NOTES ON ÉTALE COHOMOLOGY

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Abