the stalk \( x^*U \) of an object \( U \in \mathcal{X} \) is computed by first pulling back to the localization of \( \mathcal{X} \) at \( x^* \) and then taking global sections on the local ∞-topos. The chief example of a local ∞-topos is the étale ∞-topos of a strictly henselian local ring (Example 6.7.4). As with local rings in algebraic geometry, often questions about ∞-topoi with enough points can be reduced to problems about local ∞-topoi. This is the main reason we need the theory of local ∞-topoi; to prove a basechange theorem for oriented fiber products (Theorem 7.1.7) in Chapter 7 by reduction to the local case.

The ∞-toposic theory follows the 1-toposic story very closely; as such, a number of items in this chapter are likely known to experts. Notably, Urs Schreiber has studied local ∞-topoi [110, §3.2].

In §6.1 we begin by discussing a more general class of geometric morphisms that contains the global sections geometric morphism of a local ∞-topos. Section 6.2 then specializes to the study of local ∞-topoi. Section 6.3 explains how to to use oriented fibre products to localize an ∞-topos at a point. In Section 6.4 we prove a compatibility between oriented fibre products and localizations. In algebraic geometry, the spectrum of the strictly henselian local ring \( \mathcal{O}_{X, x}^\text{sh} \) of a scheme \( X \) at a geometric point \( \bar{x} \to X \) can be written as a limit over all étale neighborhoods of \( \bar{x} \); Section 6.5 proves that the localization of an ∞-topos at a point can be described in exactly the same way (Proposition 6.5.3). Using this description, in Section 6.6 we show that the localization of a bounded coherent ∞-topos at a point is coherent (Lemma 6.6.4). Section 6.7 concludes by collecting geometric examples of localizations.

6.1 Quasi-equivalences

As a precursor, we begin by discussing the ∞-toposic generalisation of the notion of a connected geometric morphism [65, p. 525]. In the homotopical setting, the term ‘connected’ (and its variants) doesn’t seem appropriate. Instead, we elect for the distinct term quasi-equivalence.

6.1.1 Definition. A geometric morphism \( f_* : X \to Y \) of ∞-topoi is a quasi-equivalence if the pullback functor \( f^* \) is fully faithful.

6.1.2. Every geometric morphism of ∞-topoi factors as the composite of a quasi-equivalence followed by an algebraic geometric morphism [HTT, Proposition 6.3.6.2]. Moreover, this factorisation is unique up to (canonical) equivalence.

If \( f_* \) is a quasi-equivalence, then \( f^* \) is fully faithful, whence we deduce the following.

6.1.3 Lemma. Let \( f_* : X \to Y \) be a quasi-equivalence of ∞-topoi, and let \( \eta : \text{id}_Y \to f_*, f^* \) denote the unit of the adjunction \( f^* \dashv f_* \). Then the canonical natural transformation

\[
\Gamma_{Y_*} \eta : \Gamma_{Y_*} f_* f^* \to \Gamma_{X_*} = \Gamma_{X_*} f^*
\]
is an equivalence (Notation 3.1.7).

6.1.4. If \( f_* : X \to Y \) is a quasi-equivalence of \( \infty \)-topoi, then by composing the canonical natural transformation \( \Gamma_{Y,*} \to \Gamma_{X,*} f^* \) with \( \Gamma_Y^* \), Lemma 6.1.3 ensures that the canonical natural transformation

\[
\eta : \Gamma_{Y,*} \to \Gamma_{X,*} f^* \Gamma_Y^* = \Gamma_{X,*} \Gamma_X^*
\]

is an equivalence in \( \text{Pro}(S)^\text{op} \subset \text{Fun}(S, S) \). In particular, \( f_* \) is a shape equivalence (Definition 4.2.4).

6.1.5. As noted in [HTT, Remark 7.1.6.12], an \( \infty \)-topos \( X \) has trivial shape if and only if the geometric morphism \( X \to S \) is a quasi-equivalence. However, in general a shape equivalence of \( \infty \)-topoi need not be a quasi-equivalence.

6.1.6 Example. Let \( X \) be a scheme. By [15, Lemma 5.1.2], the natural geometric morphism \( X_{	ext{proét}} \to X_{	ext{ét}} \) from the proétale \( \infty \)-topos of \( X \) to the étale \( \infty \)-topos of \( X \) is a quasi-equivalence, hence a shape equivalence.

6.2 Local \( \infty \)-topoi

Now we specialise to local \( \infty \)-topoi. Recall that a geometric morphism \( f_* : X \to Y \) is essential if \( f^* \) admits a left adjoint \( f_! : X \to Y \).

6.2.1 Definition. We say that a geometric morphism \( f_* : X \to Y \) of \( \infty \)-topoi is coëssential if \( f_* \) admits a right adjoint \( f^! : Y \to X \). In this case, the functor \( f^! \) and its left adjoint \( f_* \) define a geometric morphism \( f^! : Y \to X \) called the centre of \( f_* \).

The next definition generalises what are sometimes called local geometric morphisms in the 1-topos theory literature [65, §C.3.6; 66]. We instead choose terminology that syncs with the algebro-geometric terminology for local rings and doesn’t conflict with other uses of the term ‘local’ in higher category theory.

6.2.2 Definition. We say that geometric morphism \( f_* : X \to Y \) of \( \infty \)-topoi exhibits \( X \) as local over \( Y \) if \( f_* \) is both coëssential and a quasi-equivalence.

An \( \infty \)-topos \( X \) is local if \( X \) is local over \( S \). In this case we simply call \( \Gamma_X^* : S \to X \) the centre of \( X \).

6.2.3. Please observe that a geometric morphism of \( \infty \)-topoi \( f_* : X \to Y \) exhibits \( X \) as local over \( Y \) if and only if the functor \( f_* \) admits a fully faithful right adjoint \( f^! \). Equivalently, \( X \) is local over \( Y \) if and only if \( f_* \) admits a section \( f^! \) in the \((\infty, 2)\)-category of \( \infty \)-topoi.

6.2.4. Let \( X \) be an \( \infty \)-topos. Note that if the global sections functor \( \Gamma_* : X \to S \) admits a right adjoint \( \Gamma^! : S \to X \), then \( \Gamma^! \) is automatically fully faithful, whence \( X \) is local.

Consequently, by the Adjoint Functor Theorem and (6.2.4), an \( \infty \)-topos \( X \) is local if and only if the terminal object \( 1_X \in X \) is completely compact.

6.2.5 Remark. Let \( X \) be a 1-topos with corresponding 1-localic \( \infty \)-topos \( X' \). Then \( X' \) is local over \( S \) if and only if the global sections functor \( (X')_{\leq 0} = X \to \text{Set} \) admits a right adjoint.
6.2.6 Lemma. Let \( \mathcal{X} \) be a local \( \infty \)-topos. Then \( \mathcal{X} \) has homotopy dimension \( \leq 0 \). In particular, \( \mathcal{X} \) has cohomological dimension \( \leq 0 \).

Proof. By [HTT, Lemma 7.2.1.7], it suffices to show that \( \Gamma_{X,*} : \mathcal{X} \to \mathcal{S} \) preserves effective epimorphisms; this follows from the assumption that \( \Gamma_{X,*} \) is a left adjoint. The second statement is a consequence of [HTT, Corollary 7.2.2.30].

6.2.7 Definition. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be local \( \infty \)-topoi with centres \( \Gamma_X \) and \( \Gamma_Y \), respectively. A geometric morphism \( f_* : \mathcal{X} \to \mathcal{Y} \) is a local geometric morphism if the canonical natural morphism

\[
f_* \Gamma_X \to \Gamma_Y
\]

adjoint to \( \Gamma_Y \eta : \Gamma_X \to \Gamma_Y f^* \) is an equivalence. Write \( \text{Top}_{loc}^\infty \subset \text{Top}_{loc}^\infty \) for the (non-full) subcategory whose objects are local \( \infty \)-topoi and whose morphisms are local geometric morphisms.

If \( \mathcal{X} \) is a local \( \infty \)-topos, then the centre of \( \mathcal{X} \) is an initial object of the \( \infty \)-category \( \text{Pt}(\mathcal{X}) \); in fact, more is true.

6.2.8 Notation. Let \( f_* : \mathcal{X} \to \mathcal{Y} \) and \( f'_* : \mathcal{X}' \to \mathcal{Y} \) be geometric morphisms of \( \infty \)-topoi. Write

\[
\text{Fun}_{\mathcal{Y}}(\mathcal{X}, \mathcal{X}') = \text{Fun}_{\mathcal{Y}}(\mathcal{X}, \mathcal{X}') \times_{\text{Fun}(\mathcal{X}, \mathcal{Y})} \{f_*\}
\]

for the \( \infty \)-category of geometric morphisms \( \mathcal{X} \to \mathcal{X}' \) over \( \mathcal{Y} \).

6.2.9 Lemma. Let \( f_* : \mathcal{X} \to \mathcal{Y} \) be a geometric morphism that exhibits \( \mathcal{X} \) as local over \( \mathcal{Y} \) with centre \( f_! \). Then \( f_! \) is a terminal object of the \( \infty \)-category \( \text{Fun}_{\mathcal{Y}}(\mathcal{X}, \mathcal{X}) \).

Proof. Let \( g_* : \mathcal{Y} \to \mathcal{X} \) be a geometric morphism over \( \mathcal{Y} \). Then

\[
\text{Map}_{\text{Fun}_{\mathcal{Y}}(\mathcal{Y}, \mathcal{X})}(g_*, f_!) = \text{Map}_{\text{Fun}_{\mathcal{Y}}(\mathcal{Y}, \mathcal{X})}(f_* g_*, \text{id}_Y) = \text{Map}_{\text{Fun}_{\mathcal{Y}}(\mathcal{Y}, \mathcal{X})}(\text{id}_Y, \text{id}_Y) = * .
\]

Like local rings in algebraic geometry, local \( \infty \)-topoi provide a convenient way to compute stalks: take global sections after pulling back to an appropriate local \( \infty \)-topos.

6.2.10 Lemma. Let \( p_* : \mathcal{W} \to \mathcal{X} \) be a geometric morphism of \( \infty \)-topoi. Assume that where \( \mathcal{W} \) is local with centre \( w_* \), and write \( x_* = p_* w_* \). Then \( x^* = \Gamma_{W,*} p^* \).

Proof. Since \( w^* = \Gamma_{W,*} \), we see that

\[
x^* = (p w)^* = w^* p^* = \Gamma_{W,*} p^* .
\]

We shall soon see (Definition 6.3.7 and (6.3.8)) that for any \( \infty \)-topos \( \mathcal{X} \) and any point \( x_* \in \text{Pt}(\mathcal{X}) \), there is a geometric morphism \( p_* : \mathcal{W} \to \mathcal{X} \) in which \( \mathcal{W} \) is local with centre \( w_* \) and \( x_* = p_* w_* \) (and is, moreover, universal with this property).

Local geometric morphisms are also stable under pullback, though we do not use this fact in an integral way in the present paper.
6.2.11. Consider a pullback square of ∞-topoi

\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{f_*} & Y \\
\downarrow g_* & & \downarrow g_* \\
X & \xrightarrow{f_*} & Z,
\end{array}
\]

where \(g_*\) exhibits \(Y\) as local over \(Z\) with centre \(g^!\). By the universal property of the pullback, the identity on \(X\) and the geometric morphism \(g^! f_* : X \to Y\) induce a geometric morphism

\[
g^! := (\text{id}_X, g^! f_*) : X \to X \times_Z Y
\]

such that \(g_* g^! = \text{id}_X\) and \(f_* g^! = g^! f_*\). Using the universal property of the pullback and the fact that \(g_*\) is exhibits \(Y\) as local over \(Z\), one easily checks that the functor \(g^!\) is right adjoint to \(g_*\), so that \(g_*\) exhibits \(X \times_Z Y\) as local over \(X\) with centre \(g^!\).

6.3 Nearby cycles & localisations

We now show that the evanescent ∞-topos (Example 5.4.10) provides a wealth of local ∞-topoi. Then, following Deligne as well as Johnstone–Moerdijk [66, Definition 3.1], we use the evanescent ∞-topos to construct the localisation of an ∞-topos at a point.

A site-theoretic proof of the following result (originally stated without proof by Lau- mon [74, 3.2]) is given in [61, Exposé XI, Proposition 4.4]. The reliance on sites renders the proof given in [61, Exposé XI] inadequate in the context of ∞-topoi; luckily the work of Emily Riehl and Dominic Verity [99] permits us to employ simple 2-categorical arguments.

6.3.1 Proposition. Let \(f_* : X \to Z\) be a geometric morphism of ∞-topoi. Then:

1. The nearby cycles functor \(\Psi_{f, *} : X \to X \times_Z Z\) is right adjoint to the projection \(\text{pr}_{1,*} : X \times_Z Z \to X\).

2. The functor \(\Psi_{f,*}\) is fully faithful. Hence the geometric morphism \(\text{pr}_{1,*}\) exhibits \(X \times_Z Z\) as local over \(X\) with centre \(\Psi_{f,*}\).

Proof. Recall that for any ∞-topos \(E\), the functor

\[
\text{Fun}_* (E, -)^{op} : \text{Top}_\infty \to \text{Cat}_{\infty, \delta}^{\text{op}}
\]

carries oriented fibre products in \(\text{Top}_\infty\) to oriented fibre products in \(\text{Cat}_{\infty, \delta}^{\text{op}}\) (5.4.7). Thus the proof of [99, Proposition 3.4.6] works perfectly, giving the oriented fibre product in \(\text{Top}_\infty\) the necessary ‘weak universal property’ (as Riehl and Verity call it) to apply [99, Lemma 3.5.9], proving both (6.3.1.1) and (6.3.1.2).

The dual notion to being local over an ∞-topos naturally appears as the property satisfied by the second projection from the coevanescent ∞-topos.
6.3.2 Definition. A geometric morphism \( f_* : X \to Y \) of \( \infty \)-topoi exhibits \( X \) as colocal over \( Y \) if \( f_* \) is a quasi-equivalence and \( f^* \) admits a left exact left adjoint \( f_! : X \to Y \). In this case, the functor \( f^* \) and its left adjoint \( f_! \) define a geometric morphism \( f^* : Y \to X \) called the cocentre of \( f_* \).

6.3.3. In the setting of 1-topoi, Johnstone [65, Theorem C.3.6.16] uses the term totally connected for what we call colocal. Again, such lingo is inapt in our context.

The following is dual to Proposition 6.3.1.

6.3.4 Proposition. Let \( g_* : Y \to Z \) be a geometric morphism of \( \infty \)-topoi. Then:

(6.3.1.1) The conearby cycles functor \( \Psi^d_\ast : Y \to Z \tilde{\times}_Z Y \) is left adjoint to the projection \( \text{pr}_2^* : Z \tilde{\times}_Z Y \to Y \).

(6.3.1.2) The functor \( \Psi^d_\ast = \text{pr}_2^* \) is fully faithful. Hence the geometric morphism \( \text{pr}_2^* \) exhibits \( Z \tilde{\times}_Z Y \) as colocal over \( Y \) with cocentre \( \Psi^d_\ast \).

6.3.5. A geometric morphism \( f_* : X \to Y \) that exhibits \( X \) as colocal over \( Y \) always satisfies the étale projection formula

\[
f_!(f^*(X) \times_{f^*(Z)} Y) = X \times_{Z} f_!(Y)
\]

of [HTT, Proposition 6.3.5.11]. However, the geometric morphism \( f_* \) will almost never be étale; \( f_! \) is conservative if and only if \( f_* \) is an equivalence.

6.3.6 Example. For any \( \infty \)-topos \( X \) the diagonal functor

\[
\psi(\text{id}_X, \text{id}_X, \text{id})_* : X \to X \tilde{\times}_X X = \text{Path}(X)
\]

is both the nearby and conearby cycles functor. Combining Propositions 6.3.1 and 6.3.4, we deduce that we have a chain of (left exact) adjoints

\[
\xymatrix{ \text{Path}(X) & X \\
\text{pr}_1^* & \text{pr}_1^* \ar[ur] & \text{pr}_2^* \\
\text{pr}_2^* & \ar[ur]}
\]

In particular, the geometric morphisms \( \text{pr}_1^*, \text{pr}_2^* : \text{Path}(X) \to X \) are shape equivalences.

Now we define the localisation of an \( \infty \)-topos at a point as an evanescent \( \infty \)-topos; for this please recall Notation 3.1.7.

6.3.7 Definition. Let \( X \) be an \( \infty \)-topos and \( x_* : S \to X \) a point of \( X \). The localisation of \( X \) at \( x_* \) is the evanescent \( \infty \)-topos

\[
X_{(x)} := \tilde{x}_* \tilde{x}_* X.
\]

We write \( \ell_{x,*} : X_{(x)} \to X \) for the second projection \( \text{pr}_2^* : \tilde{x}_* \tilde{x}_* X \to X \).
6.3.8. Let $X$ be an $\infty$-topos and $x_*$ a point of $X$. By Proposition 6.3.1, the $\infty$-topos $X_{(x)}$ is local with centre $\Psi_{x_*} : S \to X_{(x)}$. By Lemma 6.2.10, for every object $F \in X$ we can compute the stalk at $x$ via the familiar formula

$$x^* F \cong \Gamma(X_{(x)}; \ell_*^x F).$$

6.3.9 Notation. Write $\text{Top}_{\infty, *} = \text{Top}_{\infty, S}$ for the $\infty$-category of $\infty$-topoi equipped with a topos-theoretic point. The assignment $(X, x_*) \mapsto X_{(x)}$ defines a functor

$$\text{Top}_{\infty,*} \to \text{Top}_{\infty}.$$ 

In the other direction, the assignment $X \mapsto (X, \Gamma_X)$ defines a fully faithful functor

$$\text{Top}_{\infty} \to \text{Top}_{\infty,*}.$$ 

6.3.10 Proposition. Let $X$ be a local $\infty$-topos with centre $x_*$. Then the geometric morphism $\ell_{x_*} : X_{(x)} \to X$ is an equivalence.

Proof. Let $\eta : \text{id}_X \to x_* \Gamma_{X,*}$ be the unit of the adjunction $\Gamma_{X,*} \dashv x_*$. Then the oriented square

\[
\begin{array}{ccc}
X & \xrightarrow{\Gamma_{x_*}} & X \\
\downarrow{\eta} & \searrow{\eta} & \downarrow{\text{id}_X} \\
\widetilde{x} & \longrightarrow & X
\end{array}
\]

exhibits $X$ as the oriented fibre product $\widetilde{x} \times_X X$. \qed

6.3.11 Corollary. The fully faithful functor $\text{Top}_{\infty} \to \text{Top}_{\infty,*}$ admits a right adjoint given by the assignment $(X, x_*) \mapsto X_{(x)}$.

6.4 Compatibility of oriented fibre products with localisations

In this section we prove that the formation oriented fibre products is compatible with localisations of $\infty$-topoi. First we note that taking path $\infty$-topoi commutes with the formation of oriented fiber products.

6.4.1 Lemma. Let $f_* : X \to Z$ and $g_* : Y \to Z$ be geometric morphisms of $\infty$-topoi. Then we have a natural equivalence

$$\text{Path}(X \times_Z Y) \cong \text{Path}(X) \times_{\text{Path}(Z)} \text{Path}(Y).$$

Proof. Since the path $\infty$-topos construction is a right adjoint $\text{Top}_{\infty} \to \text{Top}_{\infty}$, we have natural equivalences

$$\text{Path}(X \times_Z Y) = \text{Path}(X \times_Z \text{Path}(Z) \times_Z Y)$$

$$= \text{Path}(X) \times_{\text{Path}(Z)} \text{Path}((\text{Path}(Z)) \times_{\text{Path}(Z)} \text{Path}(Y))$$

$$= \text{Path}(X) \times_{\text{Path}(Z)} \text{Path}(Y).$$ \qed
6.4.2 Proposition. Let \( f_* : (X, x_*) \to (Z, z_*) \) and \( g_* : (Y, y_*) \to (Z, z_*) \) be morphisms of pointed \( \infty \)-topoi, so that there is an induced point \( x_* \overset{\sim}{\to} z_* \). Then we have a natural equivalence

\[
(X \times_Z Y)_{(x_*, z_*, y_*)} = X_{(x_*)} \times_{Z_{(z_*)}} Y_{(y_*)}.
\]

Proof. Consider the diagram \( \Lambda^2_2 \to \text{Fun}(\Lambda^2_2, \text{Top}_\infty) \) defined by the diagram

\[
\begin{array}{ccc}
\text{Path}(X) & \stackrel{\text{Path}(f_*)}{\longrightarrow} & \text{Path}(Z) \\
\downarrow^{pr_1,*} & & \downarrow^{pr_1,*} \\
X & \stackrel{f_*}{\longrightarrow} & Z \\
\downarrow^{x_*} & & \downarrow^{z_*} \\
S & \stackrel{S}{\longrightarrow} & S
\end{array}
\]

(6.4.3)

where we have displayed objects of \( \text{Fun}(\Lambda^2_2, \text{Top}_\infty) \) horizontally, and morphisms in \( \text{Fun}(\Lambda^2_2, \text{Top}_\infty) \) vertically. First taking the (vertical) limit of the diagram (6.4.3) in \( \text{Fun}(\Lambda^2_2, \text{Top}_\infty) \)

we obtain the cospan

\[
X_{(x_*)} \stackrel{f_*}{\longrightarrow} Z_{(z_*)} \leftarrow g_* Y_{(y_*)}.
\]

Then taking the oriented fibre product of this cospan yields \( X_{(x_*)} \times_{Z_{(z_*)}} Y_{(y_*)} \). On the other hand, by Lemma 6.4.1, first forming the oriented fibre product horizontally then taking pullbacks vertically yields the localization

\[
(X \times_Z Y)_{(x_*, z_*, y_*)}.
\]

The claim now follows from the fact that the formation of oriented fibre products commutes with limits (5.4.3).

6.5 Localisation à la Grothendieck–Verdier

In order to get our hands on geometric examples of localised \( \infty \)-topoi, we give another description of \( X_{(x_*)} \) that is akin to Grothendieck and Verdier’s original (1-toposic) definition of the localisation as a limit over étale neighbourhoods of \( x_* \) in \( X \) [SGA 4\text{ii}, Exposé VI, 8.4.2].

6.5.1 Definition. Let \( (X, x_*) \) be a pointed \( \infty \)-topos. The \( \infty \)-category of étale neighbourhoods of \( x_* \) is the pullback

\[
\begin{array}{ccc}
\text{Nbd}(x) & \longrightarrow & S_* \\
\downarrow & & \downarrow \\
X & \longrightarrow & S
\end{array}
\]

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formed in $\text{Cat}_{\infty, \delta}$.  
By [HTT, Corollary 6.3.5.6 & Remark 6.3.5.7] the $\infty$-category $\text{Nbd}(x)$ is equivalent to the full subcategory of $((\text{Top}_{\infty, x})_{/x, x_\ast})$ spanned by those objects $(E, e_\ast) \to (X, x_\ast)$ with the property that the geometric morphism $E \to X$ is étale.

Please note that $\text{Nbd}(x)$ is an inverse $\infty$-category.

To provide the limit description of the localisation as well as the familiar colimit formula for the stalk $x^\ast$, we need to take limits of diagrams indexed by the (not necessarily $\delta_0$-small) $\infty$-category $\text{Nbd}(x)$. Happily the exact same cofinality argument given in [SGA 4ii, Exposé IV, 6.8] works in the setting of higher topoi, showing that $\text{Nbd}(x)$ admits a limit-cofinal $\delta_0$-small subcategory.

**6.5.2 Construction.** Let $X$ be a $\infty$-topos and $x_\ast \in \text{Pt}(X)$. Then by the Yoneda lemma the stalk functor $x^\ast : X \to S$ can be computed as the filtered colimit

$$x^\ast \cong \text{colim}_{(U, u) \in \text{Nbd}(x)} \text{Map}_X(U, -).$$

The assignment $(U, u) \mapsto X_{/U}$ defines a functor $E_x : \text{Nbd}(x) \to \text{Top}_{\infty, /X}$. Moreover, the natural forgetful functor $\text{Top}_{\infty, /E_x} \to \text{Top}_{\infty, /X}$ is a right fibration. We write $\lim_{(U, u) \in \text{Nbd}(x)} X_{/U}$ for the limit in $\text{Top}_{\infty, /X}$ (equivalently, in $\text{Top}_{\infty}$) of the diagram $E_x$.

By Recollection 3.1.9=[HTT, Corollary 6.3.5.6], specifying a geometric morphism

$$X' \to \lim_{(U, u) \in \text{Nbd}(x)} X_{/U}$$

is equivalent to specifying a geometric morphism $p_\ast : X' \to X$ along with a global section

$$s \in \Gamma_{X'}(p_\ast) \left( \lim_{(U, u) \in \text{Nbd}(x)} p^\ast U \right) = \lim_{(U, u) \in \text{Nbd}(x)} \Gamma_{X'} p^\ast U .$$

Since $X_{(x)}$ is the localisation of $X$ at $x_\ast$, we have a natural equivalence $x^\ast = \Gamma_{X_{(x)}}(\ell^\ast_x)$ (6.3.8). Thus for $U \in X$, we obtain a natural equivalence

$$\lim_{(U, u) \in \text{Nbd}(x)} x^\ast(U) = \Gamma_{X_{(x)}}(\ell^\ast_x(U)).$$

The global sections $u \in x^\ast(U)$ for $(U, u) \in \text{Nbd}(x)$ together define a global section $s \in \lim_{(U, u) \in \text{Nbd}(x)} x^\ast(U)$.

This provides a comparison geometric morphism

$$c_{x, \ast} : X_{(x)} \to \lim_{(U, u) \in \text{Nbd}(x)} X_{/U}$$

over $X$.

**6.5.3 Proposition.** Let $X$ be an $\infty$-topos and $x_\ast$ a point of $X$. Then the comparison geometric morphism $c_{x, \ast} : X_{(x)} \to \lim_{(U, u) \in \text{Nbd}(x)} X_{/U}$ of Construction 6.5.2 is an equivalence.
Proof. We wish to show that $c_{x,*} : X(x) \rightarrow \lim_{(U, u) \in \text{Nbd}(x)} X_U$ induces an equivalence

$$\text{Top}_{\text{co}/X(x)} \simeq \text{Top}_{\text{co}/E_x}.$$ 

Since both projections onto $\text{Top}_{\text{co}/X}$ are right fibrations, we are reduced to showing that for every object $p_* : X' \rightarrow X$ of $\text{Top}_{\text{co}/X}$ the induced map on fibres of these right fibrations is an equivalence. By Recollection 3.1.9=[HTT, Corollary 6.3.5.6] the fibre of the right fibration $\text{Top}_{\text{co}/E_x} \rightarrow \text{Top}_{\text{co}/X}$ over $p_* : X' \rightarrow X$ is given by

$$\{p_*\} \times_{\text{Top}_{\text{co}/X}} \text{Top}_{\text{co}/E_x} = \lim_{(U, u) \in \text{Nbd}(x)} \Gamma_{X',*} p^*(U).$$

On the other hand, we have equivalences

$$\{p_*\} \times_{\text{Top}_{\text{co}/X}} \text{Top}_{\text{co}/X(x)} = \lim_{(U, u) \in \text{Nbd}(x)} \Gamma_{X',*} p^*(U).$$

By the colimit formula for the stalk (Construction 6.5.2), we have natural equivalences

$$\text{Map}_{\text{Fun}(X, S)}(x^*, \Gamma_x p^*) = \text{Map}_{\text{Fun}(X, S)}\left(\lim_{(U, u) \in \text{Nbd}(x)} \Gamma_{X',*} p^*(U)\right).$$

Unwinding definitions, we see that the induced map on fibres

$$\{p_*\} \times_{\text{Top}_{\text{co}/X}} \text{Top}_{\text{co}/X(x)} \rightarrow \{p_*\} \times_{\text{Top}_{\text{co}/X}} \text{Top}_{\text{co}/E_x}$$

is an equivalence. 

\section{6.6 Coherence of localisations}

In this section we use the Grothendieck–Verdier description of the localisation to deduce that $X(x)$ is bounded coherent when $X$ is. Please note that this is not automatic from Lemma 5.6.6, as points of bounded coherent $\infty$-topoi need not be coherent in general.

\subsection*{6.6.1.} Let $f : U \rightarrow V$ be a morphism between coherent objects of an $\infty$-topos $X$. Then the geometric morphism $f_* : X_U \rightarrow X_V$ is coherent.

\subsection*{6.6.2 Lemma.} Let $X$ be a bounded $\infty$-topos and $U \in X_{\text{co}}$ a truncated object of $X$. Then the over $\infty$-topos $X_{U/}$ is bounded.

Proof. Indeed, if $U$ is $n$-truncated, and if $X$ is $N$-localic for some $N \geq n$, then $X_{U/}$ in $N$-localic as well (Example 3.2.8). The claim now follows by exhibiting $X$ as an inverse limit of localic $\infty$-topoi. 

\hfill $\square$
6.6.3. Let $X$ be a bounded coherent $\infty$-topos and $x_*$ a point of $X$. Then the full subcategory
$$\text{Nbd}_{\text{coh}}^\infty(x) \subset \text{Nbd}(x)$$
consisting of those neighbourhoods $(U, u)$ such that $U$ is a truncated coherent object of $X$ is limit-cofinal in $\text{Nbd}(x)$. Thus Proposition 6.5.3, (6.6.1), and Lemma 6.6.2 together show that
$$X_{(x)} = \lim_{(U, u) \in \text{Nbd}_{\text{coh}}^\infty(x)} X_{/U}$$
is an inverse limit in $\text{Top}_\infty$ of bounded coherent $\infty$-topoi and coherent geometric morphisms.

From Corollary 3.9.4=[SAG, Corollary A.8.3.3] we deduce the following.

6.6.4 Lemma. Let $X$ be a bounded coherent $\infty$-topos and $x_*$ a point of $X$. Then the localisation $X_{(x)}$ is bounded coherent and the geometric morphism $\ell_{x_*} : X_{(x)} \to X$ is coherent.

6.7 Geometric examples of localisations

Now we turn to examples of local $\infty$-topoi coming from algebraic geometry. For these examples, please recall Remark 6.2.5.

6.7.1 Example ([66, Example 1.2(a)]). Let $W$ be a topological space and $s \in W$ a special point in the sense that the only open set of $W$ containing $s$ is $W$ itself. Then it is immediate that the functor $\tilde{W} \to S$ given by taking the stalk at $s$ is equivalent to the global sections functor. Hence the $\infty$-topos $\tilde{W}$ is local with centre $x_* : S \to \tilde{W}$.

6.7.2 Subexample ([SGA 4_1, Exposé VI, 8.4.6]). Let $A$ be a local ring with maximal ideal $m$. Then the point $m$ of the Zariski space $\text{Spec}(A)_{\text{zar}}$ is special. Hence the Zariski $\infty$-topos $\text{Spec}(A)_{\text{zar}}$ is local. Moreover, if $\phi : A \to A'$ is a local homomorphism of local rings, then the induced geometric morphism of Zariski $\infty$-topoi $\text{Spec}(A')_{\text{zar}} \to (\text{Spec } A)_{\text{zar}}$ is a local geometric morphism.

6.7.3 Example ([SGA 4_1, Exposé VI, 8.4.4]). Let $X$ be a scheme and $x \in X$. Then the localisation of the Zariski $\infty$-topos of $X$ at the point $x$ is the Zariski $\infty$-topos of the local ring $O_{X,x}$.

6.7.4 Example. Let $X$ be a scheme, and let $x \to X$ be a point with image $x_0 \in X_{\text{zar}}$. Suppose $x$ is a geometric point in the sense that the residue field $\kappa(x)$ is a separable closure of $\kappa(x_0)$. Then the localisation of the étale $\infty$-topos of $X$ at $x$ is the étale $\infty$-topos of the strict localisation $X_{(x)} := \text{Spec } O_{X,x}^\text{sh}$. That is,
$$X_{(x)} = (X_{(x)})_\text{et}.$$

More generally, for any point $x \to X$, the evanescent $\infty$-topos $x_\text{et} \tilde{X}_{X,x} X_{\text{et}}$ admits an analogous description. Write $O_{X,x}^\text{sh}$ for the hensilization of the local ring $O_{X,x}$, and let
$$A_x \supset O_{X,x_0}^\text{sh}$$
denote the unramified extension of $O_{X,x_0}^k$ with residue field the separable closure of $\kappa(x_0)$ in $\kappa(x)$. Then there is an equivalence of co-topoi

$$x_{\mu} \sim_{X_{\mu}} X_{\mu} \simeq (\text{Spec } A_{\mu})_{\text{ét}}.$$
7 Basechange conditions for oriented fibre products

The goal of this chapter is to prove a basechange result for oriented fibre products of bounded coherent ∞-topoi (Theorem 7.1.7). Our result provides a nonabelian refinement of a basechange result of Ofer Gabber [61, Exposé XI, Théorème 2.4] as well as one of Moerdijk and Jacob Vermeulen [84, Theorem 2(i)]. This basechange result is essential to our décollage approach to stratified higher topoi in Chapter 8. So that we can first introduce the basechange theorem in question, a detailed overview of this chapter appears at the end of §7.1.

7.1 Basechange transformations & basechange conditions

We begin by recalling the basechange natural transformation associated to an oriented square of ∞-categories. We are mostly concerned with the ‘left’ basechange transformation, but have one situation in which we need to consider the ‘right’ basechange transformation, so we introduce them both here.

7.1.1 Definition. Consider an oriented square of ∞-categories:

\[
\begin{array}{c}
A \\
\downarrow q^* \\
\downarrow \sigma \\
B \\
\downarrow f^* \\
C \\
\downarrow g^* \\
D \\
\end{array}
\]

Assume that the functors \( f^* \) and \( q^* \) admit left adjoints \( f_* \) and \( q_* \), respectively. Write \( \varepsilon_f : f^* f_* \to \text{id}_B \) for the counit and \( \eta_q : \text{id}_C \to q_* q^* \) for the unit. The left basechange transformation associated to the oriented square (7.1.2) is the composition

\[
\text{BC}_f : f^* g_* f^* g_* q_* q^* f^* f_* p_* q^* f_* f_* p_* q_* \xrightarrow{\eta_f \eta_q} \text{id}_B.
\]

We say that the square (7.1.2) is left adjointable or satisfies the left basechange condition if the natural transformation \( \text{BC}_f \) is an equivalence.

(7.1.1.2) Assume that the functors \( p_* \) and \( g_* \) admit right adjoints \( p^! \) and \( g^! \), respectively. Write \( \varepsilon_p : p_* p^! \to \text{id}_B \) for the counit and \( \eta_g : \text{id}_C \to g^! g_* \) for the unit. The right basechange transformation associated to the oriented square (7.1.2) is the composition

\[
\text{BC}_g : q_* p^! q_* p^! g^! g_* p^! g^! g_* p^! g^! g_* p^! g^! g_* p^! \xrightarrow{\eta_g \eta_p} \text{id}_B.
\]

7.1.2 Remark. In classical category theory, the adjointability of a commutative square of 1-categories is often referred to as the Beck–Chevalley condition, and the basechange transformations are often referred to as Beck–Chevalley transformations [84; 85, Chapter I, §3].
7.1.4. Please observe that given oriented squares of $\infty$-categories

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow \sigma & & \downarrow \sigma' \\
A' & \rightarrow & B'
\end{array}
\quad \begin{array}{ccc}
A & \leftarrow & B \\
\downarrow \sigma & & \downarrow \sigma' \\
A' & \leftarrow & B'
\end{array}
\]

in which the horizontal functors all admit left adjoints, the basechange morphism of the outer oriented rectangle is equivalent to natural transformation given by the composite of the basechange morphisms

\[
\begin{array}{ccc}
A & \leftrightarrow & B \\
\downarrow \text{BC}_\sigma & & \downarrow \text{BC}_{\sigma'} \\
A' & \leftrightarrow & B'
\end{array}
\]

The purpose of this chapter is to generalize the Theorem of Gabber–Illusie and Moerdijk–Vermeulen that oriented fibre product squares of coherent ordinary toposi satisfy the basechange condition. However, the $\infty$-toposic generalization is a bit more subtle: exactly because coherent geometric morphisms between bounded coherent $\infty$-topoi only preserve colimits of uniformly truncated filtered diagrams and not all filtered colimits (Corollary 3.10.5), oriented fibre product squares of bounded coherent $\infty$-topoi only satisfy the weaker truncated basechange condition.

7.1.5 Definition. We say that an oriented square of $\infty$-topoi and geometric morphisms

\[
\begin{array}{ccc}
W & \rightarrow & Y \\
\downarrow p_* & & \downarrow g_* \\
X & \rightarrow & Z
\end{array}
\]

satisfies the truncated basechange condition if for every truncated object $F \in Y_{\infty}$, the basechange morphism $\text{BC}_\sigma(F) : f^* g_*(F) \rightarrow \text{pr}_1 \text{pr}_2^*(F)$ is an equivalence in $X$.

The following theorem the main result of this chapter:

7.1.7 Theorem. Let $f_* : X \rightarrow Z$ and $g_* : Y \rightarrow Z$ be coherent geometric morphisms between bounded coherent $\infty$-topoi. Then the oriented fibre product square

\[
\begin{array}{ccc}
X \hat{\times}_Z Y & \rightarrow & Y \\
\downarrow \text{pr}_2 & & \downarrow g_* \\
X \rightarrow & \rightarrow & Z
\end{array}
\]

satisfies the truncated basechange condition.
By passing to 1-localic co-topoi in Theorem 7.1.7, we deduce Moerdijk and Vermeulen’s 1-toposic basechange condition [84, Theorem 2(i)].

7.1.9 Corollary. Let \( f_* : X \to Z \) and \( g_* : Y \to Z \) be coherent geometric morphisms between coherent 1-topoi. Then the oriented fibre product square of 1-topoi

\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{pr_2^*} & Y \\
\downarrow{pr_1^*} & \searrow{g_*} & \\
X & \xrightarrow{f_*} & Z
\end{array}
\]

satisfies the left basechange condition. That is, the basechange natural transformation \( f^* g_* \to pr_1^* pr_2^* \) is an isomorphism.

Proof. Write \( X' \), \( Y' \), and \( Z' \) for the 1-localic co-topoi associated to \( X \), \( Y \), and \( Z \) respectively. Combining the equivalence between coherent 1-localic co-topoi and coherent 1-topoi (Proposition 3.6.11) with Theorem 7.1.7 shows that the oriented fibre product square of co-topoi

\[
\begin{array}{ccc}
X' \times_{Z'} Y' & \xrightarrow{pr_2^*} & Y' \\
\downarrow{pr_1^*} & \searrow{g_*} & \\
X' & \xrightarrow{f_*} & Z'
\end{array}
\]

satisfies the truncated basechange condition. We conclude by restricting to 0-truncated objects and applying Lemma 5.4.13.

We now give an overview of the rest of the chapter. The chapter is broken into two parts: §§7.2 to 7.4 provide examples and applications of Theorem 7.1.7 that do no require understanding its proof, and §§7.5 to 7.7 are dedicated to the proof of Theorem 7.1.7.

Section 7.2 provides some example situations where the basechange condition for oriented fibre products can be easily verified. Section 7.3 provides some example applications of Theorem 7.1.7; notably we generalize [SGA 1, Exposé X, Corollaire 1.7; 22, Theorem 5.3] by showing that if \( X \) and \( Y \) are coherent schemes over a separably closed field \( k \) and \( Y \) is proper, then the profinite étale homotopy type of \( X \times_{\text{Spec} k} Y \) coincides with the product of the profinite étale homotopy types of \( X \) and \( Y \). In §7.4 we investigate the stable consequences of Theorem 7.1.7 and deduce a generalization of the derived categories basechange theorem for oriented fibre products of Gabber–Illusie [61, Exposé XI, Théorème 2.4].

We then embark on our proof of Theorem 7.1.7, which is inspired by the proof of the Gabber–Illusie basechange theorem. Just like how the proof of the proper basechange theorem in étale cohomology reduces to the case where two of the schemes involved are spectra of local rings, our proof of Theorem 7.1.7 reduces to the case where the co-topoi \( X \) and \( Z \) are local. In Section 7.5 we prove that fibre product squares obtained by pulling back along a localization \( \ell_{x,*} : X_{(x)} \to X \) satisfy the truncated basechange condition (Proposition 7.5.1); this is one of the key ingredients that allows us to reduce the proof of Theorem 7.1.7 to the case where \( X \) and \( Z \) are local. Section 7.6 discusses the functoriality of oriented fibre products in oriented morphisms of cospans that we
need to deduce Theorem 7.1.7 from the contents of Section 7.5. In Section 7.7 we put everything together and prove Theorem 7.1.7.

7.2 Examples of the basechange condition

In this section we provide a few examples of (oriented) squares that are easily seen to satisfy the basechange condition. None of these examples are used in the sequel. The first two examples are due to an observation of Gabber [61, Exposé XI, Remarque 4.9].

7.2.1 Example. Let \( f : X \to Z \) be a geometric morphism of \( \infty \)-topoi. Then from the equivalence \( \Psi_f^* = \text{pr}_1^* : X \times_Z Z \to X \) and the fact that \( \text{pr}_2^* \Psi_f = f * \) (Proposition 6.3.1), we have equivalences

\[
\text{pr}_1^* \text{pr}_2^* = \Psi_f^* \text{pr}_2^* \cong f^*.
\]

From this we deduce the left basechange condition for the defining oriented square of the evanescent \( \infty \)-topos:

\[
\begin{array}{ccc}
X \times_Z Z & \xrightarrow{\text{pr}_2^*} & Z \\
\downarrow \text{pr}_1^* & & \downarrow \\
X & \xrightarrow{f_*} & Z.
\end{array}
\]

7.2.2 Example. Dually, let \( g : Y \to Z \) be a geometric morphism of \( \infty \)-topoi. From Proposition 6.3.4 we see that the defining oriented square of the coëvanescent \( \infty \)-topos \( Z \times_Z Y \) satisfies the left basechange condition.

As noted by Johnstone–Moerdijk [66, Remark 2.5], pullbacks along local geometric morphisms also satisfy the basechange condition.

7.2.3 Example. Consider a pullback square of \( \infty \)-topoi

\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{f_*} & Y \\
\downarrow \tilde{g}_* & & \downarrow g_* \\
X & \xrightarrow{f_*} & Z,
\end{array}
\]

where \( g_* \) exhibits \( Y \) as local over \( Z \) with centre \( g^! \). Then by (6.2.11) the geometric morphism \( \tilde{g}_* \) exhibits \( X \times_Z Y \) as local over \( X \) and the center \( \tilde{g}^! \) of \( \tilde{g}_* \) satisfies \( f_* \tilde{g}^! = g^! f_* \).

We have adjunctions

\[
f^* g_* \cong g^! f_* \quad \text{and} \quad \tilde{g}_* f^* \cong \tilde{f}_* \tilde{g}^!,
\]

so the equivalence \( f_* \tilde{g}^! = g^! f_* \) shows that \( f^* g_* = \tilde{g}_* f^* \). From this equivalence which we deduce the left basechange condition for the square (7.2.4).
7.2.5 Example. Let $f_* : X \to Z$ and $g_* : Y \to Z$ be geometric morphisms of \inftopoi. Decompose the oriented fibre product $X \times_Z Y$ as an iterated pullback

\[
\begin{array}{ccc}
X & \xrightarrow{f_*} & Z \\
\downarrow & & \downarrow \\
Y & \xrightarrow{g_*} & Y \\
\end{array}
\]

It follows from Example 7.2.3 that local geometric morphisms are proper [HTT, Definition 7.3.1.4]. Assume that $g_*$ is a proper geometric morphism. Then by applying Example 7.2.1 to the lower right square of (7.2.6), Examples 6.3.6 and 7.2.3 to the lower left square of (7.2.6), and the properness of $g_*$ to the top squares of (7.2.6), we deduce that the three pullback squares in (7.2.6) and the oriented square all satisfy the left basechange condition, and that $\text{pr}_{1,*} : X \times_Z Y \to X$ is a proper geometric morphism.

7.3 Applications of the basechange theorem for oriented fiber products

In this section we give a number of applications of our basechange theorem (Theorem 7.1.7).

7.3.1 Example. Let $f_* : X \to Z$ and $g_* : Y \to Z$ be geometric morphisms of \inftopoi, and assume that $X$ and $Y$ are bounded coherent and $Z$ is Stone. Then by Corollary 4.4.15=[SAG, Corollary E.3.1.2], $f_*$ and $g_*$ are automatically coherent. Since $Z$ is Stone, Proposition 9.1.1 shows that

\[
X \times_Z Y = X \times Y.
\]

Hence by Theorem 7.1.7 we see that the (unoriented) pullback square

\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{\text{pr}_{2,*}} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f_*} & Z \\
\end{array}
\]

satisfies the truncated basechange condition.

7.3.3 Subexample. Set $Z = S$ in Example 7.3.1, so that $f_* = \Gamma_{X,*}$ and $g_* = \Gamma_{Y,*}$. Since left exact functors preserve truncated objects, we see that for any truncated space $K$ the natural morphism

\[
\Gamma_{X,*}\Gamma_{Y,*}(K) \to \Gamma_{X,*}\text{pr}_{1,*}\Gamma_{Y,*}(K)
\]
in $S$ is an equivalence. Hence the natural morphism
\[ \Pi_{\infty}(X) \times \Pi_{\infty}(Y) \to \Pi_{\infty}(X \times Y) \]
of prospaces becomes an equivalence after protruncation. Since the composition monoidal structure and cartesian monoidal structure on $\text{Pro}(S)$ coincide on the full subcategory $S_{\text{pr}}^\infty$ of profinite spaces (Recollection 2.8.2), we deduce that
\[ \tilde{\Pi}_{\infty}(X \times Y) \Rightarrow \tilde{\Pi}_{\infty}(X) \times \tilde{\Pi}_{\infty}(Y) \].
Combining this with Corollary 4.3.7 we see that the profinite shape $\tilde{\Pi}_{\infty} : \text{Top}_{bc}^\infty \to S_{\text{pr}}^\infty$ preserves both inverse limits and finite products.

7.3.4 Example. Let $k$ be a separably closed field and let $X$ and $Y$ be $k$-schemes. Assume that $X$ is coherent and $Y$ is proper over $k$. Then combining Chough's work generalizing the proper basechange theorem in étale cohomology to the nonabelian setting [22, Theorem 5.3] with Subexample 7.3.3, we see that the natural geometric morphism
\[ (X \times \text{Spec } k \ Y)_{\text{et}} \to X_{\text{et}} \times (\text{Spec } k)_{\text{et}} = X_{\text{et}} \times Y_{\text{et}} \]
induces an equivalence on profinite shapes. Equivalently, the natural geometric morphism $(X \times \text{Spec } k \ Y)_{\text{et}} \to X_{\text{et}} \times Y_{\text{et}}$ induces an equivalence on lisse local systems (Corollary 4.4.17=[SAG, Corollary E.2.3.3]).

7.4 Stable consequences of nonabelian basechange

Let $R$ be a commutative ring and
\[ \begin{array}{ccc}
X \times_Z Y & \xrightarrow{pr_{1,*}} & Y \\
\downarrow{pr_{1,\sigma}} & & \downarrow{g_*} \\
X & \xrightarrow{f_*} & Z
\end{array} \]
an oriented fiber product square of coherent 1-topoi and coherent geometric morphisms. Gabber and Illusie proved the following stable variant of Theorem 7.1.7: for any object $F \in D(Y; R)$ that is bounded-above with respect to the natural $t$-structure on $D(Y; R)$ (Recollection 7.4.9), the basechange morphism
\[ f^* g_* (F) \to pr_{1,*} pr_{2,*}^* (F) \]
is an equivalence [61, Exposé XI, Théorème 2.4]. In this section, we explain how to deduce this result of Gabber–Illusie from Theorem 7.1.7. We also show that the result holds more generally when $X, Y$, and $Z$ are bounded coherent $\infty$-topoi and $R$ is replaced by a connective $E_1$-ring spectrum (Proposition 7.4.11 and Example 7.4.13).

The proof ultimately reduces to the fact that the basechange morphisms are compatible with the forgetful functors from sheaves of $R$-module spectra to sheaves of spaces; this fact is elementary, but we could not locate it elsewhere in the literature. To explain this fact, we begin by recalling the basics of stabilization and sheaves of $R$-module spectra. The reader familiar with this basic fact or more interested in the stable consequences of Theorem 7.1.7 but not their proofs is encouraged to skip ahead to Proposition 7.4.11.
7.4.1 Recollection (stabilization [HA, Definition 1.4.2.8]). Write $S_{\text{fin}} \subset S$ for the $\infty$-category of finite spaces: the smallest full subcategory of $S$ containing the terminal object and closed under finite colimits. Let $C$ be an $\infty$-category with finite limits. Recall that the stabilization of $C$ is the full subcategory

$$\text{Sp}(C) \subset \text{Fun}(S_{\text{fin}}^*, C)$$

spanned by those functors that preserve the terminal object and carry pushout squares in $S_{\text{fin}}^*$ to pullback squares in $C$. Also recall that the functor

$$\Omega_\infty^C : \text{Sp}(C) \to C$$

is defined by evaluation on the 0-sphere $S^0 \in S_{\text{fin}}^*$.

Write $\mathbf{Sp} = \text{Sp}(S)$ for the $\infty$-category of spectra. If $C$ is presentable, then the stabilization $\text{Sp}(C)$ is equivalent to the tensor product of presentable $\infty$-categories $C \otimes \mathbf{Sp}$ [HA, Example 4.8.1.23].

7.4.2. Let $F : C \to D$ be a left exact functor between $\infty$-categories with finite limits. Then post-composition with $F$ defines a functor $\text{Sp}(F) : \text{Sp}(C) \to \text{Sp}(D)$ on stabilizations. When not confusing, we simply denote this induced functor $\text{Sp}(C) \to \text{Sp}(D)$ by $F$. It is immediate from the definitions that the square

$$\begin{array}{ccc}
\text{Sp}(C) & \xrightarrow{\text{Sp}(F)} & \text{Sp}(D) \\
\Omega_\infty^C & \downarrow & \downarrow \Omega_\infty^D \\
C & \xrightarrow{F} & D
\end{array}$$

canonically commutes.

7.4.3. Let $F : C \rightleftarrows D : G$ be an adjunction between $\infty$-categories with finite limits, and assume that $F$ is left exact. Then the functor $F : \text{Sp}(C) \to \text{Sp}(D)$ is left adjoint to $G : \text{Sp}(D) \to \text{Sp}(C)$.

Now let us explain the sense in which stabilization is compatible with basechange morphisms.

7.4.4. Let $F, F' : C \to D$ be left exact functors between $\infty$-categories with finite limits, and let $\sigma : F \to F'$ be a natural transformation. Then pointwise application of $\sigma$ defines a natural transformation $\text{Sp}(\sigma) : \text{Sp}(F) \to \text{Sp}(F')$. When not confusing, we simply denote this natural transformation $\text{Sp}(F) \to \text{Sp}(F')$ by $\sigma$.

It is immediate from the definitions that the natural transformation $\text{Sp}(\sigma)$ is compatible with $\sigma$ in the following sense: we have a natural identification $\Omega_\infty^D \text{Sp}(\sigma) = \sigma \Omega_\infty^C$ of natural transformations

$$F \Omega_\infty^C = \Omega_\infty^D \text{Sp}(F) \to \Omega_\infty^D \text{Sp}(F') = F' \Omega_\infty^C.$$
7.4.5. Consider an oriented square of $\infty$-categories and left exact functors:

\[
\begin{array}{ccc}
A & \xrightarrow{q_*} & C \\
B & \xrightarrow{f_*} & D \\
\end{array}
\]

where $p_*$ is a geometric morphism. Assume that the functors $f_*$ and $q_*$ admit left exact left adjoints $f^*$ and $q^*$, respectively. From (7.4.4) we see that we have a natural identification

\[
\Omega^\infty_B \text{BC}_{\text{Sp}(o)} = \text{BC}_o \Omega^\infty_C
\]

of natural transformations

\[
f^* g_* \Omega^\infty_C = \Omega^\infty_B \text{Sp}(f^*) \text{Sp}(g_*) \to \Omega^\infty_B \text{Sp}(p_*) \text{Sp}(q^*) = p_* q^* \Omega^\infty_C
\]

Now we generalize to coefficients in any connective $E_1$-ring spectrum.

7.4.6 Notation. Let $X$ be an $\infty$-topos and $R$ a connective $E_1$-ring spectrum. Write:

- $\text{LMod}_R$ for the $\infty$-category of left $R$-module spectra.
- $\text{D}(X; R) = X \otimes \text{LMod}_R$ for the $\infty$-category of sheaves of (left) $R$-modules on $X$.
- $u_X : \text{D}(X; R) \to \text{Sp}(X)$ for the forgetful functor.

Given a geometric morphism $f_* : X \to Z$, we simply write

\[
f_* : \text{D}(X; R) \to \text{D}(Z; R)
\]

for the induced right adjoint functor with left exact left adjoint. Note that the induced functors on sheaves of $R$-module spectra commute with the forgetful functors in the sense that we have canonical identifications

\[
u_Z f_* = f_* u_X \quad \text{and} \quad u_Z f^* = f^* u_X.
\]

The analogues of (7.4.4) and (7.4.5) hold true when we forget from sheaves of $R$-module spectra to sheaves of spectra. The important point is the following:

7.4.7. Given an oriented square of $\infty$-topoi and geometric morphisms

\[
\begin{array}{ccc}
W & \xrightarrow{q_*} & Y \\
X & \xrightarrow{f_*} & Z \\
\end{array}
\]

we have a natural identification

\[
u_X \text{BC}_o = \text{BC}_o \nu_Y
\]

of natural transformations

\[
f^* g_* \nu_Y = u_X f^* g_* \to u_X p_* q^* = p_* q^* u_Y.
\]
Finally, to state the main results of this section, let us recall the natural $t$-structure on $D(\mathcal{X}; R)$.

7.4.8 Convention. We use homological indexing conventions for our $t$-structures. If $D$ is a stable $\infty$-category with a $t$-structure, then the shift $G \mapsto G[1]$ is suspension, and we write $D_{\geq n} = D_{\geq 0}[n]$ and $D_{\leq n} = D_{\leq 0}[n]$.

7.4.9 Recollection ([SAG, Proposition 1.3.2.7]). Let $\mathcal{X}$ be an $\infty$-topos. Recall that the stabilization $\text{Sp}(\mathcal{X})$ has a natural $t$-structure $(\text{Sp}(\mathcal{X})_{\geq 0}, \text{Sp}(\mathcal{X})_{\leq 0})$ defined by saying that $F \in \text{Sp}(\mathcal{X})_{\leq 0}$ if and only if $\Omega^\infty_{\mathcal{X}} F$ is a 0-truncated object of $\mathcal{X}$. Consequently, for each integer $n \geq 0$, an object $F$ of $\text{Sp}(\mathcal{X})$ is in $\text{Sp}(\mathcal{X})_{\leq n}$ if and only if $\Omega^\infty_{\mathcal{X}} F$ is an $n$-truncated object of $\mathcal{X}$.

Let $R$ be a connective $E_1$-ring spectrum. There is a natural $t$-structure on $D(\mathcal{X}; R)$ given by setting

$$D(\mathcal{X}; R)_{\geq 0} = u_{\mathcal{X}}^{-1}(\text{Sp}(\mathcal{X})_{\geq 0}) \quad \text{and} \quad D(\mathcal{X}; R)_{\leq 0} = u_{\mathcal{X}}^{-1}(\text{Sp}(\mathcal{X})_{\leq 0}).$$

7.4.10 Notation. Let $\mathcal{S}$ be a stable $\infty$-category with $t$-structure $(\mathcal{S}_{\geq 0}, \mathcal{S}_{\leq 0})$. We write

$$\mathcal{S}_{<\infty} := \bigcup_{n \in \mathbb{Z}} \mathcal{S}_{\leq n}$$

for the full subcategory of $\mathcal{S}$ spanned by the $t$-bounded-above objects.

We are now ready to prove our refinement of the basechange theorem of Gabber–Illusie. The proof proceeds in two steps. First we note that it suffices to check the claim in the ‘universal’ case where $R$ is the sphere spectrum. We then show that, in this case, the claim follows from the truncated basechange condition at the level of $\infty$-topoi.

7.4.11 Proposition. Let

$$(7.4.12) \quad \begin{array}{ccc}
W & \xrightarrow{q_*} & Y \\
\downarrow p_* & & \downarrow g_* \\
X & \xrightarrow{f_*} & Z.
\end{array}$$

be an oriented square of $\infty$-topoi. If $(7.4.12)$ satisfies the truncated basechange condition, then for any $E_1$-ring spectrum $R$, the left basechange morphism associated to the oriented square

$$\begin{array}{ccc}
D(W; R) & \xrightarrow{q_*} & D(Y; R) \\
\downarrow p_* & & \downarrow g_* \\
D(X; R) & \xrightarrow{f_*} & D(Z; R).
\end{array}$$

of stable $\infty$-categories is an equivalence when restricted to $D(Y; R)_{<\infty} \subset D(Y; R)$.

Proof. Since the forgetful functor $u_X : D(\mathcal{X}; R) \to \text{Sp}(\mathcal{X})$ is conservative, it suffices to show that for all $F \in D(Y; R)_{<\infty}$, the morphism

$$u_X \text{BC}(F) : u_X f^* g_* (F) \to u_X p_* q^* (F)$$

is an equivalence.
is an equivalence. By (7.4.7), we see that the morphism \( u_X \) \( BC(F) \) is equivalent to the morphism

\[
BC(u_Y F) : f^* g_* (u_Y F) \rightarrow p_* q^* (u_Y F)
\]

in \( \text{Sp}(X) \).

To see that \( BC(u_Y F) \) is an equivalence, we need to show that for each integer \( n \in \mathbb{Z} \), the morphism

\[
\Omega^\infty_{\infty - n} BC(u_Y F) : \Omega^\infty_{\infty - n} f^* g_* (u_Y F) \rightarrow \Omega^\infty_{\infty - n} p_* q^* (u_Y F)
\]

is an equivalence. Since both the left and right adjoint in a geometric morphism of \( \infty \)-topoi are left exact, applying (7.4.5) we see that the morphism \( \Omega^\infty_{\infty - n} BC(u_Y F) \) is equivalent to the morphism

\[
BC(\Omega^\infty_{\infty - n} u_Y F) : f^* g_* (\Omega^\infty_{\infty - n} u_Y F) \rightarrow p_* q^* (\Omega^\infty_{\infty - n} u_Y F).
\]

The assumption that \( F \in D(Y; R)_{<\infty} \) is t-bounded-above guarantees that for all integers \( n \in \mathbb{Z} \), the object \( \Omega^\infty_{\infty - n} u_Y F \) is truncated. Since the square (7.4.12) satisfies the truncated basechange condition, we see that \( BC(\Omega^\infty_{\infty - n} u_Y F) \) is an equivalence. This completes the proof.

7.4.13 Example. Let \( f_* : X \rightarrow Z \) and \( g_* : Y \rightarrow Z \) be coherent geometric morphisms between bounded coherent \( \infty \)-topoi and let \( R \) be a connective \( E_1 \)-ring spectrum. Theorem 7.1.7 and Proposition 7.4.11 show that the left basechange morphism associated to the oriented square

\[
\begin{array}{ccc}
D(X \times_Z Y; R) & \xrightarrow{p_1^*} & D(Y; R) \\
\downarrow & & \downarrow g_* \\
D(X; R) & \xrightarrow{f_*} & D(Z; R)
\end{array}
\]

of stable \( \infty \)-categories is an equivalence when restricted to \( D(Y; R)_{<\infty} \subset D(Y; R) \).

7.5 Localisations & the truncated basechange condition

The remainder of the chapter is dedicated to actually proving Theorem 7.1.7. In this section we prove the following basechange result, which ultimately allows us to reduce to proving Theorem 7.1.7 in the case where \( X \) and \( Z \) are local and \( f_* \) is a local geometric morphism.

7.5.1 Proposition. Let \( p_* : W \rightarrow X \) be a coherent geometric morphism between bounded coherent \( \infty \)-topoi. Then for any point \( x_* \) of \( X \), the pullback square

\[
\begin{array}{ccc}
\tilde{X} \times_X W & \xrightarrow{\tilde{d}} & W \\
\downarrow & & \downarrow p_* \\
X_{(x)} & \xrightarrow{\iota_*} & X
\end{array}
\]

satisfies the truncated basechange condition.
To prove Proposition 7.5.1, we use the Grothendieck–Verdier description of the localisation (Proposition 6.5.3) and the (obvious) fact that pullbacks along étale geometric morphisms satisfy base change condition to reduce to a general result on inverse limits (Proposition 7.5.5).

7.5.2 Lemma. Let \( f_* : E \to X \) and \( p_* : W \to X \) be geometric morphisms of \( \infty \)-topoi. If \( f_* \) is étale, then the pullback square

\[
\begin{array}{ccc}
E \times_X W & \xrightarrow{j} & W \\
\downarrow & & \downarrow p_* \\
E & \xrightarrow{f_*} & X
\end{array}
\]

satisfies the left base change condition.

We fix some useful notation for the result.

7.5.3 Notation. Let \( W, X : I \to \text{Top}_\infty \) be diagrams of \( \infty \)-topoi. For each morphism \( \alpha : j \to i \) in \( I \), we write

\[ e_{\alpha,*} : W_j \to W_i \quad \text{and} \quad f_{\alpha,*} : X_j \to X_i \]

for the transition morphisms. For each \( i \in I \), we write

\[ \xi_{i,*} : \lim_{i \in I} W_i \to W_i \quad \text{and} \quad \pi_{i,*} : \lim_{i \in I} X_i \to X_i \]

for the projections. In addition, we assume that for each morphism \( \alpha : j \to i \) in \( I \), the functors

\[ e_{\alpha,*} : W_j \to W_i \quad \text{and} \quad f_{\alpha,*} : X_j \to X_i \]

almost preserve filtered colimits (Definition 3.10.2).

7.5.4. Most importantly, the assumptions of Notation 7.5.3 are valid for inverse systems of bounded coherent \( \infty \)-topoi and coherent geometric morphisms (Corollary 3.10.5).

7.5.5 Proposition. Keep the assumptions of Notation 7.5.3. Let \( p : W \to X \) be a natural transformation, each of whose components \( p_{i,*} : W_i \to X_i \) is almost preserves filtered colimits. If for each morphism \( \alpha : j \to i \) in \( I \), the square

\[
\begin{array}{ccc}
W_j & \xrightarrow{e_{\alpha,*}} & W_i \\
\downarrow p_{j,*} & & \downarrow p_{i,*} \\
X_j & \xrightarrow{f_{\alpha,*}} & X_i
\end{array}
\]

satisfies the truncated base change condition, then for each \( i \in I \) the induced square

\[
\begin{array}{ccc}
\lim_{i \in I} W_j & \xrightarrow{\xi_{i,*}} & W_i \\
\downarrow \lim_{i \in I} p_{j,*} & & \downarrow p_{i,*} \\
\lim_{i \in I} X_j & \xrightarrow{\pi_{i,*}} & X_i
\end{array}
\]

satisfies the truncated base change condition.
Proof. Since $I$ is inverse, for each $i \in I$, the forgetful functor $I_{ji} \to I$ is limit-cofinal [HTT, Example 5.4.5.9 & Lemma 5.4.5.12]. Thus we may without loss of generality assume that $I$ admits a terminal object $1$ and that $i = 1$. Writing $q_{\ast} := \lim_{i \in I} p_{i, \ast}$, we see that we have reduced to showing that the square

$$\begin{array}{ccc}
\lim_{i \in I} W_{i} & \xrightarrow{\xi_{i, \ast}} & W_{1} \\
q_{\ast} \downarrow & & \downarrow p_{1, \ast} \\
\lim_{i \in I} X_{i} & \xrightarrow{\pi_{i, \ast}} & X_{1}
\end{array}$$

(7.5.7)

satisfies the truncated base change condition.

Inverse limits in $\text{Top}_{\mathcal{C}, \delta_{j}}$ (Theorem 3.1.1 in [HTT, Theorem 6.3.3.1]), so an object of the limit of a diagram $Y : I \to \text{Top}_{\mathcal{C}}$ is specified by a compatible system $\{U_{i}\}_{i \in I}$ of objects $U_{i} \in Y_{i}$ along with, for each $\alpha : j \to i$ in $I$, an equivalence $\phi_{\alpha} : g_{\alpha, \ast}(U_{j}) = U_{i}$, where $g_{\alpha, \ast} : Y_{j} \to Y_{i}$ is the transition morphism. Thus for $U \in W_{1}$ we have

$$q_{\ast}\xi_{i, \ast}(U) = \{p_{i, \ast},\xi_{i, \ast}, \xi_{i, \ast}(U)\}_{i \in I},$$

and

$$\pi_{i, \ast}^{1} p_{1, \ast}(U) = \{\pi_{1, \ast}^{1},\pi_{1, \ast}^{1}, \pi_{1, \ast}^{1}(U)\}_{i \in I}.$$ 

It therefore suffices to show that for each $i \in I$, the natural morphism

$$\pi_{i, \ast}^{1} p_{1, \ast}(U) \xrightarrow{\pi_{i, \ast}^{1} f_{i, \ast}^{\ast}} \pi_{i, \ast}^{1}$$

induced by the basechange morphism $\text{BC} : \pi_{i, \ast}^{1} p_{1, \ast} \to \pi_{i, \ast}^{1} q_{\ast} \xi_{i, \ast}$ is an equivalence when restricted to $(W_{1})_{\mathcal{C}, \delta_{j}}$.

For $i \in I$ and $\alpha : i \to 1$ the unique morphism, we simply write $f_{i, \ast} = f_{i, \ast}$ and $e_{i, \ast} = e_{i, \ast}$. Note that for every truncated object $W \in (W_{1})_{\mathcal{C}, \delta_{j}}$, we have equivalences

$$\pi_{i, \ast}^{1} p_{1, \ast}(U) = \pi_{i, \ast}^{1} f_{i, \ast}^{\ast} p_{1, \ast}(U) = \text{colim}_{\alpha \in (I_{j})_{\mathcal{C}, \delta_{j}}} f_{i, \ast}^{\ast} p_{1, \ast}(U)$$

(Corollary 4.3.4)

$$= \text{colim}_{[a : j \to i] \in (I_{j})_{\mathcal{C}, \delta_{j}}} f_{i, \ast}^{\ast} p_{j, \ast} e_{a}^{-1} e_{a}^{\ast}(U)$$

$$= \text{colim}_{[a : j \to i] \in (I_{j})_{\mathcal{C}, \delta_{j}}} p_{j, \ast} e_{a}^{-1} e_{a}^{\ast}(U),$$

where the third equivalence is by the assumption that the square (7.5.6) satisfies the truncated basechange condition. In addition, Corollary 4.3.4 and the fact that $\xi_{i, \ast} f_{i, \ast}^{\ast} = \xi_{i, \ast}$ give equivalences

$$p_{i, \ast} \left( \text{colim}_{\alpha \in (I_{j})_{\mathcal{C}, \delta_{j}}} e_{\alpha, \ast} e_{\alpha}^{-1} e_{\alpha}^{\ast}(U) \right) = p_{i, \ast} \xi_{i, \ast} e_{\alpha, \ast} e_{\alpha}^{-1} e_{\alpha}^{\ast}(U)$$

for every truncated object $U \in (W_{1})_{\mathcal{C}, \delta_{j}}$. As left exact functors preserve $n$-truncatedness for all $n \geq -2$, and each $p_{i, \ast}$ almost preserves filtered colimits by assumption, we see that for every truncated object $U$ of $W_{1}$, the natural morphism

$$\text{colim}_{\alpha \in (I_{j})_{\mathcal{C}, \delta_{j}}} p_{i, \ast} e_{\alpha, \ast} e_{\alpha}^{-1} e_{\alpha}^{\ast}(U) \to p_{i, \ast} \left( \text{colim}_{\alpha \in (I_{j})_{\mathcal{C}, \delta_{j}}} e_{\alpha, \ast} e_{\alpha}^{-1} e_{\alpha}^{\ast}(U) \right)$$

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is an equivalence. This provides an equivalence

\[ \pi_1, \pi_1^* p_{1*,*} (U) \Rightarrow p_{1*,*} \xi_1^* (U) . \]

To conclude, note that the equivalence (7.5.8) is homotopic to \( \pi_1, \) BC(U). \( \square \)

*Proof of Proposition 7.5.1.* Combine Lemma 7.5.2 and Proposition 7.5.5; note that the hypotheses of Proposition 7.5.5 are valid by (6.6.3) and Corollary 3.10.5 (cf. Corollary 3.9.4=[SAG, Corollary A.8.3.3]). \( \square \)

### 7.6 Functoriality of oriented fibre products in oriented diagrams

In this section we discuss the functoriality of the oriented fibre product in oriented diagrams of cospans. Then we use this additional functoriality to construct some unexpected extra adjoints to the second projection from the oriented fibre product (Proposition 7.6.6). In nice cases, this provides a way to check that the basechange morphism becomes an equivalence after passing to stalks (Lemma 7.6.9); this is key to our proof of Theorem 7.1.7.

The main results of this section generalise and refine results of Gabber–Illusie [61, Exposé XI, Proposition 2.3].

#### 7.6.1. Suppose that we are given a diagram of \( \infty \)-topoi and natural transformations

\[
\begin{array}{ccc}
X & \xrightarrow{f_*} & Z & \xleftarrow{g_*} & Y \\
\downarrow{\eta} & & \downarrow{\theta} & & \downarrow{\epsilon} \\
X' & \xrightarrow{f'_*} & Z' & \xleftarrow{g'_*} & Y'.
\end{array}
\]

Then by the universal property of the oriented fibre product \( X' \times_{Z'} Y' \), the diagram

\[
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{pr_2, *} & Y \\
\downarrow{pr_1, *} & & \downarrow{\epsilon} \\
X & \xrightarrow{f_*} & Z & \xleftarrow{g_*} & Y' \\
\downarrow{\eta} & & \downarrow{\theta} & & \downarrow{\epsilon} \\
X' & \xrightarrow{f'_*} & Z' & \xleftarrow{g'_*} & Y'.
\end{array}
\]

(functorially) induces a geometric morphism \( X \times_Z Y \to X' \times_{Z'} Y' \). We simply denote this geometric morphism by

\[ x_*, y_* : X \times_Z Y \to X' \times_{Z'} Y' , \]
leaving the natural transformations \( \eta \) and \( \theta \) implicit. Please note that the geometric morphism \( x_* \xrightarrow{\kappa} y_* \) satisfies the obvious relations

\[
\text{pr}_{1,*} \circ (x_* \xrightarrow{\kappa} y_*) = x_* \text{pr}_{1,*} \quad \text{and} \quad \text{pr}_{2,*} \circ (x_* \xrightarrow{\kappa} y_*) = y_* \text{pr}_{2,*}.
\]

7.6.2. Suppose that we are given a commutative diagram of \( \infty \)-topoi

\[
\begin{array}{ccc}
X & \xrightarrow{f_*} & Z \\
\downarrow{z_*} & & \downarrow{y_*} \\
X' & \xrightarrow{f'_*} & Z' \\
\end{array}
\]

and assume that \( x_*, y_*, \) and \( z_* \) are coëssential with centres \( x^i, y^i, \) and \( z^i \), respectively. Using the right basechange morphisms with respect to the adjunctions \( x_\bigcirc x^i, y_* \bigcirc y^i, \) and \( z_* \bigcirc z^i \) (7.1.1.2), we obtain a pair of oriented squares

\[
\begin{array}{ccc}
X' & \xrightarrow{f'_*} & Z' \\
\downarrow{z'_*} & & \downarrow{y'_*} \\
X & \xrightarrow{f_*} & Z \\
\end{array}
\]

Note that the natural transformation in the left-hand square of (7.6.4) points in the wrong direction to apply (7.6.1).

7.6.5. Keep the notations of (7.6.2), and additionally assume that the natural transformation in the left-hand square of (7.6.4) is an equivalence, so that \( f_* \bigcirc x^i \xrightarrow{\kappa} z^i f'_* \). Then by the functoriality of the oriented fibre product in oriented diagrams (7.6.1), the diagram (7.6.4) defines a geometric morphism \( x^i \xrightarrow{\kappa} z^i ; y^i : X' \xrightarrow{\kappa} Z' Y' \to X \xrightarrow{\kappa} Y \).

7.6.6 Proposition. With the notations and assumptions of (7.6.5), the geometric morphism

\[
x_* \xrightarrow{\kappa} y_* : X \xrightarrow{\kappa} Y \to X' \xrightarrow{\kappa} Z' Y' \to X \xrightarrow{\kappa} Z Y
\]

is coëssential with centre \( x^i \xrightarrow{\kappa} z^i ; y^i : X' \xrightarrow{\kappa} Z' Y' \to X \xrightarrow{\kappa} Z Y \).

We now explain a particular application of Proposition 7.6.6 that allows us to show that if \( f_* : X \to Z \) is a local geometric morphism of local \( \infty \)-topoi and \( g_* : Y \to Z \) is any geometric morphism, then the second projection exhibits \( X \xrightarrow{\kappa} Z Y \) as local over \( Y \).

7.6.7. Let \( f_* : X \to Z \) be a local geometric morphism of local \( \infty \)-topoi with centres \( x_* \) and \( z_* \), respectively, and let \( g_* : Y \to Z \) be a geometric morphism of \( \infty \)-topoi. Note that all of the vertical geometric morphisms in the commutative diagram of \( \infty \)-topoi
exhibit the top ∞-topoi as local over the bottom ∞-topoi. Since $f_*$ is a local geometric morphism, applying the discussion of (7.6.2) shows that we are in the situation of (7.6.5). That is to say $x_*, z_*$, and $\text{id}_Y$ induce a geometric morphism

$$x_*, \tilde{\kappa}_z, \text{id}_Y : Y = S \tilde{\kappa}_Y \to X \tilde{\kappa}_X Y.$$  

The following is our generalisation of [61, Exposé XI, Proposition 2.3]. Note that this generalisation is not just ∞-toposic: in our version we don’t need to take stalks.

7.6.8 Lemma. With the notations of (7.6.7), the second projection $\text{pr}_{2,*} : X \tilde{\kappa}_Z Y \to Y$ exhibits $X \tilde{\kappa}_Z Y$ as local over $Y$ with centre

$$x_*, \tilde{\kappa}_z, \text{id}_Y : Y = S \tilde{\kappa}_Y \to X \tilde{\kappa}_X Y.$$  

Proof. The fact that $\text{pr}_{2,*}$ is coëssential with centre $x_*, \tilde{\kappa}_z, \text{id}_Y$ is immediate from Proposition 7.6.6. The full faithfulness of $x_*, \tilde{\kappa}_z, \text{id}_Y$ follows from the equivalence

$$\text{pr}_{2,*} \circ (x_*, \tilde{\kappa}_z, \text{id}_Y) = \text{id}_Y.$$  

In the setting of Lemma 7.6.8, we deduce that the basechange morphism becomes an equivalence after taking its stalk at the centre of $X$.

7.6.9 Lemma. Consider an oriented square of ∞-topoi

$$\begin{array}{ccc}
W & \xrightarrow{g_*} & Y \\
P_* \downarrow & \searrow \alpha & \downarrow g_* \\
X & \xrightarrow{f_*} & Z,
\end{array}$$

where $g_*$ is a quasi-equivalence, $X$ and $Z$ are local with centres $x_*$ and $z_*$, respectively, and $f_*$ is a local geometric morphism. Then the natural transformation

$$x^* \text{BC}_\alpha : x^* f^* g_* \to x^* p_* q^*$$

is an equivalence.

Proof. We prove the stronger claim that $x^* f^* g_* = x^* p_* q^*$ and the space of natural transformations $x^* f^* g_* \to x^* p_* q^*$ is contractible. Since $Z$ is local we have equivalences

$$x^* f^* g_* = z^* g_* = \Gamma_{Z,*} g_* = \Gamma_{Y,*}.$$  

Since $X$ is local and $q_*$ is a quasi-equivalence, applying Lemma 6.1.3 we have equivalences

$$x^* p_* q^* = \Gamma_{X,*} p_* q^* = \Gamma_{W,*} q^* = \Gamma_{Y,*}.$$  

Thus both $x^* f^* g_*$ and $x^* p_* q^*$ are equivalent to the global sections functor on $Y$. We are now done since $\Gamma_{Y,*}$ is corepresented by the terminal object of $Y$.  

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7.7 Proof of the basechange condition for oriented fibre products

This section is devoted to the proof of Theorem 7.1.7.

**Proof of Theorem 7.1.7.** Write $\text{BC} : f^* g_* \to \text{pr}_{1,*} \text{pr}_{2,*}$ for the left basechange natural transformation of the oriented fibre product square (7.1.8). Notice that since $X$ is bounded coherent, left exact functors preserve truncated objects, and morphisms between truncated objects are truncated, Deligne Completeness (3.11.12) shows that to prove the claim it suffices to show that for every point $x_* \in \text{Pt}(X)$ and truncated object $F \in Y_{\text{coh}}$, the morphism

$$x^* \text{BC}(F) : x^* f^* g_*(F) \to x^* \text{pr}_{1,*} \text{pr}_{2,*}(F)$$

is an equivalence in $S$. We prove this by localizing $X$ at the point $x_*$ and reducing to the case where $X$ and $Z$ are local and $f_*$ is a local geometric morphism; the claim then follows from Lemma 7.6.9.

To reduce to the local case, fix a point $x_* \in \text{Pt}(X)$, define $z_* : = f_! x_*$, and let $f_* : X(x) \to Z(z)$ be the induced geometric morphism on localisations. To simplify notation we write

$$W = X \times_Z Y, \quad W(x) = X(x) \times_X W, \quad \text{and} \quad Y(z) = Z(z) \times_Z Y.$$

Consider the cube

$$\begin{array}{c}
X(x) \xrightarrow{f_*} Z(z) \\
\downarrow \text{pr}_{2,*} \quad \downarrow \text{pr}_{1,*} \\
W(x) \xrightarrow{p_*} Y(z) \xrightarrow{q_*} Y \\
\uparrow f_* \quad \uparrow g_* \\
X \xrightarrow{f_*} Z, \\
\end{array}$$

formed by pulling back the back vertical face along the bottom horizontal face. In the cube (7.7.1), the front vertical face is an oriented square, the back vertical face is an oriented fibre product square, all other vertical faces are commutative, and the side faces are pullback squares. Moreover, the cube satisfies the following property:

\((\ast)\) The natural transformation between the right adjoints given by the composite of the back and left faces of (7.7.1) is equivalent to the natural transformation given by the composite of the front and right faces of (7.7.1).

We claim that the front vertical face of (7.7.1) is an oriented fibre product square. To see this, note that by Proposition 6.5.3, the compatibility of the oriented fibre product with limits (5.4.3), the compatibility of oriented fibre products with étale geometric...
morphisms (Proposition 5.7.5), and Corollary 5.7.6, we have equivalences

\[
X(x) \xrightarrow{\sim} Z(y) \xrightarrow{\sim} \left( \lim_{U \in \text{Nbd}(x)} X_U / U \right) \xrightarrow{\sim} \left( \lim_{V \in \text{Nbd}(y)} Y_V / V \right)
\]

which are both equivalences when restricted to \(\alpha\), of the cube (7.7.1), shows that the natural transformation \(\alpha\) commutes, under identification of left adjoints, \(\text{BC}^R\) defines a natural transformation

\[
\tilde{f}^* \text{BC}^R : \ell_x^* f^* g_* \to \tilde{g}_*^* \ell_z^*.
\]

Let \(\alpha^R\) be the composite

\[
\alpha^R : x^* f^* g_* \xrightarrow{\sim} \Gamma_{X(x)}^* \tilde{f}^* \ell_z^* g_* \xrightarrow{\text{BC}^R} \Gamma_{X(x)}^* \tilde{g}_*^* \ell_z^*.
\]

where the left-hand equivalence is by Lemma 6.2.10 and the fact that \(z^* = x^* f^*\). By Proposition 7.5.1, \(\text{BC}^R\) is an equivalence when restricted to \(Y_{<\infty}\); therefore \(\alpha^R\) is also an equivalence when restricted to \(Y_{<\infty}\). Similarly, since the top horizontal face of (7.7.1) commutes, under identification of left adjoints, \(\text{BC}^L\) defines a natural transformation

\[
\text{BC}^L \: \ell_x^* p^* \to p_* q^* \ell_z^*.
\]

Let \(\alpha^L\) be the composite

\[
\alpha^L : x^* p^* \xrightarrow{\sim} \Gamma_{X(x)}^* \ell_x^* p^* \xrightarrow{\text{BC}^L} \Gamma_{X(x)}^* p_* q^* \ell_z^*.
\]

where the left-hand equivalence is ensured by Lemma 6.2.10. By Proposition 7.5.1, the natural transformation \(\text{BC}^L\) is an equivalence when restricted to \(W_{<\infty}\). Since the functor \(p^*\) is left exact we see that \(\alpha^L\) is an equivalence when restricted to \(Y_{<\infty}\).

Write \(\text{BC}^F : \tilde{f}^* \tilde{g}_* \to p_* q^*\) for the basechange morphism for the front vertical face of the cube (7.7.1). Since \(q_* : W_{<\infty} \to Y_{<\infty}\) exhibits \(W_{<\infty}\) as local over \(Y_{<\infty}\), Lemma 7.6.9 shows that the natural transformation

\[
\Gamma_{X(x)}^* \text{BC}^F : \Gamma_{X(x)}^* \tilde{f}^* \tilde{g}_* \to \Gamma_{X(x)}^* p_* q^*.
\]
is an equivalence. Since $\alpha^R$ and $\alpha^L$ are equivalences when restricted to $Y_{\infty}$, to complete the proof it suffices to show that the square

$$
\begin{array}{ccc}
x^* f^* g_* & \xrightarrow{\alpha^R} & \Gamma_{X_{(0)},*} \tilde{f}^* \tilde{g}_* \tilde{\ell}^*_z \\
x^* \tilde{\beta} & & \Gamma_{X_{(0)},*} \tilde{\epsilon}^*_z \\
x^* \text{pr}_1^*, \text{pr}_2^* & \xrightarrow{\alpha^L} & \Gamma_{X_{(0)},*} q^* \tilde{\ell}^*_z
\end{array}
$$

commutes. This is immediate from the property ($*$) combined with (7.1.4). □
Part III
Stratified higher topos theory

In this part, we import the theory of stratifications into higher topos theory (Chapter 8). In Chapter 9 we introduce a class of bounded coherent $\infty$-topoi called spectral $\infty$-topoi. These are the bounded coherent stratified $\infty$-topoi all of whose strata are Stone $\infty$-topoi. The chief example of a spectral $\infty$-topos is the étale $\infty$-topos of a coherent scheme (Example 9.2.4). We then prove our $\infty$-Categorical Hochster Duality Theorem (Theorem 9.3.1) which shows that the $\infty$-category of profinite stratified spaces is equivalent to the $\infty$-category of spectral $\infty$-topoi. In Chapter 10 we use $\infty$-Categorical Hochster Duality to provide a stratified refinement of the profinite shape – the profinite stratified shape, and provide stratified refinement of the main results on the profinite shape discussed in §4.4.
8 Stratified higher topoi

We now introduce stratifications in the setting of higher topoi. We study the embedding of finite posets and spectral topological spaces into ∞-topoi. Just as with stratified spaces, a ∞-topos stratified over a finite poset can be specified in two ways: first, as a single morphism of ∞-topoi and second, as a décollage. We show that these two ways of representing a stratified ∞-topos are equivalent. We extend the former description to define stratifications of ∞-topoi over a spectral topological space. We prove that any coherent ∞-topos admits a canonical such stratification, and we study the embedding of profinite stratified spaces into stratified ∞-topoi. In the following chapter, we provide an intrinsic characterisation of the stratified ∞-topoi that arise in this manner.

8.1 Higher topoi attached to finite posets & spectral topological spaces

8.1.1. A sheaf on a finite poset $P$ (with its Alexandroff topology – Definition 1.1.3) is determined by its values on the principal open sets, which coincide with its stalks. Precisely, the principal opens form a basis for the topology on $P$ and $\tilde{P}$, and the assignment $p \mapsto P_{\geq p}$ is a fully faithful functor $P \hookrightarrow \text{Open}(P)$ which induces an equivalence $\tilde{P} = \text{Sh}(\text{Open}(P)) \Rightarrow \text{Fun}(P, S)$ (Example 3.12.15). In particular, the co-topos $\tilde{P}$ is both 0-localic and Postnikov complete [SAG, §A.7.2].

8.1.2. If $P$ is a finite poset, then $\tilde{P}$ is a coherent ∞-topos (Example 3.7.1), and a sheaf $F$ on $P$ is $n$-coherent if and only if all of the stalks of $F$ have finite homotopy sets in degrees $m \leq n$.

8.1.3. The assignment $P \mapsto \tilde{P}$ extends to a functor $\text{TSp} \rightarrow \text{Pro} \rightarrow \text{Top}_\infty$ which we also denote by $S \mapsto \tilde{S}$. Thus if $S = \{P_\alpha\}_{\alpha \in A}$ is an inverse system of finite posets, then

$$\tilde{S} = \lim_{\alpha \in A} \tilde{P}_\alpha$$

in $\text{Top}_\infty$. That is, by [HTT, Theorem 6.3.3.1], $\tilde{S}$ is equivalent to the ∞-category with objects collections $\{F_\alpha\}_{\alpha \in A}$ of functors $F_\alpha : P_\alpha \rightarrow S$ along with compatible identifications of $F_\alpha$, with the right Kan extension of $F_\alpha$ along $P_\alpha \rightarrow P_{\alpha'}$ for any morphism $\alpha \rightarrow \alpha'$ in $A$. In particular, $\tilde{S}$ is 0-localic.

If we think of $S$ as a spectral topological space, the 0-topos (locale) $\text{Open}(S)$ is the limit of the 0-topoi $\text{Open}(P)$ over $\text{FC}(S)$. Thus we have an equivalence of 0-localic co-topoi

$$\tilde{S} = \lim_{P \in \text{FC}(S)} \tilde{P}.$$  

Since $\tilde{S}$ is coherent (Example 3.7.1), the ∞-pretopos $\tilde{S}^{\text{coh}}$ of truncated coherent objects of $\tilde{S}$ can be identified with the filtered colimit $\text{colim}_{P \in \text{FC}(S)} \tilde{P}$ over the category...
FC(S) of finite constructible stratifications \( S \to P \), along the relevant restriction functors (§3.9).

Recall that if \( f : S' \to S \) is a quasicompact continuous map of spectral topological spaces, then the induced geometric morphism \( f_* : \tilde{S}' \to \tilde{S} \) is coherent (Example 3.7.1).

8.1.4. If \( S \) is a spectral topological space, then the \( \infty \)-category of points of \( \tilde{S} \) is equivalent to the materialisation of \( S \) (regarded as a profinite poset), viz.,

\[
\text{Pt}(\tilde{S}) = \text{mat}(S).
\]

Thus the points of \( \tilde{S} \) are precisely the points of \( S \) equipped with the specialisation partial ordering.

8.2 Stratifications over finite posets

There are a number of ways to describe stratified \( \infty \)-topoi, but let us focus upon the most elementary description – a straightforward generalisation of the notion of a stratified topological space (Definition 1.2.1).

8.2.1 Definition. For any finite poset \( P \) and any \( \infty \)-topos \( \mathcal{X} \), a **stratification of \( \mathcal{X} \) by \( P \)** – or, more briefly, a **\( P \)-stratification of \( \mathcal{X} \)** – is a geometric morphism of \( \infty \)-topoi \( f_* : \mathcal{X} \to \tilde{\mathcal{P}} \). We define the \( \infty \)-category \( \text{StrTop}_{\infty, P} \) of \( P \)-stratified \( \infty \)-topoi as the over-category \( \text{Top}_{\infty, /\tilde{\mathcal{P}}} \).

8.2.2 Notation. Let \( P \) be a finite poset, and let \( \mathcal{X} \) be a \( P \)-stratified \( \infty \)-topos. For any open subset \( U \subseteq P \), we abuse notation and write \( U \) also for the corresponding open of \( \tilde{\mathcal{P}} \), and we write \( \mathcal{X}_U := \mathcal{X} / f_* U \cong \mathcal{X} \times \tilde{\mathcal{P}} \tilde{U} \subseteq \mathcal{X} \) for the corresponding open subtopos. (Here the fibre product is formed in \( \text{Top}_{\infty, \text{Cat}} \).) Dually, if \( Z \subseteq P \) is closed, then we write \( \mathcal{X}_Z := \mathcal{X} \setminus f_* (P \setminus Z) = \mathcal{X} \setminus \tilde{\mathcal{P}} \tilde{Z} \subseteq \mathcal{X} \) for the corresponding closed subtopos, so that if \( U \) and \( Z \) are complementary, then one exhibits \( \mathcal{X} \) as a recollement of \( \mathcal{X}_Z \) and \( \mathcal{X}_U \).

In particular, for any point \( p \in P \), we write

\[
\mathcal{X}_{\leq p} := \mathcal{X}_{P_{\leq p}} \quad \text{and} \quad \mathcal{X}_{\geq p} := \mathcal{X}_{P_{\geq p}}
\]
as well as

\[
\mathcal{X}_{< p} := \mathcal{X}_{P_{< p}} \quad \text{and} \quad \mathcal{X}_{> p} := \mathcal{X}_{P_{> p}}.
\]
More generally, if $\Sigma \subseteq P$ is any subset, then we write

$$X_\Sigma := X \times_P \Sigma$$

for the fibre product formed in $\text{Top}_{\infty}$. So we define the $p$-th stratum as the fibre product in $\text{Top}_{\infty}$:

$$X_p = X_{\geq p} \times_X X_{\leq p},$$

which is an open subtopos of the closed subtopos $X_{\leq p} \subseteq X$ as well as a closed subtopos of the open subtopos $X_{\geq p} \subseteq X$.

8.2.3 Definition. Let $P$ be a finite poset. We say that a $P$-stratified $\infty$-topos $f_* : X \to \bar{P}$ is constructive if and only if for any point $p \in P$ and any quasicompact open $V \in \text{Open}(X)$, the $\infty$-topos $X_{\geq p} \times_X X_{/V}$ is coherent. We say that a constructive stratification $f_* : X \to \bar{P}$ is coherent constructive if $X$ is a coherent $\infty$-topos, and we say that $f_*$ is bounded coherent constructive if $X$ is a bounded coherent $\infty$-topos. Proposition 5.1.8=[DAG XIII, Proposition 2.3.22] shows that a stratification $f_* : X \to \bar{P}$ is coherent constructive if and only if $X$ is coherent and the geometric morphism $f_*$ is coherent. We write $\text{StrTop}_{\infty, \text{bcc}} \subset \text{StrTop}_{\infty}$ for the subcategory whose objects are bounded coherent constructive stratified $\infty$-topoi and whose morphisms are coherent geometric stratified morphisms:

$$\text{StrTop}_{\infty, \text{bcc}} := \text{Fun}([1], \text{Top}_{\infty, \text{bc}}) \times_{\text{Fun}([1], \text{Top}_{\infty, \text{bc}})} \text{Pos}^{\text{fin}}.$$

8.2.4. Let $P$ be a finite poset. Since $\bar{P}$ is $0$-localic, it follows that a $P$-stratification of an $\infty$-topos $X$ is tantamount to the data of a morphism of $0$-topoi (locales) $\text{Open}(X) \to \text{Open}(P)$, where $\text{Open}(X)$ is the $0$-topos of $(−1)$-truncated objects of $X$, and $\text{Open}(P) = \text{Open}(\bar{P})$ is the $0$-topos of open subsets of $P$. Thus one obtains an equivalence of $\infty$-categories

$$\text{StrTop}_{\infty, P} = \text{Top}_{\infty, \text{bc}} \times_{\text{Top}_{0, \text{bc}}/\text{Open}(P)} \text{Pos}^{\text{fin}}.$$

One may speak of a stratification of an $n$-topos for any $n \in \mathbb{N}$ (as well as the $\infty$-category $\text{StrTop}_n$), and it is tantamount to a stratification of the corresponding $n$-localic $\infty$-topoi:

$$\text{StrTop}_{n, P} = \text{Top}_{n, \text{bc}} \times_{\text{Top}_{0, \text{bc}}/\text{Open}(P)} \text{Pos}^{\text{fin}}.$$

8.2.5 Example. A $[0]$-stratified $\infty$-topos is nothing more than an $\infty$-topos.

8.2.6 Example. Rephrasing (5.1.3), a $[1]$-stratified $\infty$-topos $X \to \bar{[1]}$ is tantamount to a recollement of $\infty$-topoi. If $X$ is coherent, the stratification is constructible if and only if the open subtopos $X_1$ is quasicompact.

8.2.7. To generalise the previous example, let $P$ be a finite poset. We claim that the data of a $P$-stratified $\infty$-topos determines and is determined by a suitable colax functor from $P^{\text{op}}$ to a double $\infty$-category of $\infty$-topoi and left exact functors.

To make a precise assertion, we shall say that a locally cocartesian fibration $X \to P^{\text{op}}$ is left exact if each fibre $X_p$ admits all finite limits, and for any $p \leq q$ in $P$, the
functor $X_q \to X_p$ is left exact. Now left exact locally cocartesian fibrations $X \to P^{\text{op}}$ whose fibres are $\infty$-topoi organise themselves into a $\infty$-category $\text{LocCocart}_{\text{lex, top}}^{\text{loc, top}}$. Then it seems likely that one can produce an equivalence of $\infty$-categories

$$\text{LocCocart}_{\text{lex, top}}^{\text{loc, top}} \cong \text{StrTop}_{\text{co, P}},$$

natural in $P$. To prove this would involve a diversion into a simplicial thicket that is unnecessary for our work here; we therefore leave this matter for a later paper.

8.2.8 Example. The $\infty$-topos $\tilde{P}$, equipped with the identity stratification, is itself terminal in $\text{StrTop}_{\text{co, P}}$.

8.2.9 Example. If $P$ is a finite poset, and $\text{TSp}^{\text{ober}}$ denotes the 1-category of sober topological spaces, then the assignment $W \mapsto \tilde{W}$ is a fully faithful functor $\text{TSp}^{\text{ober}} / P \to \text{StrTop}_{\text{co, P}}$.

8.2.10 Example. Let $P$ be a finite poset, and $f : \Pi \to P$ a $P$-stratified space (Definition 2.1.5); i.e., $f$ is a conservative functor. In light of the equivalence $\tilde{P} = \text{Fun}(P, S)$, let us abuse notation slightly and write

$$\tilde{\Pi} = \text{Fun}(\Pi, S)$$

for the $\infty$-topos of functors $\Pi \to S$; then right Kan extension along $f$ is a morphism of $\infty$-topoi

$$f_* : \tilde{\Pi} \to \tilde{P},$$

whence $\tilde{\Pi}$ is a $P$-stratified $\infty$-topos. For any point $p \in P$, the $p$-th stratum of $\tilde{\Pi}$ is canonically identified with the $\infty$-topos $\tilde{\Pi}_p = \text{Fun}(\Pi_p, S)$.

The assignment $\tilde{\Pi} \mapsto \tilde{\Pi}$ defines a functor $\text{Str} \to \text{StrTop}_{\infty}$ over $\text{Pos}$.

8.2.11 Subexample. Let $P$ be a finite poset, and let $X$ be a conically $P$-stratified topological space [HA, Definition A.5.5]. Then we obtain the $P$-stratified space

$$\Pi_{(\infty,1)}(X; P) := \text{Sing}_{\text{gp}}(X)$$

and thus the $P$-stratified $\infty$-topos $\tilde{\Pi}_{(\infty,1)}(X; P)$. If $X$ is hereditarily paracompact and locally of singular shape, then in light of [HA, §A.4], the stratum $\tilde{\Pi}_{(\infty,1)}(X; P)_p$ over any point $p \in P$ is equivalent to the $\infty$-category of locally constant sheaves on $X_p$. In light of [HA, §A.9], the $\infty$-topos $\tilde{\Pi}_{(\infty,1)}(X; P)$ is equivalent to the $\infty$-category of formally constructible sheaves on $X$ — i.e., those sheaves whose restrictions to each stratum $X_p$ are locally constant.

8.2.12 Lemma. Let $P$ be a finite poset and $\Pi$ be a $\pi$-finite $P$-stratified space. Then the stratification $\tilde{\Pi} \to \tilde{P}$ is bounded coherent constructible.

Proof. By definition $\tilde{\Pi}$ is $n$-localic for some $n \in \mathbb{N}$. Moreover, the truncated coherent objects of $\tilde{\Pi}$ are those functors $\Pi \to S$ that are valued in $\pi$-finite spaces. We concludes that $\tilde{\Pi}$ is coherent. Since this is true for $\Pi$, it is true for any open therein, whence $\tilde{\Pi} \to \tilde{P}$ is constructible. 

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8.3 Gluing squares

We now use the truncated basechange theorem for oriented fibre products (Theorem 7.1.7) to study oriented squares of bounded coherent ∞-topoi that are both oriented fibre product squares and oriented pushouts. These gluing squares are essential to our décollement approach to stratified higher topoi in §8.4.

8.3.1 Definition. A gluing square is an oriented square

\[
\begin{array}{ccc}
W & \xrightarrow{q_*} & U \\
\downarrow{p_*} & \searrow{j_*} & \downarrow{j_*} \\
Z & \xrightarrow{i_*} & X \\
\end{array}
\]

in which:

- every ∞-topos is bounded coherent;
- every geometric morphism is coherent;
- the natural geometric morphism \(Z \cup_{bc} U \to X\) is an equivalence (Construction 5.2.6);
- the natural geometric morphism \(W \to Z \times_X U\) is an equivalence (Definition 5.4.1).

We call the oriented fibre product \(W\) the link of the gluing square, or the deleted tubular neighbourhood of \(Z\) inside \(X\).

8.3.2 Construction. Let \(X\) be a bounded coherent ∞-topos, \(j_* : U \hookrightarrow X\) a quasicompact open subtopos, and write \(i_* : Z \hookrightarrow X\) for the closed complement of \(U\). Consider the oriented fibre product square

\[
\begin{array}{ccc}
Z \times_X U & \xrightarrow{pr_1,*} & U \\
\downarrow{pr_1,*} & \searrow{j_*} & \downarrow{j_*} \\
Z & \xleftarrow{i_*} & X \\
\end{array}
\]

(8.3.3)

The ∞-topos \(X\) is the bounded coherent recollement \(Z \cup_{bc} U\). Indeed, the truncated basechange theorem (Theorem 7.1.7) ensures that the basechange morphism

\[BC_{\sigma} : i^* j_* \to pr_{1,*} pr_{2,*}\]

becomes an equivalence after restriction to \(U_{coh}^{\mathrm{coh}}\). So Proposition 5.1.15 applies, whence (8.3.3) is a gluing square.

Dually, let \(W\), \(Z\), and \(U\) be bounded coherent ∞-topoi, and let \(p_* : W \to Z\) and \(q_* : W \to U\) be geometric morphisms. Forming the bounded coherent oriented pushout
\[ X := Z \, \mathcal{U}^{W}_{bc} \, U, \text{ we obtain a square} \]

\[
\begin{array}{c}
\begin{array}{ccc}
W & \xrightarrow{q_{\ast}} & U \\
\downarrow p_{\ast} & \searrow & \downarrow j_{\ast} \\
Z & \xrightarrow{i_{\ast}} & Z \, \mathcal{U}^{W}_{bc} \, U .
\end{array}
\end{array}
\]

\[(8.3.4)\]

We thus obtain a geometric morphism \( \psi(p_{\ast},q_{\ast},\sigma)_{\ast} : W \to Z \mathcal{X} U \), and if \( \psi(p_{\ast},q_{\ast},\sigma)_{\ast} \) is an equivalence, then the square \((8.3.4)\) is a gluing square.

The full subcategory of \( \text{Fun}([1] \times [1], \text{Top}_{bc}^{\infty}) \) spanned by the gluing squares is equivalent to the (non-full) subcategory of \( \text{Fun}([1], \text{Cat}_{\omega,\delta}) \) whose objects are bounded coherent gluing functors between bounded coherent \( \infty\)-topoi and whose morphisms \( \phi \to \phi' \) are squares

\[
\begin{array}{c}
\begin{array}{ccc}
U & \xrightarrow{\phi} & Z \\
\downarrow f_{\ast} & \leftarrow & \downarrow g_{\ast} \\
U' & \xrightarrow{\phi'} & Z'
\end{array}
\end{array}
\]

in which \( f_{\ast} \) and \( g_{\ast} \) are coherent geometric morphisms.

**8.3.5 Warning.** If the coherence assumptions are removed, then Construction 8.3.2 does not recover \( X \) as an oriented pushout of \( Z \) and \( U \) along \( Z \, \mathcal{X} \, U \). To see this, let \( X := [0,1] \) be the usual closed interval, \( Z := \{0\} \), and \( U := X \setminus Z \) the open complement of \( Z \). Then the oriented fibre product \( \bar{Z} \, \mathcal{X} \, \bar{U} \) is the initial \( \infty\)-topos \( \emptyset \). The oriented pushout of \( \bar{Z} \) and \( \bar{U} \) along \( \emptyset \) is the coproduct \( \bar{Z} \cup \bar{U} \) in \( \text{Top}_{\omega}^{\infty} \), however the \( \infty\)-topos \( \bar{Z} \cup \bar{U} \) is not equivalent to \( \bar{X} \). The main problem here is that the \( \infty\)-topoi \( \bar{U} \) and \( \bar{X} \) are not coherent.

**8.3.6 Example.** Let

\[
\begin{array}{c}
\begin{array}{ccc}
Z & \xleftarrow{p} & W \xrightarrow{q} U \\
\end{array}
\end{array}
\]

be a span of profinite spaces, and write \( X \) for the profinite \([1]\)-stratified space corresponding to the profinite spatial décollage \((8.3.7)\). Now we form the Stone \( \infty\)-topoi

\[ W := \bar{W}, \quad Z := \bar{Z}, \quad \text{and} \quad U := \bar{U}, \]

and we form the bounded coherent oriented pushout \( X := Z \, \mathcal{U}^{W}_{bc} \, U \):

\[
\begin{array}{c}
\begin{array}{ccc}
W & \xrightarrow{q_{\ast}} & U \\
\downarrow p_{\ast} & \searrow & \downarrow j_{\ast} \\
Z & \xrightarrow{i_{\ast}} & \bar{X}.
\end{array}
\end{array}
\]

\[(8.3.8)\]

Since \( \bar{X} \) is the recollement of \( Z \) and \( U \) with gluing functor that agrees with \( p_{\ast}q^{\ast} \) when restricted to truncated objects (Theorem 7.1.7), and \( \bar{X} \) is bounded coherent, the natural
geometric morphism $X \to \bar{X}$ is an equivalence (Lemma 5.1.14 and Proposition 5.1.15). Now we compute

$Z \times_X U = \text{Mor}_{[\Sigma]}([1], X) = \text{Map}_{[1]}([1], X) = \bar{W} = W$.

Thus the square (8.3.8) is in fact a gluing square.

### 8.4 Toposic décollages

In analogy with the construction of the spatial décollage attached to a stratified space (Construction 2.7.1), we can attach to a stratified ∞-topos what we call its (toposic) décollage. Whereas a stratified ∞-topos consists of strata that are glued together, its décollage is the result of pulling these strata apart while retaining the linking information necessary to reconstruct the stratified ∞-topos.

#### 8.4.1 Definition

Let $P$ be a finite poset. We say that a functor $D : \text{sd}^{op}(P) \to \text{Top}^{bc}_{\infty}$ is a décollage over $P$ if and only if the following conditions are satisfied.

- If $p_0, p_1 \in P$ are two points such that $p_0 < p_1$, then the square

  \[ \begin{array}{ccc}
  D\{p_0 < p_1\} & \longrightarrow & D\{p_1\} \\
  \downarrow & & \downarrow \\
  D\{p_0\} & \leftarrow & D\{p_0\} \cup D\{p_0 < p_1\} D\{p_1\}
  \end{array} \]

  is a gluing square.

- For any string $\{p_0 < \cdots < p_m\} \subseteq P$, the geometric morphism to the fibre product of co-topoi

  \[ D\{p_0 < \cdots < p_m\} \to D\{p_0 < p_1\} \times_{D\{p_1\}} D\{p_1 < p_2\} \times_{D\{p_2\}} \cdots \times_{D\{p_{m-1}\}} D\{p_{m-1} < p_m\} \]

  is an equivalence.

We write $\text{Déc}_P(\text{Top}^{bc}_{\infty}) \subseteq \text{Fun}(\text{sd}^{op}(P), \text{Top}^{bc}_{\infty})$ for the full subcategory spanned by the décollages over $P$.

It seems likely that a décollage over $P$ can be thought of as a suitable category internal to $\text{Top}^{bc}_{\infty}$ along with a conservative functor to $P$. Making such an interpretation precise and helpful is a task that lies outside the scope of this work.

#### 8.4.2. If $D : \text{sd}^{op}(P) \to \text{Top}^{bc}_{\infty}$ is a décollage over $P$, and if $p, q \in P$ are points with $p < q$, then for the sake of typographical brevity, let us here write

\[ D\{p\} \cup D\{q\} = D\{p\} \cup_{D^{\leq q}} D\{q\}. \]

The two conditions of Definition 8.4.1 together specify, for any string $\{p_0 < \cdots < p_m\} \subseteq P$, an equivalence

\[ D\{p_0 < \cdots < p_m\} \Rightarrow D\{p_0\} \bar{\times}_{D\{p_0\}} D\{p_1\} \bar{\times}_{D\{p_0\}} \cdots \bar{\times}_{D\{p_0\}} D\{p_m\}, \]

which we will call the Segal equivalence.
8.4.3 Example. The terminal object of $\text{Décc}(\text{Top}^bc_\infty)$ is the constant functor $\text{sd}^{op}(P) \to \text{Top}^bc_\infty$ whose value is the co-topos $S$.

8.4.4 Construction. Consider the 1-category $\int \text{sd}^{op}$ of Construction 2.6.5, whose objects are pairs $(P, \Sigma)$ consisting of a poset $P$ and a string $\Sigma \subseteq P$. Recall that the assignment $(P, \Sigma) \mapsto P$ is a cocartesian fibration $\int \text{sd}^{op} \to \text{Pos}$ whose fibre over a poset $P$ is the poset $\text{sd}^{op}(P)$.

We write

$$\text{Pair}_{\text{Pos}}(\int \text{sd}^{op}, \text{Top}^bc_\infty)$$

for the simplicial set over $\text{Pos}$ defined by the following universal property: for any simplicial set $K$ over $\text{Pos}$, one demands a bijection

$$\text{Mor}_{\text{Set}/\text{Pos}}(K, \text{Pair}_{\text{Pos}}(\int \text{sd}^{op}, \text{Top}^bc_\infty)) \cong \text{Mor}_{\text{Set}}(K \times \text{Pos} \int \text{sd}^{op}, \text{Top}^bc_\infty),$$

natural in $K$. By [HTT, Corollary 3.2.2.13], the functor $\text{Pair}_{\text{Pos}}(\int \text{sd}^{op}, \text{Top}^bc_\infty) \to \text{Pos}$ is a cartesian fibration whose fibre over a poset $P$ is the $\infty$-category $\text{Fun}(\text{sd}^{op}(P), \text{Top}^bc_\infty)$.

Now let $\text{Décc}(\text{Top}^bc_\infty) \subset \text{Pair}_{\text{Pos}}(\int \text{sd}^{op}, \text{Top}^bc_\infty)$ denote the full subcategory spanned by the pairs $(P, D)$ in which $D$ is a topoisic décollage over $P$. Since $\text{Décc}(\text{Top}^bc_\infty)$ contains all the cartesian edges, the functor $\text{Décc}(\text{Top}^bc_\infty) \to \text{Pos}$ is a cartesian fibration.

8.5 The nerve of a stratified $\infty$-topos

8.5.1 Construction. Let $P$ be a finite poset, and let $f_* : \mathcal{X} \to \bar{P}$ be a $P$-stratified $\infty$-topos. Then for any monotonic map $\phi : \mathcal{Q} \to P$, we define the $\infty$-topos of sections of $\mathcal{X}$ over $\mathcal{Q}$ as the pullback of $\infty$-topoi

$$\text{Mor}_{\text{Set}/\text{Pos}}(K, \text{Pair}_{\text{Pos}}(\int \text{sd}^{op}, \text{Top}^bc_\infty)) \equiv \text{Mor}_{\text{Set}}(K \times \text{Pos} \int \text{sd}^{op}, \text{Top}^bc_\infty),$$

natural in $K$. By [HTT, Corollary 3.2.2.13], the functor $\text{Pair}_{\text{Pos}}(\int \text{sd}^{op}, \text{Top}^bc_\infty) \to \text{Pos}$ is a cartesian fibration whose fibre over a poset $P$ is the $\infty$-category $\text{Fun}(\text{sd}^{op}(P), \text{Top}^bc_\infty)$.

We thus obtain a functor $\mathcal{N}_P(\mathcal{X}) : \text{sd}^{op}(P) \to \text{Top}_{\infty}$ that carries a string $\Sigma \subseteq P$ to the $\infty$-topos

$$\mathcal{N}_P(\mathcal{X})(\Sigma) := \text{Mor}_{\bar{Q}}(\bar{Q}, \mathcal{X} \times_P \bar{Q}).$$

The $\infty$-topos $\text{Mor}_{\bar{Q}}(\bar{Q}, \mathcal{X})$ depends only on the pullback $\mathcal{X} \times_P \bar{Q}$:

$$\text{Mor}_{\bar{Q}}(\bar{Q}, \mathcal{X}) \cong \text{Mor}_{\bar{Q}}(\bar{Q}, \mathcal{X} \times_P \bar{Q}).$$

We thus obtain a functor $\mathcal{N}_P(\mathcal{X}) : \text{sd}^{op}(P) \to \text{Top}_{\infty}$ that carries a string $\Sigma \subseteq P$ to the $\infty$-topos

$$\mathcal{N}_P(\mathcal{X})(\Sigma) := \text{Mor}_{\bar{Q}}(\bar{Q}, \mathcal{X}).$$

For any string $\{p_0 < \cdots < p_m\} \subseteq P$, we thus obtain an identification

$$\mathcal{N}_P(\mathcal{X})(\{p_0 < \cdots < p_m\}) = \mathcal{X}_{p_0} \hat{\times} \mathcal{X}_{p_1} \hat{\times} \cdots \hat{\times} \mathcal{X}_{p_m}.$$

In particular, if $\mathcal{X}$ is bounded coherent constructible (Definition 8.2.3), then the functor $\mathcal{N}_P(\mathcal{X})$ is a décollage over $P$. We call $\mathcal{N}_P(\mathcal{X})$ the nerve of the $P$-stratified $\infty$-topos $\mathcal{X}$, and we call $\mathcal{N} : \text{StrTop}^bc_\infty \to \text{Décc}(\text{Top}^bc_\infty)$ over $\text{Pos}$ the nerve functor.
8.5.2 Example. Let $P$ be a finite poset, and $\Pi$ a $P$-stratified space. Then one has an identification
\[ N_P(\Pi) = \tilde{N}_P(\Pi), \]
natural in $P$ and $\Pi$, since for any string $\Sigma \subseteq P$, one has
\[ \text{Mor}_P(\Sigma, \Pi) \cong \tilde{\text{Map}}_P(\Sigma, \Pi), \]
via the natural morphism.

We now proceed to demonstrate that the nerve is an equivalence of $\infty$-categories.

8.5.3 Theorem. For any finite poset $P$, the nerve functor $N_P : \text{StrTop}^{bc}_{\infty, P} \to \text{Déc}_P(\text{Top}^{bc}_{\infty})$ is an equivalence of $\infty$-categories.

Proof. We begin by reducing to the case in which $P$ is a nonempty, finite, totally ordered set. To make this reduction, we note that $P \cong \text{colim}_{\Sigma \in \text{sd}(P)} \Sigma$, whence $\tilde{P}$ is the limit $\tilde{\Sigma}$ in $\text{Cat}_{\infty, h}$ (which is the colimit in $\text{Top}^{\infty}$) and moreover
\[ \text{sd}^{\tilde{P}}(P) = \text{colim}_{\Sigma \in \text{sd}(P)} \text{sd}^{\tilde{\Sigma}}. \]

From this we deduce that
\[ \text{StrTop}^{bc}_{\infty, P} = \text{colim}_{\Sigma \in \text{sd}(P)} \text{StrTop}^{bc}_{\infty, \Sigma} \quad \text{and} \quad \text{Déc}_P(\text{Top}^{bc}_{\infty}) = \text{colim}_{\Sigma \in \text{sd}(P)} \text{Déc}_\Sigma(\text{Top}^{bc}_{\infty}), \]
which provides our reduction.

Now when $P = [n] := \{0 \leq \cdots \leq n\}$ is a nonempty totally ordered finite set, we construct an inverse
\[ U_n : \text{Déc}_{[n]}(\text{Top}^{bc}_{\infty}) \to \text{StrTop}^{bc}_{\infty, [n]} \]
to the nerve functor $N_n : N_{[n]} \to \text{StrTop}^{bc}_{\infty, [n]}$ by forming the iterated bounded coherent oriented pushout:
\[ U_n(D) := D[0] \cup_{D[2]} D[1] \cup_{D[3]} D[2] \cdots \cup_{D[n]} D[n], \]
equipped with its canonical geometric morphism to
\[ [n] = U_n(S), \]
which is visibly coherent.

The universal properties of the iterated bounded coherent oriented pushout and the iterated oriented pullback provide natural transformations $U_nN_n \to \text{id}$ and $\text{id} \to N_nU_n$. We aim to show that these natural transformations are equivalences.

We prove that the natural morphisms $U_nN_n \to \text{id}$ and $\text{id} \to N_nU_n$ are equivalences by induction on $n$. The base case $n = 0$ is obvious. Now assume that $n \geq 1$ and that the natural morphism $U_{n-1}N_{n-1} \to \text{id}$ is an equivalence; we prove that the natural morphism $U_nN_n \to \text{id}$ is an equivalence. If $X$ is a bounded coherent $\infty$-topos with a
constructible stratification \( X \to \overline{[n]} \), then consider the recollement of \( X \) given by \( X_{\leq n-1} \) and \( X_n \). We thus have a gluing square

\[
\begin{array}{c}
X_{\leq n-1} \xrightarrow{p} X_n \xrightarrow{j} X \\
\downarrow \sigma \quad \downarrow j_s \\
X_{\leq n-1} \xrightarrow{i_s} X
\end{array}
\]

As a result, we compute:

\[
U_n N_n(X) = U_{n-1} N_{n-1}(X_{\leq n-1}) \cup^X_{bc} X_n \\
= X_{\leq n-1} \cup^X_{bc} X_n \\
= X ,
\]

as desired.

Now assume that the natural morphism \( \text{id} \to N_{n-1} U_{n-1} \) is an equivalence; we prove that the natural morphism \( \text{id} \to N_n U_n \) is an equivalence. Let \( D : \text{sd}^{op}([n]) \to \text{Top}_{\infty} \) be a toposic dêcollage; we need to show that for every string \( \Sigma \subset [n] \), the natural morphism \( D(\Sigma) \to N_n U_n(D)(\Sigma) \) is an equivalence. There are two cases to consider: \( \Sigma \neq [n] \) and \( \Sigma = [n] \). If \( \Sigma \neq [n] \), then there exists an element \( k \in [n] \) such that \( k \notin \Sigma \). Then applying the inductive hypothesis we see that the map \( D(\Sigma) \to N_n U_n(D)(\Sigma) \) factors as a composite of equivalences

\[
D(\Sigma) = (D|_{\text{sd}^{op}([n]-\{k\})})(\Sigma) \\
\Rightarrow N_{[n]-\{k\}} U_{[n]-\{k\}} (D|_{\text{sd}^{op}([n]-\{k\})})(\Sigma) \\
= N_n U_n(D)(\Sigma) .
\]

In the case that \( \Sigma = [n] \), note that the morphism \( D([n]) \to N_n U_n(D)([n]) \) is homotopic to the Segal equivalence

\[
D[0 \leq \cdots \leq n] \Rightarrow D[0] \underset{\check{v}_d(D)}{\times} D[1] \underset{\check{v}_d(D)}{\times} \cdots \underset{\check{v}_d(D)}{\times} D[n] ,
\]

whence our claim. \( \square \)

### 8.6 Stratifications over spectral topological spaces

**8.6.1 Definition.** For any spectral topological space \( S \), an \( S \)-stratified \( \infty \)-topos is a morphism of \( \infty \)-topoi \( X \to \mathcal{S} \). We write \( \text{StrTop}_{\infty,S} \) for the \( \infty \)-category \( \text{Top}_{\infty,S} \) of \( S \)-stratified \( \infty \)-topoi.

We define

\[
\text{StrTop}_{\infty} = \text{Fun}([1], \text{Top}_{\infty}) \times^{\text{Fun}([1], \text{Top}_{\infty})} \text{TSp}^{\text{dec}} ,
\]

so that the fibre over \( S \) can be identified with \( \text{StrTop}_{\infty,S} \).
8.6.2. If $S$ is a spectral topological space, then one has
\[
\text{StrTop}_{\infty,S} = \text{Top}_{\infty} \times_{\text{Top}_0/\text{Open}(S)} \text{Top}_0,
\]
where $\text{Open}(S)$ is the locale of open subsets of $S$.

In the case of stratifications over spectral topological spaces, we employ notations as in Notation 8.2.2.

8.6.3 Notation. Let $S$ be a spectral topological space, and let $X$ be a $S$-stratified $\infty$-topos. For any open subset $U \subseteq S$, we abuse notation and write $U$ also for the corresponding open of $S$, and we write
\[
X_U := X_{f^*U} = X \times_{\overline{S}} U \subseteq X
\]
for the corresponding open subtopos. (Here the fibre product is formed in $\text{Top}_{\infty}$.) Dually, if $Z \subseteq S$ is closed, then we write
\[
X_Z = X_{f^*(S-Z)} = X \times_{\overline{S}} \overline{Z} \subseteq X
\]
for the corresponding closed subtopos, so that if $U$ and $Z$ are complementary, then one exhibits $X$ as a recollement of $X_Z$ and $X_U$.

More generally, for any subspace $W \subset S$, we write
\[
X_W = X \times_{\overline{S}} \overline{W}.
\]
In particular, for any point $s \in S$ we define the $s$-th stratum as the fibre product in $\text{Top}_{\infty}$:
\[
X_s = X \times_{\overline{S}} \{s\} \subseteq X.
\]

The key finiteness condition for stratifications over spectral topological spaces is bounded coherent constructibility.

8.6.4 Definition. If $X$ is an $\infty$-topos and $S$ is a spectral topological space. A stratification $f_* : X \to \overline{S}$ is constructible if and only if, for any quasicompact open $U \subseteq S$ and any quasicompact open $V \in \text{Open}(X)$, the $\infty$-topos
\[
X_U \times_X X_{f^*V} = X_{f^*(U) \times V}
\]
is coherent. We say that a constructible stratification $f_* : X \to \overline{S}$ is coherent constructible if $X$ is a coherent $\infty$-topos, and we say that $f_*$ is bounded coherent constructible if $X$ is a bounded coherent $\infty$-topos.

8.6.5 Lemma. Let $S$ be a spectral topological space and $f_* : X \to \overline{S}$ be an $S$-stratified $\infty$-topos. If $X$ is coherent, then the stratification $f_*$ is constructible if and only if $f_*$ is a coherent geometric morphism.
**Proof.** If $f_*$ is coherent, then since quasicompact opens in $X$ are coherent [SAG, Remark A.2.3.5] and coherent objects of $X$ are closed under finite products, $f_*$ is a constructible stratification.

For the other direction, assume that $f_*$ is a constructible stratification. By Corollary 3.4.5, to show that $f_*$ is coherent it suffices to show that $f^*$ carries truncated coherent objects of $\tilde{S}$ to coherent objects of $X$. Let $F \in \tilde{S}_{\text{coh}}$ be a truncated coherent object; then there exists a finite constructible stratification $S \to \tilde{P}$ such that $F$ is the pullback of a truncated coherent object of $\tilde{P}$. Thus, for every point $p \in \tilde{P}$, the restriction $f^*(F)|_{X_p}$ is lisse. By Proposition 5.1.8=[DAG XIII, Proposition 2.3.22] it follows that $F$ is coherent.

**8.6.6 Notation.** Let $S$ be a spectral topological space. We define the $\infty$-category of coherent constructible $S$-stratified $\infty$-topoi as the overcategory

$$\text{StrTop}_{\text{co}, S}^{\text{cc}} := \text{Top}_{\text{co}, S}^{\text{coh}}.$$ 

We write $\text{StrTop}_{\text{co}, S}^{\text{bcc}} \subset \text{StrTop}_{\text{co}, S}^{\text{cc}}$ for the full subcategory spanned by the bounded coherent constructible $S$-stratified $\infty$-topoi.

More generally, we define

$$\text{StrTop}_{\text{co}}^{\text{cc}} := \text{Fun}([1], \text{Top}_{\text{co}}^{\text{coh}}) \times_{\text{Fun}([1], \text{Top}_{\text{coh}}^{\text{coh}})} \text{TSpc}_{\text{sp}}^{\text{sp}},$$

so that the fibre over $S$ can be identified with $\text{StrTop}_{\text{co}, S}^{\text{cc}}$. We write $\text{StrTop}_{\text{co}}^{\text{bcc}} \subset \text{StrTop}_{\text{co}}^{\text{cc}}$ for the full subcategory spanned by those objects $X \to S$ where $X$ is a bounded $\infty$-topos.

**8.7 The natural stratification of a coherent $\infty$-topos**

It turns out that any coherent $\infty$-topos $X$ has a canonical profinite stratification: the 0-topos (=locale) $\text{Open}(X)$ is the locale of a spectral topological space. This provides a fully faithful embedding of the $\infty$-category of coherent $\infty$-topoi into that of coherent constructible stratified $\infty$-topoi.

To explain this point, let us first recall the equivalence between coherent locales and spectral topological spaces.

**8.7.1 Recollection.** Let $A$ be a locale. An object $a \in A$ is quasicompact\(^{26}\) if and only if for every subset $S \subset A$ such that $\bigsqcup_{s \in S} s = a$, there exists a finite subset $S_0 \subset S$ such that $\bigsqcup_{s \in S_0} s = a$.

One says that $A$ is coherent if and only if $A$ is coherent in the sense of Definition 3.3.1. Proposition 3.5.6 shows that this is the case if and only if the following conditions are satisfied:

- The quasicompact elements of $A$ form a sublattice of $A$: the maximal element $1_A \in A$ is quasicompact and binary products (=meets) of quasicompact elements are quasicompact.

\(^{26}\)Such elements are sometimes called finite; see [67, Chapter II, §3.1].
The quasicompact elements of $A$ generate $A$: every element $a \in A$ can be written as a coproduct (=join) $a = \bigsqcup_{s \in S} s$, where $S \subset A$ is a subset consisting of quasicompact elements of $A$.

A morphism $A \to A'$ between coherent locales is coherent if and only if the corresponding map of posets $A' \to A$ sends quasicompact elements to quasicompact elements.

We write $\text{Top}^{\text{coh}}_0$ for the category of coherent locales and coherent morphisms between them (cf. Corollary 3.6.12).

8.7.2 Example. Let $X$ be an $\infty$-topos. Then an open $U \in \text{Open}(X)$ is a quasicompact element of the locale $\text{Open}(X)$ if and only if $U$ is a quasicompact (i.e., 0-coherent) object of the $\infty$-topos $X$.

The following three results are immediate from the definitions and Example 8.7.2.

8.7.3 Lemma. For any 1-coherent $\infty$-topos $X$, the locale $\text{Open}(X)$ is coherent.

8.7.4 Lemma. Let $f_\ast : X \to Y$ be a coherent geometric morphism between coherent $\infty$-topoi. Then the induced morphism $\text{Open}(X) \to \text{Open}(Y)$ of coherent locales is coherent.

8.7.5 Corollary. Let $S$ be a spectral topological space and $f_\ast : X \to S$ an $S$-stratified co-topos. If $X$ is coherent, then $f_\ast$ is a constructible stratification if and only if the induced morphism of coherent locales $\text{Open}(X) \to \text{Open}(S)$ is coherent.

The following classical result is an important recognition principle for coherent locales.

8.7.6 Proposition ([67, Chapter II, §§3.3–3.4]). The functor $\text{Open} : \text{TSpec}^{\text{spec}} \to \text{Top}^0_0$ given by sending a spectral topological space $S$ to its locale of open subsets factors through $\text{Top}^{\text{coh}}_0$ and defines an equivalence of categories

$$\text{Open} : \text{TSpec}^{\text{spec}} \simeq \text{Top}^{\text{coh}}_0.$$  

8.7.7 The functor $\text{Open} : \text{TSpec}^{\text{spec}} \simeq \text{Top}^{\text{coh}}_0$ has an explicit inverse $\text{Top}^{\text{coh}}_0 \simeq \text{TSpec}^{\text{spec}}$ given by taking the topological space of points of a locale; see [67, Chapter II, §1.3].

8.7.8 Notation. Lemma 8.7.4 and Proposition 8.7.6 provide a functor

$$S : \text{Top}^{\text{coh}}_\infty \overset{\text{Open}}{\longrightarrow} \text{Top}^{\text{coh}}_0 \longrightarrow \text{TSpec}^{\text{spec}},$$

which we denote by $S$. By definition, the 0-localic reflection of a coherent $\infty$-topos $X$ is given by the $\infty$-topos of sheaves on the spectral topological space $S(X)$. Thus $X$ comes equipped with a natural $S(X)$-stratification $X \to S(X)$.

The localisation $\text{Top}^{\text{coh}}_\infty \simeq \text{Top}^{\text{coh}}_0$ thus restricts to a localisation $\text{Top}^{\text{coh}}_\infty \simeq \text{Top}^{\text{coh}}_0$.

8.7.9 Lemma. For any coherent $\infty$-topos $X$, the natural stratification $f_\ast : X \to S(X)$ is constructible (Definition 8.6.4).

Proof. Clear from Corollary 8.7.5 and the fact that $f_\ast : X \to S(X)$ induces an equivalence of locales $\text{Open}(X) \simeq \text{Open}(S(X)) = \text{Open}(S(X))$. 

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8.7.10. The source functor $\text{StrTop}_{\infty}^{cc} \to \text{Top}_{\infty}^{coh}$ admits a fully faithful left adjoint, given by the assignment

$X \mapsto [X \to \tilde{S}(X)]$.

The essential image of this left adjoint is the full subcategory spanned by those coherent constructible stratified $\infty$-topoi $X \to \tilde{S}$ such that the stratification induces an equivalence of locales $\text{Open}(X) \simeq \text{Open}(S)$.

The source functor $\text{StrTop}_{\infty}^{cc} \to \text{Top}_{\infty}^{coh}$ also admits a fully faithful right adjoint, which carries a coherent $\infty$-topos $X$ to $X$ again, equipped with the essentially unique stratification over $S = \{0\}$.

8.8 Stratified spaces & profinite stratified spaces as stratified $\infty$-topoi

We now extend the functor $\text{Str}_{\pi} \to \text{StrTop}_{\infty}$ given by $\pi \mapsto \tilde{\pi}$ to a functor on profinite stratified spaces.

8.8.1 Notation. Denote by $\lambda: \text{Str} \to \text{StrTop}_{\infty}$ the left exact functor over $\text{Pos}$ that is defined by the assignment $\pi \mapsto \tilde{\pi}$. For each poset $P$, we consider also the functor on fibres $\lambda_P: \text{Str}_P \to \text{StrTop}_{\infty,P}$.

In light of Example 8.5.2, if $P$ is finite, then the diagram

$$
\begin{array}{ccc}
\text{Str}_P & \xrightarrow{\lambda_P} & \text{StrTop}_{\infty,P} \\
\downarrow \text{N}_P & & \downarrow \text{N}_P \\
\text{Déc}(S_P) & \xrightarrow{\lambda_{P_{\infty}}} & \text{Déc}(\text{Top}_{\infty}^b)
\end{array}
$$

commutes, where the vertical functors are equivalences (Definition 2.4.3, (2.7.5), and Theorem 8.5.3).

We now show that the functor $\lambda$ is fully faithful. We first describe stratified geometric morphisms $X \to \tilde{P}$ in a more familiar fashion. Let us begin with the case in which the base poset is trivial.

8.8.2. In light of Recollection 3.1.9=HTT, Corollary 6.3.5.6, if $P$ is an essentially $\delta_0$-small space, then one has an equivalence

$\text{Map}_{\text{Pro}(S)}(\Pi_{\infty}(X), P) \simeq \text{Fun}_{\infty}(X, \tilde{P})$,

where $\Pi_{\infty}(X)$ is the shape prospace $\Gamma_{X,*}^\infty: S \to S$ (Definition 4.2.1). In particular, $\text{Fun}_{\infty}(X, \tilde{P})$ is an essentially $\delta_0$-small $\infty$-groupoid.

In this case, one also deduces that if $P$, $P'$ are two essentially $\delta_0$-small spaces, then the natural map $\text{Map}_{\infty}(P', P) \to \text{Map}_{\text{Top}_{\infty}}(P', \tilde{P})$ is an equivalence.

Now we extend this result to the context of $P$-stratified $\infty$-topoi.

8.8.3 Notation. Let $P$ be a finite poset, and let $f_*: X \to \tilde{P}$ and $g_*: Y \to \tilde{P}$ be $P$-stratified $\infty$-topoi. Let us write

$\text{Fun}_{P,*}(X,Y) := \text{Fun}_{\infty}(X,Y) \times_{\text{Fun}_{\infty}(X,\tilde{P})} \{f_*\}$.
The mapping space $\text{Map}_{\text{StrTop}_{\infty,\mathcal{P}}}(\mathcal{X}, \mathcal{Y})$ is the interior of $\text{Fun}_{\mathcal{P},*}(\mathcal{X}, \mathcal{Y})$.

If $\mathcal{X}$ and $\mathcal{Y}$ are bounded coherent and constructibly stratified, then in light of Theorem 8.5.3, one has an equivalence of $\infty$-categories

$$\text{Fun}_{\mathcal{P},*}(\mathcal{X}, \mathcal{Y}) \simeq \int_{\mathcal{S} \in \text{sd}^{\text{op}}(\mathcal{P})} \text{Fun}_{*}(\mathcal{N}_{\mathcal{P}}(\mathcal{X})(\mathcal{S}), \mathcal{N}_{\mathcal{P}}(\mathcal{Y})(\mathcal{S})).$$

This implies the following.

8.8.4 Proposition. Let $\mathcal{P}$ be a finite poset and $\mathcal{X}$ a bounded coherent constructible $\mathcal{P}$-stratified $\infty$-topos. Then for any $\pi$-finite $\mathcal{P}$-stratified space $\mathfrak{v}$, one has a natural equivalence

$$\text{Fun}_{\mathcal{P},*}(\mathcal{X}, \mathfrak{v}) \simeq \int_{\mathcal{S} \in \text{sd}^{\text{op}}(\mathcal{P})} \text{Map}_{\text{Pro}(\mathcal{S})}(\Pi_{\infty}(\mathcal{N}_{\mathcal{P}}(\mathcal{X})(\mathcal{S})), \mathcal{N}_{\mathcal{P}}(\mathfrak{v})(\mathcal{S})).$$

Since the right hand side is a $\delta_0$-small limit of $\delta_0$-small $\infty$-groupoids, we obtain the following.

8.8.5 Corollary. Let $\mathcal{P}$ be a finite poset and $\mathcal{X}$ a bounded coherent constructible $\mathcal{P}$-stratified $\infty$-topos. Then for any $\pi$-finite $\mathcal{P}$-stratified space $\mathfrak{v}$, the $\infty$-category $\text{Fun}_{\mathcal{P},*}(\mathcal{X}, \mathfrak{v})$ is an essentially $\delta_0$-small $\infty$-groupoid.

Additionally, the full faithfulness of $\lambda_{(0)}$ now implies the following.

8.8.6 Corollary. For any finite poset and any two $\pi$-finite $\mathcal{P}$-stratified spaces $\mathfrak{v}$ and $\mathfrak{v}'$, the functor

$$\text{Map}_{\mathcal{P}}(\mathfrak{v}', \mathfrak{v}) \rightarrow \text{Fun}_{\mathcal{P},*}(\mathfrak{v}', \mathfrak{v})$$

is an equivalence. That is, the functor $\lambda_{\mathcal{P}}$ is a fully faithful functor $\text{Str}_{\pi,\mathcal{P}} \hookrightarrow \text{StrTop}_{\infty,\mathcal{P}}^{\text{bcc}}$.

Finally, we obtain:

8.8.7 Corollary. If $\mathcal{P}$ is a finite poset, then for any bounded coherent constructible $\mathcal{P}$-stratified $\infty$-topos $\mathcal{X}$ and filtered diagram $\mathcal{P} : A \rightarrow \text{Str}$ of $\pi$-finite $\mathcal{P}$-stratified spaces, the natural map

$$\text{colim}_{\alpha \in A} \text{Map}_{\text{StrTop}_{\infty,\mathcal{P}}}(X, \mathcal{I}_\alpha) \rightarrow \text{Map}_{\text{StrTop}_{\infty,\mathcal{P}}}(X, \text{colim}_{\alpha \in A} \mathcal{I}_\alpha)$$

is an equivalence.

8.8.8. The functor $\lambda : \text{Str}_{\pi} \hookrightarrow \text{StrTop}_{\infty,\mathcal{P}}^{\text{bcc}}$ is left exact. To see this, we combine two facts. First, the functor $\text{Pos}^{\text{fin}} \rightarrow \text{Top}_{\infty,\mathcal{P}}^{\text{bc}}$ given by $P \mapsto \mathcal{P}$ is left exact. Second, for any finite poset $\mathcal{P}$, the functor

$$\lambda_{\mathcal{P}} : \text{Str}_{\pi,\mathcal{P}} \rightarrow \text{StrTop}_{\infty,\mathcal{P}}^{\text{bcc}},$$

when regarded as a functor $\text{Déc}_{\mathcal{P}}(S_n) \rightarrow \text{Décc}_{\mathcal{P}}(\text{Top}_{\infty,\mathcal{P}}^{\text{bc}})$, is equivalent to composition with $\lambda_{(0)}$, so it too is left exact.
8.8.9 Construction. Since bounded coherent constructible stratified co-topoi are closed under the formation of inverse limits, we can now apply (0.3.5) and extend $\lambda$ to a functor

$$\hat{\lambda} : \text{Str}^\wedge_n \to \text{StrTop}^{bc}_{\infty}$$

over $\text{TSpc}^{\text{spec}}$, which we write as the assignment $\Pi \mapsto \Pi$. Let us caution that if $S$ is a spectral topological space and $\Xi$ is a profinite $S$-stratified space, then although $S$ determines and is determined by the $\text{mat}(S)$-stratified space $\text{mat}(\Pi)$, the co-topoi $\Pi$ and $\text{mat}(\Pi)$ are quite different in general. The latter is always a presheaf $\infty$-category, but the former is typically not.

8.8.10 Proposition. The functor $\hat{\lambda}$ is fully faithful. In particular, if $S$ is a spectral topological space, then we obtain a fully faithful functor $\text{Str}^\wedge_n, S \hookrightarrow \text{StrTop}^{bc}_{\infty, S}$.

Proof. First we treat the case in which $S = P$ is a finite poset. In this case, in light of the equivalences

$$\text{Str}^\wedge_n, P \simeq \text{Déc}(S^{\wedge}_n) \quad \text{and} \quad \text{StrTop}^{bc}_{\infty, P} \simeq \text{Déc}(\text{Top}^{bc}_\infty)$$

of Construction 2.8.8 and Theorem 8.5.3, it suffices to prove that the functor

$$\text{Déc}(S^{\wedge}_n) \to \text{Déc}(\text{Top}^{bc}_\infty)$$

given by the objectwise application of $\hat{\lambda}_{[0]} : \text{Str}^\wedge_n \to \text{Top}^{bc}_\infty$ is fully faithful. This follows as in Corollary 8.8.6 from the full faithfulness of the functor $S^{\wedge}_n \to \text{Top}^{bc}_\infty$.

Now for a more general spectral topological space $S$, the identifications

$$\text{Str}^\wedge_{n, S} = \lim_{P \in \text{FC}(S)} \text{Str}^\wedge_{n, P} \quad \text{and} \quad \text{StrTop}^{bc}_{\infty, S} = \lim_{P \in \text{FC}(S)} \text{StrTop}^{bc}_{\infty, P},$$

the first of which is Proposition 2.5.11 and the latter of which is obvious, together complete the proof.

8.8.11 Proposition. Let $P$ be a finite poset. Then the essential image of the functor

$$\text{Déc}(S^{\wedge}_n) \to \text{Déc}(\text{Top}^{bc}_\infty)$$

given by the objectwise application of $\hat{\lambda}_{[0]} : \text{Str}^\wedge_n \to \text{Top}^{bc}_\infty$ is the full subcategory

$$\text{Déc}(\text{Top}^{\text{str}}_\infty) \subset \text{Déc}(\text{Top}^{bc}_\infty)$$

spanned by those décollages over $P$ that carry each string to a Stone $\infty$-topos.

Proof. The only nontrivial point to verify is that indeed $\hat{\lambda}$ carries décollages in profinite spaces to décollages in Stone $\infty$-topoi. This follows from Example 8.3.6.

The essential image of $\hat{\lambda}$ can be characterised as the $\infty$-category of spectral $\infty$-topoi, to which we shall now turn.
9 Spectral higher topoi

In this chapter, we define the notion of a spectral $\infty$-topos. The idea is that, on one hand, these are the kinds of $\infty$-topoi that arise as the étale $\infty$-topoi of coherent schemes, and on the other, these will turn out to be precisely the $\infty$-topoi that arise as $\tilde{\Pi}$ for some profinite stratified space $\Pi$.

Section 9.1 begins by showing that in an oriented fibre product of bounded coherent $\infty$-topoi $X \times_Z Y$, if $X$ and $Y$ are Stone, then $X \times_Z Y$ is Stone; this is key to understand the links in our décollage approach to spectral $\infty$-topoi developed in §9.2. Section 9.3 states and proves our $\infty$-Categorical Hochster Duality Theorem (Theorem E), which provides an equivalence between profinite stratified spaces and spectral $\infty$-topoi. Section 9.4 is dedicated to the study of constructible sheaves in the setting of stratified $\infty$-topoi; for spectral $\infty$-topoi the constructible sheaves coincide with the truncated coherent objects.

9.1 Stone $\infty$-topoi & oriented fibre products

In this section we prove two useful facts about oriented fibre products involving Stone $\infty$-topoi.

9.1.1 Proposition. Let $f_* : X \rightarrow Z$ and $g_* : Y \rightarrow Z$ be geometric morphisms of $\infty$-topoi. If $Z$ is Stone, then the natural geometric morphism $X \times_Z Y \rightarrow X \times_Z Y$ is an equivalence.

Proof. It suffices to show that the projections $\text{pr}_{1,*}, \text{pr}_{2,*} : \text{Path}(Z) \rightarrow Z$ are equivalences. Since $Z$ is Stone, by Lemma 5.6.6 the $\infty$-topos $\text{Path}(Z)$ is bounded coherent, and Theorem 4.4.10=[SAG, Theorem E.3.4.1] shows that the $\infty$-category $\text{Pt}(Z)$ is an $\infty$-groupoid. Thus $\text{Pt}(\text{Path}(Z)) = \text{Fun}(\{1\}, \text{Pt}(Z))$ is an $\infty$-groupoid as well, and again appealing to Theorem 4.4.10=[SAG, Theorem E.3.4.1] we conclude that $\text{Path}(Z)$ is Stone. The claim now follows from the fact that $\text{pr}_{1,*}$ and $\text{pr}_{2,*}$ are shape equivalences (Example 6.3.6).

9.1.2 Proposition. Let $X$ and $Y$ be Stone $\infty$-topoi, $Z$ a bounded coherent $\infty$-topos, and $f_* : X \rightarrow Z$ and $g_* : Y \rightarrow Z$ coherent geometric morphisms. Then the oriented fibre product $X \times_Z Y$ is a Stone $\infty$-topos.

Proof. By Lemma 5.6.6 the $\infty$-topos $X \times_Z Y$ is bounded coherent, so by Theorem 4.4.10=[SAG, Theorem E.3.4.1] it suffices to prove that the $\infty$-category $\text{Pt}(X \times_Z Y)$ is an $\infty$-groupoid. In light of Lemma 5.4.8 we have $\text{Pt}(X \times_Z Y) = \text{Pt}(X) \downarrow_{\text{Pt}(Z)} \text{Pt}(Y)$, so the fact that $\text{Pt}(X)$ and $\text{Pt}(Y)$ are $\infty$-groupoids implies that the $\infty$-category $\text{Pt}(X \times_Z Y)$ is as well.

9.2 Spectral $\infty$-topoi & toposic décollages

In this section we define the $\infty$-toposic generalisation of spectral topological spaces relevant for our $\infty$-Categorical Hochster Duality Theorem (Theorem 9.3.1).

9.2.1 Definition. Let $S$ be a spectral topological space. An $S$-stratified $\infty$-topos $X \rightarrow S$ is a spectral $S$-stratified $\infty$-topos if and only if the following conditions are satisfied.
The ∞-topos $X$ is bounded and coherent.

The stratification by $S$ is constructible.

For every point $s \in S$, the stratum $X_s$ is a Stone ∞-topos.

We write $\text{StrTop}^\text{spec}_\infty \subset \text{StrTop}^\text{bcc}_{\infty,S}$ for the full subcategory spanned by the spectral $S$-stratified ∞-topoi.

More generally, write $\text{StrTop}^\text{spec}_\infty \subset \text{StrTop}^\text{bcc}_\infty$ for the full subcategory whose objects are spectral ∞-topoi and whose morphisms are squares

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
\tilde{S}' & \longrightarrow & \tilde{S}
\end{array}
$$

of coherent geometric morphisms. We observe that the pullback of a spectral ∞-topos along the geometric morphism induced by a quasicompact continuous map is again spectral, whence the functor $\text{StrTop}^\text{spec}_\infty \to \text{TSp}^\text{spec}_\infty$ is a cartesian fibration.

9.2.2 Example. Let $\Pi \to S$ be a profinite stratified space (Definition 2.5.7). Then $\tilde{\Pi}$ is a spectral ∞-topos, as the fibres $\tilde{\Pi}_s = \tilde{\Pi}_s$ are Stone ∞-topoi.

9.2.3. In Theorem 9.3.1, we will prove the central ∞-Categorical Hochster Duality Theorem, which states that every spectral ∞-topos is of the form $\tilde{\Pi}$ for some profinite stratified space.

9.2.4 Example. Let $X$ be a coherent scheme. Write $X_{\text{zar}}$ for its underlying Zariski spectral topological space, and let $X_{\text{et}}$ denote its coherent, 1-localic étale ∞-topos. Since $\text{Open}(X_{\text{et}}) \equiv \text{Open}(X_{\text{zar}})$, the natural stratification of the coherent ∞-topos $X_{\text{et}}$ from §8.7 is given by the natural geometric morphism $X_{\text{et}} \to X_{\text{zar}}$. For any point $x \in X_{\text{zar}}$, the stratum $(X_{\text{et}})_x$ is identified with $(\text{Spec } \kappa(x))_{\text{et}}$, which is the Stone ∞-topos $\tilde{\text{B}G}_{\kappa(x)}$. Consequently $X_{\text{et}}$ is a spectral ∞-topos.

9.2.5 Proposition. Let $S$ be a spectral topological space, and let $X$ be a bounded coherent constructible $S$-stratified ∞-topos. Then $X$ is spectral if and only if the functor

$$
\text{Pt}(X) \to \text{Pt}(\tilde{S}) = \mat(S)
$$

exhibits $\text{Pt}(X)$ as a mat($S$)-stratified space.

Proof. This follows directly from Theorem 4.4.10=[SAG, Theorem E.3.4.1].

9.2.6. Let $P$ be a finite poset. We now consider the nerve of a spectral $P$-stratified ∞-topos $X \to \tilde{P}$. Since each stratum $X_p$ is Stone, it follows from Proposition 9.1.2 that for any string $\{p_0 \leq \cdots \leq p_n\} \subseteq P$, the value

$$
\mathcal{N}_P(X)_{\{p_0 \leq \cdots \leq p_n\}} = X_{p_0} \times_X X_{p_1} \times_X \cdots \times_X X_{p_n}
$$

exhibits $\text{Pt}(X)$ as a mat($S$)-stratified space.
is a Stone ∞-topos. Consequently, we deduce that the equivalence
\[ N_P : \text{StrTop}_{\omega,P}^{bc} \Rightarrow \text{Déc}_P(\text{Top}_{\omega}^{bc}) \]
restricts to an equivalence between the ∞-category of spectral $P$-stratified ∞-topoi and the full subcategory $\text{Déc}_P(\text{Top}_{\omega}^{\text{Stn}}) \subset \text{Déc}_P(\text{Top}_{\omega}^{bc})$ spanned by those décollages over $P$ that carry each string to a Stone ∞-topos.

9.2.7 Lemma. Let $P$ be a finite poset. Then the nerve equivalence
\[ N_P : \text{StrTop}_{\omega,P}^{bc} \Rightarrow \text{Déc}_P(\text{Top}_{\omega}^{bc}) \]
restricts to an equivalence $\text{StrTop}_{\omega,P}^{\text{spec}} \Rightarrow \text{Déc}_P(\text{Top}_{\omega}^{\text{Stn}})$.

9.3 Hochster duality for higher topoi

In (1.3.6) we described Hochster duality as a cube of dualities: the equivalence of 1-categories between profinite posets and spectral topological spaces restricts on one hand to an equivalence of 1-categories between profinite sets and Stone spaces, and on the other to an equivalence of 1-categories between finite posets and finite topological spaces. Our objective now is to exhibit the analogous cube for higher topoi:

\[
\begin{align*}
S_\pi & \Rightarrow \text{Top}_{\omega}^{\text{fin}} \\
S_{\pi}^{\omega} & \Rightarrow \text{Top}_{\omega}^{\text{Stn}} \\
\text{Str}_{\pi} & \Rightarrow \text{StrTop}_{\omega}^{\text{fin}} \\
\text{Str}_{\pi}^{\omega} & \Rightarrow \text{StrTop}_{\omega}^{\text{spec}}
\end{align*}
\]

where the vertical fully faithful functors are given by equipping an object with the trivial stratification. The top face of this cube was established by Lurie [SAG, Appendix E]. We must now address the bottom face, more precisely the equivalence $\text{Str}_{\pi}^{\omega} \Rightarrow \text{StrTop}_{\omega}^{\text{spec}}$.

9.3.1 Theorem (∞-Categorical Hochster Duality). Let $S$ be a spectral topological space. Then the functor
\[ \tilde{\lambda}_S : \text{Str}_{\pi,S}^{\omega} \Rightarrow \text{StrTop}_{\omega,S}^{\text{spec}} \]
given by the assignment $\Pi \mapsto \bar{\Pi}$ is an equivalence of ∞-categories. Consequently, the functor
\[ \tilde{\lambda} : \text{Str}_{\pi}^{\omega} \Rightarrow \text{StrTop}_{\omega}^{\text{spec}} \]
is an equivalence of ∞-categories.
Proof. Since $\lambda$ is fully faithful (Proposition 8.8.10) and preserves inverse limits, it suffices to prove that for any finite poset $P$, the fully faithful functor $\lambda : \text{Str}^{sfc}_{c,P} \to \text{StrTop}^{sfc}_{c,P}$ is essentially surjective.

This now follows from the conjunction of Lemma 9.2.7 and Proposition 8.8.11. □

The back face of the cube is now just a restriction of the front face: we define $\text{Top}^{fin}_{\infty}$ as the full subcategory of $\text{Top}^\infty$ spanned by the essential image of the fully faithful functor $\text{S}_\pi \hookrightarrow \text{Top}^\infty$ given by $\Pi \mapsto S/\Pi = \text{Fun}(\Pi, S)$. Then $\text{StrTop}^{fin}_{\infty}$ is the $\infty$-category of bounded coherent constructible $\infty$-topoi over a finite poset $P$ such that for every point $p \in P$, the $\infty$-topos $X_p$ is in $\text{Top}^{fin}_{\infty}$.

9.4 Constructible sheaves

The truncated coherent objects of a Stone $\infty$-topos are exactly the lisse sheaves (Recollection 4.4.11). This turns out to be a defining property of Stone $\infty$-topoi (Proposition 4.4.14=[SAG, Proposition E.3.1.1]). In the same manner, the truncated coherent objects of a spectral $\infty$-topos are exactly the constructible sheaves, to which we now turn.

9.4.1 Definition. Let $P$ be a finite poset and $X$ a $P$-stratified $\infty$-topos. An object $F \in X$ is formally constructible (or formally $P$-constructible if disambiguation is called for) if and only if, for any point $p \in P$, the restriction $F|_{X_p} = e_p^* F \in X_p$ is a local system, where $e_{p,*} : X_p \hookrightarrow X$ is the inclusion of the $p$-th stratum.

We say that $F$ is constructible (or $P$-constructible) if and only if the following pair of conditions is satisfied:

- The object $F$ is formally constructible.
- For any point $p \in P$, the restriction $F|_{X_p} \in X_p$ is lisse.

9.4.2. This notion of constructibility depends upon the whole structure of the stratified $\infty$-topos, not only upon the underlying $\infty$-topos.

9.4.3. For any finite poset $P$ and $P$-stratified $\infty$-topos $X \to \overline{P}$, the $\infty$-category of constructible sheaves on $X$ is given by the pullback of $\infty$-categories:

$$
\begin{array}{ccc}
X^{P-\text{cons}} & \longrightarrow & \prod_{p \in P} X_{p}^{\text{lisse}} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \prod_{p \in P} e_p^* \prod_{p \in P} X_p 
\end{array}
$$

where here $\prod_{p \in P} X_p$ is the product in $\text{Cat}_{\infty,\delta_1}$. Lemmas 3.8.4 and 3.8.5 now show that $X^{P-\text{cons}}$ is an $\infty$-pretopos (Definition 3.8.2) and the inclusion $X^{P-\text{cons}} \hookrightarrow X$ is a morphism of $\infty$-pretopoi.

The pullback functor in a geometric morphism of $\infty$-topoi preserves lisse objects (see Recollection 4.4.11); in the same manner, the pullback of a morphism of stratified $\infty$-topoi preserves constructible objects.
9.4.4 Lemma. Let \( f : P \to Q \) be a morphism of finite posets, and let \( X \to \overline{P} \) and \( Y \to \overline{Q} \) be stratified \( \infty \)-topoi. Then for any geometric morphism \( q_* : X \to Y \) over \( f_* : \overline{P} \to \overline{Q} \), the pullback \( q^* : Y \to X \) sends \( \mathcal{Q} \)-constructible objects of \( Y \) to \( \mathcal{P} \)-constructible objects of \( X \). Hence \( q^* \) restricts to a morphism of \( \mathcal{C} \)-pretopoi

\[
q^* : \mathcal{Y}^{\mathcal{Q}\text{-cons}} \to \mathcal{X}^{\mathcal{P}\text{-cons}}.
\]

Proof. Let \( F \in \mathcal{Y}^{\mathcal{Q}\text{-cons}} \) be a \( \mathcal{Q} \)-constructible object of \( Y \). Then for any point \( p \in P \), the restriction \( F|_{Y_{\{p\}}} \) is lisse, so since the pullback in a geometric morphism preserves lisse objects, we see that the object \( q^*(F)|_{X_p} \) is lisse. Hence \( q^*(F) \) is \( \mathcal{P} \)-constructible.

The fact that \( q^* : \mathcal{Y}^{\mathcal{Q}\text{-cons}} \to \mathcal{X}^{\mathcal{P}\text{-cons}} \) is a morphism of \( \mathcal{C} \)-pretopoi is immediate from (9.4.3).

9.4.5 Proposition. Let \( P \) be a finite poset and \( X \to \overline{P} \) a \( \mathcal{P} \)-stratified \( \infty \)-topos. Then the \( \mathcal{C} \)-pretopos \( \mathcal{X}^{\mathcal{P}\text{-cons}} \) is bounded (Definition 3.8.8).

Proof. If \( P = \emptyset \), then the claim is obvious, so assume that \( P \) is nonempty. We prove the claim by induction on the rank of \( P \).

In the base case where \( P \) has rank 0, \( P \) is discrete, so \( X \) is finite the coproduct of \( \infty \)-topoi \( \coprod_{p \in P} X_p \) (which is the product \( \prod_{p \in P} X_p \) in \( \mathbf{Cat}_{\infty, \delta_1} \)). Thus \( \mathcal{X}^{\mathcal{P}\text{-cons}} \) is the product of \( \infty \)-categories:

\[
\mathcal{X}^{\mathcal{P}\text{-cons}} = \prod_{p \in P} \mathcal{X}_p^{\text{lisse}}. \]

By Theorem 4.4.16=[SAG, Theorem E.2.3.2], for all \( p \in P \) the \( \mathcal{C} \)-pretopos \( \mathcal{X}_p^{\text{lisse}} \) is bounded; the finiteness of \( P \) and Lemma 3.8.11 now show that \( \mathcal{X}^{\mathcal{P}\text{-cons}} \) is also bounded.

For the induction step, let \( n \geq 0 \) be a natural number and assume that the claim holds for all finite posets \( P \) of rank \( n \) and \( \mathcal{P} \)-stratified \( \infty \)-topoi \( X \to \overline{P} \). Let \( P \) be a finite poset of rank \( n + 1 \), and write \( M \subset P \) for the full subposet spanned by the minimal elements of \( P \). Then \( M \) is discrete and closed in \( P \). Write \( U := P \setminus M \) for the open complement of \( M \) in \( P \). Then \( U \) is a poset of rank \( n \). Moreover, since \( \overline{P} \) is the recollement of \( \overline{M} \) and \( \overline{U} \), the \( \mathcal{P} \)-stratified \( \infty \)-topos \( X \) is the recollement of \( X_M \) and \( X_U \). An object \( F \in X \) is \( \mathcal{P} \)-constructible if and only if \( F|_{X_M} \) and \( F|_{X_U} \) are both constructible, from which we deduce that \( \mathcal{X}^{\mathcal{P}\text{-cons}} \) is the oriented fibre product of \( \infty \)-categories

\[
\mathcal{X}^{\mathcal{P}\text{-cons}} = \mathcal{X}_M^{\mathcal{M}\text{-cons}} \times_{\mathcal{X}_M^{\mathcal{U}\text{-cons}}} \mathcal{X}_U^{\mathcal{U}\text{-cons}}. \]

Since \( M \) is a poset of rank 0 and \( U \) is a poset of rank \( n \), by the induction hypothesis both \( \mathcal{X}_M^{\mathcal{M}\text{-cons}} \) and \( \mathcal{X}_U^{\mathcal{U}\text{-cons}} \) are bounded \( \mathcal{C} \)-pretopoi. To conclude that the \( \mathcal{C} \)-pretopos \( \mathcal{X}^{\mathcal{P}\text{-cons}} \) is a bounded, note that by (5.1.2) every object of \( \mathcal{X}^{\mathcal{P}\text{-cons}} \) is truncated and by (0.4.2) the \( \infty \)-category \( \mathcal{X}^{\mathcal{P}\text{-cons}} \) is \( \delta_0 \)-small.

9.4.6 Definition. Let \( S \) be a spectral topological space and \( X \) an \( S \)-stratified \( \infty \)-topos. We say that an object \( F \in X \) is formally constructible (or formally \( S \)-constructible) if and only if there exist a finite poset \( P \) and a constructible stratification \( S \to P \) of proposets such that \( F \) is formally \( P \)-constructible. We say that \( F \) is constructible (or \( S \)-constructible) if and only if there exist a poset \( P \) and a finite constructible stratification \( S \to P \) of proposets such that \( F \) is \( P \)-constructible.
For any spectral topological space $\mathcal{S}$ and any $\mathcal{S}$-stratified $\infty$-topos $\mathcal{X} \rightarrow \tilde{\mathcal{S}}$, we denote by $X^{\mathcal{S}\text{-cons}} \subseteq \mathcal{X}$ (respectively, by $X^{\mathcal{S}\text{-cons}} \subseteq \mathcal{X}$) the full subcategory spanned by the formally constructible objects (respectively, the constructible objects).

9.4.7. For any spectral topological space $\mathcal{S}$ and $\mathcal{S}$-stratified $\infty$-topos $\mathcal{X} \rightarrow \tilde{\mathcal{S}}$, the $\infty$-category of constructible sheaves on $\mathcal{X}$ is thus a filtered colimit of $\infty$-categories:

$$X^{\mathcal{S}\text{-cons}} \cong \colim_{P \in FC(S)} X^{P\text{-cons}}.$$  

Thus Lemma 9.4.4 and Proposition 9.4.5 combined with Proposition 3.9.1=[SAG, Proposition A.8.3.1] show that $X^{\mathcal{S}\text{-cons}}$ is a bounded $\infty$-pretopos. Moreover, (9.4.3) shows that the inclusion $X^{\mathcal{S}\text{-cons}} \hookrightarrow \mathcal{X}$ is a morphism of $\infty$-pretopoi.

From Lemma 9.4.4 we now immediately deduce the following.

9.4.8 Lemma. Let $f : S \rightarrow T$ be a quasicompact continuous map of spectral topological spaces, and let $\mathcal{X} \rightarrow \tilde{\mathcal{S}}$ and $\mathcal{Y} \rightarrow \tilde{T}$ be stratified $\infty$-topoi. Then for any geometric morphism $q^* : \mathcal{Y} \rightarrow \mathcal{X}$ over $f_* : \tilde{S} \rightarrow \tilde{T}$, the pullback $q^* : \mathcal{Y} \rightarrow \mathcal{X}$ sends $T$-constructible objects of $\mathcal{Y}$ to $S$-constructible objects of $\mathcal{X}$. Hence $q^*$ restricts to a morphism of $\infty$-pretopoi

$$q^* : Y^{T\text{-cons}} \rightarrow X^{S\text{-cons}}.$$  

We now turn to the relationship between coherence and constructibility in $\infty$-topoi stratified by a spectral topological space.

9.4.9 Recollection. Let $\mathcal{S}$ be a spectral topological space. The collection of constructible subsets of $\mathcal{S}$ is the smallest collection of subsets of $\mathcal{S}$ containing all quasicompact open subsets and closed under taking finite intersections and complements.

9.4.10 Lemma. Let $\mathcal{S}$ be a spectral topological space, and let $\mathcal{X}$ be an $\mathcal{S}$-stratified $\infty$-topos. Then an object $F$ of $\mathcal{X}$ is constructible if and only if, for every point $s \in \mathcal{S}$, there exists a constructible subset $W \subseteq \mathcal{S}$ containing $s$ such that $F|_{\mathcal{X}W}$ is lisse.

Proof. The ‘only if’ direction is clear. Conversely, assume that every point of $\mathcal{S}$ is contained in such a constructible set. Hence the collection $\{W_{\alpha} : \alpha \in \Lambda\}$ of constructible subsets of $\mathcal{S}$ such that $F|_{\mathcal{X}W_{\alpha}}$ is lisse is a cover of $\mathcal{S}$ by constructible subsets. Since the constructible topology on $\mathcal{S}$ is quasicompact, it follows that there exists a finite subcover $\{W_{\alpha} : \alpha \in \Lambda'\}$. Select a finite constructible stratification $\mathcal{S} \rightarrow P$ of $\mathcal{S}$ such that for every $p \in P$, there exists an $\alpha \in \Lambda'$ such that the stratum $S_p \subseteq W_{\alpha}$. Now $F$ is $P$-constructible. \(\square\)

9.4.11 Lemma. Let $\mathcal{S}$ be a spectral topological space, and $\mathcal{X} \rightarrow \tilde{\mathcal{S}}$ a coherent $\mathcal{S}$-stratified $\infty$-topos. Then every constructible object of $\mathcal{X}$ is truncated and coherent. If $\mathcal{X}$ is also bounded and every truncated and coherent object of $\mathcal{X}$ is constructible, then $\mathcal{X}$ is spectral.

Proof. For the first statement, let $F \in X^{\mathcal{S}\text{-cons}}$, and let $\mathcal{S} \rightarrow P$ be a finite constructible stratification such that for every point $p \in P$, the restriction $F|_{X_p}$ is lisse. By Proposition 5.1.8=[DAG XIII, Proposition 2.3.22] it follows that $F$ is coherent. If each $F|_{X_p}$ is $N$-truncated, then $F$ is $N$-truncated.
For the second statement, if every truncated coherent object of \( X \) is constructible and \( X \) is bounded, then \( X = \text{Sh}_{\text{eff}}(X^\text{\text{-cons}}) \). For any point \( s \in S \), one thus has an equivalence \( X_s = \text{Sh}_{\text{eff}}(X^\text{\text{-cons}}) \), which is a Stone \( \infty \)-topos. Thus \( X \) is spectral.

\[ 9.4.12 \text{ Proposition. If } S \text{ is a spectral topological space, and } X \text{ is a spectral } S\text{-stratified } \infty \text{-topos } X, \text{ then every truncated and coherent object of } X \text{ is constructible.} \]

\text{Proof. Let } F \text{ be a truncated coherent object of } X, \text{ and } s \in S \text{ a point. We wish to show that there exists a constructible subset of } W \subset S \text{ containing } s \text{ such that } F|_{X_w} \text{ is lisse (Lemma 9.4.10). Passing to the closure of } s, \text{ it suffices to assume that } S \text{ is irreducible, and } s \text{ is its generic point.}

\text{Since } F|_{X_w} \text{ is lisse, it follows from Lemma 4.4.12=[SAG, Proposition E.2.7.7] that there exists a full subcategory } E \subset S, \text{ spanned by finitely many } \pi\text{-finite spaces and a unique geometric morphism } g_* : X_s \rightarrow S/E; \text{ and an equivalence } \epsilon : F|_{X_w} \Rightarrow g^*(I), \text{ where } I \text{ is the inclusion functor } E \hookrightarrow S. \text{ Now since } S/E \text{ is cocompact as an object of } \text{Top}_{\text{co}}, \text{ (Lemma 4.4.13) and } X_s \text{ is identified with the limit } \lim_{W}^\to X_w \text{ over constructible subsets } W \subset S \text{ containing } s, \text{ it follows that for some such } W, \text{ one may factor } g_* \text{ through a geometric morphism } g_{W,*} : X_W \rightarrow S/E. \text{ Now since } X_w^{\text{coh}} = \colim W X_W^{\text{coh}_{\text{co}}}, \text{ we shrink } W \text{ as needed to ensure that there exists an equivalence } \epsilon : F|_{X_w} \Rightarrow g_W^*(I), \text{ and conclude that } F \text{ is lisse on } W. \]

\[ 9.4.13 \text{ Example. If } X \text{ is a coherent scheme, then the truncated coherent objects of } X_{\text{et}} \text{ are precisely the constructible sheaves of spaces. This is the nonabelian analogue of the well-known result that for a finite ring } A, \text{ the compact objects of the } \infty\text{-category } \text{Sh}_{\text{et}}(X; D(A)) \text{ of étale sheaves of } A\text{-complexes on } X \text{ coincide with the derived } \infty\text{-category of constructible } A\text{-sheaves [35, Proposition 2.2.6.2].}

\text{We have shown that the } \infty\text{-category } \text{Str}^A_{\text{co}} \text{ of profinite stratified spaces is equivalent to the } \infty\text{-category } \text{StrTop}_{\text{co}}^{\text{prec}}, \text{ which is in turn a full subcategory of } \text{StrTop}_{\text{co}}^{A,\text{prec}} \text{ of bounded coherent constructible stratified } \infty\text{-topoi. This last } \infty\text{-category is a non-full subcategory of } \text{StrTop}_{\text{co}}^{A}, \text{ however. Just as how every geometric morphism between Stone } \infty\text{-topoi is coherent (Corollary 4.4.15=[SAG, Corollary E.3.1.2]), the subcategory}

\text{StrTop}_{\text{co}}^{\text{prec}} \subset \text{StrTop}_{\text{co}}^{A}

\text{is full, as we shall now explain.}

\[ 9.4.14 \text{ Proposition. Let } f : S \rightarrow T \text{ be a quasicompact continuous map of spectral topological spaces, let } X \rightarrow S \text{ be a coherent constructible stratified } \infty\text{-topos, and let } Y \rightarrow T \text{ be a spectral } \infty\text{-topos. Then any geometric morphism } q_* : X \rightarrow Y \text{ over } f_* : S \rightarrow T \text{ is coherent.}

\text{Proof. By Corollary 3.4.5 it suffices to show that if } F \in X \text{ is truncated and coherent, then } p^* F \text{ is coherent. By Proposition 9.4.12}

\[ X_S^{\text{cons}} = X^{\text{coh}}_{\text{co}, S} \]

so the claim now follows from the facts that } q^* \text{ preserves constructibility (Lemma 9.4.8) and the } S'\text{-constructible objects of } X' \text{ are truncated coherent (Lemma 9.4.11).} \]
9.4.15 Corollary. The subcategory \( \text{StrTop}^{\text{spec}}_{\infty} \subset \text{StrTop}^\wedge_{\infty} \) is full.

9.4.16 Construction. Let \( S \) be a spectral topological space, and \( X \) an \( S \)-stratified \( \infty \)-topos. By [SAG, Proposition A.6.4.4], the fully faithful inclusion \( X^{S\text{-cons}} \twoheadrightarrow X \) of \( \infty \)-pre-topoi extends (essentially uniquely) to a geometric morphism \( X \to \text{Sh}_{\text{eff}}(X^{S\text{-cons}}) \) over \( S \). By construction, the \( S \)-stratified \( \infty \)-topos \( X^{S\text{-spec}} := \text{Sh}_{\text{eff}}(X^{S\text{-cons}}) \) is spectral. Furthermore, \( X^{S\text{-spec}} \) is the universal spectral \( S \)-stratified \( \infty \)-topos receiving a geometric morphism over \( S \) from \( X \). Thus the assignment

\[
X \mapsto X^{S\text{-spec}}
\]

provides a relative left adjoint to the inclusion \( \text{StrTop}^{\text{spec}}_{\infty} \hookrightarrow \text{StrTop}^\wedge_{\infty} \) over \( \text{TSp}^{\text{spec}}_{\infty} \), which we call the spectrification. This is the stratified analogue of the Stone reflection ([Theorem 4.4.16]).

9.4.17 Example. When \( S = [n] \), the spectrification of a bounded coherent \( \infty \)-topos \( X \) equipped with a constructible stratification by \( [n] \) can be identified as an iterated bounded coherent oriented pushout:

\[
X^{[n]\text{-spec}} \cong X_0^{\text{Str}} \bigcup_{X_1} (X_0 \times_{X_1} X_1)^{\text{lim}} \cdots \bigcup_{X_n} (X_{n-1} \times_{X_n} X_n)^{\text{lim}} X_n^{\text{Str}}.
\]

9.4.18 Construction. Thanks to the existence of the spectrification functor, we deduce the forgetful functor \( \text{StrTop}^{\text{spec}}_{\infty} \to \text{TSp}^{\text{spec}}_{\infty} \) is a cocartesian fibration (as well as a cartesian fibration): for any quasicompact continuous map \( f : S' \to S \) and any spectral \( S' \)-stratified \( \infty \)-topos \( X \), the stratified geometric morphism \( X \to X^{S\text{-spec}} \) is a cocartesian edge over \( f \).

9.4.19 Lemma. Let \( S \) be a spectral topological space. Then the natural functor

\[
\text{StrTop}^{\text{spec}}_{\infty, S} \to \lim_{P \in \text{FC}(S)} \text{StrTop}^{\text{spec}}_{\infty, P}
\]

is an equivalence of \( \infty \)-categories.

Proof. The formation of the limit is an inverse. \( \square \)
10 Profinite stratified shape

In this chapter we investigate the inverse to the equivalence of \(\infty\)-categories

\[ \hat{\lambda}: \text{Str}_n^\infty \Rightarrow \text{StrTop}_\infty^{\text{spec}} \]

provided by \(\infty\)-Categorical Hoschster Duality. This inverse equivalence provides a stratified refinement of the profinite shape (Example 10.1.6).

In Section 10.1 we introduce this inverse, which we call the \textit{profinite stratified shape}. To justify this language, Section 10.2 shows that, up to protruncation, the shape of a spectral \(\infty\)-topos can be recovered from its profinite stratified shape by inverting all morphisms in the ‘pro’ sense. Section 10.3 shows that the materialization of the profinite stratified shape of a spectral \(\infty\)-topos \(X\) recovers the \(\infty\)-category Pt\((X)\) of points of \(X\) and proves the analogues of the basic operations relating profinite spaces and Stone \(\infty\)-topoi discussed in §4.4 in the stratified setting. Section 10.5 provides a van Kampen Theorem that allows one to compute the profinite shape of a spectral stratified \(\infty\)-topos in terms of the profinite shapes of strata and links.

10.1 The definition of the profinite stratified shape

10.1.1 Construction. We have constructed (Theorem 9.3.1) an equivalence of \(\infty\)-categories

\[ \hat{\lambda}: \text{Str}_n^\infty \Rightarrow \text{StrTop}_\infty^{\text{spec}} \]

over \(\text{TSpc}^{\text{spec}}\), given by the assignment \(\Pi \mapsto \tilde{\Pi}\). The further inclusion

\[ \text{StrTop}_\infty^{\text{spec}} \hookrightarrow \text{StrTop}_\infty^\hat{\lambda} \]

admits a left adjoint, given by spectrification (Construction 9.4.16). We therefore obtain an adjunction

\[ \hat{\Pi}_{(\infty,1)}: \text{StrTop}_\infty^\hat{\lambda} \rightleftarrows \text{Str}_n^\infty: \hat{\lambda} \]

in which the left adjoint carries a stratified \(\infty\)-topos \(X \rightarrow S\) to the profinite \(S\)-stratified space that as a left exact accessible functor \(\text{Str}_n^S \rightarrow S\) is given by

\[ \Pi \mapsto \text{Map}_{\text{StrTop}_\infty^\hat{\lambda}}(X, \Pi) . \]

Over any spectral topological space \(S\), we obtain an adjunction

\[ \hat{\Pi}_{(\infty,1)}^{S_{\infty}}: \text{StrTop}_\infty^{\hat{\lambda}^S} \rightleftarrows \text{Str}_n^{S_{\infty}}: \hat{\lambda}^S \]

over \(S\).

10.1.2 Example. For any spectral topological space \(S\) and any profinite \(S\)-stratified space \(\Pi\), we have \(\hat{\Pi}_{(\infty,1)}^{S_{\infty}}(\Pi) = \Pi\).

10.1.3 Example. The functor \(\hat{\Pi}_{(\infty,1)}^{[0]}\) is the profinite shape of Definition 4.4.2.

10.1.4 Definition. Let \(S\) be a spectral topological space, and let \(X \rightarrow S\) be an \(S\)-stratified \(\infty\)-topos. Then we call the profinite \(S\)-stratified space \(\hat{\Pi}_{(\infty,1)}^{S_{\infty}}(X)\) the \textit{\(S\)-stratified homotopy type} of \(X\).
10.1.5. Since left adjoints compose, if \( \eta : S' \to S \) is a quasicompact continuous map of spectral topological spaces, then there is a natural equivalence

\[
\eta_! \tilde{\Pi}^S_{(\infty,1)} \Rightarrow \tilde{\Pi}^S_{(\infty,1)}.
\]

10.1.6 Example. For any bounded coherent constructible \( S \)-stratified \( \infty \)-topos \( \mathcal{X} \), the homotopy type \( \tilde{\Pi}^\infty_\mathcal{X}(X) \) is the classifying profinite space of the profinite \( \infty \)-category \( \tilde{\Pi}^\infty_{(\infty,1)}(X) \); thus the stratification on \( X \) gives rise to a delocalisation of its homotopy type.

Combining \( \infty \)-Categorical Hochster Duality (Theorem 9.3.1) with Proposition 9.4.12 we deduce the Exodromy Equivalence stated as Theorem B in the introduction.

10.1.7 Theorem (Exodromy Equivalence for Stratified \( \infty \)-Topoi). Let \( S \) be a spectral topological space and \( \mathcal{X} \) an \( S \)-stratified \( \infty \)-topos. Then the unit

\[
X \to \text{Fun}(\tilde{\Pi}^S_{(\infty,1)}(X), S)
\]

of the adjunction to profinite stratified spaces restricts to an equivalence

\[
\text{Fun}(\tilde{\Pi}^S_{(\infty,1)}(X), S) \cong X^{S_{\text{cons}}}
\]

The Exodromy Equivalence implies the following stable variant:

10.1.8 Theorem. Let \( R \) be a finite ring, \( S \) be a spectral topological space, and \( \mathcal{X} \) an \( S \)-stratified \( \infty \)-topos. Then there is a natural equivalence

\[
\text{Fun}(\tilde{\Pi}^S_{(\infty,1)}(X), \text{Perf}(R)) \cong D_{\text{cons}}(\mathcal{X}; R).
\]

10.2 Recovering the protruncated shape from the profinite stratified shape

In Example 10.1.6 we saw how to recover the profinite shape \( \tilde{\Pi}^\infty_\mathcal{X}(X) \) of a spectral stratified \( \infty \)-topos \( \mathcal{X} \) from its profinite stratified shape \( \tilde{\Pi}^\infty_{(\infty,1)}(X) \) by 'inverting all morphisms' in a suitable sense. This delocalisation result essentially comes for free from the functoriality of the profinite stratified shape. In this section prove a stronger delocalisation result (Theorem 10.2.3): the profinite stratified shape is a delocalisation of the protruncated shape.\(^{27}\)

The equivalence \( \text{Str}_n^\wedge = \text{StrTop}_{\text{co}}^{\text{spec}} \) provided by \( \infty \)-categorical Hochster Duality (Theorem 9.3.1) provides a way to recover the shape of a spectral \( \infty \)-topos from its profinite stratified shape, via the composite

\[
\text{Str}_n^\wedge \longrightarrow \text{StrTop}_{\infty}^{\text{spec}} \longrightarrow \text{Top}_{\infty}^{\text{bc}} \longrightarrow \Pi_{\infty} \longrightarrow \text{Pro}(S),
\]

where the middle functor functor forgets the stratification. There is another functor \( E : \text{Str}_n^\wedge \to \text{Pro}(S) \) that doesn't require the use of \( \infty \)-topoi, namely, the extension to pro\( \infty \)-objects of the composite

\[
\text{Str}_n^\wedge \longrightarrow \text{Cat}_{\infty} \xrightarrow{E} S,
\]

\(^{27}\)The contents of this section originally appeared in a short preprint by the third-named author [43].
where the first functor forgets the stratification and the second functor sends an \(\infty\)-category \(C\) to the \(\infty\)-groupoid \(E(C)\) obtained by inverting every morphism in \(C\) (Notation 2.2.1). It follows formally that these two functors agree on \(\text{Str}_\pi\):

10.2.1 Lemma. The square

\[
\begin{array}{ccc}
\text{Str}_\pi & \xrightarrow{\tilde{\lambda}} & \text{StrTop}^\text{spec}_{\infty} \\
\downarrow & & \downarrow \Pi_{\infty} \\
S & \xrightarrow{\perp} & \text{Pro}(S)
\end{array}
\]

commutes.

Proof. By the definition of the equivalence \(\tilde{\lambda}: \text{Str}_\pi^\lambda \Rightarrow \text{StrTop}^\text{spec}_{\infty}\) (Theorem 9.3.1), the following square commutes

\[
\begin{array}{ccc}
\text{Str}_\pi & \xrightarrow{\tilde{\lambda}} & \text{StrTop}^\text{spec}_{\infty} \\
\downarrow & & \downarrow \\
\text{Cat}_{\infty} & \xrightarrow{\text{Fun}(-,S)} & \text{Top}_{\infty}
\end{array}
\]

where the vertical functors forget stratifications. Combining this with Example 4.2.3 proves the claim. \(\square\)

10.2.2. Since the functor \(E: \text{Str}_\pi^\lambda \rightarrow \text{Pro}(S)\) preserves inverse limits, Lemma 10.2.1 provides a natural transformation

\[
\theta: \Pi_{\infty} \circ \tilde{\lambda} \rightarrow E.
\]

10.2.3 Theorem. The natural transformation

\[
\tau_{\infty} \theta: \Pi_{\infty} \circ \tilde{\lambda} \rightarrow \tau_{\infty} E
\]

of functors \(\text{Str}_\pi^\lambda \rightarrow \text{Pro}(S_{\infty})\) is an equivalence.

Proof. Since the forgetful functor \(\text{StrTop}^\text{spec}_{\infty} \rightarrow \text{Top}_{\infty}^\text{bc}\) preserves inverse limits, Corollary 4.3.7 implies that the truncated shape \(\Pi_{\infty} : \text{StrTop}^\text{spec}_{\infty} \rightarrow \text{Pro}(S_{\infty})\) preserves inverse limits. Both \(\tau_{\infty}\) and \(E\) preserve inverse limits, hence their composite

\[
\tau_{\infty} E: \text{Str}_\pi^\lambda \rightarrow \text{Pro}(S_{\infty})
\]

preserves inverse limits. The claim now follows from the fact that \(\theta\) is an equivalence when restricted to \(\text{Str}_\pi\) (Lemma 10.2.1) and the universal property of the \(\infty\)-category \(\text{Str}_\pi^\lambda\) of profinite stratified spaces. \(\square\)
10.3 Points & materialisation

We now provide a stratified refinement of (4.4.7), which allows us to prove a ‘Whitehead Theorem’ for profinite stratified spaces, and effectively speak of $n$-truncated profinite stratified spaces via materialisation.

10.3.1. Let $S$ be a spectral topological space, and let $X$ be an $S$-stratified $\infty$-topos. The $\infty$-category of points of $X$ is

$$\text{Pt}(X) = \text{Fun}_*(S, X)^{op} = \text{Fun}_{\text{StrTop}^{\infty,\ast}_*(\{0\}, X)^{op}}.$$

Since $\Pi_{(\infty,1)}(\{0\}) = \ast$, applying $\Pi_{(\infty,1)}$ yields a natural functor

$$\text{Pt}(X) \rightarrow \text{Fun}_{\text{Str}^{\infty,\ast}_*(\ast, \Pi_{(\infty,1)}(X)) = \text{mat}\Pi_{(\infty,1)}(X).$$

In the case where $X$ is a spectral $\infty$-topos, then $\infty$-Categorical Hochster Duality (Theorem 9.3.1) implies the following stratified refinement of (4.4.7).

10.3.2 Lemma. If $X$ is a spectral $\infty$-topos, then the natural morphism

$$\text{Pt}(X) \rightarrow \text{mat}\Pi_{(\infty,1)}(X)$$

of stratified spaces is an equivalence.

Now we can deduce a stratified refinement of Whitehead's Theorem for profinite spaces (Theorem 4.4.8=[SAG, Theorem E.3.1.6]).

10.3.3 Theorem (Profinite Stratified Whitehead Theorem). The materialisation functor $\text{mat} : \text{Str}^\infty_n \rightarrow \text{Str}$ is conservative.

Proof. Let $f : \Pi \rightarrow \Pi'$ be a morphism in $\text{Str}^\infty_n$ and assume that $\text{mat}(f)$ is an equivalence in $\text{Str}$. Write $f_* : \Pi \rightarrow \Pi'$ for the induced morphism of spectral $\infty$-topoi. From Lemma 10.3.2 we deduce that

$$\text{Pt}(f_* : \Pi \rightarrow \Pi)$$

is an equivalence of $\infty$-categories. Conceptual Completeness (Theorem 3.11.2=[SAG, Theorem A.9.0.6]) implies that $f_*$ is an equivalence of $\infty$-topoi. The full faithfulness of the functor $\Pi \rightarrow \Pi$ completes the proof.

We can employ the Profinite Stratified Whitehead Theorem to study the Postnikov tower of profinite stratified spaces.

10.3.4 Definition. Let $n \in \mathbb{N}$. A profinite stratified space $\Pi \rightarrow S$ is $n$-truncated if and only if $\Pi$ can be exhibited as an inverse limit of finite $n$-truncated $\pi$-finite stratified spaces. Equivalently, if we extend $h_n : \text{Str}_n \rightarrow \text{Str}$ to an inverse-limit preserving functor $h_n : \text{Str}_n^\infty \rightarrow \text{Str}_n^\infty$, then an $n$-truncated profinite space is one in the essential image of $h_n$.

We write $(\text{Str}^\infty_n)^{\text{sa}} \subset \text{Str}_n^\infty$ for the full subcategory spanned by the $n$-truncated profinite stratified spaces.
10.3.5 Lemma. Let \( n \in \mathbb{N} \), and let \( S \) be a spectral topological space. Then a profinite stratified space \( \Pi \to S \) is \( n \)-truncated if and only if, for all \( s, t \in \text{mat}(S) \) with \( s \leq t \), the induced morphism
\[
N_{\text{mat}(S)}(\Pi)(s, t) \to \Pi_s \times \Pi_t
\]
is an \((n-1)\)-truncated morphism of \( S_n^0 \).

**Proof.** If \( \Pi \) is exhibited as a sequence \( \{\Pi_{\alpha} \to P_{\alpha}\}_{\alpha \in A} \) of \( n \)-finite \( n \)-truncated stratified spaces, then express \( s \) and \( t \) as sequences \( \{s_{\alpha}\}_{\alpha \in A} \) and \( \{t_{\alpha}\}_{\alpha \in A} \) of points. So the sequence
\[
\left\{N_{P_{\alpha}}(\Pi_{\alpha})|s_{\alpha}, t_{\alpha}\rangle \to \Pi_{s_{\alpha}} \times \Pi_{t_{\alpha}}\right\}_{\alpha \in A},
\]
which exhibits the morphism \( N_{\text{mat}(S)}(\Pi)(s, t) \to \Pi_s \times \Pi_t \) of \( S_n^0 \), is \((n-1)\)-truncated.

Conversely, assume that \( \Pi \) is exhibited as a sequence \( \{\Pi_{\alpha} \to P_{\alpha}\}_{\alpha \in A} \) of \( n \)-finite stratified spaces, and that for any \( s, t \in \text{mat}(S) \) with \( s \leq t \), the morphism \( N_{\text{mat}(S)}(\Pi)(s, t) \to \Pi_s \times \Pi_t \) of \( S_n^0 \) is \((n-1)\)-truncated. Now consider \( \text{h} \gamma \Pi = \{\text{h}_{\gamma} \Pi_{\alpha} \to P_{\alpha}\}_{\alpha \in A} \) and the natural morphism \( \Pi \to \text{h} \gamma \Pi \). To see that this morphism is an equivalence, we may pass to the materialisation by Theorem 10.3.3, where it is obvious. 

10.3.6 Lemma. Let \( n \in \mathbb{N} \). A profinite stratified space \( \Pi \to S \) is \( n \)-truncated if and only if \( \text{mat}(\Pi) \in \text{Str} \) is \( n \)-truncated in the sense of Definition 2.3.4.

**Proof.** For \( s, t \in \text{mat}(S) \) with \( s \leq t \), we have
\[
\text{mat}(N_{\text{mat}(S)}(\Pi)(s, t)) = N_{\text{mat}(S)}(\text{mat}(\Pi))(s, t).
\]
By Proposition 4.4.9=SAG, Proposition E.4.6.1 and the fact that materialisation is a right adjoint, we see that a profinite stratified space \( \Pi \) is \( n \)-truncated if and only if the morphism
\[
N_{\text{mat}(S)}(\text{mat}(\Pi))(s, t) \to \text{mat}(\Pi)_s \times \text{mat}(\Pi)_t
\]
is an \((n-1)\)-truncated morphism of spaces, which is true if and only if \( \text{mat}(\Pi) \) is \( n \)-truncated in the sense of Definition 2.3.4.

Under \( \infty \)-Categorical Hochster Duality (Theorem 9.3.1) \( n \)-localic spectral stratified \( \infty \)-topoi correspond to \( n \)-truncated profinite stratified spaces:

10.3.7 Proposition. Let \( X \) be a spectral \( \infty \)-topos and \( n \in \mathbb{N} \). Then the following are equivalent:

- The \( \infty \)-topos \( X \) is \( n \)-localic.
- The \( \infty \)-category \( \text{Pt}(X) \) of points of \( X \) is an \( n \)-category.
- The profinite stratified shape \( \Pi_{(\infty,1)}(X) \) is an \( n \)-truncated profinite stratified space.

**Proof.** If \( X \) is \( n \)-localic, then the \( \infty \)-category \( \text{Pt}(X) \) is an \( n \)-category, which shows that
\[
\text{mat}\Pi_{(\infty,1)}(X) = \text{Pt}(X)
\]
is an \( n \)-category (Lemma 10.3.2). Applying Lemma 10.3.6 we see that \( \Pi_{(\infty,1)}(X) \) is an \( n \)-truncated profinite stratified space.
If $\widehat{\Pi}_{(\infty,1)}(X)$ is an $n$-truncated profinite stratified space, then $\widehat{\Pi}_{(\infty,1)}(X)$ can be exhibited as an inverse system $\{\Pi_a\}_{a \in A}$ of $n$-truncated $\pi$-finite stratified spaces. Thus

$$X = \lim_{a \in A} \Pi_a$$

is an $n$-localic $\infty$-topos.

10.3.8. Combining the preceding with ordinary Stone Duality between profinite sets and Stone topological spaces, the functor $Pt: (\text{Str}_\pi^{m} \leq 1) \to \text{Cat}$ factors through a fully faithful functor $(\text{Str}_\pi^{m} \leq 1) \to (\text{Top}_{\text{bc}} \text{Stn})$ from the 2-category of 1-truncated profinite stratified spaces to the 2-category of category objects in the category of Stone topological spaces. The essential image of this functor is spanned by the layered category objects – i.e., the ones in which every endomorphism is an isomorphism.

10.4 Stratified homotopy types via décollages

To identify the functor $\widehat{\Pi}_{(\infty,1)}$ in terms of the usual homotopy type $\widehat{\Pi}_{\infty}$, we can pass to the décollage over $P$.

10.4.1 Construction. Let $P$ be a finite poset. Let us consider the functor

$$\tilde{\lambda}^\text{dec}_P: \text{Déc}_P(S^\infty_n) \to \text{Déc}_P(\text{Top}_{\text{bc}}^{bc})$$

given by composition with $\lambda_{(0)}$, so that a profinite spatial décollage $D: \text{sd}^P(P) \to S^\infty_n$ is carried to the toposic décollage $\Sigma \mapsto \tilde{D}(\Sigma)$. We have seen (Proposition 8.8.11) that this is a fully faithful functor whose essential image is $\text{Déc}_P(\text{Top}_{\text{bc}}^{\text{Stn}})$.

In the other direction, let us consider the functor

$$\Pi_{\infty}^{\text{predic}, P}: \text{Déc}_P(\text{Top}_{\text{bc}}^{\text{bc}}) \to \text{Fun}(\text{sd}^P(P), S^\infty_n)$$

given by composition with the profinite shape functor $\Pi_{\infty}$, so that a toposic décollage $D: \text{sd}^P(P) \to \text{Top}_{\text{bc}}^{\text{bc}}$ is carried to the functor $\Sigma \mapsto \Pi_{\infty} D(\Sigma)$. We can then compose this with the Segalification functor – that is, the left adjoint to the fully faithful functor $\text{Déc}_P(S^\infty_n) \to \text{Fun}(\text{sd}^P(P), S^\infty_n)$ – to obtain a functor

$$\Pi_{\infty}^{\text{dec}, P}: \text{Déc}_P(\text{Top}_{\text{bc}}^{\text{bc}}) \to \text{Déc}_P(S^\infty_n)$$

that is left adjoint to $\tilde{\lambda}^\text{dec}_P$.

The difficulty here is that the functor $\Pi_{\infty}^{\text{dec}, P}$ is very inexplicit, because it involves Segalification. To address this, we have the following.

10.4.2 Theorem. Let $P$ be a finite poset. If $X \to P$ is a spectral $P$-stratified $\infty$-topos, then the functor $\Sigma \mapsto \Pi_{\infty} D(\Sigma)$ is already a profinite spatial décollage; that is, the Segalification morphism

$$\Pi_{\infty}^{\text{predic}, P}(X) \to \Pi_{\infty}^{\text{dec}, P}(X)$$

is an equivalence in $\text{Fun}(\text{sd}^P(P), S^\infty_n)$. 

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Proof. It suffices to prove that for every string \( \Sigma = \{ p_0 \leq \cdots \leq p_n \} \subset P \), the natural morphism

\[
f_{\Sigma} : \tilde{\Pi}_{\infty}(X_{p_0} \overset{\sim}{\times} \cdots \overset{\sim}{\times} X_{p_n}) = \tilde{\Pi}_{\infty} \operatorname{Mor}_{P}(\tilde{\Sigma}, X) \to \operatorname{Map}_{P}(\Sigma, \tilde{\Pi}_{(\infty,1)}(X))
\]

in \( S_\wedge \) is an equivalence. By Whitehead’s Theorem for profinite spaces (Theorem 4.4.8 = [SAG, Theorem E.3.1.6]), it suffices to prove that the materialisation \( \operatorname{mat}(f_{\Sigma}) \) is an equivalence. Since \( X \) is spectral, we have a natural equivalence

\[
\operatorname{mat} \tilde{\Pi}_{\infty}(X_{p_0} \overset{\sim}{\times} \cdots \overset{\sim}{\times} X_{p_n}) \simeq \operatorname{Pt}(X_{p_0} \overset{\sim}{\times} \cdots \overset{\sim}{\times} X_{p_n}).
\]

Similarly, since \( \Sigma \) is constant as a pro-object and \( X \) is spectral, by Whitehead’s Theorem for profinite stratified spaces (Theorem 10.3.3) we have natural equivalences

\[
\operatorname{mat} \operatorname{Map}_{P}(\Sigma, \tilde{\Pi}_{(\infty,1)}(X)) \simeq \operatorname{Map}_{P}(\Sigma, \operatorname{mat} \tilde{\Pi}_{(\infty,1)}(X)) \simeq \operatorname{Map}_{P}(\Sigma, \operatorname{Pt}(X)).
\]

By the universal property of the iterated oriented fibre product \( X_{p_0} \overset{\sim}{\times} \cdots \overset{\sim}{\times} X_{p_n} \), we have a natural identification

\[(10.4.3) \quad \operatorname{Map}_{P}(\Sigma, \operatorname{Pt}(X)) \cong \operatorname{Pt}(X_{p_0} \overset{\sim}{\times} \cdots \overset{\sim}{\times} X_{p_n}).\]

To complete the proof, note that the materialisation \( \operatorname{mat}(f_{\Sigma}) \) is equivalent to the morphism (10.4.3). \( \square \)

10.4.4 Example. Let \( P \) be a finite poset, and let \( X \to \tilde{P} \) be a spectral \( P \)-stratified \( \infty \)-topos. It follows from Theorem 10.4.2 that, for any point \( p \in P \), the \( p \)-th stratum \( \tilde{\Pi}_{(\infty,1)}(X)_p \) is equivalent to the homotopy type \( \tilde{\Pi}_{\infty}(X_p) \).

10.4.5 Example. Let \( P \) be a finite poset, and let \( X \to \tilde{P} \) be a spectral \( P \)-stratified \( \infty \)-topos. It follows from Theorem 10.4.2 that, for any points \( p, q \in P \) with \( p < q \), the link \( \operatorname{Map}_{P}(\{ p \leq q \}, \tilde{\Pi}_{(\infty,1)}(X)) \) between the \( p \)-th and \( q \)-th strata of \( \tilde{\Pi}_{(\infty,1)}(X) \) is equivalent to the homotopy type \( \tilde{\Pi}_{\infty}(X_p \overset{\sim}{\times} X_q) \) of the link.

10.4.6 Example. Let \( P \) be a finite poset, and \( X \) a spectral \( P \)-stratified \( \infty \)-topos. For any points \( p, q \in P \) with \( p \leq q \), write

\[
i_{pq,*} : X_p \hookrightarrow X_{\{p \leq q\}} \quad \text{and} \quad j_{pq,*} : X_q \hookrightarrow X_{\{p \leq q\}}
\]

for the closed and open immersions of strata, respectively. Then the base change Theorem (Theorem 7.1.7) ensures that the décollage

\( \tilde{\Pi}^{dec,P}(X) : \text{sd}^P(P) \to S_{\wedge} \)

carries any string \( \{ p_0 \leq \cdots \leq p_n \} \subset P \) to the profinite space \( S_{\wedge} \to S \) given by the composite

\[
\Gamma_{X_{p_0}} \circ i_{p_0} \circ i_{p_0} \circ i_{p_1} \circ j_{p_1} \circ j_{p_1} \circ i_{p_2} \circ j_{p_2} \circ \cdots \circ i_{p_{n-1}} \circ j_{p_{n-1}} \circ i_{p_n} \circ \Gamma_{X_{p_n}}.
\]
10.5 Van Kampen Theorem

If $P$ is a poset and $\eta: P \to \{0\}$ then the ‘invert everything’ functor $\Pi \mapsto E(\Pi)$ from $P$-stratified spaces to spaces, regarded as a functor from spatial décollages to spaces, is given by the formation of the colimit. That is, if $\Pi \to P$ is a $P$-stratified space, then one has an equivalence

$$E(\Pi) = \text{colim}_{\Sigma \in \text{sd}^d(P)} N_P(\Pi)(\Sigma).$$

The ‘invert everything’ functor extends to a functor $\Pi \mapsto E(\Pi)$ from profinite $P$-stratified spaces to profinite spaces, and this formula is precisely the same in that context. The compatibility (10.1.5) therefore provides a colimit description of the homotopy type of a stratified $\infty$-topos:

10.5.1 Proposition (van Kampen Theorem). Let $P$ be a finite poset, and let $X \to \tilde{P}$ be a spectral $P$-stratified $\infty$-topos. Then the profinite shape of $X$ is given by the colimit

$$\check{\Pi}_\infty(X) = \text{colim}_{\Sigma \in \text{sd}^d(P)} \check{\Pi}_\infty(N_P(X)(\Sigma))$$

in profinite spaces.

10.5.2 Example. If $X$ is a spectral $\infty$-topos exhibited as a recollement $Z \cup^\phi U$ of Stone $\infty$-topoi $Z$ and $U$, then one has the formula

$$\check{\Pi}_\infty(X) = \check{\Pi}_\infty(Z \cup^\phi U)$$

in profinite spaces.

10.5.3 Example. Let $n \in \mathbb{N}$, and let $X \to \underline{n}$ be a spectral $[n]$-stratified $\infty$-topos. Then $\check{\Pi}_\infty(X)$ can be exhibited as the colimit of a punctured $(n+1)$-cube $\text{sd}^d([n]) \to S^\infty$ given by

$$\{p_0 < \cdots < p_k\} \mapsto \check{\Pi}_\infty(X_{p_0} X_{p_1} \cdots X_{p_k}).$$
Part IV

Stratified étale homotopy theory

In this part we use the profinite stratified shape of Chapter 10 to give a refinement of the étale homotopy theory of Artin–Mazur–Friedlander. We first recall how to define the étale homotopy type from the ∞-categorical perspective, as well as the main theorems in étale homotopy theory (Chapter 11). We then study the profinite stratified shape of the étale ∞-topos of coherent schemes in detail (Chapter 12). In particular, we provide a concrete description in terms the profinite Galois categories introduced in the Introduction (preceding Theorem A). We conclude with Chapter 14 where we discuss Grothendieck’s anabelian conjectures and use a theorem of Voevodsky to prove a strong reconstruction theorem for schemes in characteristic 0 in terms of profinite Galois categories (Theorem A=Theorem 14.4.7).
11 Aide-mémoire on étale homotopy types

In this chapter we recall how to situate the étale homotopy type of Artin–Mazur–Friedlander in the \( \infty \)-categorical setting. We also provide some example computations of étale homotopy types.

Section 11.1 recalls the definition of the étale homotopy type via shape theory. In Section 11.2 we give some sample computations and uses of the étale homotopy type. Section 11.3 recalls the monodromy equivalence for lisse étale sheaves in terms of the profinite étale homotopy type. Section 11.4 explains how étale homotopy theory works for simplicial schemes. Section 11.5 recalls Artin and Mazur’s formulation of the Riemann Existence Theorem in terms of the étale homotopy type, and §11.6 gives a quick proof of the étale van Kampen Theorem.

11.1 Artin & Mazur’s étale homotopy types of schemes

From an \( \infty \)-categorical perspective, there are \emph{a priori} four étale shapes to contemplate:

11.1.1 Definition. Let \( X \) be a scheme. We write:

\[
\Pi^{\text{ét}}_{\infty}(X) = \Pi_{\infty}(X_{\text{ét}})
\]

for the shape of the 1-localic étale \( \infty \)-topos of \( X \),

\[
\Pi^{\text{ét, hyp}}_{\infty}(X) = \Pi_{\infty}(X'_{\text{hyp}})
\]

for the shape of the hypercomplete étale \( \infty \)-topos of \( X \),

\[
\tilde{\Pi}^{\text{ét}}_{\infty}(X) = \tilde{\Pi}_{\infty}(X_{\text{ét}})
\]

for the profinite shape of the 1-localic \( \infty \)-topos of \( X \), and

\[
\tilde{\Pi}^{\text{ét, hyp}}_{\infty}(X) = \tilde{\Pi}_{\infty}(X'_{\text{hyp}})
\]

for the profinite shape of the hypercomplete étale \( \infty \)-topos of \( X \).

11.1.2. As a special case of Example 4.2.8, we see that the natural morphism

\[
\Pi^{\text{ét, hyp}}_{\infty}(X) \to \Pi^{\text{ét}}_{\infty}(X)
\]

becomes an equivalence after protruncation. In particular, the natural morphism

\[
\Pi^{\text{ét, hyp}}_{\infty}(X) \to \tilde{\Pi}^{\text{ét}}_{\infty}(X)
\]

is an equivalence. We simply write

\[
\Pi^{\text{ét}}_{\infty}(X) = \Pi^{\text{ét, hyp}}_{\infty}(X) = \tilde{\Pi}^{\text{ét}}_{\infty}(X)
\]

for the protruncated shape of the étale \( \infty \)-topos and

For a locally noetherian scheme \( X \), Artin and Mazur [7, §9] constructed a pro-object in the homotopy category of spaces called the \emph{étale homotopy type} of \( X \). Friedlander [34, §4] later refined this construction, producing a pro-object in simplicial sets called the \emph{étale topological type} of \( X \). The image of the étale topological type in \( \text{Pro}(h_1 S) \) agrees with the étale homotopy type of Artin–Mazur [34, Proposition 4.5]. Hoyois [56, §5] has shown that Friedlander’s étale topological type corepresents the shape of the hypercomplete étale \( \infty \)-topos of \( X \):
11.1.3 Theorem ([56, Corollary 5.6]). Let \( X \) be a locally noetherian scheme. Then the étale topological type of \( X \) corepresents the shape \( \Pi_{\text{co}}^{\text{ét}}(X) \) of the hypercomplete étale \( \infty \)-topos of \( X \).

11.1.4. We refer to the shape \( \Pi_{\text{co}}^{\text{ét}}(X) \) of the étale \( \infty \)-topos as the étale shape. Call the protruncated shape \( \Pi_{\text{co}}^{\text{ét}}<\infty(X) \) the protruncated étale shape, and call the profinite shape \( \hat{\Pi}_{\text{co}}^{\text{ét}}(X) \) the profinite étale shape.

In many examples, the protruncated étale shape is already profinite:

11.1.5 Theorem ([DAG XIII, Theorem 3.6.5; 7, Theorem 11.1; 34, Theorem 7.3]). Let \( X \) be a connected noetherian scheme that is geometrically unibranch. Then the protruncated étale shape of \( X \) is profinite; that is, the natural morphism

\[
\Pi_{\text{co}}^{\text{ét}}(X) \to \hat{\Pi}_{\text{co}}^{\text{ét}}(X)
\]

is an equivalence.

11.1.6 Question. Let \( X \) be a connected noetherian scheme that is geometrically unibranch. Even in simple cases, we do not at this point have a very good understanding of the kind of information that is contained in the étale shape \( \Pi_{\text{co}}^{\text{ét}}(X) \) but not in the other invariants. In this paper, we are content to focus our attention on the profinite homotopy types and their stratified variants.

11.1.7. Let \( X \) be a scheme and \( x \to X \) a geometric point of \( X \). Then \( x \) induces a point of the prospace \( \Pi_{\text{co}}^{\text{ét}}(X) \). The \( n \)th extended étale homotopy progroup of \( X \) is the progroup

\[
\pi_{\text{ét}}^{\text{et}}(X, x) = \pi_n(\Pi_{\text{co}}^{\text{ét}}(X), x).
\]

In particular, the progroup \( \pi^{\text{et}}_1(X, x) \) is the groupe fondamentale élargi of [SGA 3\text{ii}, Exposé X, §6]; see [7, Corollary 10.7]. The usual étale fundamental group of [SGA 1, Exposé V, §7] is the profinite completion of \( \pi^{\text{et}}_1(X, x) \). Moreover, the usual étale fundamental group of \( X \) is isomorphic to the profinite fundamental group \( \pi_1(\hat{\Pi}_{\text{co}}^{\text{ét}}(X), x) \) [7, Corollary 3.9]. We denote the usual (profinitley complete) étale fundamental group by \( \hat{\pi}^{\text{et}}_1(X, x) \).

11.2 Examples

In this section we provide some example computations of étale shapes.

11.2.1 Example. Let \( k \) be a field, and \( k^{\text{sep}} \supset k \) a separable closure of \( k \). Write \( G_k \) for the absolute Galois group of \( k \) with respect to the separable closure \( k^{\text{sep}} \supset k \). The choice of separable closure of \( k \) provides an identification

\[
\hat{\Pi}_{\text{co}}^{\text{ét}}(\text{Spec } k) = BG_k
\]

11.2.2 Example. Since \( \text{Spec } Z \) has no unramified étale covers, the étale fundamental group of the \( \text{Spec } Z \) is trivial. Moreover, for all integers \( i \geq 1 \) and \( n \geq 2 \), the étale cohomology group \( H^i_{\text{ét}}(\text{Spec } Z; Z/n) \) is trivial (see [80; 102]). The Universal Coefficient Theorem and Hurewicz Theorem imply that the profinite étale shape \( \hat{\Pi}_{\text{co}}^{\text{ét}}(\text{Spec } Z) \) of \( \text{Spec } Z \) is trivial (cf. [7, §4]). Since \( Z \) is a noetherian domain, Theorem 11.1.5 applies, hence the protruncated étale shape \( \Pi_{\text{co}}^{\text{ét}}(\text{Spec } Z) \) of \( \text{Spec } Z \) is trivial.
11.2.3 Example. Let \( k \) be an algebraically closed field of characteristic 0 and \[ C = \text{Spec}(k[x, y]/(y^2 - x^3 - x^2)) \]
the nodal cubic. Then there is a noncanonical identification \( \Pi_{\text{et}}^{\leq}(C) = B\mathbb{Z} \).

Since the group \( \mathbb{Z} \) is good in the sense of Serre [113, p. 16], the profinite étale shape is given by \( \Pi_{\text{et}}^{\leq}(C) = B\mathbb{Z} \) [SAG, Warning E.4.3.4; 95, Theorem 3.14].

11.2.4 Example. Let \( C \) be a smooth connected curve over a field. If \( C \) is affine or of positive genus, then the protruncated étale homotopy type \( \Pi_{\text{et}}^{\leq}(X) \) is 1-truncated [104, Proposition 15; 105, Lemma 2.7(a)]. Thus we have a noncanonical identification \( \Pi_{\text{et}}^{\leq}(C) \simeq B\hat{\pi}_{\text{et}}^1(C) \).

11.2.5 Example (see Theorem 11.5.3). Let \( S^2 \in S \) denote the 2-sphere. There an equivalence \( \Pi_{\text{et}}^{\leq}(P^1_C) = (S^2)^{\wedge}_{\pi} \).

11.2.6 Example ([53, Theorem 1]). Let \( k \) be an algebraically closed field of positive characteristic and let \( X \) be a smooth \( k \)-variety. Then \( \Pi_{\text{et}}^{\leq}(X) = \ast \) if and only if \( X \) is isomorphic to \( \text{Spec} k \).

11.2.7 Example (Example 7.3.4). Let \( k \) be a separably closed field, and let \( X \) and \( Y \) be coherent \( k \)-schemes. If \( Y \) is proper, then the natural morphism of profinite spaces
\[ \Pi_{\text{et}}^{\leq}(X \times_{\text{Spec} k} Y) \rightarrow \Pi_{\text{et}}^{\leq}(X) \times \Pi_{\text{et}}^{\leq}(Y) \]
is an equivalence.

The following two examples are from Piotr Achinger’s remarkable work on \( K(\pi, 1) \)-schemes in positive characteristic [2].

11.2.8 Example. Let \( p \) be a prime number. Achinger showed that if \( X \) is a connected affine \( F_p \)-scheme, then the profinite étale homotopy type \( \Pi_{\text{et}}^{\leq}(X) \) is 1-truncated [2, Theorem 1.1]. This is in stark contrast with the case of schemes in characteristic zero.

11.2.9 Example. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). By Michel Raynaud’s proof of Abhyankar’s Conjecture [97], a finite group \( G \) arises as a quotient of the profinite group \( \pi_{\text{et}}^1(A_k^1) \) if and only if \( G \) is a quasi-\( p \)-group (i.e., \( G \) has no nontrivial quotient of order prime to \( p \)). More generally, it follows from Raynaud’s work that for \( n \geq 1 \), a finite group \( G \) arises as a quotient of \( \pi_{\text{et}}^1(A_k^n) \) if and only if \( G \) is a quasi-\( p \)-group. Even though the étale fundamental groups of \( A_k^1 \) and \( A_k^n \) abstractly have the same finite quotients, Achinger showed that for positive integers \( m \neq n \), the étale fundamental groups \( \pi_{\text{et}}^1(A_k^m) \) and \( \pi_{\text{et}}^1(A_k^n) \) are not isomorphic as profinite groups [2, Proposition 7.6].

Example 11.2.9 demonstrates how ‘large’ étale fundamental groups of \( F_p \)-schemes tend to be. One might interpret Examples 11.2.8 and 11.2.9 by saying that the étale fundamental group of a connected affine \( F_p \)-scheme is so large that it contains all of the homotopical information of the scheme.
11.3 Monodromy

Specialising Proposition 4.4.18 to the case of the étale co-topos of a scheme shows that lisse étale sheaves are the same as representations of the profinite étale shape:

11.3.1 Proposition. Let $X$ be a scheme. The unit $X^\text{ét} \to X^\text{Stn \text{ét}}$ restricts to an equivalence

$$\text{Fun}(\hat{\Pi}^\text{ét}_\infty(X), S_\pi) \cong X^\text{lisse \text{ét}}.$$ 

This generalises the classical fact that the profinite étale fundamental groupoid

$$\hat{\Pi}^\text{ét}_1(X) \cong \tau_{\leq 1} \hat{\Pi}^\text{ét}_\infty(X)$$

classifies lisse étale sheaves of sets (see Example 4.4.19).

11.4 Friedlander’s étale homotopy of simplicial schemes

Let $X_\ast$ be a simplicial scheme. Then there is also an étale topological type of $X_\ast$. The étale topological type for simplicial schemes was originally constructed by Eric Friedlander [34, §4] and was later refined by David Cox [27], Daniel Isaksen [63], Ilan Barnea and Tomer Schlank [11], David Carchedi [18], and Chang-Yeon Chough [20; 21]. Thanks to work of Cox [27, Theorem III.8], Isaksen [63, §3, Theorem 11], and Chough [21, Proposition 3.2.13], the étale topological type of $X_\ast$ can be defined as the colimit of the simplicial prospace $[m] \mapsto \Pi^\text{ét, hyp}_\infty(X_m)$. Again, from an oo-categorical perspective, there are variations on this notion:

11.4.1 Definition. Let $X_\ast$ be a simplicial scheme. We define:

- The étale shape of $X_\ast$ is the geometric realization

$$\Pi^\text{ét}_\infty(X_\ast) := \text{colim}_{[m] \in \Delta^\text{op}} \Pi^\text{ét}_\infty(X_m)$$

of the simplicial prospace $[m] \mapsto \Pi^\text{ét}_\infty(X_m)$.

- Friedlander’s étale topological type of $X_\ast$ is the geometric realization

$$\Pi^\text{ét, hyp}_\infty(X_\ast) := \text{colim}_{[m] \in \Delta^\text{op}} \Pi^\text{ét, hyp}_\infty(X_m)$$

of the simplicial prospace $[m] \mapsto \Pi^\text{ét, hyp}_\infty(X_m)$.

11.4.2. Since truncation is a left adjoint, from (11.1.2) we deduce that the natural morphism of prospaces

$$\Pi^\text{ét, hyp}_\infty(X_\ast) \to \Pi^\text{ét}_\infty(X_\ast)$$

becomes an equivalence after truncation. Hence after profinite completion as well.

11.4.3. We can extend the functor that assigns a scheme its étale co-topos to simplicial schemes by left Kan extension; then the étale co-topos of a simplicial scheme $X_\ast$ is given by the geometric realisation

$$X_{\ast, \text{ét}} := \text{colim}_{[m] \in \Delta^\text{op}} X_{m, \text{ét}}$$
in \( \text{Top}_{\infty} \). Since the shape is a left adjoint, we see that the shape of the \( \infty \)-topos \( X_{*, \text{ét}} \) coincides with the étale shape \( \Pi^{\infty}_{\text{ét}}(X_*) \). Since hypercomplete \( \infty \)-topoi are closed under colimits in \( \text{Top}_{\infty} \), Friedlander’s étale topological type coincides with the shape of the hypercomplete \( \infty \)-topos given by the geometric realisation of the simplicial hypercomplete \( \infty \)-topos \( [m] \mapsto X_{m, \text{ét}}^{\text{hyp}} \).

### 11.5 Riemann Existence Theorem

In this section we recall the Artin–Mazur–Friedlander Riemann Existence Theorem (Theorem 11.5.3); this states that the profinite étale shape of a scheme of finite type over the complex numbers agrees with the profinite completion of the homotopy type of its underlying analytic space.

#### 11.5.1 Notation.

Write \( \mathbb{C} \) for the field of complex numbers and \( \text{Sch}^\text{ft}_{/\mathbb{C}} \) for category of schemes of finite type over \( \mathbb{C} \) and finite type morphisms between them. We write \((-)^{\text{an}} : \text{Sch}^\text{ft}_{/\mathbb{C}} \to \text{TSpc} \) for the analytification functor: this carries a scheme \( X \) of finite type over \( \mathbb{C} \) to \( X(\mathbb{C}) \) equipped with the complex analytic topology.

#### 11.5.2 Recollection.

Let \( X \) be a scheme finite type over \( \mathbb{C} \). In [SGA 4\text{iii}, Exposé XI, 4.0], Artin defines a natural geometric morphism of \( 1 \)-topoi

\[
\varepsilon_{X,*} : (\overline{X^{\text{an}}})_{\leq 0} \to X_{\text{ét}, \leq 0}
\]

from the \( 1 \)-topos of sheaves of sets on \( X^{\text{an}} \) to the \( 1 \)-topos of sheaves of sets on the étale site of \( X \). The geometric morphism \( \varepsilon_{X,*} \) extends to a natural geometric morphism of \( 1 \)-localic \( \infty \)-topoi

\[
\varepsilon_{X,*} : \overline{X^{\text{an}}} \to X_{\text{ét}}.
\]

The naturality can be encoded as a functor \( \text{Sch}^\text{ft}_{/\mathbb{C}} \to \text{Fun}([1], \text{Top}_{\infty}) \): if \( f : X \to Y \) is a finite type morphism of finite type \( \mathbb{C} \)-schemes, then there is a natural equivalence

\[
f^* \varepsilon_{X,*} \simeq \varepsilon_{Y,*} f^* \cdot f^! .
\]

In light of Theorem 11.1.3, the Riemann Existence Theorem proved by Artin–Mazur [7, Theorem 12.9] and later Friedlander [34, Theorem 8.6] asserts that \( \overline{X^{\text{an}}} \) and \( X_{\text{ét}} \) have the same profinite shape, when regarded as pro-objects of the homotopy category of spaces. One may refine the Artin–Mazur–Friedlander equivalence to an equivalence in the \( \infty \)-category of profinite spaces (cf. [18, Proposition 4.12; 21, §4]). Indeed, the Théorème de Comparaison [SGA 4\text{iii}, Exposé XI, Théorèmes 4.3 & 4.4] can be employed to provide an equivalence between the \( \infty \)-category of lisse étale sheaves of spaces on \( X \) and that of lisse sheaves of spaces on \( X^{\text{an}} \), whence we obtain the following.

#### 11.5.3 Theorem (Riemann Existence).

Let \( X \) be a scheme finite type over \( \mathbb{C} \). Then \( \varepsilon_{X,*}^\wedge \) restricts to an equivalence \( X^{\text{lisse}}_{\text{ét}} \simeq (\overline{X^{\text{an}}}^{\text{lisse}})_{\infty} \) of \( \infty \)-categories of lisse sheaves. Equivalently, \( \varepsilon_{X,*}^\wedge \) induces an equivalence of profinite spaces

\[
(X^{\text{an}})^{\wedge} = \Pi_{\infty}(\overline{X^{\text{an}}}) \simeq \Pi_{\infty}(X_{\text{ét}}) .
\]
11.6  Van Kampen Theorem for étale shapes

In this section we prove a van Kampen Theorem from étale shapes (Corollary 11.6.6). We deduce this from the fact that the functor that sends a scheme to its étale co-topos satisfies Nisnevich excision (Proposition 11.6.3).

11.6.1 Definition. We call a pullback square of schemes

\[
\begin{array}{ccc}
U' & \longrightarrow & X' \\
\downarrow j & & \downarrow p \\
U & \longleftarrow & X
\end{array}
\]

\hspace{1cm}

(11.6.2)

an elementary Nisnevich square if \( j \) is an open immersion, \( p \) is étale, and \( p \) induces an isomorphism \( p^{-1}(X \setminus U) \cong X \setminus U \). Here the closed complement \( X \setminus U \) of \( U \) is given the reduced structure.

David Rydh’s general descent theorem \([101, \text{Theorem A}]\) implies that the formation of the étale 1-topos sends elementary Nisnevich squares to pushout squares of 1-topoi. The same is true for étale \( \infty \)-topoi, though this is not implied by Rydh’s result because 1-localic co-topoi are not closed under colimits in \( \text{Top}_{\infty} \). As in Rydh’s theorem, this can be deduced from étale descent (combined with Morel and Voevodsky’s theorem characterizing Nisnevich sheaves as presheaves satisfying Nisnevich excision \([\text{SAG}, \text{Theorem } 3.7.5.1; \text{86}, \S 3, \text{Proposition 1.16}]\)), but the following proposition provides an elementary proof.

11.6.3 Proposition. Given an elementary Nisnevich square of schemes (11.6.2), the induced square of étale \( \infty \)-topoi

\[
\begin{array}{ccc}
U'_{\text{ét}} & \longrightarrow & X'_{\text{ét}} \\
\downarrow j_{\text{ét}} & & \downarrow p_{\text{ét}} \\
U_{\text{ét}} & \longleftarrow & X_{\text{ét}}
\end{array}
\]

\hspace{1cm}

(11.6.4)

is a pushout square and pullback square in \( \text{Top}_{\infty} \). The same is true after passing to hypercomplete étale co-topoi.

Proof. The fact that the (11.6.4) is a pullback is immediate from the fact that \( j \) is an open immersion; the same is true for hypercomplete étale co-topoi since hypercompletion is a right adjoint.

Let \( \mathcal{Y} : X_{\text{ét}} \to X_{\text{sh}} \) denote the Yoneda embedding of étale site of \( X \) to the étale co-topos. Note that if \( Y \in X_{\text{ét}} \) is a scheme étale over \( X \), then the natural geometric morphism \( Y_{\text{ét}} \to (X_{\text{ét}})/\mathcal{Y}(Y) \) is equivalence. Since colimits in an co-topos are van Kampen\footnote{A colimit in an co-category \( C \) with pullbacks is \textit{van Kampen} if the functor \( C^{op} \to \text{Cat}_{\infty} \) given by \( c \mapsto C_{c}^{op} \) transforms it into a limit in \( \text{Cat}_{\infty} \). A presentable co-category \( C \) is an co-topos if and only if colimits in \( C \) are van Kampen; see [HTT, Proposition 5.5.3.13, Theorem 6.1.3.9(3), & Proposition 6.3.2.3; 58].}
and $\mathcal{A}(X)$ is the terminal object of $X_{\text{et}}$, it thus suffices to prove that the pullback square

$\begin{array}{c}
\mathcal{A}(U') \\
\downarrow \\
\mathcal{A}(U)
\end{array} \begin{array}{c}
\mathcal{A}(X') \\
\downarrow \\
\mathcal{A}(X)
\end{array}$

(11.6.5)

in $X_{\mathcal{A}}$ is also a pushout. In this case, the fact that truncated objects are hypercomplete implies that the same is true in $X_{\text{hyp}}$. The fact that (11.6.5) is a pullback square is immediate from [SAG, Proposition 2.5.2.1(3)], the hypotheses of which are valid because (11.6.2) is an elementary Nisnevich square.

Proposition 11.6.3 immediately implies the classical 'excision' theorem in étale cohomology [81, Chapter III, Proposition 1.27]. Since the shape is a left adjoint, the following van Kampen Theorem for the étale shape is immediate. This generalizes a theorem of Isaksen [63, §2, Theorem 8].

**11.6.6 Corollary.** Given an elementary Nisnevich square of schemes (11.6.2), the induced squares

$\begin{array}{c}
\Pi_{S_1}(U') \\
\downarrow \\
\Pi_{S_1}(U)
\end{array} \begin{array}{c}
\Pi_{S_1}(X') \\
\downarrow \\
\Pi_{S_1}(X)
\end{array}$

and

$\begin{array}{c}
\Pi_{S_1}^{\text{hyp}}(U') \\
\downarrow \\
\Pi_{S_1}^{\text{hyp}}(U)
\end{array} \begin{array}{c}
\Pi_{S_1}^{\text{hyp}}(X') \\
\downarrow \\
\Pi_{S_1}^{\text{hyp}}(X)
\end{array}$

are pushout squares in $\text{Pro}(S)$.

**11.6.7.** Since protruncation and profinite completion are left adjoints, Corollary 11.6.6 show that the protruncated and profinite étale shapes send elementary Nisnevich squares to pushout squares in $\text{Pro}(S_{\text{co}})$ and $S_{\mathcal{A}}$, respectively. In particular, Proposition 11.6.3 (and [SAG, Proposition 2.5.2.1]) immediately imply Misamore's 'étale van Kampen Theorem' [82, Corollaries 6.5 & 6.6] in the case of schemes. See also [17; 115, §5; 121].
12 Galois categories

In this chapter we study the profinite stratified shape of étale ∞-topoi of coherent schemes, which we call the stratified étale homotopy type. The stratified étale homotopy type of a coherent scheme $\mathcal{X}$ turns out to have a very explicit description: it is the profinite category $\text{Gal}(\mathcal{X})$ from the Introduction.

Section 12.1 defines the stratified étale homotopy type and shows that it coincides with the ‘Galois category’ from the Introduction. Section 12.2 gives some sample computations of the stratified étale homotopy type. Section 12.3 demonstrates how some properties of schemes can be detected on the level of their Galois categories. Section 12.4 shows that the Galois category of the strict localization of a scheme at a point is an undercategory, and, dually, the Galois category of the strict normalization of a scheme at a point is an overcategory. Section 12.5 explains how to recover (up to protruncation) the étale homotopy type of a coherent scheme $\mathcal{X}$ from $\text{Gal}(\mathcal{X})$ by inverting all morphisms. Section 12.6 provides a stratified refinement of the Riemann Existence Theorem (Theorem 11.5.3) in the setting of finite type schemes over the complex numbers. In §12.7 we finish the chapter with a van Kampen theorem for Galois categories.

12.1 Galois categories of schemes

12.1.1 Notation. Recall that for a coherent scheme $\mathcal{X}$, we let $\text{FC}(\mathcal{X})$ denote the 1-category of nondegenerate, finite, constructible stratifications of the spectral topological space $\mathcal{X}^{\text{zar}}$. We abuse notation and write merely $P$ for an object $\mathcal{X}^{\text{zar}} \rightarrow P$ of this category, leaving the structure morphism implicit. The 1-category $\text{FC}(\mathcal{X})$ is, up to equivalence, a poset in which $P \leq Q$ if and only if $P$ refines $Q$; that is, $P \leq Q$ if and only if $\mathcal{X}^{\text{zar}} \rightarrow Q$ factors through $\mathcal{X}^{\text{zar}} \rightarrow P$. The spectral topological space $\mathcal{X}^{\text{zar}}$ corresponds under Hochster Duality to the profinite poset $\{P\}_{P \in \text{FC}(\mathcal{X})}$.

12.1.2 Notation. We write $\text{Sch}$ for the 1-category of coherent schemes (0.6.2).

12.1.3 Definition. Let $\mathcal{X}$ be a coherent scheme. Then we write

$$\text{Gal}(\mathcal{X}) := \Pi_{(\infty,1)}^{\text{ét}}(\mathcal{X}_{\text{ét}}).$$

We call this the Galois category of $\mathcal{X}$. This is a functor $\text{Gal}: \text{Sch} \rightarrow \text{Str}_\pi$.

More generally, if $P$ is a nondegenerate, finite, constructible stratification, then we may therefore define

$$\text{Gal}(\mathcal{X}/P) := \Pi_{(\infty,1)}(\mathcal{X}_{\text{ét}}/P).$$

12.1.4. We obtain a diagram

$$\text{Gal}(\mathcal{X}/-) : \text{FC}(\mathcal{X}) \rightarrow \text{Str}_\pi$$

of localisations.

12.1.5 Construction. Let $\mathcal{X}$ be a coherent scheme. The $\mathcal{X}^{\text{zar}}$-stratified ∞-topos $\mathcal{X}_{\text{ét}}$ is spectral. Our ∞-Categorical Hochster Duality Theorem (Theorem 9.3.1) implies that $\mathcal{X}_{\text{ét}} = \text{Gal}(\mathcal{X})$, and thus

$$\mathcal{X}^{\text{const}}_{\text{ét}} = \text{Fun}(\text{Gal}(\mathcal{X}), S_\pi).$$
Here, at last, is the Exodromy Equivalence. If $X$ and $Y$ are coherent schemes, then the natural map
\[ \text{Map}_{\text{Top}_\infty}(X_{\text{ét}}, Y_{\text{ét}}) \to \text{Map}_{\text{Str}_\infty}(\text{Gal}(X), \text{Gal}(Y)) \]
is an equivalence.

Since the $\infty$-topos $X_{\text{ét}}$ is 1-localic, the profinite stratified space $\text{Gal}(X)$ is 1-truncated (Proposition 10.3.7). By Lemma 10.3.2 we have a natural equivalence of categories
\[ \text{mat}(\text{Gal}(X)) = \text{Pt}(X_{\text{ét}}). \]

The Grothendieck School [SGA 4\text{II}, Exposé VIII, Théorème 7.9] provides the following description of the category $\text{Pt}(X_{\text{ét}})$ of points of the étale $\infty$-topos of $X$: an object is a geometric point $x \to X$, and given geometric points $x \to X$ and $y \to X$, the set $\text{Map}_{\text{Pt}(X_{\text{ét}})}(x, y)$ is identified with the set of lifts
\[ X_{(x)} \quad \longleftarrow \quad y \]
of the geometric point $y$ to the strict localization $X_{(x)}$ of $X$ at $x$. In other words, the 1-category $\text{mat}(\text{Gal}(X))$ agrees with the underlying 1-category denoted $\text{Gal}(X)$ in the Introduction preceding Theorem A.

Regarding $\text{Gal}(X)$ can be regarded as a category object in the category of Stone topological spaces via (10.3.8), we see that topology on $\text{Gal}(X)$ is precisely the one described in the introduction.

Theorem 10.1.8 implies the following stable variant of the Exodromy Theorem for schemes.

**12.1.6 Corollary.** Let $X$ be a coherent scheme and $R$ a finite ring. Then there is a natural equivalence
\[ \text{D}_{\text{cons}}(X_{\text{ét}}; R) \cong \text{Fun}(\text{Gal}(X), \text{Perf}(R)). \]

**12.2 Examples**

We now provide some example computations of profinite Galois categories.

**12.2.1 Example.** Let $X$ be a coherent scheme, and consider $X$ with its trivial $\{0\}$-stratification. As a special case of (10.1.5), $\text{Gal}(X/\{0\})$ recovers the usual profinite étale shape of $X$: there is a canonical equivalence
\[ \text{Gal}(X/\{0\}) \cong \tilde{\Pi}^\text{et}_\infty(X). \]

**12.2.2 Example.** Let $S = \text{Spec } A$ be the spectrum of a discrete valuation ring $A$, with closed point $s$ and generic point $\eta$. Then $S^{\text{zar}} \equiv \{1\}$, so $S_{\text{dr}}$ is a spectral $\infty$-topos that is naturally $[1]$-stratified, with closed stratum $s_{\text{dr}}$ and open stratum $\eta_{\text{dr}}$. 171
Write \( S^h \) and \( S^{sh} \) for the spectra of the henselisation \( A^h \) and the strict henselisation \( A^{sh} \) of \( A \), and write \( \eta^h \) and \( \eta^{sh} \) for the spectra of the fraction field \( K^h \) of \( A^h \) and the fraction field \( K^{sh} \) of \( A^{sh} \).

In this case, please observe that the evanescent \( \infty \)-topos \( S^\overset{\text{ét}}{\times} S^\overset{\text{ét}}{\text{ét}} \) can be identified with the étale \( \infty \)-topos \( S^\overset{\text{h}}{\text{ét}} \) (Example 6.7.4), and the oriented fibre product \( S^\overset{\text{ét}}{\times} S^\overset{\text{ét}}{\eta} \) can be identified with the étale \( \infty \)-topos \( \eta^h \).

Now we have the following (noncanonical) identifications of profinite spaces:

\[
\hat{\Pi}^\infty(\eta^h) \cong \text{BG}_K, \quad \hat{\Pi}^\infty(\eta^{sh}) \cong \text{BD}_A, \quad \hat{\Pi}^\infty(S^h) \cong \text{BG}_k,
\]

where \( G_K \) and \( G_k \) are the absolute Galois groups of \( K \) and \( k \), the subgroup \( D_A \subseteq G_K \) is the decomposition group, and \( I_A \subseteq D_A \) is the inertia group.

We thus identify, noncanonically, the corresponding profinite décollage \( N^1[\text{Gal}(S)] \) over \([1]\) as the functor \( \text{sd}^\text{op}(S^h) \rightarrow S^h_k \) given by the diagram

\[
\text{BG}_k \leftarrow \text{BD}_A \rightarrow \text{BG}_K.
\]

12.2.3 Example (Knots and primes). If \( A \) is a number ring with fraction field \( K \), then \( \text{Gal}(\text{Spec } A) \) is a profinite category with (isomorphism classes of) objects the prime ideals of \( A \). For each nonzero prime ideal \( \mathfrak{p} \in \text{Spec } A \), the automorphisms of \( \mathfrak{p} \) can be identified with the absolute Galois group \( G_{k(\mathfrak{p})} \) of the finite field \( k(\mathfrak{p}) \). Thus the profinite étale shape of \( \text{Spec } A \) is stratified by the various closed strata, each of which is an embedded circle – i.e., a knot \( G_{k(\mathfrak{p})} \). The open complement of each \( G_{k(\mathfrak{p})} \) is a \( G_{p(\mathfrak{p})} \), where

\[
G_p := \hat{\pi}^\text{ét}_1(\text{Spec } (A) \sim \mathfrak{p})
\]

is the automorphism group of the maximal Galois extension of \( K \) that is ramified at most only at \( \mathfrak{p} \) and the infinite primes. Enveloping each knot is a tubular neighbourhood, given by \( \text{Gal}(\text{Spec } A^{sh}_\mathfrak{p}) \), so that the deleted tubular neighbourhood of \( G_{k(\mathfrak{p})} \) is a \( G_{k(\mathfrak{p})} \).

12.3 Sieves & cosieves of Galois categories

One can read off various facts about schemes from their Galois categories. In this section and the next, we begin to propose a dictionary between schemes and their profinite Galois categories.\(^{29}\) We continue this endeavour in Chapter 14, as the dictionary is strongest between profinite Galois categories and perfectly reduced schemes (Definition 14.2.2).

The following is immediate.

12.3.1 Proposition. A monomorphism \( U \hookrightarrow X \) of coherent schemes is an open immersion if and only if the induced functor \( \text{Gal}(U) \to \text{Gal}(X) \) is equivalent to the inclusion of a cosieve.

Dually, a monomorphism \( Z \hookrightarrow X \) of coherent schemes is a closed immersion if and only if \( \text{Gal}(Z) \to \text{Gal}(X) \) is equivalent to the inclusion of a sieve.

An interval in an \( \infty \)-category \( C \) is a full subcategory \( D \subseteq C \) such that a morphism \( d \to d' \) of \( D \) factors through an object \( c \) of \( C \) only if \( c \) lies in \( D \).

\(^{29}\)This dictionary first appeared in a preprint of the first-named author [12].
12.3.2 Corollary. A monomorphism $W \hookrightarrow X$ of coherent schemes is a locally closed immersion if and only if the induced functor $\text{Gal}(W) \to \text{Gal}(X)$ is equivalent to the inclusion of an interval.

12.3.3 Corollary. A coherent scheme $X$ is local if and only if $\text{Gal}(X)$ contains a weakly initial object – i.e., an object from which every object receives a morphism. Dually, a coherent scheme $X$ is irreducible if and only if $\text{Gal}(X)$ contains a weakly terminal object – i.e., an object to which every object sends a morphism.

12.3.4. For any coherent scheme $X$ and any point $x_0 \in X^{\text{zar}}$, the Galois category of the localisation is the fibre product

$$\text{Gal}(X_{(x_0)}) = \text{Gal}(X) \times_{X^{\text{zar}}} X^{\text{zar}}_{x_0}.$$ 

Dually, for any point $y_0 \in X^{\text{zar}}$, the Galois category of the closure $X^{(y_0)}$ of $y_0$ (with the reduced subscheme structure, say) is the fibre product

$$\text{Gal}(X^{(y_0)}) = \text{Gal}(X) \times_{X^{\text{zar}}} X^{\text{zar}}_{y_0}.$$ 

12.4 Undercategories & overcategories of Galois categories

In this section we extend our dictionary by showing that undercategories correspond to localisations, while overcategories correspond to normalisations (Corollary 12.4.4).

12.4.1 Notation. If $x \to X$ is a point of a scheme $X$, then we write $O^{h}_{X,x_0}$ for the henselisation of the local ring $O_{X,x_0}$, and we write

$$O^{h}_{X,x} \supseteq O^{h}_{X,x_0}$$

for the unique extension of henselian local rings that on residue fields reduces to the field extension $\kappa \supseteq \kappa(x_0)$, where $\kappa$ is the separable closure of $\kappa(x_0)$ in $\kappa(x)$. We will also write

$$X_{(x)} := \text{Spec}(O^{h}_{X,x}).$$

We call $X_{(x)}$ the localisation of $X$ at $x$ (Example 6.7.4). The scheme $X_{(x)}$ is the limit of the factorisations $x \to U \to X$ in which $U \to X$ is étale.

If $x \to X$ is a geometric point, then $O^{h}_{X,x}$ is the strict henselisation of $O_{X,x_0}$, and $X_{(x)}$ is the strict localisation of $X$ at $x$.

Dually, if $y \to X$ is a geometric point, then we write $X^{(y_0)}$ for the reduced subscheme structure on the Zariski closure of $y_0$, and we write $X^{(y)}$ for the normalisation of $X^{(y_0)}$ under $\text{Spec}\kappa(y)^{\text{alg}}$. We call $X^{(y)}$ the strict normalisation of $X$ at $y$.

12.4.2. Absolutely integrally closed schemes are integral normal schemes whose function field is algebraically closed. In other words, an absolutely integrally closed scheme is a strict normalisation $X^{(y)}$ of a scheme at a geometric point $y \to X$. This class of schemes has a number of curious properties:

- If $Z$ is absolutely integrally closed, then for any point $z_0 \in Z^{\text{zar}}$, the local ring $O_{Z,z_0}$ is strictly henselian [111, Proposition 2.6].
If $Z$ is absolutely integrally closed, then the étale topos and the Zariski topos of $Z$ coincide, so that $\text{Gal}(Z) = Z^{\text{zar}}_{\text{Zar}}$ [111, Corollary 2.5]. In other words, $\text{Gal}(Z)$ is a profinite poset with a terminal object.

If $Z$ is absolutely integrally closed and $W$ is irreducible, then any integral morphism $W \to Z$ is radicial [111, Lemma 2.3]. Thus any integral surjection $W \to Z$ is a universal homeomorphism.

If $Z$ is absolutely integrally closed, then the poset $\text{Gal}(Z) \cong Z^{\text{zar}}_{\text{Zar}}$ has all finite nonempty joins [112, Theorem 2.1].

Here now is the basic observation, which follows more or less immediately from the limit descriptions of the strict localisation and the strict normalisation:

12.4.3 Proposition. Let $X$ be a coherent scheme, and let $x \to X$ and $y \to X$ be two geometric points of $X$. The following profinite sets are in (canonical) bijection:

- the set $\text{Map}_{\text{Gal}(X)}(x, y)$ of morphisms $x \to y$ in $\text{Gal}(X)$;
- the set $\text{Mor}_X(y, X(x))$ of lifts of $y$ to the strict localisation $X(x)$;
- the set $\text{Mor}_X(x^{\text{alg}}, X(y))$ of lifts of $x$ to the strict normalisation $X(y)$.

We may thus describe the over- and undercategories of Galois categories:

12.4.4 Corollary. Let $X$ be a coherent scheme, and let $x \to X$ and $y \to X$ be geometric points of $X$. Then we have

$$
\text{Gal}(X)_{x/y} = \text{Gal}(X(x)) \quad \text{and} \quad \text{Gal}(X)_Y = \text{Gal}(X(y))
$$

The first sentence is originally due to Grothendieck [SGA 4_{1/2}, Exposé VIII, Corollaire 7.6].

12.4.5 Corollary. Let $X$ be a coherent scheme. Then $\text{Gal}(X)$ is equivalent to both of the following full subcategories of $X$-schemes:

- the full subcategory spanned by the strict localisations of $X$, and
- the full subcategory spanned by the strict normalisations of $X$.

Since $\text{Gal}(X(y)) = X^{(y)\text{zar}}$, it follows that Galois categories are of a very particular sort:

12.4.6 Corollary. Let $X$ be a coherent scheme. For any geometric point $y \to X$, the overcategory $\text{Gal}(X)_{y}$ is a profinite poset with all finite nonempty joins. In particular, every morphism of $\text{Gal}(X)$ is a monomorphism.

12.4.7 Definition. Let $X$ be a scheme. Then a witness is a absolutely integrally closed valuation ring $V$ and a morphism $\gamma : \text{Spec} V \to X$. If $p_0$ is the initial object of $\text{Gal}(V)$ and $p_{\infty}$ is the terminal object of $\text{Gal}(V)$, then we say that $\gamma$ witnesses the map $\gamma(p_0) \to \gamma(p_{\infty})$ of $\text{Gal}(X)$.

12.4.8. Any morphism $x \to y$ of $\text{Gal}(X)$ has a witness: you can always find a local morphism $\text{Spec} V \to (X^{(y)})(x)$ that induces an isomorphism of function fields.
12.5 Recovering the protruncated étale shape

Since \( \text{Gal}(X) \) is the profinite stratified shape of a coherent topos, the fact that the profi-
nite stratified shape is a delocalisation of the protruncated shape \( \text{(Theorem 10.2.3)} \) im-
mEDIATELY implies the following:

\[ \text{12.5.1 Theorem. Let } X \text{ be a coherent scheme. Then there is a natural natural map of } \]

prospaces \( \theta_X : \Pi^{\text{et}}_{\infty}(X) \to E(\text{Gal}(X)) \).

Moreover, \( \theta_X \) induces an equivalence on protruncations. As a consequence:

- For each integer \( n \geq 1 \) and geometric point \( x \to X \), we have natural isomorphisms of progroups
  \[ \pi^{\text{et}}_n(X, x) \cong \pi_n(\text{E}(\text{Gal}(X)), x), \]
  where \( \pi^{\text{et}}_n(X, x) \) is the \( n \)th homotopy progroup of the étale shape \( \Pi^{\text{et}}_{\infty}(X) \) of \( X \) \( \text{(11.1.7)} \).

- For any ring \( R \), there is an equivalence of \( \infty \)-categories between local systems of \( R \)-modules on \( X \) that are uniformly bounded both below and above and continuous functors \( \text{Gal}(X) \to D_b(\mathcal{R}) \) that carry every morphism to an equivalence.

12.6 Stratified Riemann Existence Theorem

We now use the Artin Comparison Theorem to prove a straified refinement of the Rie-
mann Existence Theorem of Artin–Mazur–Friedlander \( \text{(Theorem 11.5.3)} \), giving a com-
parison between étale and analytic stratified homotopy types for schemes of finite type
over the complex numbers. To do so, we invoke the critical basechange result of Artin.
A straightforward Postnikov argument permits us to reformulate Artin’s theorem as fol-
lows.

\[ \text{12.6.1 Theorem (Artin Comparison; [SGA 4_1, Exposé XVI, Théorème 4.1]). Let } f : X \to Y \text{ be a finite type morphism of finite type } \mathbb{C} \text{-schemes, and let } F \in \mathcal{X}_{\text{ét}} \text{ be a constructible sheaf. Then the natural basechange morphism} \]

\[ \epsilon^* f^{\text{ét}} F \to f^* \epsilon^* F \]
is an equivalence, where \( \epsilon_* : \overline{X}^{\text{an}} \to X_{\text{ét}} \) is the geometric morphism of \( \text{Recollection 11.5.2} \).

\[ \text{12.6.2 Construction. If } X \text{ is a scheme of finite type over } \mathbb{C}, \text{ then the topological space } X^{\text{an}} \text{ admits the evident profinite stratification } X^{\text{an}} \to X^{\text{zar}}, \text{ and } \epsilon_* \text{ is an } X^{\text{zar}} \text{-stratified geometric morphism.} \]

\[ \text{If } X^{\text{zar}} \to P \text{ is a finite constructible stratification, then the topological space } X^{\text{an}} \text{ also inherits a stratification } X^{\text{an}} \to P, \text{ which is conical.} \]

\[ \text{On each stratum } X_{p}, \text{ the functor } \epsilon^* \text{ restricts to a functor } \mathcal{X}_{p, \text{ét}}^{\text{lisse}} \to (\overline{X_p})^{\text{lisse}} \text{ (which is an equivalence by Theorem 11.5.3), whence we obtain a functor} \]

\[ \epsilon^* : \mathcal{X}_{\text{ét}}^{P, \text{cons}} \to (\overline{X^{\text{an}}})^{P, \text{cons}} \]
which in turn induces a $P$-stratified geometric morphism
\[ \varepsilon^P_\ast : \text{Sh}^\text{eff}(\overline{\text{X}^\text{an}}_{\overline{P} \text{-cons}}) \to \text{Sh}^\text{eff}(\overline{X}^P_{\text{cons}}) \]
of spectral $P$-stratified oo-topoi.

Please note that we also have the Exodromy Equivalence for stratified topological spaces (Subexample 8.2.11), which provides an equivalence
\[ \Pi_{(\infty,1)}(X^\text{an}; P) \cong \Pi_{(\infty,1)}(\text{Sh}^\text{eff}(\overline{X}^\text{an})_{\overline{P} \text{-cons}}; P) \]
between the profinite completion (in the stratified sense) of the exit-path oo-category of $X^\text{an}$ and the profinite stratified shape of $\text{Sh}^\text{eff}(\overline{X}^\text{an})_{\overline{P} \text{-cons}}$.

**12.6.3 Proposition (Stratified Riemann Existence).** Let $X$ be a scheme of finite type over $\mathbb{C}$, and let $X^\text{zar} \to P$ be a finite constructible stratification. Then the geometric morphism $\varepsilon^P_\ast$ is an equivalence. Consequently, $\varepsilon_\ast$ induces an equivalence
\[ \Pi_{(\infty,1)}(X^\text{an}; P) \cong \Pi_{(\infty,1)}(\text{Sh}^\text{eff}(\overline{X}^\text{an}_{\overline{P} \text{-cons}}; P) \rightarrow \Pi_{(\infty,1)}(X; P) \]
of profinite $P$-stratified spaces.

**Proof.** On strata, $\varepsilon^P_\ast$ is an equivalence by the Riemann Existence Theorem. For any point $p \in P$, let us write $X^L_p$ for the Stone oo-topos $\text{Sh}^\text{eff}(\overline{X}^\text{an}_{\overline{P} \text{-cons}}) = \text{Sh}^\text{eff}(X^\text{lie}_p, \text{et})$. For any points $p < q$, the geometric morphism $\varepsilon^P_\ast$ on the link from $p$ to $q$ is a geometric morphism of Stone oo-topoi
\[ X^L_p \text{Sh}^\text{eff}(\overline{X}^\text{an}_{\overline{P} \text{-cons}}) \rightarrow X^L_q \text{Sh}^\text{eff}(\overline{X}^\text{an}_{\overline{P} \text{-cons}}) . \]
To see that this is an equivalence, since the oriented fibre product is invariant under localisations of the corner (Example 5.4.11), we may assume that $P = \lbrace p \leq q \rbrace$, in which case $\text{Sh}^\text{eff}(\overline{X}^\text{an}_{\overline{P} \text{-cons}})$ and $\text{Sh}^\text{eff}(X^\text{lie}_p, \text{et})$ are each bounded coherent recollements of $X^L_p$ and $X^L_q$. Therefore it suffices to prove that the gluing functors coincide on truncated coherent objects. That is, one needs to confirm that the natural transformation
\[ \varepsilon^* i_{\text{et}}^! j^* \to i_{\text{an}}^* j_{\ast \text{an}}^* \varepsilon^* \]
is an equivalence when restricted to $(X^L_q)_{\text{coh}}^\text{lie} = X^\text{lie}_q, \text{et}$. This now follows from the Artin Comparison Theorem (Theorem 12.6.1) and naturality of $\varepsilon^*$. \qed

Passing to the limit over finite stratifications, we obtain the following.

**12.6.4 Corollary.** Let $X$ be a scheme of finite type over $\mathbb{C}$. Then $\varepsilon_\ast$ induces an equivalence
\[ \Pi_{(\infty,1)}(X^\text{an}; X^\text{zar}) \cong \Pi_{(\infty,1)}(X) . \]
12.7 Van Kampen Theorem for Galois categories

Let $X$ be a coherent scheme. A nonempty closed subset $Z \subset X$ with nonempty quasi-compact open complement $U \subset X$ specifies a nondegenerate constructible stratification $X^{\text{zar}} \to [1]$, and we may – in an overindulgence of abusive notation – write

$$\text{Gal}(X; Z) \coloneqq \text{Gal}(X/[1]),$$

which is a profinite $[1]$-stratified space. Its décollage is the functor $\text{sd}^{op}([1]) \to S^*_\lambda$ given by the diagram

$$\hat{\Pi}^\text{ét}_{\omega}(Z) \leftarrow \hat{\Pi}^\text{ét}_{\omega}(Z_{\text{et}} \times_X U_{\text{et}}) \to \hat{\Pi}^\text{ét}_{\omega}(U).$$

(Note that any subscheme structure on $Z$ will do, as nilimmersions don’t affect the étale $\infty$-topos.) The profinite space $\hat{\Pi}^\omega_{\text{et}}(Z_{\text{et}} \times_X U_{\text{et}})$ is the deleted tubular neighbourhood of $\hat{\Pi}^\omega_{\text{et}}(Z)$ in $\hat{\Pi}^\omega_{\text{et}}(X)$.

When $Z = \{z\}$ with $\kappa(z)$ separably closed, one may identify the deleted tubular neighbourhood of $\hat{\Pi}^\omega_{\text{et}}(Z) = \ast$ in $\hat{\Pi}^\omega_{\text{et}}(X)$ with the profinite étale shape of the punctured Milnor ball $X^\text{sh}_{(z)} \setminus \{z\}$.

When $X$ is a curve over a field $k$ and $Z = \{z\}$ is a rational point, we obtain an identification of the deleted tubular neighbourhood with the classifying space of ‘the’ profinite decomposition group $D_z \subset \hat{\pi}_1^\text{ét}(U)$. More generally, we may regard the deleted tubular neighbourhood $\hat{\Pi}^\omega_{\text{et}}(Z_{\text{et}} \times_X U_{\text{et}})$ as a kind of generalised ‘decomposition homotopy type’.

Our van Kampen theorem (Proposition 10.5.1) exhibits an equivalence of profinite spaces

$$\hat{\Pi}^\omega_{\text{et}}(X) \cong \hat{\Pi}^\omega_{\text{et}}(Z) \cup \hat{\Pi}^\omega_{\text{et}}(U),$$

which functions in the same manner as Friedlander’s van Kampen theorem [34, Proposition 15.6].

12.7.1. Cox [25; 26] also developed a deleted tubular neighbourhood for schemes, which is what appears in Friedlander’s formulation of the van Kampen Theorem. It can be shown that Cox’s deleted tubular neighbourhood and our toposic version have equivalent protruncated homotopy types.

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13 Extending exodromy: stacks & ℓ-adic sheaves

One limitation of our study of constructible sheaves is that we have effectively restricted our attention to finite coefficients: if $X$ is a coherent scheme and $\Lambda$ is a finite ring, our Exodromy Theorem identifies the $\infty$-category of continuous $\Lambda$-linear representations of the profinite 1-category $\text{Gal}(X)$ with the $\infty$-category of constructible sheaves of $\Lambda$-modules. (This sentence may be taken in either the derived or underived sentence.) However, most of the serious applications of constructible sheaves and their cohomology arise when $\Lambda$ is a more general topological ring, such as $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, or $\overline{\mathbb{Q}}_\ell$. Consequently, it is desirable to extend our Exodromy Theorem to include these topological rings. To do this, we have to solve two problems:

1. First, it is not a priori clear how to incorporate the topology on $\Lambda$ in the $\infty$-category $\text{Perf}(\Lambda)$. We will address this by introducing pyknotic structures – called condensed structures by Clausen and Scholze. Pyknotic structures generalise common topological structures in a way that is well-adapted to the study of algebraic structures. Using this formalism, if $C$ and $D$ are two pyknotic $\infty$-categories, we can meaningfully speak of the the $\infty$-category $\text{Fun}^{\infty}(C, D)$ of continuous functors $C \to D$.

2. Second, constructible sheaves of $\Lambda$-modules are not generally sheaves for the étale topology, but the proétale topology of Bhatt and Scholze. One expects a version of our Exodromy Theorem for $\infty$-topoi that closely resemble the proétale $\infty$-topos of a scheme; we have not been able to obtain such a result. Instead, we do something more modest: we identify situations in which the theorem can be extended along limits and filtered colimits.

The formulation and proof of the following two results will occupy most of this chapter.

13.0.1 Notation. Let $X$ be a coherent scheme.

- Let $\Lambda$ be a noetherian ring that is complete with respect to the topology defined by an ideal $I \subset \Lambda$. We write $D_{\text{cons}}(X_{\text{proét}}; \Lambda)$ for the constructible derived $\infty$-category of proétale sheaves of $\Lambda$-modules on $X$ [15, Definition 6.5.1].

- Let $\ell$ be a prime number and $E$ an algebraic extension of $\mathbb{Q}_\ell$. If $X$ is topologically noetherian, we write $D_{\text{cons}}(X_{\text{proét}}; E)$ for the constructible derived $\infty$-category of proétale sheaves of $E$-modules on $X$ [15, Definition 6.8.8].

13.0.2 Theorem. Let $X$ be a coherent scheme, $\Lambda$ be a noetherian ring, and $I \subset \Lambda$ an ideal. Assume that $\Lambda$ is complete with respect to the $I$-adic topology and that for each integer $n \geq 1$, the quotient ring $\Lambda/I^n$ is finite. Then there is an equivalence of $\infty$-categories

$$D_{\text{cons}}(X_{\text{proét}}; \Lambda) \cong \text{Fun}^{\infty}(\text{Gal}(X), \text{Perf}(\Lambda)).$$

Similarly, if $E$ is an algebraic extension of $\mathbb{Q}_\ell$, we write $D_{\text{cons}}(X_{\text{proét}}; E)$ for the derived $\infty$-category of constructible $E$-sheaves on $X$ [15, Definition 6.8.8].
13.0.3 Theorem. Let $X$ be a topologically noetherian scheme (thus automatically coherent), $\ell$ be a prime number, and $E$ be an algebraic field extension of $\mathbb{Q}_\ell$. Then there is an equivalence of \(\infty\)-categories

\[
D_{\mathrm{cons}}(X_{\mathrm{proet}}; E) \simeq \mathrm{Fun}^\mathrm{cts}(\mathrm{Gal}(X), \mathrm{Perf}(E)).
\]

Section 13.1 sets up the basics of pyknotic spaces and \(\infty\)-categories and explains what we mean by \(\mathrm{Fun}^\mathrm{cts}\). Armed with this, in Sections 13.2 and 13.3 we explain how to embed profinite spaces into pyknotic spaces, and profinite stratified spaces into pyknotic \(\infty\)-categories. Section 13.4 shows that functors out of $\mathrm{Gal}(X)$ in the ‘pro’ sense agree with continuous functors in the pyknotic sense. This gives a reinterpretation of the Exodromy Theorem with finite coefficients in terms of the pyknotic formalism. Section 13.5 then uses a small portion of the pyknotic formalism to extend the Exodromy Theorem to a large class of stacks (Proposition 13.5.17). With all of this in place, in Section 13.6 we explain how the Exodromy Theorem with profinite coefficients (Theorem 13.0.2) follows from Exodromy with finite coefficients (see Theorem 13.6.8). Finally Section 13.7 is dedicated to extending Exodromy with profinite coefficients to coefficients in an algebraic extension of $\mathbb{Q}_\ell$ (Theorem 13.0.3); this step is not entirely formal and involves an analysis of truncation conditions in the pyknotic world (see Theorem 13.7.8).

13.1 Pyknotic spaces & pyknotic higher categories

To begin, we describe elements of the pyknotic formalism; for more details, we refer the reader to [14; 107; 108; 109].

13.1.1 Construction. Stone duality identifies the category $\mathrm{Stn}$ of Stone topological spaces with the category $\mathrm{Pro}(\mathrm{Set}^{\delta_1})$ of profinite sets. The subcategory $E \subseteq \mathrm{Stn}$ consisting of effective epimorphisms is an (1-)presite structure on $\mathrm{Stn}$. We write $\mathrm{eff} := \tau_E$ for the resulting finitary topology, the effective epimorphism topology.

13.1.2 Definition. A pyknotic space is a hypersheaf $\mathrm{Stn}^{\mathrm{op}} \to S_{\delta_1}$ for the effective epimorphism topology. We write

\[ \mathrm{Pyk}(S) := \mathrm{Sh}_{\mathrm{eff}}^\mathrm{hyp}(\mathrm{Stn}; S_{\delta_1}) \]

for the \(\infty\)-category of pyknotic spaces.

13.1.3 Warning. The category $\mathrm{Stn}$ is $\delta_1$-small, but it is not $\delta_0$-small. Moreover, there does not exist a cofinal $\delta_0$-small set of covering sieves of an object for the effective epimorphism topology. Consequently, when we speak of hypersheaves on $\mathrm{Stn}$ for the effective epimorphism topology, we have to consider hypersheaves valued in $\delta_1$-small spaces. The result will be a large \(\infty\)-topos, that is, a left exact $\delta_1$-accessible localisation of an \(\infty\)-category $\mathrm{Fun}(C^{\mathrm{op}}, S_{\delta_1})$ of presheaves of $\delta_1$-small spaces on a $\delta_1$-small \(\infty\)-category $C$.

Large \(\infty\)-topoi work exactly as do \(\infty\)-topoi, except that everything has to be shifted one universe up. For example, if $X$ is bounded and coherent as a large \(\infty\)-topos, then $X^{\mathrm{coh}}$ is only $\delta_1$-small.

An alternative to working with large \(\infty\)-topoi is considering instead only the accessible hypersheaves $\mathrm{Stn}^{\mathrm{op}} \to S_{\delta_0}$. These are what Clausen and Scholze call condensed.
spaces. These do not form a ∞-topos (large or small), but in many ways the ∞-category of accessible sheaves is relatively well behaved.

13.1.4. The large ∞-topos Pyk(S) is hypercomplete, coherent, and locally coherent [SAG, Propositions A.2.2.2 & A.3.1.3].

Let us identify two further generating sites for Pyk(S) – one larger and one smaller.

13.1.5 Notation. Let TSp denote the 1-category of δ₀-small topological spaces. Define full subcategories

\[ \text{Proj} \subset \text{Comp} \subset TSp \]

as follows:

- **Comp** is spanned by the **compacta** – i.e., compact Hausdorff topological spaces.
- **Proj** is spanned by the **projective compacta** – i.e., compact Hausdorff topological spaces that are extremally disconnected [38; 67, Chapter III, §3.7].

Equivalently, a topological space is a projective compactum if and only if it can be exhibited as the retract of the Stone–Čech compactification \( \beta(S) \) of some set \( S \). In particular, \( \text{Proj} \subset \text{Stn} \).

13.1.6. For every compactum \( K \), there is a natural surjection \( \beta(K^{dsc}) \to K \) from the Stone–Čech compactification of the discrete topological space \( K^{dsc} \) with underlying set \( K \) to \( K \) (cf. [106, Remark 2.8]). Hence the subcategories \( \text{Stn} \subset \text{Comp} \) and \( \text{Proj} \subset \text{Comp} \) are bases for the effective epimorphism topology on \( \text{Comp} \) (Definition 3.12.2). Therefore, restriction of presheaves defines equivalences of ∞-categories

\[ \text{Sh}_{hyp}^{\text{eff}}(\text{Comp}; S_{δ₀}) \cong \text{Sh}_{hyp}^{\text{eff}}(\text{Stn}; S_{δ₀}) \cong \text{Sh}_{hyp}^{\text{eff}}(\text{Proj}; S_{δ₀}) \]

with inverses given by right Kan extension (Proposition 3.12.11).

13.1.7 Warning. Since the 1-sites \( \text{Comp} \) and \( \text{Stn} \) have finite limits and the inclusion \( \text{Stn} \hookrightarrow \text{Comp} \) preserves finite limits, from Corollary 3.12.14 we deduce that restriction defines an equivalence of 1-localic topoi

\[ \text{Sh}_{\text{eff}}(\text{Comp}; S_{δ₀}) \Rightarrow \text{Sh}_{\text{eff}}(\text{Stn}; S_{δ₀}) \].

However, as pointed out to us by Dustin Clausen and Peter Scholze, since the 1-site \( \text{Proj} \) of projective compacta does not have finite limits, and restriction only defines an equivalence

\[ \text{Sh}_{\text{eff}}^{\text{hyp}}(\text{Comp}; S_{δ₀}) \Rightarrow \text{Sh}_{\text{eff}}^{\text{hyp}}(\text{Proj}; S_{δ₀}) \]

on topos of hypersheaves.

13.1.8. Since \( \text{Proj} \subset \text{Comp} \) consists of projective objects of \( \text{Comp} \), the Čech nerve of any surjection in \( \text{Proj} \) is a split simplicial object. Hence by [SAG, Proposition A.3.3.1] we see that a functor

\[ F : \text{Proj}^{op} \to S_{δ₀} \]
is a sheaf with respect to the effective epimorphism topology if and only if $F$ carries coproducts in $\text{Proj}$ to products in $\mathcal{S}$. That is to say, the category $\text{Pyk}(\mathcal{S})$ is equivalent to the $\infty$-category of functors $\text{Proj}^{\text{op}} \to \mathcal{S}_{\text{δ}}$ that carry finite coproducts of projective compacta to products.

From this description, it is essentially immediate that the $\infty$-topos $\text{Sh}_{\text{eff}}(\text{Proj}; \mathcal{S}_{\text{δ}})$ is Postnikov complete, whence we obtain an equivalence

$$\text{Sh}_{\text{hyp}}^{\text{hyp}}(\text{Comp}; \mathcal{S}_{\text{δ}}) \simeq \text{Sh}_{\text{eff}}(\text{Proj}; \mathcal{S}_{\text{δ}})$$

(cf. [14, §2.4]). That is to say, the $\infty$-category $\text{Pyk}(\mathcal{S})$ is the nonabelian derived $\infty$-category or animation of the category $\text{Proj}$ [HTT, §5.5.8; 19, §5.1]

13.1.9 Definition. Let $\mathcal{C}$ be a $\infty$-category with all finite products. The $\infty$-category $\text{Pyk}(\mathcal{C})$ of pyknotic objects of $\mathcal{C}$ is the full subcategory of $\text{Fun}(\text{Proj}^{\text{op}}, \mathcal{C})$ spanned by those functors that carry finite coproducts of projective compacta to products in $\mathcal{C}$.

13.1.10 Construction. Let $\mathcal{C}$ be a $\delta_1$-presentable $\infty$-category. The global sections functor $\text{Pyk}(\mathcal{C}) \to \mathcal{C}$ is given by evaluation at the one-point compactum $\ast$. For any pyknotic object $Y$, we write $Y^{\text{and}} \equiv Y(\ast)$. We call $Y^{\text{and}}$ the underlying object of $Y$.

Left adjoint to this is the constant sheaf functor $\mathcal{C} \to \text{Pyk}(\mathcal{C})$ that carries an object $X \in \mathcal{C}$ to what we will call the discrete pyknotic object $X^{\text{disc}}$ attached to $X$. This pyknotic object can be described explicitly: $X^{\text{disc}}$ is given by the assignment

$$K \mapsto X^K := \text{colim}_{I \in \text{Set}_{\text{fin}}} \prod_{i \in I} X.$$ 

The underlying space functor also admits a right adjoint $X \mapsto X^{\text{indisc}}$. For an object $X \in \mathcal{C}$, the sheaf $X^{\text{indisc}} : \text{Proj}^{\text{op}} \to \mathcal{C}$ is given by the assignment

$$K \mapsto X^{|K|} := \prod_{k \in |K|} X,$$

i.e., the product of copies of $X$ indexed by the underlying set $|K|$ of the topological space $K$. We call $X^{\text{indisc}}$ the indiscrete pyknotic object attached to $X$.

Both the discrete and indiscrete functors are fully faithful, so that

$$(X^{\text{disc}})^{\text{and}} = (X^{\text{indisc}})^{\text{and}} = X.$$

Accordingly, we say that a pyknotic object in the essential image of $X \mapsto X^{\text{disc}}$ is discrete, and a pyknotic object in the essential image of $X \mapsto X^{\text{indisc}}$ is indiscrete.

13.1.11 Warning. The centre $X \mapsto X^{\text{indisc}}$ is not the only point of the $\infty$-topos $\text{Pyk}(\mathcal{S})$. Let $X$ be a topological space $X$. Define pyknotic set $P_X$ by sending $K$ to the quotient of the set continuous maps $K \to X$ by the locally constant maps:

$$P_X(K) := \text{Map}_{\text{TSpec}}(K, X)/ \text{Map}_{\text{TSpec}}^{\text{lc}}(K, X).$$

If $X$ is nonempty, then the pyknotic set $P_X$ has underlying set $\ast$; thus if $X$ is neither empty nor $\ast$, then $P_X$ is a nontrivial pyknotic structure on the point. See [STK, Tag 0991].
13.1.12 Example. For any finite set \( J \), the discrete pyknotic set \( J^{\text{disc}} \) is the sheaf \( K \mapsto \text{Map}_{\text{Proj}}(J, K) \) represented by \( J \). If \( \{ J_\alpha \}_{\alpha \in \mathcal{A}} \) is an inverse system of finite sets, then the limit
\[
\lim_{\alpha \in \mathcal{A}} J^{\text{disc}}_\alpha
\]
is the sheaf represented by the Stone topological space \( \lim_{\alpha \in \mathcal{A}} J_\alpha \); this is not discrete. Cf. (13.2.11).

13.1.13 Example. Write \( \text{TSpc}^{\text{cg}} \subset \text{TSpc} \) for the full subcategory spanned by the compactly generated topological spaces. Then the functor \( \text{TSpc}^{\text{cg}} \to \text{Pyk}(\text{Set}) \) given by the assignment
\[
T \mapsto [K \mapsto \text{Map}_{\text{TSpc}}(K, T)]
\]
is a fully faithful right adjoint. See [14, Example 2.1.6; 108, Proposition 1.7].

13.1.14 Definition. Let \( C \) and \( D \) be pyknotic \( \infty \)-categories. The\( \infty \)-category of continuous functors \( C \to D \) is the end
\[
\text{Fun}^{ct}(C, D) = \int_{K \in \text{Proj}} \text{Fun}(C(K), D(K)).
\]
This is part of the natural enrichment of \( \text{Pyk}(\text{Cat}_{\infty}) \) over \( \text{Cat}_{\infty} \).

13.2 Profinite spaces as pyknotic spaces

The goal of this section is to show that not only profinite sets embed into \( \text{Pyk}(S) \), but profinite spaces actually embed into \( \text{Pyk}(S) \). The first approach one might have to showing this is to try to show that \( \pi \)-finite spaces are cocompact when regarded as discrete pyknotic spaces, so that the extension \( S^n_\pi \to \text{Pyk}(S) \) of the discrete functor \( (\cdot)^{\text{disc}}: S_n \to \text{Pyk}(S) \) is fully faithful. Unfortunately, this approach is destined for failure: finite sets aren’t even cocompact objects of \( \text{Pyk}(S) \), as the following counterexample shows.

13.2.1 Counterexample. The finite set \( \{0, 1\} \) with two elements is not cocompact when regarded as a discrete pyknotic space. Since the embedding of compactly generated topological spaces into \( \text{Pyk}(\text{Set}) \) preserves limits, and the image of a finite set \( S \) under the Yoneda embedding \( \text{Pro}(\text{Set}^{\text{fin}}) \to \text{Pyk}(\text{Set}) \) coincides with its image under the embedding \( \text{TSpc}^{\text{cg}} \to \text{Pyk}(\text{Set}) \), to see that \( \{0, 1\} \) is not cocompact in \( \text{Pyk}(\text{Set}) \), it suffices to prove that the discrete topological space \( \{0, 1\} \) is not cocompact in \( \text{TSpc}^{\text{cg}} \). To see this, let \( s: N \to N \) be the successor function \( n \mapsto n + 1 \) and consider the diagram of discrete topological spaces
\[
\cdots \hookrightarrow N \xrightarrow{s} N \xrightarrow{s} N.
\]
We claim that the map of sets
\[
(13.2.2) \quad \text{colim}_n \text{Map}(N, \{0, 1\}) \to \text{Map}(\lim_n N, \{0, 1\})
\]
is not a bijection. To see this, note that the limit \( \lim_n N \) empty, so \( \text{Map}(\lim_n N, \{0, 1\}) \) has cardinality 1. On the other hand, \( \text{Map}(N, \{0, 1\}) \) is the powerset \( P(N) \) of \( N \), and the colimit \( \text{colim}_n P(N) \) along the inverse image maps \( s^{-1}: P(N) \to P(N) \) has infinite cardinality.
The approach we take to show that profinite spaces embed into Pyk(S) is somewhat indirect: we show that profinite sets form a basis for the effective epimorphism topology on \( S^\alpha_n \), so that hypersheaves on \( \text{Pro}(\text{Set}^{fin}) \) and \( S^\alpha_n \) coincide (Proposition 3.12.11). The Yoneda embedding provides the desired embedding \( S^\alpha_n \hookrightarrow \text{Pyk}(S) \).

In order to get this approach off the ground, we first need to talk about the effective epimorphism topology on \( S^\alpha_n \). The existence of this topology is immediate from [SAG, Proposition A.3.2.1 & Theorem E.6.3.1].

13.2.3 Proposition. Write \( E \subseteq S^\alpha_n \) for the subcategory of those morphisms \( e : X \to Y \) in \( S^\alpha_n \) that can be written as an inverse limit of morphisms \( e_\alpha : X_\alpha \to Y_\alpha \) where each \( e_\alpha \) is an effective epimorphism of \( \pi \)-finite spaces. Then \( E \) defines a \( \infty \)-presite structure on \( S^\alpha_n \). We write \( \text{eff} \) \( = \tau_E \) for the resulting finitary topology, the effective epimorphism topology.

13.2.4. From [SAG, Proposition A.3.3.1] it follows that the effective epimorphism topology on \( S^\alpha_n \) is subcanonical. Moreover, since the Yoneda embedding \( S^\alpha_n \hookrightarrow \text{Sh}_{\text{eff}}(S^\alpha_n; \text{S}^\delta_1) \) preserves \( \delta_0 \)-small limits, truncated objects of an \( \infty \)-topos are hypercomplete, and hypercomplete objects are closed under limits, the Yoneda embedding factors through \( \text{Sh}_{\text{hyp}}(S^\alpha_n; \text{S}^\delta_1) \).

Now we show that \( \text{Pro}(\text{Set}^{fin}) \subset S^\alpha_n \) is a basis for the effective epimorphism topology (in the sense of Definition 3.12.2). In fact, we show that every object of \( S^\alpha_n \) admits an effective epimorphism from a profinite set. This requires a number of preliminaries.

13.2.5. Note that for every space \( U \) there exists an effective epimorphism \( \pi_0(U) \twoheadrightarrow U \). In particular, \( \text{Set}^{fin} \subset S^\alpha_n \) is a basis for the effective epimorphism topology and every object admits a cover by a single object of \( \text{Set}^{fin} \).

Since we must contend with proöbjects, it isn’t immediate from (13.2.5) that every profinite space admits an effective epimorphism from a profinite set. To show this, we’ll use the fact that we can always arrange to index a proöbject by a particularly nice poset.

13.2.6 Definition. We say that a poset \( A \) is down-finite if for every element \( \alpha \in A \), the set \( \{ \beta \in A \mid \beta \leq \alpha \} \) is finite.

13.2.7 Lemma ([SAG, Lemma E.1.6.4]). Let \( A' \) be a filtered poset. Then there exists a colimit-cofinal map of posets \( A \to A' \), where \( A \) is a down-finite filtered poset.

13.2.8 Construction. If \( A \) is a down-finite poset, then there exists a map of posets
\[
\text{rk} : A \to \mathbb{N}
\]
called the rank which is determined by the following requirement: for each \( \alpha \in A \), the number \( \text{rk}(\alpha) \) is the smallest natural number not equal to \( \text{rk}(\beta) \) for \( \beta < \alpha \) (cf. [HA, Remark A.5.17]). In particular, \( \text{rk}(\alpha) = 0 \) if and only if \( \alpha \) is a minimal element of \( A \).

13.2.9 Proposition. For every object \( X \in S^\alpha_n \), there exists an cover \( Y \to X \) for the effective epimorphism topology on \( S^\alpha_n \) for which \( Y \in \text{Pro}(\text{Set}^{fin}) \). In particular, \( \text{Pro}(\text{Set}^{fin}) \subset S^\alpha_n \) is a basis for the effective epimorphism topology on \( S^\alpha_n \).
Proof. To simplify notation, write $C = S_\pi$ and $D = \text{Set}^{\aleph_0}$. Let $\{X_\alpha\}_{\alpha \in A^{\aleph_0}}$ be an object of $\text{Pro}(C)$. We without loss of generality assume that $A$ is a down-finite filtered poset (Lemma 13.2.7). We construct a morphism $e : \{Y_\alpha\}_{\alpha \in A^{\aleph_0}} \to \{X_\alpha\}_{\alpha \in A^{\aleph_0}}$ in $\text{Pro}(C)$ such that for each $\alpha \in A$, the morphism $e_\alpha : Y_\alpha \to X_\alpha$ is an effective epimorphism and $Y_\alpha \in D$. We construct this inductively on the rank of elements of $A$. For each $n \in \mathbb{N}$, write

$$A_{\leq n} = \{\alpha \in A \mid \text{rk}(\alpha) \leq n\}.$$  

First, for each element $\alpha \in A$ with $\text{rk}(\alpha) = 0$ (i.e., minimal element of $A$), appealing to (13.2.5), choose an effective epimorphism $e_\alpha : Y_\alpha \twoheadrightarrow X_\alpha$ where $Y_\alpha \in D$.

For the induction step, suppose that we have defined a functor $Y : A_{\leq n}^{op} \to D$ along with a natural effective epimorphism $e : Y \to X|_{A_{\leq n}^{op}}$; we now extend $Y$ to $A_{\leq n+1}$ as follows. For each $\alpha \in A$ with $\text{rk}(\alpha) = n+1$, consider the pulled-back effective epimorphism

$$\bigwedge_{\beta < \alpha, \text{rk}(\beta) = n} X_\alpha \times_{X_\beta} Y_\beta \twoheadrightarrow X_\alpha.$$  

Then by construction the functor $Y : A_{\leq n}^{op} \to D$ extends to a functor $Y : A_{\leq n+1}^{op} \to D$ equipped with a natural effective epimorphism $e : Y \to X|_{A_{\leq n+1}^{op}}$, as desired.  

As an immediate consequence of Proposition 3.12.11, we obtain the desired equivalence on hypersheaves:

13.2.10 Corollary. Restriction of presheaves along the inclusion $\text{Pro}(\text{Set}^{\aleph_0}) \hookrightarrow S_\pi^n$ defines an equivalence of large $\infty$-topoi

$$\text{Sh}_{\text{hyp}}^\pi(S_\pi^n; S_\delta_1) \simeq \text{Pyk}(S)$$

with inverse given by right Kan extension.

13.2.11. We finish this section by showing that the restricted Yoneda embedding $\chi : S_\pi \hookrightarrow \text{Pyk}(S)$ agrees with the discrete functor $\Gamma^\ast : S_\pi \hookrightarrow \text{Pyk}(S)$. To see this, first note that the global sections functor $\Gamma^\ast : \text{Pyk}(S) \to S$ is given by the composite

$$\text{Pyk}(S) \xrightarrow{(-)_{S_\pi}} \text{Sh}_{\text{eff}}(S_\pi) \xrightarrow{\text{Map}_S(-, Y)} S$$

of restriction along the inclusion $S_\pi \hookrightarrow S_\pi^n$ with evaluation at the terminal object. The inverse equivalence $S \Rightarrow \text{Sh}_{\text{eff}}(S_\pi)$ is given by sending a space $Y$ to the functor

$$\text{Map}_S(-, Y) : S_\pi^n \to S.$$  

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Hence we have natural equivalences
\[
\text{Map}_{\text{Pyk}(S)}(\mathbb{K}(K), X) \cong X(K) = X|_{S^K}(K) \\
\cong \text{Map}_S(K, X) \\
\cong \text{Map}_{\text{Pyk}(S)}(\Gamma^*(K), X).
\]

Since the Yoneda embedding \(\mathbb{K} : \mathcal{S} \to \text{Pyk}(S)\) preserves inverse limits, we see that the Yoneda embedding is the extension of the discrete functor \(\Gamma^* : \mathcal{S} \to \text{Pyk}(S)\) to proobjects.

13.3 Profinite stratified spaces as pyknotic \(\infty\)-categories

Our next goal is to show that profinite layered \(\infty\)-categories embed into \(\text{Pyk}(\text{Cat}_{\infty})\). We'll deduce this from the fact that profinite spaces embed into \(\text{Pyk}(S)\) by regarding profinite layered \(\infty\)-categories as complete Segal objects of \(\mathcal{S}^\Lambda\). This perspective will also allow us to easily deduce that profinite layered \(\infty\)-categories are 'almost compact' when regarded as pyknotic \(\infty\)-categories; this almost compactness property a key ingredient in our proof of Theorem 13.0.3 (see §13.7).

13.3.1 Definition. Let \(D\) be an \(\infty\)-category with finite limits. We say that a simplicial object \(F : \Delta^\text{op} \to D\) is a category object of \(D\) if \(F\) satisfies the Segal condition:

\[(13.3.1.1)\] For every integer \(k \geq 1\), the natural morphism
\[
F_k \to F[0 < 1] \times_{F[1]} F[1 < 2] \times_{F[2]} \cdots \times_{F[k-1]} F[k-1 < k]
\]
is an equivalence in \(D\).

We say that a category object \(F : \Delta^\text{op} \to D\) is a complete Segal object if, in addition, \(F\) satisfies the following completeness condition:

\[(13.3.1.2)\] The natural morphism
\[
F_0 \to F_3 \times_{F[0 < 2]} F_1
\]
is an equivalence in \(D\).

Write
\[
\text{CO}(D) \subset \text{Fun}(\Delta^\text{op}, D)
\]
for the full subcategory spanned by the category objects and \(\text{CS}(D) \subset \text{CO}(D)\) for the full subcategory spanned by the complete Segal objects.

13.3.2. Joyal and Tierney [68] showed that the nerve construction defines an equivalence
\[
N : \text{Cat}_{\infty} \to \text{CS}(S)
\]
from the \(\infty\)-category of \(\infty\)-categories to the \(\infty\)-category of complete Segal spaces. For each integer \(n \geq 0\), the nerve restricts to an equivalence
\[
N : \text{Cat}_n \to \text{CS}(S_{\leq n-1})
\]
between \(n\)-categories and complete Segal objects in \((n-1)\)-truncated spaces.

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13.3.3. See [SAG, §A.8.2] for more on category objects.

13.3.4. Let $D$ be an $\infty$-category with finite limits. Then the full subcategories

$$\text{CS}(D) \subset \text{CO}(D) \subset \text{Fun}(\Delta^\op, D)$$

are closed under all limits that exist in $D$.

13.3.5. Let $\mathcal{X}$ be an $\infty$-topos. Since finite limits commute with filtered colimits in $\infty$-topoi, the full subcategories

$$\text{CS}(\mathcal{X}) \subset \text{CO}(\mathcal{X}) \subset \text{Fun}(\Delta^\op, \mathcal{X})$$

are closed under limits and filtered colimits. In particular $\text{CS}(\mathcal{X})$ and $\text{CO}(\mathcal{X})$ are presentable and the inclusions into $\text{Fun}(\Delta^\op, \mathcal{X})$ admit left adjoints.

For future use, we’ll record a few facts about the interaction between the formation of pyknotic objects and complete Segal objects. All are immediate from the definitions.

13.3.6 Notation. Let $B$ and $D$ be $\infty$-categories with finite products. We write $\text{Fun}×(B, D) \subset \text{Fun}(B, D)$ for the full subcategory spanned by the functors that preserve finite products.

13.3.7 Lemma. Let $B$, $C$, and $D$ be $\infty$-categories, and assume that $B$ and $D$ have finite products. Then the natural equivalence of $\infty$-categories

$$\text{Fun}(B, \text{Fun}(C, D)) \simeq \text{Fun}(C, \text{Fun}(B, D))$$

restricts to an equivalence

$$\text{Fun}×(B, \text{Fun}(C, D)) \simeq \text{Fun}(C, \text{Fun}×(B, D)) .$$

13.3.8 Corollary. Let $B$ be an $\infty$-category with products and $D$ an $\infty$-category with finite limits. Then the natural equivalence of $\infty$-categories

$$\text{Fun}×(B, \text{Fun}(\Delta^\op, D)) \simeq \text{Fun}(\Delta^\op, \text{Fun}×(B, D))$$

restrict to equivalences

$$\text{Fun}×(B, \text{CO}(D)) \simeq \text{CO}(\text{Fun}×(B, D)) \quad \text{and} \quad \text{Fun}×(B, \text{CS}(D)) \simeq \text{CS}(\text{Fun}×(B, D)) .$$

13.3.9 Example. Corollary 13.3.8 provides equivalences

$$\text{Pyk}(\text{CO}(S)) \simeq \text{CO}(\text{Pyk}(S)) \quad \text{and} \quad \text{Pyk}(\text{Cat}_{n\omega}) \simeq \text{CS}(\text{Pyk}(S)) .$$

In light of (13.3.2), for each integer $n \geq 0$, Corollary 13.3.8 provides an equivalence

$$\text{Pyk}(\text{Cat}_n) \simeq \text{CS}(\text{Pyk}(S_{\leq n-1})) .$$

13.3.10 Example. Let $C$ and $D$ be pyknotic $\infty$-categories. As a complete Segal space, the $\infty$-category $\text{Fun}^{\text{ct}}(C, D)$ of continuous functors $C \to D$ is given by the assignment

$$[n] \mapsto \text{Map}_{\text{Pyk}(\text{Cat}_{n\omega})}(C \times [n]^{\text{disc}}, D) = \int_{K \in \text{Proj}} \text{Map}_{\text{Cat}_{n\omega}}((C \times [n]^{\text{disc}})(K), D(K)) .$$
The nerve provides an embedding \( \text{N} : \text{Lay}_n \hookrightarrow \text{CS}(S_n) \) of \( \pi \)-finite layered \( \infty \)-categories into complete Segal objects in \( \pi \)-finite spaces. In light of Example 13.3.9, the embedding \( S_n^\wedge \hookrightarrow \text{Pyk}(S) \) provided by Corollary 13.2.10 extends to an embedding \( \text{CS}(S_n^\wedge) \hookrightarrow \text{Pyk}(\text{Cat}_\infty) \) on the level of complete Segal objects. Hence, in order to provide the desired embedding \( \text{Pro}(\text{Lay}_n) \hookrightarrow \text{Pyk}(\text{Cat}_\infty) \), it suffices to provide an embedding \( \text{Pro}(\text{CS}(S_n)) \hookrightarrow \text{CS}(S_n^\wedge) \).

To provide this last embedding, it suffices to show that the subcategory \( \text{CS}(S_n^\wedge) \subset \text{CS}(S_n^\wedge) \) consists of cocompact objects. We'll show this by combining the fact that \( S_n^\wedge \subset S_n \) consists of cocompact objects with the fact that a category object in truncated spaces is always right Kan-extended from a finite subcategory of \( \Delta^{op} \).

13.3.11 Notation. Let \( m \) be a positive integer. We write \( \Delta_{\leq m} \subset \Delta \) for the full subcategory spanned by the objects \([0], [1], \ldots, [m] \).

13.3.12 Definition. Let \( D \) be an \( \infty \)-category with finite limits and \( m \geq 1 \) an integer. An \( m \)-skeletal category object of \( D \) is a functor \( F : \Delta^{op}_{\leq m} \rightarrow D \) satisfying the Segal condition: for every integer \( 0 \leq k \leq m \), the natural morphism

\[
F_k \to F[0 < 1] \times_{F[1]} F[1 < 2] \times_{F[2]} \cdots \times_{F[k-1]} F[k-1 < k]
\]

is an equivalence in \( D \). We write

\[
\text{CO}_{\leq m}(D) \subset \text{Fun}(\Delta_{\leq m}^{op}, D)
\]

for the full subcategory spanned by the \( m \)-skeletal category objects.

13.3.13. Let \( X \) be an \( \infty \)-topos and \( m \geq 1 \) an integer. Then the restriction functor

\[
(-)|_{\Delta_{\leq m}^{op}} : \text{CO}(X) \rightarrow \text{CO}_{\leq m}(X)
\]

commutes with all limits and filtered colimits.

We now prove the relevant cocompactness result that we need to see that profinite stratified spaces embed into pyknotic \( \infty \)-categories.

13.3.14 Proposition. Let \( X \) be an \( \infty \)-category with finite limits, \( n \geq -2 \) an integer, and \( C \in \text{CO}(X) \) a category object of \( X \). If the natural map

\[
C[0 < 1] \to C[0] \times C[1]
\]

is \( n \)-truncated and for each \( I \in \Delta \), the object \( C(I) \) of \( X \) is cocompact, then \( C \) is a cocompact object of \( \text{CO}(X) \).

Proof of Proposition 13.7.9. Let \( D : A \rightarrow \text{CO}(X) \) be a cofiltered diagram. By [SAG, Proposition A.8.2.6], the object \( C \) is right Kan extended from \( \Delta_{\leq n+2}^{op} \), hence we see that

\[
\colim_{\alpha \in A^{op}} \text{Map}_{\text{CO}(X)}(D_\alpha, C) = \colim_{\alpha \in A^{op}} \text{Map}_{\text{CO}(\Delta_{\leq n+2}^{op})}(D\alpha|_{\Delta_{\leq n+2}^{op}}, C|_{\Delta_{\leq n+2}^{op}}).
\]
From the end description of mapping spaces in a functor category, the fact that finite limits commute with filtered colimits in \( S \), and the cocompactness of \( C(I) \) for each \( I \in \Delta^\op \), we see that we have equivalences

\[
\colim_{\alpha \in A^\op} \Map_{\CO(X)}(D_\alpha, C) \cong \colim_{I \in \Delta_{\leq n+2}} \Map_X(D_\alpha(I), C(I))
\]

\[
\cong \int_{I \in \Delta_{\leq n+2}} \colim_{\alpha \in A^\op} \Map_X(D_\alpha(I), C(I))
\]

\[
\cong \int_{I \in \Delta_{\leq n+2}} \Map_X(\lim_{\alpha \in A^\op} D_\alpha(I), C(I))
\]

\[
= \Map_{\CO(X)}(\lim_{\alpha \in A^\op} D_\alpha, C).
\]

**13.3.15 Corollary.** Every object in the image of the Yoneda embedding \( \CO(S_\pi) \hookrightarrow \CO(S_\pi^\circ) \) is cocompact.

**13.3.16 Example.** From Corollary 13.3.15 and [HTT, Proposition 5.3.5.11] we see that the proextension of the Yoneda embedding \( \CS(S_\pi) \hookrightarrow \CS(S_\pi^\circ) \) defines a fully faithful functor

\[ \Pro(\CS(S_\pi)) \hookrightarrow \CS(S_\pi^\circ) \]

that preserves inverse limits. In particular, the composite

\[ (13.3.17) \quad \Pro(\text{Lay}_\pi) \hookrightarrow \Pro(\CS(S_\pi)) \hookrightarrow \CS(S_\pi^\circ) \hookrightarrow \Pyk(\Cat_{\infty}) \]

is fully faithful and preserves inverse limits.

Since the embedding (13.3.17) preserves inverse limits and agrees with the discrete functor \((-)_{\text{disc}} : \text{Lay}_\pi \hookrightarrow \Pyk(\Cat_{\infty})\) when restricted to \( \pi \)-finite layered \( \infty \)-categories (13.2.11), it is the extension to proöbjects of the discrete functor.

**13.4 Comparison between notions of continuous functors**

Let \( X \) be a scheme. The original formulation of the Exodromy Theorem for schemes says that functors \( \text{Gal}(X) \rightarrow S_\pi \) in the ‘pro’ sense are the same things as constructible sheaves of spaces on \( X \) (Construction 12.1.5). The goal of this section is to give a pyknotic formulation of this theorem. To do this, we show that functors \( \text{Gal}(X) \rightarrow S_\pi \) in the ‘pro’ sense are the same as continuous functors \( \text{Gal}(X) \rightarrow S_{\pi_{\text{disc}}} \) in the pyknotic sense; here we regarded \( \text{Gal}(X) \) as a pyknotic category under the embedding of Chapter 10. Appealing to the stable version of the Exodromy Theorem (Corollary 12.1.6), we prove the analogous claim where \( S_\pi \) is replaced by the derived \( \infty \)-category of a finite ring \( R \).

The important property shared by \( S_\pi \) and \( \text{Perf}(R) \) that allows us to reduce the claim to Proposition 13.3.14 is that all of the mapping spaces in these \( \infty \)-categories are \( \pi \)-finite.

**13.4.1 Definition.** We say that an \( \infty \)-category \( C \) is **locally \( \pi \)-finite** if for all objects \( X, Y \in C \), the mapping space \( \Map_C(X, Y) \) is \( \pi \)-finite. We say that a locally \( \pi \)-finite \( \infty \)-category \( C \) is **\( \pi \)-finite** if \( C \) has finitely many objects up to equivalence.
13.4.2 Examples.

- The $\infty$-category $S_\pi$ of $\pi$-finite spaces is locally $\pi$-finite [SAG, Remark E.2.6.4].
- For any finite ring $R$, the $\infty$-category $\text{Perf}(R)$ of perfect complexes over $R$ is locally $\pi$-finite.
- If $\Pi$ is a $\pi$-finite stratified space, then the $\infty$-category $\Pi$ is $\pi$-finite.

13.4.3 Lemma. Let $C$ be a locally $\pi$-finite $\infty$-category and $\Pi = \{\Pi_\alpha\}_{\alpha \in A}$ a profinite layered $\infty$-category. Then the natural functor

$$\text{colim}_{\alpha \in A^{op}} \text{Fun}(\Pi_\alpha, C) \to \text{Fun}^{ct}(\Pi, C^{\text{disc}})$$

is an equivalence.

Proof. For each finite set $F$ of equivalence classes of objects of $C$, write $C_F \subset C$ for the full subcategory closed under equivalence spanned by the objects in $F$. Since $C$ is locally $\pi$-finite and $F$ is finite, the $\infty$-category $C_F$ is $\pi$-finite. Moreover, $C$ is equivalent to the filtered union $\bigcup_F C_F$ over finite subsets of equivalence classes of objects ordered by inclusion. Since the pyknotic set $\pi_0(\Pi)$ is quasicompact and $\pi_0(C)$ is discrete, we see that every continuous functor $\Pi \to C^{\text{disc}}$ factors through $C_F^{\text{disc}} \subset C^{\text{disc}}$ for some finite subset $F$ of equivalence classes of objects of $C$. Hence we have an identification

$$\text{Fun}^{ct}(\Pi, C^{\text{disc}}) = \bigcup_F \text{Fun}^{ct}(\Pi, C^{\text{disc}}_F).$$

Since each $\infty$-category $C_F$ is $\pi$-finite, each $\Pi_\alpha$ has finitely many objects up to equivalence, and colimits commute, from Proposition 13.3.14 we see that

$$\bigcup_F \text{Fun}^{ct}(\Pi, C^{\text{disc}}) = \bigcup_F \text{colim}_{\alpha \in A^{op}} \text{Fun}^{ct}(\Pi^{\text{disc}}_\alpha, C^{\text{disc}}_F) = \bigcup_F \text{colim}_{\alpha \in A^{op}} \text{Fun}(\Pi_\alpha, C_F) = \text{colim}_{\alpha \in A^{op}} \text{Fun}(\Pi_\alpha, C). \qedhere$$

13.4.4 Example. Let $\Pi$ be a profinite layered $\infty$-category. Lemma 13.4.3 provides an equivalence

$$\text{Fun}(\Pi, S_\pi) = \text{Fun}^{ct}(\Pi, S_\pi^{\text{disc}}).$$

Moreover, for any finite ring $R$, Lemma 13.4.3 provides an equivalence

$$\text{Fun}(\Pi, \text{Perf}(R)) = \text{Fun}^{ct}(\Pi, \text{Perf}(R)^{\text{disc}}).$$

The following is immediate from Corollary 12.1.6 and Lemma 13.4.3.

13.4.5 Theorem. Let $R$ be a finite ring, and let $X$ be a coherent scheme. Then there is a natural equivalence of $\infty$-categories

$$D_{\text{cons}}(X_{\text{proét}}, R) = \text{Fun}^{ct}(\text{Gal}(X), \text{Perf}(R)^{\text{disc}}).$$
13.4.6. Let $R$ be a finite ring, and let $X$ be a coherent scheme. Attached to any constructible sheaf $F$ of $R$-complexes on $X$, we have an associated exodromy representation

$$\rho_F : \text{Gal}(X) \to \text{Perf}(R)^{\text{disc}}$$

that is sufficient to reconstruct $F$.

13.5 Fibred Galois categories and exodromy for simplicial schemes and stacks

In this section we extend our notion of Galois categories to simplicial schemes and stacks, and we prove our Exodromy Theorem in this context.\footnote{The material from this section first appeared – with a slightly different set-up – in a preprint of the first- and third-named authors [13].}

13.5.1 Recollection ([HTT, Definition 6.3.1.6]). Let $M$ be an $\infty$-category. A functor $p : X \to M$ is a topos fibration if $p$ is a bicartesian fibration, for each $m \in M$ the fiber $X_m$ is an $\infty$-topos, and for each morphism $f : n \to m$ of $M$, the induced pullback functor $f^* : X_n \to X_m$ is left exact.

13.5.2 Definition. Let $M$ be an $\infty$-category. A bounded coherent topos fibration $X \to M$ is a topos fibration in which each fiber $X_m$ is a bounded coherent $\infty$-topos, and for every morphism $f : n \to m$ of $M$, the induced geometric morphism $f_* : X_m \to X_n$ is coherent. A spectral topos fibration $X \to S$ is a bounded coherent topos fibration in which each fiber $X_m$ is a spectral $\infty$-topos (for the canonical profinite stratification of §8.7).

13.5.3. The usual straightening/unstraightening equivalence restricts to an equivalence between the $\infty$-category of bounded coherent (respectively, spectral) topos fibrations $X \to M$ and the $\infty$-category of functors from $M^{\text{op}}$ to the $\infty$-category of bounded coherent (resp., spectral) $\infty$-topoi (cf. [HTT, Proposition 6.3.1.7]).

For a bounded coherent topos fibration $X \to M$ we write $X_{\text{coh}}^{\leq \infty} \subseteq X$ for the full subcategory spanned by the objects that are truncated and coherent in their fibre. Then $X_{\text{coh}}^{\leq \infty} \to M$ is a cocartesian fibration that is classified by a functor from $M$ to the category of bounded $\infty$-pretopoi [SAG, Definition A.7.4.1 & Theorem A.7.5.3].

13.5.4 Example. If $X_*$ is a simplicial coherent scheme, then the fibred topos $X_{*,\text{et}} \to \Delta$ is a spectral topos fibration.

A fibred form of $\infty$-Categorical Hochster Duality is what allows us to construct fibred Galois categories. To prove this fibred form of $\infty$-Categorical Hochster Duality, we need to make sense of $\infty$-categories fibred in profinite stratified spaces.

13.5.5 Definition. Let $M$ be an $\infty$-category. We say that a functor $f : \Pi \to M$ is an $\infty$-category over $M$ fibred in layered $\infty$-categories if $f$ is a catesian fibration whose fibres are layered $\infty$-categories. We write $\text{Lay}_{/M}^{\text{cat}}$ for the $\infty$-category of $\infty$-categories over $M$ fibred in layered $\infty$-categories.

More generally, an $\infty$-category over $M$ fibred in profinite layered $\infty$-categories is a pyknotic object $\Pi$ in $\infty$-categories over $M$ such that:
for every projective compactum $K$, the functor $\Pi(K) \to M$ is a cartesian fibration, and

- for every object $m \in M$, the pyknotic $\infty$-category $\Pi_m := \Pi \times_M \{m\}$ is a profinite layered $\infty$-category.

We write $\text{Lay}^{\text{cart} \land \pi, /}_M$ for the $\infty$-category of $\infty$-categories over $M$ fibred in profinite layered $\infty$-categories.

### 13.5.6 Warning.
One might also contemplate the $\infty$-category $\text{Pro}(\text{Lay}^{\text{cart} \land \pi, /}_M)$ of proobjects in the full subcategory $\text{Lay}^{\text{cart} \land \pi, /}_M \subseteq \text{Lay}^{\land \pi, /}_M$ spanned by those cartesian fibrations whose fibres are $\pi$-finite layered $\infty$-categories. This is generally not equivalent to the $\infty$-category of categories over $M$ fibred in profinite layered $\infty$-categories. Under straightening/unstraightening, the $\infty$-category $\text{Lay}^{\text{cart} \land \pi, /}_M$ is equivalent to the $\infty$-category $\text{Fun}(M^{\text{op}}, \text{Lay}^{\land \pi})$, whereas $\text{Pro}(\text{Lay}^{\text{cart} \land \pi, /}_M)$ is equivalent to the $\infty$-category $\text{Pro}(\text{Fun}(M^{\text{op}}, \text{Lay}^{\pi}))$. These coincide when $M$ is a finite poset [HTT, Proposition 5.3.5.15], but otherwise typically do not coincide.

### 13.5.7.
Let $M$ be an $\infty$-category. By $\infty$-Categorical Hochster Duality, the $\infty$-category of spectral topos fibrations over $M$ is equivalent to the $\infty$-category $\text{Lay}^{\text{cart} \land \pi, /}_M$. Let us make the equivalence explicit. If $\mathcal{X} \to M$ is a spectral topos fibration, then we define an $\infty$-category over $M$ fibred in layered $\infty$-categories $\tilde{\Pi}^{\text{cart}}_{\infty, 1}(\mathcal{X}) \to M$ as follows. An object of $\tilde{\Pi}^{\text{cart}}_{\infty, 1}(\mathcal{X})$ is a pair $(m, v)$, where $m \in M$ and $v_* : S \to X_m$ is a point. A morphism $(m, v) \to (n, \xi)$ is a morphism $f : m \to n$ of $M$ and a natural transformation $v_* \to f_* \xi_*$. The $\infty$-category $\tilde{\Pi}^{\text{cart}}_{\infty, 1}(\mathcal{X})$ fibred in layered $\infty$-categories admits a canonical fibrewise profinite structure; the fibre $\tilde{\Pi}^{\text{cart}}_{\infty, 1}(\mathcal{X})_s$ over an object $s \in M$ is the profinite stratified shape $\tilde{\Pi}^{\pi}_{\infty, 1}(X_s)$.

In the other direction, if $\Pi \to M$ is an $\infty$-category over $M$ fibred in profinite layered $\infty$-categories, then let $X_\emptyset \to M$ denote the cocartesian fibration in which the objects are pairs $(m, F)$ consisting of an object $m \in M$ and a functor $F : \Pi_m \to S_m$, and a morphism $(f, \phi) : (m, F) \to (n, G)$ consists of a morphism $f : m \to n$ of $M$ and a natural transformation $\phi : f_* F \to G$. Then $(\tilde{\Pi})^{\text{coh}}_{\infty}$ is equivalent to the subcategory of $X_\emptyset$ whose objects are those pairs $(m, F)$ in which $F$ is continuous and whose morphisms are those pairs $(f, \phi)$ in which $\phi$ is continuous.

### 13.5.8 Construction.
If $M$ is an $\infty$-category and $Y$ is a bounded coherent topos, then the projection $Y \times M \to M$ is a bounded coherent topos fibration. The assignment $Y \mapsto Y \times M$ defines a functor from the $\infty$-category of bounded coherent topoi to the
∞-category of bounded coherent topos fibrations over $M$. This functor admits a left adjoint, which we denote by $|-|_M$. At the level of ∞-pretopoi, $(|X|_M)^{coh}$ is equivalent to the ∞-category of cocartesian sections of $X^{coh}_c \to M$, i.e., the limit of the corresponding functor $M \to \text{preTop}^b_{\text{cof}}$.

Now we arrive at the main topos-theoretic result.

**13.5.9 Proposition.** Let $M$ be an ∞-category, and let $X \to M$ be a spectral topos fibration. Then the ∞-pretopos $(|X|_M)^{coh}$ is equivalent to the ∞-category of functors

$$F : \Pi^{M}_{(\infty,1)}(X) \to S$$

with the following properties.

(13.5.9.1) The functor $F$ carries every cartesian edge to an equivalence.

(13.5.9.2) For any object $m \in M$, the restriction $F|_{\Pi^{M}_{(\infty,1)}(X_m)}$ is continuous.

(13.5.9.3) The functor $F$ is uniformly truncated in the following sense: there exists an $N \in \mathbb{N}$ such that for any object $(m, v) \in \Pi^{M}_{(\infty,1)}(X)$, the space $F(m, v)$ is $N$-truncated.

**Proof.** The ∞-pretopos $(|X|_M)^{coh}$ can be identified with the ∞-category of cocartesian sections of $X^{coh}_c \to M$. The description of (13.5.7) completes the proof. $\square$

Please note that the last condition of Proposition 13.5.9 is automatic if $M$ has only finitely many connected components (e.g., $M = \Delta$).

Finally, since the profinite stratified shape is a delocalisation of the protruncated shape (Theorem 10.2.3) we deduce the following:

**13.5.10 Proposition.** Let $M$ be an ∞-category, and let $X \to M$ be a spectral topos fibration. Then the protruncated shape of $|X|_M$ is equivalent to the protruncation of the classifying prospace of $\Pi^{M}_{(\infty,1)}(X)$.

**13.5.11 Construction** ($\Gamma^\Delta(Y_\ast)$). Let $Y_\ast$ be a simplicial coherent scheme. Denote by $\Gamma^\Delta(Y_\ast)$ the following 1-category. The objects are pairs $(m, \nu)$ consisting of an object $m \in \Delta$ and a geometric point $\nu \to Y_m$. A morphism $(m, \nu) \to (n, \xi)$ of $\Gamma^\Delta(Y_\ast)$ is a morphism $\sigma : m \to n$ of $\Delta$ and a specialisation $\nu \Rightarrow \sigma^\ast(\xi)$. This category has an obvious forgetful functor $\Gamma^\Delta(Y_\ast) \to \Delta$, which is a cartesian fibration. A morphism $(m, \nu) \to (n, \xi)$ is cartesian over $\sigma : m \to n$ in $\Delta$ if and only if the specialisation $\nu \Rightarrow \sigma^\ast(\xi)$ is an isomorphism.

Now $\Gamma^\Delta(Y_\ast)$ is a category over $\Delta$ fibred in profinite categories, and the ∞-category over $\Delta$ fibred in profinite layered ∞-categories $\Pi^{\Delta}_{(\infty,1)}(Y_\ast, \text{et})$ associated to the spectral topos fibration

$$Y_\ast, \text{et} \to \Delta$$

can be identified with the 1-category $\Gamma^\Delta(Y_\ast)$. 192
In this case, Proposition 13.5.9 implies that $(|Y_*|, \text{ét})_{\text{coh}}^{\text{sh}}$ is equivalent to the co-category of functors $\text{Gal}^b(Y_\Delta) \to S_\Delta$ that carry cartesian edges to equivalences and restrict to continuous functors $\text{Gal}^b(X_{\Delta m}) \to S_m$ for all $m \in \Delta$.

**13.5.12 Example.** If $X_*$ is a simplicial coherent scheme, then classifying pro-space of the fibrewise profinite category $\text{Gal}^b(X_\Delta)$ is equivalent to the protruncation of Friedlander étale topological type of $X_*$ (Theorem 12.5.1).

Now let us use this formalism to extend the *Exodromy Equivalence* of Theorem B to the context of simplicial schemes and thus stacks.

**13.5.13 Construction.** Write $\text{Aff}$ for the 1-category of affine schemes. We employ [HTT, Corollary 3.2.2.13] to construct an ∞-category $\text{PSh}_\text{ét}$ and a cocartesian fibration $\text{PSh}_\text{ét} \to \text{Aff}^{\text{op}}$ in which:

- The objects of $\text{PSh}_\text{ét}$ are pairs $(S, F)$ consisting of an affine scheme $S$ and a presheaf $F$ on the small étale site of $S$.
- A morphism $(S, F) \to (T, G)$ is a pair $(f, \phi)$ consisting of a morphism $f : T \to S$ and a morphism of presheaves $\phi : f^{-1}F \to G$ on the small étale site of $T$.

Define $\text{Sh}_\text{ét} \subset \text{PSh}_\text{ét}$ to be the full subcategory spanned by those pairs $(S, F)$ in which $F$ is a sheaf; then $\text{Sh}_\text{ét} \to \text{Aff}^{\text{op}}$ is a topos fibration. Define $\text{Cons}_\text{ét} \subset \text{Sh}_\text{ét}$ to be the further full subcategory spanned by those pairs $(S, F)$ in which $F$ is a constructible sheaf (Definition 9.4.1); then $\text{Cons}_\text{ét} \to \text{Aff}^{\text{op}}$ is a cocartesian fibration.

**13.5.14 Definition.** Let $X \to \text{Aff}$ be a stack, i.e., a right fibration that is classified by an accessible fpqc sheaf $\text{Aff}^{\text{op}} \to S$. A *constructible sheaf* on $X$ is a cocartesian section $F : X^{\text{op}} \to \text{Cons}_\text{ét}$ over $\text{Aff}^{\text{op}}$. We write $\text{Cons}_\text{ét}(X)$ for the co-category of constructible sheaves on $X$.

**13.5.15 Warning.** This can only be expected to be a reasonable definition for coherent stacks.

**13.5.16.** Informally, a constructible sheaf $F$ on $X$ assigns to every affine scheme $S$ over $X$ a constructible sheaf $F_S$ on $S$ and to every morphism $f : S \to T$ of affine schemes an equivalence $F_S \cong f^{*}F_T$. In other words, the co-category of constructible sheaves on $X$ is the limit of the diagram $X^{\text{op}} \to \text{Cat}_{\text{co}}$ given by the assignment $S \mapsto \text{Cons}_\text{ét}(S) = S^{\text{op}}_{\text{cons}}$.

Since $X$ is not generally a small category, it is not obvious that this limit exists in $\text{Cat}_{\text{co}}$. However, if $X$ contains a small limit-cofinal full subcategory $Y$, then the desired limit exists.

We thus conclude:
13.5.17 Proposition. Let \( p : X \to \text{Aff} \) is a stack. If \( Y_* \) is a simplicial coherent scheme presenting \( X \), then there is an equivalence between the \( \infty \)-category \( \text{Cons}_\text{ét}(X) \) and the \( \infty \)-category of continuous functors

\[
\text{Gal}^\Delta(Y_*) \to S_n
\]

that carry cartesian edges to equivalences (cf. Definition 13.1.14).

Recall that the protruncated étale topological type of a simplicial scheme \( Y_* \) can be identified with the colimit in protruncated spaces of the simplicial object that carries \( m \in \Delta \) to the protruncated étale shape of the fibres of the cartesian fibration \( \text{Gal}^\Delta(Y_*) \to \Delta \) agree with the protruncated étale shape of the schemes \( Y_m \), it follows from Proposition 13.5.10 that the protruncated shape of the total category \( \text{Gal}^\Delta(Y_*) \) is the colimit of this simplicial diagram. In other words:

13.5.18 Theorem. Let \( Y_* \) be a simplicial coherent scheme. The classifying protruncated space of \( \text{Gal}^\Delta(Y_*) \) recovers the protruncated étale topological type of \( Y_* \).

Combining this with Proposition 13.5.17 we obtain:

13.5.19 Corollary. Let \( n \in \mathbb{N} \) and let \( X \) be an Artin \( n \)-stack. If \( Y_* \) is a simplicial coherent scheme presenting \( X \), then the localisation of \( \text{Gal}^\Delta(Y_*) \) at the cartesian edges classifies constructible sheaves on \( X \).

Corollary 13.5.19 speaks only of Artin \( n \)-stacks, but of course applies just as well to any coherent fppc stack with a presentation by a simplicial coherent scheme.

13.5.20 Example. Let \( k \) be a ring, \( G \) be an affine \( k \)-group, and \( X \) be a \( k \)-scheme with an action of \( G \). Recall that the simplicial \( k \)-scheme \( \text{Bar}_{k,*}(X, G, k) \) whose \( n \)-simplices are \( X \times_{\text{Spec} k} G^m \) presents the quotient stack \( X/G \).

By Corollary 13.5.19, the category of \( G \)-equivariant constructible sheaves on \( X \) is equivalent to the category of continuous functors

\[
\text{Gal}^\Delta(\text{Bar}_{k,*}(X, G, k)) \to S_n
\]

that carry the cartesian edges to equivalences. If \( A \) is a ring, then the derived category of \( G \)-equivariant constructible sheaves of \( A \)-modules on \( X \) is equivalent to the category of continuous functors

\[
\text{Gal}^\Delta(\text{Bar}_{k,*}(X, G, k)) \to \text{Perf}(A)
\]

that carry cartesian edges to equivalences.

The objects of the category \( \text{Gal}^\Delta(\text{Bar}_{k,*}(X, G, k)) \) can be thought of as tuples

\[
([m], \Omega, x_0, g_1, \ldots, g_m)
\]

where \( [m] \in \Delta \), \( \Omega \) is a separably closed field, and

\[
x_0 : \text{Spec} \Omega \to X \quad \text{and} \quad g_1, \ldots, g_m : \text{Spec} \Omega \to G
\]

are points with the property that \((x_0, g_1, \ldots, g_m)\) is a geometric point of \( X \times_{\text{Spec} k} G^m \) such that \( \Omega \) is the separable closure of the residue field of the image of \((x_0, g_1, \ldots, g_m)\) in the Zariski space of \( X \times_{\text{Spec} k} G^m \).
13.6 Exodromy with profinite coefficients

In this section we extend the Exodromy Theorem from finite coefficients to coefficients in the ring of integers in a nonarchimedean local field. First we isolate the class of profinite rings that we're interested in.

13.6.1 Definition. Let \( \Lambda \) be ring and \( I \subset \Lambda \) an ideal. We say that \( \Lambda \) is \( I \)-profinite if:

1. (13.6.1.1) the ring \( \Lambda \) is noetherian,
2. (13.6.1.2) the ring \( \Lambda \) is complete with respect to the topology defined by the ideal \( I \),
3. (13.6.1.3) and for each integer \( n \geq 1 \), the quotient ring \( \Lambda/I^n \) is finite.

We simply say 'let \( \Lambda \) be an \( I \)-profinite ring' to mean 'let \( \Lambda \) be a ring with ideal \( I \subset \Lambda \) satisfying (13.6.1.1)–(13.6.1.3)'.

13.6.2. The reason for the noetherian hypothesis is to apply results from [15, §§6.5-6.8]; these results use the noetherianity of \( \Lambda \) to apply the Artin–Rees Lemma. The requirement that the quotients \( \Lambda/I^n \) be finite is so that we have access to the Exodromy Theorem we have proven for finite discrete rings.

13.6.3 Example. Let \( E \) be a nonarchimedean local field with ring of integers \( \mathcal{O}_E \). Write \( m_E \subset \mathcal{O}_E \) for the maximal ideal. Then \( \mathcal{O}_E \) is an \( m_E \)-profinite ring.

In particular, for any prime number \( \ell \) and prime power \( q \), the ring \( \mathbb{Z}_\ell \) is \( (\ell) \)-profinite and the ring \( \mathbb{F}_q[t] \) is \( (t) \)-profinite.

13.6.4 Definition. Let \( \Lambda \) be an \( I \)-profinite ring. We regard \( \Lambda \) as a topological ring and thereby as a pyknotic ring. The pyknotic \( \infty \)-category of perfect \( \Lambda \)-complexes is the limit

\[
\text{Perf}(\Lambda) := \lim_{n \geq 1} \text{Perf}(\Lambda/I^n)_{\text{disc}}
\]

in \( \text{Pyk}(\text{Cat}_{\infty}) \).

13.6.5. Please observe that if \( \Lambda \) is an \( I \)-profinite ring, since the underlying functor preserves limits, the underlying \( \infty \)-category \( \text{Perf}(\Lambda)^{\text{und}} \) coincides with the usual \( \infty \)-category of perfect complexes on \( \Lambda \).

More generally, by Construction 13.1.10 we see that for any projective compactum \( K \) exhibited as a profinite set \( \{K_\alpha\}_{\alpha \in \mathcal{A}^\partial} \), we have

\[
\text{Perf}(\Lambda)(K) = \lim_{n \geq 1} \text{colim}_{\alpha \in \mathcal{A}} \text{Perf}(\Lambda/I^n)^{K_\alpha}.
\]

13.6.6. It is not necessary to give such an \textit{ad hoc} definition of the pyknotic \( \infty \)-category of perfect complexes on a profinite ring. There is an intrinsic definition for a general pyknotic ring, but to develop this material here would take us too far afield.

In the case of a \( I \)-profinite ring, the more intrinsic definition recovers the definition given here.
13.6.7. Let \( X \) be a coherent scheme, and let \( \Lambda \) be an \( I \)-profinite ring. Recall [15, Definition 6.5.1] that the \( \infty \)-category of constructible \( \Lambda \)-complexes on \( X \) can be identified as the limit of \( \infty \)-categories
\[
\mathcal{D}_{\text{cons}}(X_{\text{proét}}; \Lambda) = \lim_{n \geq 1} \mathcal{D}_{\text{cons}}(X_{\text{ét}}; \Lambda/I^n).
\]
In other words, a constructible \( \Lambda \)-complex on \( X \) is a prosystem \( \{ F_n \}_{n \geq 1} \) consisting of constructible \( (\Lambda/I^n) \)-complexes \( F_n \) on \( X \) along with coherent identifications \( F_m = F_n \otimes_{\Lambda/I^n} (\Lambda/I^m) \) for all \( m \leq n \).

13.6.8 Theorem. Let \( X \) be a coherent scheme, and let \( \Lambda \) be an \( I \)-profinite ring. Then there is a natural equivalence of \( \infty \)-categories
\[
\mathcal{D}_{\text{cons}}(X_{\text{proét}}; \Lambda) \cong \text{Fun}^{\text{cts}}(\text{Gal}(X), \text{Perf}(\Lambda)).
\]
Proof. This follows by taking limits of the equivalence
\[
\mathcal{D}_{\text{cons}}(X_{\text{proét}}; \Lambda/I^n) \cong \text{Fun}^{\text{cts}}(\text{Gal}(X), \text{Perf}(\Lambda/I^n)).
\]
of Corollary 12.1.6.

13.6.9. Let \( X \) be a coherent scheme, and let \( \Lambda \) be an \( I \)-profinite ring. Attached to any constructible sheaf \( F \) of \( \Lambda \)-complexes on \( X \), we have an associated exodromy representation
\[
\rho_F : \text{Gal}(X) \to \text{Perf}(\Lambda)
\]
that is sufficient to reconstruct \( F \).

13.6.10 Warning. For an \( I \)-profinite ring \( \Lambda \), the pyknotic \( \infty \)-category \( \text{Perf}(\Lambda) \) is not discrete, and it is not generally the case that an exodromy representation \( \rho_F : \text{Gal}(X) \to \text{Perf}(\Lambda) \) factors through a quotient of \( \text{Gal}(X) \) with only finitely many isomorphism types. For instance, Bhatt and Scholze give an example of a constructible \( \mathbb{Z}_\ell \)-complex on a non-Noetherian scheme that is not lisse on the strata of any finite stratification [15, Example 6.6.12].

If \( X \) is a topologically noetherian scheme, Bhatt and Scholze demonstrate that this problem does not arise. To explain, this, let us briefly recall the basics of the constructible topology on a spectral topological space.

13.6.11 Recollection. Let \( S \) be a spectral topological space. The constructible topology on \( S \) is the topology on the underlying set of \( S \) generated by the constructible subsets of \( S \) (Recollection 9.4.9). We write \( S^c \) for the set \( S \) equipped with the constructible topology. The topological space \( S^c \) is a Stone topological space.

The constructible topology admits a description in terms of pro-objects: if we exhibit \( S \) as a profinite set \( \{ P_\alpha \}_{\alpha \in A} \), then the profinite set \( S^c \) is given by the inverse system \( \{ \cup P_\alpha \}_{\alpha \in A} \) of the underlying sets of the posets \( P_\alpha \). In particular, the assignment \( S \mapsto S^c \) is a right adjoint to the inclusion \( \text{Stn} \hookrightarrow \text{TSp}^\text{pro} \).

If \( X \) is a coherent scheme, then notice that the profinite set \( \pi_1 \text{Gal}(X) \) coincides with the set \( X^{\text{zar}} \) equipped with the constructible topology. If \( \Lambda \) is an \( I \)-profinite ring, then
13.6.12 Lemma ([15, Proposition 6.6.11]). Let $X$ be a topologically noetherian scheme, and let $\Lambda$ be an $I$-profinite ring. Then for any constructible sheaf $\mathcal{F}$ of $\Lambda$-complexes on $X$, the continuous map $\pi_0 \rho_F : \mathcal{X}^{zar, c} \to \pi_0 \text{Perf}(\Lambda)$ factors through a finite (discrete) quotient of $\mathcal{X}^{zar, c}$.

13.7 Exodromy with $\ell$-adic coefficients

The goal of this section is to use the Exodromy Theorem with $\mathbb{Z}_\ell$-coefficients (Theorem 13.6.8) to prove the Exodromy Theorem with $\mathbb{Q}_\ell$- or $\overline{\mathbb{Q}}_\ell$-coefficients. For this we begin by developing the basics of how to regard $\text{Perf}(\mathbb{Z}_\ell)$ as a pyknotic object in stable $\infty$-categories with $t$-structures and $t$-exact functors. Recall that we use homological indexing conventions for our $t$-structures (Convention 7.4.8).

13.7.1 Recollection. Let $\{C_\alpha\}_{\alpha \in A}$ be a filtered diagram of stable $\infty$-categories with $t$-structures and $t$-exact transition morphisms. Then the colimit $C := \text{colim}_{\alpha \in A} C_\alpha$ in $\text{Cat}_{\text{st}}$ admits a natural $t$-structure defined by

$$C_{\geq 0} = \text{colim}_{\alpha \in A} C_{\alpha, \geq 0} \quad \text{and} \quad C_{\leq 0} = \text{colim}_{\alpha \in A} C_{\alpha, \leq 0}.$$  

Moreover, if for each $\alpha \in A$ the $t$-structure on $C_\alpha$ is bounded, then the natural $t$-structure on $C$ is bounded as well.

13.7.2 Recollection. Let $\Lambda$ be an $I$-profinite ring. The stable $\infty$-category $\text{Perf}(\Lambda)^{\text{und}}$ carries its usual a bounded $t$-structure, defined as follows. Let $F = \{F_n\}_{n \geq 1}$ be a perfect $\Lambda$-complex. Then $F \in \text{Perf}(\Lambda)^{\text{und}}_{\geq 0}$ if and only if, for any $n \geq 1$, the perfect $(\Lambda/I^n)$-complex $F_n$ lies in $\text{Perf}(\Lambda/I^n)^{\text{und}}_{\geq 0}$. On the other hand, $F \in \text{Perf}(\Lambda)^{\text{und}}_{\leq 0}$ if and only if, for any $k > 0$, the prosystem $\{H_k(F_n)\}_{n \geq 1}$ vanishes.

13.7.3 Construction (t-structure on $\text{Perf}(\Lambda)$). Now we extend this to a $t$-structure on all the values of the pyknotic $\infty$-category $\text{Perf}(\Lambda)$ on a projective compactum $K$. To this end, for any $n \geq 1$, and any finite set $J$, we endow the $J$-fold product $\text{Perf}(\Lambda/I^n)^J$ with the product $t$-structure induced from the $t$-structure on $\text{Perf}(\Lambda/I^n)$; this $t$-structure is bounded because $J$ is finite. For any projective compactum $K = \{K_\alpha\}_{\alpha \in A^K}$, we also endow the filtered colimit

$$\text{Perf}(\Lambda/I^n)^K = \text{colim}_{\alpha \in A^K} \text{Perf}(\Lambda/I^n)^{K_\alpha}$$  

with its natural bounded $t$-structure (Recollection 13.7.1). With this definition, the assignment

$$K \mapsto \text{Perf}(\Lambda/I^n)^K$$  

is a pyknotic object in stable $\infty$-categories with bounded $t$-structures and $t$-exact functors.
For any $K$, the limit $\lim_{n \geq 1} \Perf(\Lambda/I^n)^K$ is an inverse system of right t-exact functors. Hence we may follow the example of $\Perf(\Lambda)^{und}$ and define

$$\Perf(\Lambda)(K)_{\geq 0} := \lim_{n \geq 1} (\Perf(\Lambda/I^n)^K)_{\geq 0},$$

and $\Perf(\Lambda)(K)_{<0}$ as the full subcategory of $\Perf(\Lambda)(K)$ spanned by those $F = \{F_n\}_{n \geq 1}$ such that for each $k > 0$, the prosystem $\{H_k(F_n)\}_{n \geq 1}$ vanishes. With this definition, the assignment $K \mapsto \Perf(\Lambda)(K)$ is a pyknotic object in stable $\infty$-categories with t-structures and t-exact functors.

Given integers $a \leq b$, we write $\Perf(\Lambda)_{(a,b)}$ for the subfunctor of $\Perf(\Lambda)$ given by the assignment

$$K \mapsto (\Perf(\Lambda)(K))_{(a,b)}.$$

The following is now immediate from [15, Lemma 6.5.3] and Lemma 13.6.12:

13.7.4 Lemma. Let $X$ be a topologically noetherian scheme, let $\Lambda$ be an $I$-profinite ring, and let $F$ be a constructible $\Lambda$-complex on $X$. Then the exodromy representation

$$\rho_F : \Gal(X) \to \Perf(\Lambda)$$

is bounded in the sense that there exists integers $a \leq b$ for which $\rho_F$ factors through $\Perf(\Lambda)_{(a,b)} \subset \Perf(\Lambda)$.

Now we are ready to extend to $\mathbb{Q}_\ell$ coefficients.

13.7.5 Definition. Let $\ell$ be a prime number and $E$ an algebraic extension of $\mathbb{Q}_\ell$. We define the pyknotic $\infty$-category $\Perf(E)$ as the filtered colimit of pyknotic $\infty$-categories

$$\Perf(E) := \colim_{E' \leq E} \Perf(\mathcal{O}_{E'})[\ell^{-1}],$$

over finite subextensions $\mathbb{Q}_\ell \leq E' \leq E$. Here, $C[\ell^{-1}]$ is shorthand for the filtered colimit of pyknotic $\infty$-categories

$$C[\ell^{-1}] := \colim \left( \begin{array}{c} C \\ \ell \rightarrow C \\ \ell \rightarrow C \\ \vdots \end{array} \right),$$

where $\ell : C \to C$ denotes multiplication by $\ell$.

13.7.6. Let $X$ be a topologically noetherian scheme, and let $E$ be an algebraic extension of $\mathbb{Q}_\ell$. Recall [15, Proposition 6.8.14] that the $\infty$-category of constructible $E$-complexes on $X$ can be identified as the filtered colimit

$$D_{cons}(X_{pro\acute{e}t}, E) = \colim_{E' \leq E} D_{cons}(X_{\acute{e}t}, \mathcal{O}_{E'}')[\ell^{-1}],$$

over finite subextensions $\mathbb{Q}_\ell \leq E' \leq E$.

Accordingly, we observe that $\Perf(E)^{und}$ is the usual $\infty$-category of perfect $E$-complexes. If $\xi$ is a geometric point, then $D_{cons}(\xi; E) = \Perf(E)^{und}$.

13.7.7. More generally, we can be unpack the value of the pyknotic $\infty$-category $\Perf(E)$ on a projective compactum $K = \{K_{\alpha \in A_{\ell}}^\alpha \}$ quite explicitly:

$$\Perf(E)(K) = \colim_{E' \leq E} \left( \lim_{n \geq 1} \colim_{\alpha \in A} \Perf(\mathcal{O}_{E'}/m_{E'}^n)_{K_{\alpha}} \right)[\ell^{-1}].$$

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13.7.8 Theorem. Let \( X \) be a topologically noetherian scheme, \( \ell \) a prime number, and \( E \) an algebraic extension of \( \mathbb{Q}_\ell \). Then we have an equivalence of \( \infty \)-categories

\[
D_{\text{cons}}(X_{\text{proét}}, E) \cong \text{Fun}^{\text{ct}}(\text{Gal}(X), \text{Perf}(E)) .
\]

To prove this theorem, we require some preliminaries on the almost compactness of profinitely layered \( \infty \)-categories. The formal manipulations in the proof of the following proposition are very similar to those used in the proof of Proposition 13.3.14.

13.7.9 Proposition. Let \( \mathcal{X} \) be an \( \infty \)-topos and \( C \in \text{CO}(\mathcal{X}) \) a category object of \( \mathcal{X} \). If for each \( I \in \Delta \), the object \( C(I) \) is almost compact (Definition 3.10.2), then for each integer \( n \geq -2 \) the functor

\[
\text{Map}_{\text{CO}(\mathcal{X})}(C, -) : \text{CO}(\mathcal{X}_{\leq n}) \to S
\]

preserves filtered colimits.

13.7.10. The hypotheses of Proposition 13.7.9 are satisfied if \( \mathcal{X} \) is a locally coherent \( \infty \)-topos and for each \( I \in \Delta \), the object \( C(I) \) is coherent \([\text{SAG}, \text{Corollary A.2.3.2}]\).

Proof of Proposition 13.7.9. Let \( D : A \to \text{CO}(\mathcal{X}_{\leq n}) \) be a filtered diagram. First note that by \([\text{SAG}, \text{Proposition A.8.2.6}]\), for each \( \alpha \in A \), the category object \( D_\alpha \) is right Kan extended from \( \Delta_{\leq n+2}^{\text{op}} \). Moreover, since \( \mathcal{X}_{\leq n} \subset \mathcal{X} \) is closed under filtered colimits, the colimit \( \colim_{\alpha \in A} D_\alpha \) is also right Kan extended from \( \Delta_{\leq n+2}^{\text{op}} \). Thus we see that

\[
\colim_{\alpha \in A} \text{Map}_{\text{CO}(\mathcal{X})}(C, D_\alpha) = \colim_{\alpha \in A} \text{Map}_{\text{CO}(\mathcal{X}_{\leq n})}(C|_{\Delta_{\leq n+2}^{\text{op}}}, D_\alpha|_{\Delta_{\leq n+2}^{\text{op}}})
\]

From the end description of mapping spaces in a functor category and the fact that finite limits commute with filtered colimits in \( S \), we see that we have equivalences

\[
\colim_{\alpha \in A} \text{Map}_{\text{CO}(\mathcal{X})}(C, D_\alpha) = \colim_{\alpha \in A} \int_{I \in \Delta_{\leq n+2}} \text{Map}_\mathcal{X}(C(I), D_\alpha(I))
\]

\[
\Rightarrow \int_{I \in \Delta_{\leq n+2}} \colim_{\alpha \in A} \text{Map}_\mathcal{X}(C(I), D_\alpha(I))
\]

Since the object \( C(I) \) is almost compact for each \( I \in \Delta \) and \( \colim_{\alpha \in A} D_\alpha \) is right Kan extended from \( \Delta_{\leq n+2}^{\text{op}} \), we see that

\[
\colim_{\alpha \in A} \text{Map}_{\text{CO}(\mathcal{X})}(C, D_\alpha) \Rightarrow \int_{I \in \Delta_{\leq n+2}} \text{Map}_\mathcal{X}(C(I), \colim_{\alpha \in A} D_\alpha(I))
\]

\[
= \text{Map}_{\text{CO}(\mathcal{X})}(C, \colim_{\alpha \in A} D_\alpha) .
\]

13.7.11 Corollary. Let \( \mathcal{H} \) be a profinite layered \( \infty \)-category, \( n \geq 0 \) an integer, and \( D : A \to \text{Pyk}(\text{Cat}_n) \) a filtered diagram. Then the natural functor

\[
\colim_{\alpha \in A} \text{Fun}^{\text{ct}}(\mathcal{H}, D_\alpha) \to \text{Fun}^{\text{ct}}(\mathcal{H}, \colim_{\alpha \in A} D_\alpha)
\]

is an equivalence.

Proof. In light of Example 13.3.10 and (13.7.10), note that both \( \mathcal{H} \) and \( \mathcal{H} \times [1] \) satisfy the hypotheses of Proposition 13.7.9. □
We finish this chapter by proving the Exodromy Theorem for \( \mathbb{Q}_\ell \) - and \( \overline{\mathbb{Q}}_\ell \)-coefficients.

**Proof of Theorem 13.7.8.** Note that for each projective compactum \( K \), the value \( \text{Perf}(E)(K) \) is a filtered colimit over \( t \)-exact functors. Hence combining Theorem 13.6.8, Lemma 13.7.4, and Corollary 13.7.11 we see that we have equivalences

\[
\text{D}_{\text{cont}}(X_{\text{proet}}; E) = \text{colim}_{E' \subseteq E} \text{D}_{\text{cont}}(X_{\text{et}}; O_{E'})[\ell^{-1}]
\]

\[
= \text{colim}_{E' \subseteq E} \text{Fun}^{\text{ct}}(\text{Gal}(X), \text{Perf}(O_{E'})[\ell^{-1}])
\]

\[
= \text{colim}_{n \geq 0} \text{colim}_{E' \subseteq E} \text{Fun}^{\text{ct}}(\text{Gal}(X), \text{Perf}(O_{E'})[\ell^{-1}])[-n, n]
\]

\[
= \text{Fun}^{\text{ct}}(\text{Gal}(X), \text{Perf}(E))[-n, n]
\]

\[
= \text{Fun}^{\text{ct}}(\text{Gal}(X), \text{Perf}(E))[-n, n]
\]

\[
\text{Fun}^{\text{ct}}(\text{Gal}(X), \text{Perf}(E))
\]

13.7.12. Let \( X \) be a topologically noetherian scheme, and let \( E \) be an algebraic extension of \( \mathbb{Q}_\ell \). Attached to any constructible sheaf \( F \) of \( E \)-complexes on \( X \), we have an associated exodromy representation

\[
\rho_F : \text{Gal}(X) \to \text{Perf}(E)
\]

that is sufficient to reconstruct \( F \).

For more general pyknotic rings \( \Lambda \), once one has a good pyknotic \( \infty \)-category \( \text{Perf}(\Lambda) \) of perfect \( \Lambda \)-complexes, it seems sensible simply to define constructible sheaves as continuous representations \( \text{Gal}(X) \to \text{Perf}(\Lambda) \) (even if \( X \) does not satisfy any noetherian hypotheses).

13.7.13. Recall the following well-known example of Deligne [15, Example 7.4.9]: let \( C \) be a smooth complete curve of genus at least 1 over an algebraically closed field, with two points identified. Let \( E \) be an algebraic extension of \( \mathbb{Q}_\ell \). The usual truncated \( \Pi_{\text{cont}}^\text{et}(C) \) étale homotopy type is insufficient to reconstruct \( E \)-local systems on \( C \).

However, the profinite stratified étale homotopy type does suffice to recover these local systems: the \( \infty \)-category of local systems of \( E \)-complexes on \( C \) is equivalent to the \( \infty \)-category of continuous functors \( \text{Gal}(C) \to \text{Perf}(E) \) that carry all morphisms of \( \text{Gal}(X) \) to equivalences.

The inclusion \( \text{Pyk}(S) \hookrightarrow \text{Pyk(Cat}_{\text{cont}}) \) admits a left adjoint \( \text{Epyk} = \text{Pyk}(E) \). For any coherent scheme \( X \), one can form the pyknotic étale homotopy type

\[
\Pi_{\text{pyk}}^\text{et}(X) := \text{Epyk}(\text{Gal}(X)).
\]

This pyknotic space suffices to reconstruct local systems with all the coefficient types discussed in this section. Its homotopy groups are the pyknotic étale homotopy groups \( \pi_\text{pyk}^\text{et}(X) \); one can recover the Bhatt–Scholze proétale fundamental group from \( \pi_1^\text{pyk}^\text{et}(X) \).
14 Perfectly reduced schemes & reconstruction of absolute schemes

We have shown that the étale $\infty$-topos $X_{\text{et}}$ of a coherent scheme $X$ can be reconstructed from the profinite $\infty$-category $\text{Gal}(X)$. Following Grothendieck’s Brief an Faltings [41, (8)], we can ask to what extent $X$ itself can be recovered from $X_{\text{et}}$. We first note that there are three easily-spotted obstacles to the conservativity of the functor $X \mapsto X_{\text{et}}$.

- One must restrict attention to schemes over a base with suitable finiteness conditions: for example, a nontrivial extension $\Omega \subset \Omega'$ of algebraically closed fields will give an equivalence of étale $\infty$-topoi (which are of course each trivial).
- The base must be sufficiently small: over $\mathbb{C}$, for example, any two smooth proper curves of the same genus have equivalent étale $\infty$-topoi.
- One must account for universal homeomorphisms: for example, the normalisation of the cuspidal cubic induces an equivalence of étale $\infty$-topoi. In fact, any universal homeomorphism induces an equivalence of étale $\infty$-topoi; this is the invariance topologique of the étale $\infty$-topos [SGA 1, Exposé IX, 4.10] and [SGA 4_1], Exposé VIII, 1.1].

The first two points compel us to impose serious finiteness conditions on our schemes, and this last point compels us to consider the $\infty$-category obtained from the 1-category $\text{Sch}$ of coherent schemes by inverting universal homeomorphisms. Fortunately, it is not necessary to do something excessively abstract: there is a 1-categorical colocalisation that performs this function; this is the perfection or absolute weak normalisation.

Section 14.1 analyzes the affect of universal homeomorphisms on Galois categories. Section 14.2 recalls how to characterize schemes that admit no nontrivial universal homeomorphisms. Section 14.3 shows that the subcategory of such schemes can be obtained from the category of all schemes by formally inverting the universal homeomorphisms. Section 14.4 discusses Grothendieck’s anabelian conjectures and proves Theorem A; that the Galois category is a complete invariant of a normal scheme over a finitely generated field of characteristic 0 (Theorem 14.4.7). Section 14.5 illustrates our main theorem by making explicit how to reconstruct curves from a combination of stratified-homotopy-theoretic and Galois-theoretic data. Section 14.6 explains what properties of morphisms of schemes can be detected on the level of Galois categories.

14.1 Universal homeomorphisms and equivalences of Galois categories

Now we arrive at a sensitive question: under which circumstances does a morphism of schemes induce an equivalence of étale topoi or, equivalently, of Galois categories? The well-known theorem here is Grothendieck’s invariance topologique of the étale topos [SGA 4_1], Exposé VIII, 1.1], which states that a universal homeomorphism induces an equivalence on étale topoi. Let us reprove this result with the aid of Galois categories; this will also provide us with a partial converse.

14.1.1 Proposition. Let $f : X \to Y$ be a morphism of coherent schemes. If $f$ is radicial, then every fibre of $\text{Gal}(X) \to \text{Gal}(Y)$ is either empty or a contractible groupoid. Conversely,
if $f$ is of finite type, and if every fibre of $\text{Gal}(X) \to \text{Gal}(Y)$ is either empty or a contractible groupoid, then $f$ is radicial.

Proof. If $f$ is radicial, then the map $X^{\text{zar}} \to Y^{\text{zar}}$ is an injection, and for any point $x_0 \in X^{\text{zar}}$, the map $\text{B}G_{\kappa(x_0)} \to \text{B}G_{\kappa(f(x_0))}$ on fibres is an equivalence since $\kappa(f(x_0)) \subseteq \kappa(x_0)$ is purely inseparable. So for any geometric point $y$ with image $y_0$, the fibre over $y$ is a contractible groupoid.

Conversely, if $f$ is of finite type, and if every fibre of $\text{Gal}(X) \to \text{Gal}(Y)$ is either empty or a contractible groupoid, then the map $X^{\text{zar}} \to Y^{\text{zar}}$ is an injection, so $f$ is in particular quasifinite. For any point $x_0 \in X^{\text{zar}}$, the fibres of the map $\text{B}G_{\kappa(x_0)} \to \text{B}G_{\kappa(f(x_0))}$ are all contractible, hence the map $\text{B}G_{\kappa(x_0)} \to \text{B}G_{\kappa(f(x_0))}$ is an equivalence. Now since $\kappa(f(x_0)) \subseteq \kappa(x_0)$ is a finite extension, it is purely inseparable.

14.1.2 Example. The finite type hypothesis in the second half of Proposition 14.1.1 is of course necessary, as any nontrivial extension $E \subset F$ of separably closed fields induces the identity on trivial Galois categories.

14.1.3 Corollary. Let $f : X \to Y$ be a morphism of coherent schemes. If $f$ is radicial and surjective, then every fibre of $\text{Gal}(X) \to \text{Gal}(Y)$ is a contractible groupoid. Conversely, if $f$ is of finite type, and if every fibre of $\text{Gal}(X) \to \text{Gal}(Y)$ is a contractible groupoid, then $f$ is radicial and surjective.

The following is the Valuative Criterion, along with a simple argument [STK, Tag 03K8] that allows one to extend the fraction field of the valuation ring therein.

14.1.4 Lemma. Let $f : X \to Y$ be a morphism of coherent schemes. Then the following are equivalent.

- The morphism $f$ is universally closed.
- For any absolutely integrally closed valuation ring $V$ with fraction field $K$ and (solid) commutative square
  \[
  \begin{array}{ccc}
  \text{Spec } K & \longrightarrow & X \\
  \downarrow & & \downarrow f \\
  \text{Spec } V & \longrightarrow & Y,
  \end{array}
  \]
  a (dotted) lift $\overline{\gamma} : \text{Spec } V \to X$ exists.

14.1.5. Recall that in Definition 12.4.7, we called such morphisms $\gamma$ and $\overline{\gamma}$ witnesses of morphisms in $\text{Gal}(Y)$ and $\text{Gal}(X)$. The previous lemma thus says that universal closedness is the guarantee that witnesses can be lifted when their targets can be.

14.1.6 Recollection. A functor between $\infty$-categories $f : C \to D$ is a right fibration if and only if, for any object $c \in C$, the induced functor $C_{/c} \to D_{/f(c)}$ is an equivalence of $\infty$-categories. Dually, $f$ is a left fibration if and only if $f^{\text{op}}$ is a right fibration, so that for any object $c \in C$, the induced functor $C_{/c} \to D_{/f(c)}$ is an equivalence of $\infty$-categories.

14.1.7 Proposition. Let $f : X \to Y$ be a morphism of coherent schemes. If $f$ is an integral morphism, then $\text{Gal}(X) \to \text{Gal}(Y)$ is a right fibration. Conversely, if $\text{Gal}(X) \to \text{Gal}(Y)$ is a right fibration, then $f$ is universally closed.
Proof. Assume that \( f \) is integral. Then for every geometric point \( x \to X \), the induced morphism \( X(x) \to Y(f(x)) \) is also integral, and by Schröer’s result \(^{111}\), Lemma 2.3, it is radicial as well. Hence at the level of Zariski topological spaces, \( X(x),\text{zar} \to Y(f(x)),\text{zar} \) is an inclusion of a closed subset; since source and target are each irreducible, and the inclusion carries the generic point to the generic point, it is a homeomorphism. (In fact, \( X(x) \to Y(f(x)) \) is a universal homeomorphism.) Thus

\[
\text{Gal}(X)_{/x} = \text{Gal}(X(x)) = X(x),\text{zar} \to Y(f(x)),\text{zar} = \text{Gal}(Y(f(x))) = \text{Gal}(Y)_{/f(x)}
\]

is an equivalence, whence \( \text{Gal}(X) \to \text{Gal}(Y) \) is a right fibration.

Conversely, assume that \( f \) is of finite type and that \( \text{Gal}(X) \to \text{Gal}(Y) \) is a right fibration. We employ Lemma 14.1.4 to show that \( f \) is universally closed; consider a witness \( \gamma : \text{Spec} V \to Y \) along with a diagram

\[
\begin{array}{ccc}
\text{Spec} K & \xrightarrow{\xi} & X \\
\downarrow & & \downarrow f \\
\text{Spec} V & \xrightarrow{\gamma} & Y
\end{array}
\]

in which \( K \) is the fraction field of \( V \). Let \( \psi : y \to f(\xi) \) be the morphism of \( \text{Gal}(Y) \) witnessed by \( y \), and let \( \phi : x \to \xi \) be a lift of \( \psi \) to \( \text{Gal}(X) \). We obtain a commutative square

\[
\begin{array}{ccc}
O^{\text{sh}}_{X,Y} & \xrightarrow{\gamma} & V \\
\downarrow & & \downarrow \\
O^{\text{sh}}_{X,x} & \xrightarrow{\xi} & K
\end{array}
\]

and since \( O^{\text{sh}}_{X,Y} \to O^{\text{sh}}_{X,x} \) is local, we obtain a lift \( \overline{\gamma} : O^{\text{sh}}_{X,x} \to V \) of \( \gamma \), as required. \( \square \)

A universal homeomorphism is a morphism that is radicial, surjective, and universally closed. An equivalence of categories is a right fibration with fibres contractible groupoids. We thus deduce:

14.1.8 Proposition. Let \( f : X \to Y \) be a morphism of coherent schemes. If \( f \) is a universal homeomorphism, then \( \text{Gal}(X) \to \text{Gal}(Y) \) is an equivalence. Conversely, if \( f \) is of finite type, and if \( \text{Gal}(X) \to \text{Gal}(Y) \) is an equivalence, then \( f \) is a universal homeomorphism (which is necessarily finite).

14.2 Perfectly reduced schemes

The notion of a perfect scheme is elsewhere defined only for \( \mathbb{F}_p \)-schemes. Here, we extend this notion to arbitrary reduced schemes in a way that restricts to the usual familiar notion on schemes in characteristic \( p \).

Just as a reduced scheme receives no nontrivial nilimmersions, a perfect scheme receives no nontrivial universal homeomorphisms. This is in fact a local condition that can be expressed in very concrete terms:
14.2.1 Lemma. The following are equivalent for a coherent scheme $X$.

(14.2.1.1) Any universal homeomorphism $X' \to X$ in which $X'$ is reduced is an isomorphism.

(14.2.1.2) Any universal homeomorphism $X' \to X$ admits a section.

(14.2.1.3) There exists an affine open covering $\{\text{Spec } A_i\}_{i \in I}$ of $X$ such that for every $i \in I$, the following conditions hold:

- for any $f, g \in A_i$, if $f^2 = g^3$, then there is a unique $h \in A_i$ such that $f = h^2$ and $g = h^3$; and
- for any prime number $p$ and any $f, g \in A_i$, if $f^p = p^p g$, then there is a unique element $h \in A_i$ such that $f = ph$ and $g = h^p$.

This is discussed in [STK, Tag oEUK]. See also [75, 1.4 and 1.7; 100, Appendix B; 120, Theorem 1].

14.2.2 Definition. We say that a coherent scheme $X$ is perfectly reduced – or, in the language of [100, B.1], absolutely weakly normal – if $X$ satisfies the equivalent conditions of Lemma 14.2.1. Denote by $\textbf{Sch}_{\text{perf}} \subset \textbf{Sch}$ the full subcategory spanned by the perfectly reduced schemes.

A say that a coherent scheme $X$ is seminormal if and only if there exists an affine open covering $\{\text{Spec } A_i\}_{i \in I}$ of $X$ such that for every $i \in I$ and any $f, g \in A_i$, if $f^2 = g^3$, then there is a unique $h \in A_i$ such that $f = h^2$ and $g = h^3$.

14.2.3 Example. A $\mathbb{Q}$-scheme is perfectly reduced if and only if it is seminormal.

Let $p$ be a prime number. A reduced $\mathbb{F}_p$-scheme is perfectly reduced if and only if it is perfect.

14.3 Perfection

We now show that $\textbf{Sch}_{\text{perf}}$ is the result of inverting the universal homeomorphisms in $\textbf{Sch}$. More precisely, we show that the inclusion $\textbf{Sch}_{\text{perf}} \to \textbf{Sch}$ admits a right adjoint $X \mapsto X_{\text{perf}}$ in which the counit $X_{\text{perf}} \to X$ is a universal homeomorphism. We first check that inverse limits of universal homeomorphisms are universal homeomorphisms.

14.3.1 Lemma. Let $X$ be a scheme. Let $A$ be an inverse category, and $W : A \to \textbf{Sch}_{/X}$ a diagram of $X$-schemes such that for any object $a \in A$, the structure morphism $p_a : W_a \to X$ is a universal homeomorphism. Then the natural morphism

$$p : W' = \lim_{a \in A^{op}} W_a \to X$$

is a universal homeomorphism.

Proof. All the bonding morphisms $W_a \to W_{a'}$ are universal homeomorphisms. It follows from [EGA IV, 8.3.8(i)] that $p$ is surjective. For any field $k$, the diagram

$$W(k) : A^{op} \to \text{Set}$$

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is a diagram of injections, hence for each $\alpha \in A^p$, the map $W'(k) \to W'_\alpha(k)$ is an injection; thus $p$ is a universal injection. It remains to show that $p$ is integral. Since $W$ is a diagram of affine $X$-schemes, it is enough to observe that the filtered colimit $\colim_{\alpha \in A} P_{\alpha, X} O_{W_{\alpha}}$ is an integral $O_X$-algebra.

**14.3.2 Proposition.** The inclusion $\mathbf{Sch}_{\text{perf}} \hookrightarrow \mathbf{Sch}$ admits a right adjoint, and the counit $X_{\text{perf}} \to X$ is a universal homeomorphism.

**Proof.** For any coherent scheme $X$, let $\mathbf{UH}_X \subset \mathbf{Sch}_X$ be the full subcategory spanned by the universal homeomorphisms $p: Y \to X$. The full subcategory $\mathbf{UH}_X' \subset \mathbf{UH}_X$ spanned by the finite universal homeomorphisms is limit-cofinal in $\mathbf{UH}_X$. Hence the limit of $X$-schemes

$$X_{\text{perf}} := \lim_{Y \in \mathbf{UH}_X} Y$$

exists and defines a universal homeomorphism $\varepsilon : X_{\text{perf}} \to X$. Any universal homeomorphism $Y \to X_{\text{perf}}$ admits a section, which proves that $X_{\text{perf}}$ is perfect. Moreover, if $Z$ is perfect, then for any morphism $f : Z \to X$, the pullback $Z \cong Z \times_X X_{\text{perf}} \to X_{\text{perf}}$ provides an inverse to the natural map

$$\text{Mor}_{\mathbf{Sch}}(Z, X_{\text{perf}}) \to \text{Mor}_{\mathbf{Sch}}(Z, X),$$

proving that $\varepsilon$ is a colocalisation of $\mathbf{Sch}$ relative to $\mathbf{Sch}_{\text{perf}}$. \hfill \Box

**14.3.3 Corollary.** The $\infty$-category obtained from the $1$-category $\mathbf{Sch}$ by inverting universal homeomorphisms is equivalent to $\mathbf{Sch}_{\text{perf}}$.

**14.3.4 Definition.** We call the right adjoint $X \mapsto X_{\text{perf}}$ the perfection functor or the absolute weak normalisation.

**14.3.5.** David Rydh [100, Appendix B] presented an alternative description of this functor: if $X$ is a reduced coherent scheme whose set of irreducible components is finite, or, respectively, an affine scheme, then one may form ‘the’ absolute integral closure $\overline{X}$ of $X$ [6] or, respectively, ‘the’ total integral closure $\overline{X}$ of $X$ [32; 50]. In either case, one can show that $X_{\text{perf}}$ is isomorphic to the weak normalisation of $X$ (in the sense of Andreotti–Bombieri [4, Teorema 2]) under $\overline{X} \to X$.

**14.3.6 Example.** For reduced $\mathbb{Q}$-schemes, the perfection is the seminormalisation [STK, Tag 0EUT].

**14.3.7 Example.** Let $p$ be a prime number. If $X$ is a reduced $\mathbb{F}_p$-scheme then by [16, Lemma 3.8] we have a natural isomorphism

$$X_{\text{perf}} \cong \lim \left( \cdots \xrightarrow{\text{Frob}_X} X \xrightarrow{\text{Frob}_X} X \right),$$

where $\text{Frob}_X$ is the absolute Frobenius.

**14.3.8 Definition.** Let $X$ and $Y$ be coherent schemes. A topological morphism from $X$ to $Y$ is an morphism $\phi : X_{\text{perf}} \to Y$. If $\phi$ induces an isomorphism $X_{\text{perf}} \xrightarrow{\sim} Y_{\text{perf}}$ then we call $\phi$ a topological equivalence from $X$ to $Y$. 

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14.3.9. Let $X$ and $Y$ be coherent schemes. Consider the following category $T(X, Y)$. The objects are diagrams

$$X \leftarrow X' \rightarrow Y$$

in which $X \leftarrow X'$ is a universal homeomorphism. A morphism

from $X \leftarrow X' \rightarrow Y$ to $X \leftarrow X'' \rightarrow Y$

is a commutative diagram

$\begin{array}{ccc}
X' & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f'} & Y' \\
\end{array}$

in which the vertical morphism is (of necessity) a universal homeomorphism. The nerve of the category $T(X, Y)$ is equivalent to the set $\text{Mor}(X_{\text{perf}}, Y) \cong \text{Mor}(X_{\text{perf}} Y_{\text{perf}})$ of topological morphisms from $X$ to $Y$.

14.3.10. The point now is that $\text{Gal}$, viewed as a functor from $\text{Sch}_{\text{perf}}$ to 1-categories, is conservative.

14.3.11 Definition. Let $P$ be a property of morphisms of schemes that is stable under base change and composition. We will say that a morphism $f : X \rightarrow Y$ is topologically $P$ if and only if $f$ is topologically equivalent to a morphism of schemes $f' : X' \rightarrow Y'$ with property $P$.

14.3.12. Let $P$ be a property of morphisms of schemes that is stable under base change and composition. The class of topologically $P$ morphisms is the smallest class of morphisms $P^t$ that contains $P$ and satisfies the following condition: for any commutative diagram

$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\phi} & & \downarrow{\psi} \\
X' & \xrightarrow{f'} & Y' \\
\end{array}$

in which $\phi$ and $\psi$ are universal homeomorphisms, the morphism $f$ lies in $P^t$ if and only if $f'$ does.

A morphism $f : X \rightarrow Y$ of perfectly reduced schemes is topologically $P$ precisely when $f$ factors as a universal homeomorphism $X \rightarrow X'$ followed by a morphism $X' \rightarrow Y$ with property $P$.

14.3.13 Example. A morphism $f : X \rightarrow Y$ of perfectly reduced schemes is topologically radical, surjective, universally closed, or integral if and only if $f$ is radical, surjective, universally closed, or integral (respectively).

14.3.14 Example. A morphism $f : X \rightarrow Y$ of perfectly reduced schemes is topologically étale if and only if $f$ is étale. Indeed, if $f' : X' \rightarrow Y$ is étale, then $X'$ is perfectly reduced $[100, \text{B.6(ii)}]$. 

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14.4 Grothendieck's conjecture and the proof of Theorem A

In this section we discuss the relationship between the Galois category of a coherent scheme and Grothendieck's anabelian programme.

14.4.1 Definition. We call a scheme $X$ absolute if $X$ is perfectly reduced scheme and the morphism $X \to \text{Spec} \mathbb{Z}$ is topologically essentially of finite type. Write $\text{Sch}_{\text{abs}} \subset \text{Sch}_{\text{perf}}$ for the subcategory whose objects are absolute schemes and whose morphisms are of finite type.

Chevalley’s Theorem ensures that any morphism of finite presentation between coherent schemes carries constructible sets to constructible sets. We codify this topological condition.

14.4.2 Definition. Let $S$ and $T$ be spectral topological spaces. We say that a quasicompact continuous map $f : S \to T$ is admissible if and only if the $f$ sends constructible subset of $S$ to constructible subsets of $T$.

Accordingly, we say that a morphism $\Pi' \to \Pi$ of profinite stratified spaces is admissible if and only if the induced quasicompact continuous map of spectral topological spaces $h_0 \Pi' \to h_0 \Pi$ is admissible. We write $\text{Str}^{\text{adm}} \subset \text{Str}^0$ for the subcategory whose objects are profinite stratified spaces and whose morphisms are admissible morphisms.

Likewise, if $X$ and $Y$ are bounded coherent $\infty$-topoi, we say that a coherent geometric morphism $f_* : X \to Y$ is admissible if and only if the induced quasicompact continuous map of spectral topological spaces $S(X) \to S(Y)$ is admissible (Notation 8.7.8). We write

$$\text{Top}_{\infty}^{bc, \text{adm}} \subset \text{Top}_{\infty}^{bc}$$

for the subcategory whose objects are bounded coherent $\infty$-topoi and whose morphisms are admissible geometric morphisms.

14.4.4 Conjecture. The functor

$$\text{Sch}_{\text{abs}} \to (\text{Top}_{\infty}^{bc, \text{adm}})_{/(\text{Spec} \mathbb{Z})}$$

given by the assignment $X \mapsto X_{et}$ is fully faithful. In particular, if $X$ and $Y$ are absolute schemes, then any admissible geometric morphism $X_{et} \to Y_{et}$ is induced by some morphism $X \to Y$ of finite type.

From this conjecture we may deduce a stratified anabelian result:

14.4.5 Corollary. Assume Conjecture 14.4.4; then the functor

$$\text{Sch}_{\text{abs}} \to (\text{Str}^0_{\text{adm}})_{/\text{Gal}(\text{Spec} \mathbb{Z})}$$

is fully faithful.
given by the assignment $X \mapsto \text{Gal}(X)$ is fully faithful. In particular, if $X$ and $Y$ are absolute schemes, then any admissible profinite functor $\text{Gal}(X) \to \text{Gal}(Y)$ is induced by a morphism $X \to Y$ of finite type.

An early paper of Voevodsky [119] provides a proof of Conjecture 14.4.4 for normal absolute schemes in characteristic 0.

14.4.6 Theorem ([119, Theorem 3.1]). Let $k$ be a finitely generated field of characteristic 0, and write $\text{Sch}_{\text{norm}}^k$ for the category of normal schemes of finite type over $k$. Then the functor

$$\text{Sch}_{\text{norm}}^k \to (\text{Top}_{\text{et}}^\text{bc,adm})_{/\text{Spec}^k}$$

given by the assignment $X \mapsto X_{\text{et}}$ is fully faithful.

Voevodsky also claims that his proof – with some modifications – will work when $k$ is a finitely generated field of characteristic $p$ and of transcendence degree $\geq 1$.

Voevodsky’s result combined with Conceptual Completeness (Theorem 3.11.2 = [SAG, Theorem A.9.0.6]) show that a morphism $f : X \to Y$ of such schemes is an isomorphism if and only if $f$ induces and equivalence on categories of points $\text{Pt}(X_{\text{et}}) \to \text{Pt}(Y_{\text{et}})$ of the corresponding étale $\infty$-topoi. Combining our $\infty$-categorical Hochster Duality Theorem with Voevodsky’s Theorem and our identification of $\widehat{\Pi}^\text{et}_{(\infty,1)}(X)$ with the topological category $\text{Gal}(X)$ (Construction 12.1.5), we can upgrade this conservativity result to the following strong reconstruction theorem for these schemes:

14.4.7 Theorem. Let $k$ be a finitely generated field of characteristic 0. Then for any normal $k$-schemes $X$ and $Y$ of finite type, the natural map

$$\text{Mor}^k_k(X, Y) \to \text{Mor}_{\text{BG}^k}(\text{Gal}(X), \text{Gal}(Y))$$

identifies $\text{Mor}^k_k(X, Y)$ with the subgroupoid of continuous functors $\text{Gal}(X) \to \text{Gal}(Y)$ that carry minimal objects to minimal objects.

In particular, if $X$ and $Y$ are normal $k$-schemes of finite type, and $\text{Gal}(X)$ and $\text{Gal}(Y)$ are equivalent as topological categories over $\text{BG}^k$, then $X$ and $Y$ are isomorphic.

Thus the category of normal $k$-schemes of finite type can be embedded as a subcategory of profinite categories with an action of $G_k$, as asserted in Theorem A.

The data of the map $\text{Gal}(X) \to \text{BG}^k$ is the same as a continuous $G_k$ action on the fibre, which is $\text{Gal}(X_{\overline{k}})$ for some algebraic closure $\overline{k} \supset k$. Thus by Corollary 12.6.4, a normal $k$-variety $X$ can be reconstructed from the profinite stratified space

$$\widehat{\Pi}^\text{et}_{(\infty,1)}(X_{\overline{k}}^{\text{an}}, X_{\overline{k}}^{\text{car}})$$

with its $G_k$-action.

14.5 Example: Curves

In this section, we illustrate our main theorem by making explicit how one may reconstruct a connected, smooth, complete curve over $k$ from a combination of stratified-homotopy-theoretic and Galois-theoretic data.
14.5.1 Construction. Let $n \geq 2$ be an integer. Let $X_n$ be the poset $\{0, 1, \ldots, n-1, \infty\}$, where $0, 1, \ldots, n-1$ are pairwise incomparable, and for each $i \in \{0, 1, \ldots, n-1\}$ one has $i < \infty$. Let $p_n : X_{n+1} \to X_n$ be the monotonic map defined by

$$p_n(i) = \begin{cases} i, & i \in \{0, 1, \ldots, n-1\} \\ \infty, & i \in \{n, \infty\} \end{cases}.$$ 

We thus obtain an inverse system of posets

$$\ldots \to X_4 \to X_3 \to X_2$$

whose limit $X$ in stratified topological spaces is the underlying Zariski topological space of any connected, normal curve.

Now let $g \geq 0$ be an integer. Let $C_{g,n} : sd^{op}(X_n) \to S$ be the following profinite spatial décollage over $X_n$. For any $i \in \{0, 1, \ldots, n-1\}$, set $C_{g,n}(i) = \{i\}$, and let $C_{g,n}(\infty)$ be the classifying space of the free group on generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, c_2, \ldots, c_{n-1}$.

For any $i \in \{0, 1, \ldots, n-1\}$, let $C_{g,n}(i < \infty)$ be the classifying space of the free group on a single generator $\zeta_i$. The morphisms $C_{g,n}(i < \infty) \to C_{g,n}(\infty)$ carry the generator $\zeta_i$ to

$$\begin{cases} (a_1, b_1)(a_2, b_2) \cdots (a_g, b_g)(c_1 c_2 \cdots c_{n-1})^{-1}, & i = 0 \\ c_i, & i \neq 0 \end{cases}.$$ 

Thus $C_{g,n}$ – or rather the corresponding $X_n$-stratified space – is the exit-path $\infty$-category of a closed smooth 2-manifold of genus $g$ relative to an $X_n$-stratification in which the closed strata are all points. Define a morphism of stratified spaces $f_n : C_{g,n+1} \to C_{g,n}$ over $p_n$ by carrying $a_i \mapsto a_i, b_j \mapsto b_j, c_i \mapsto c_i$ for $i \in \{0, 1, \ldots, n-1\}$, and $c_n$ to the identity. This defines an inverse system of stratified spaces

$$\ldots \to C_{g,4} \to C_{g,3} \to C_{g,2}.$$ 

After profinite completion, we obtain an inverse system of profinite stratified spaces

$$\ldots \to \widehat{C}_{g,4} \to \widehat{C}_{g,3} \to \widehat{C}_{g,2}$$

whose limit is an $X$-profinite stratified space that we will call $\widehat{C}_g$.

For the remainder of this section we fix a finitely generated field $k$ of characteristic 0 and an algebraic closure $\overline{k} \supset k$ of $k$. The following is immediate from Proposition 12.6.3.

14.5.2 Proposition. Let $C$ be a connected, smooth, complete curve over $k$ of genus $g$. Then $\text{Gal}(C_k)$ is equivalent to $\widehat{C}_g$.

Theorem 14.4.7 says that the curve $C$ can be reconstructed from the profinite stratified space $\text{Gal}(C_k) = \widehat{C}_g$ with its action of $G_k$.

To explain this point in more detail, let us make the following slightly tongue-in-cheek definition.
14.5.3 Definition. Let $k$ be a field. An *incorporeal field extension* of $k$ is a finite transitive $G_k$-set.

Galois theory shows that the assignment $E \mapsto \text{Gal}(E \otimes_k \overline{k})$ defines an equivalence from the category of finite extensions of $k$ to the category of incorporeal field extensions of $k$. We partially extend this to curves.

14.5.4 Definition. Let $k$ be a field. An *incorporeal curve over $k$ of genus $g$* is a continuous action of $G_k$ on $\hat{C}_g$.

A $k$-morphism from an incorporeal curve $(\hat{C}_{g_1}, \alpha_1)$ of genus $g_1$ to an incorporeal curve $(\hat{C}_{g_2}, \alpha_2)$ of genus $g_2$ is a $G_k$-equivariant functor

$$\hat{C}_{g_1} \to \hat{C}_{g_2}.$$ 

Let $S$ be an incorporeal field extension of $k$. An *S-point* of an incorporeal curve $(\hat{C}_g, \alpha)$ over $k$ is a $G_k$-equivariant functor $S \to \hat{C}_g$.

Incorporeal curves are completely group-theoretic objects. They amount to inverse families of free profinite groups along with actions of $G_k$.

14.5.5. Theorem 14.4.7 implies that the assignment $C \mapsto \text{Gal}(C_k)$ defines a fully faithful functor from connected, smooth, complete curves over $k$ to incorporeal curves over $k$.

Additionally, it allows one to reconstruct the points of $C$ from the corresponding incorporeal curve. For any finite extension $E \supset k$, we have a natural bijection between the set of $E$-points of $C$ and the set of $\text{Gal}(E \otimes_k \overline{k})$-points of $\text{Gal}(C_k)$.

### 14.6 Fibrations of Galois categories

We have already seen (Proposition 14.1.7) that an integral morphism of schemes induces a right fibration of Galois categories and that a morphism that induces a right fibration of Galois categories must be universally closed. Let us complete this picture.

Let us begin with an obvious characterisation of quasifinite morphisms. We will say that a functor has *finite fibres* if each of its fibres is equivalent to a finite set.

14.6.1 Lemma. Let $f : X \to Y$ be a finite type morphism of coherent schemes. Then $f$ is quasifinite if and only if $\text{Gal}(X) \to \text{Gal}(Y)$ has finite fibres.

Since proper quasifinite morphisms are finite, Proposition 14.1.7 now yields:

14.6.2 Proposition. Let $f : X \to Y$ be a morphism of coherent schemes that is separated and of finite type. Then $f$ is finite if and only if $\text{Gal}(X) \to \text{Gal}(Y)$ is a right fibration with finite fibres.

14.6.3 Proposition. Let $f : X \to Y$ be a morphism of coherent schemes. If $f$ is weakly étale, then $\text{Gal}(X) \to \text{Gal}(Y)$ is equivalent to a left fibration. Conversely, if $X$ and $Y$ are perfectly reduced, if $f$ is of finite presentation, and if $\text{Gal}(X) \to \text{Gal}(Y)$ is a left fibration with finite fibres, then $f$ is étale.
Proof. Assume that $f$ is weakly étale. Then for any geometric point $x \to X$, the morphism $X(x) \to Y(f(x))$ is an isomorphism, so the functor

$$\text{Gal}(X)_{x/} = \text{Gal}(X(x)) \to \text{Gal}(Y(f(x))) = \text{Gal}(Y)_{f(x)/}$$

is an equivalence. Thus $\text{Gal}(X) \to \text{Gal}(Y)$ is a left fibration.

Conversely, assume that $X$ and $Y$ are perfectly reduced, that $f$ is of finite presentation, and that $\text{Gal}(X) \to \text{Gal}(Y)$ is a left fibration with finite fibres. So the functor $\text{Gal}(X) \to \text{Gal}(Y)$ is classified by a continuous functor $\text{Gal}(Y) \to \text{Set}^{\text{fin}}$, which in turn corresponds to a constructible étale sheaf of finite sets on $Y$, which in particular coincides with the sheaf represented by $X$. Since the sheaf represented by $X$ is constructible, there exists an étale map $U \to Y$ and an effective epimorphism $U \to X$ of étale sheaves on $Y$. By descent, $X \to Y$ is étale. \qed

We may as well combine the last two entries in our dictionary.

14.6.4 Recollection. A Kan fibration is a functor that induces a Kan fibration on nerves. Equivalently, it is a functor that is both a left and right fibration. Equivalently, it is a functor $C \to D$ that is equivalent to the Grothendieck construction applied to a diagram of groupoids indexed on $D^{\text{op}}$ that carries every morphism to an equivalence of groupoids.

14.6.5 Proposition. Let $f : X \to Y$ be a morphism of perfectly reduced schemes that is separated and of finite presentation. Then $f$ is finite étale if and only if $\text{Gal}(X) \to \text{Gal}(Y)$ is a Kan fibration with finite fibres.

The following table provides a summary of the dictionary between perfectly reduced schemes and profinite Galois categories that we have created.
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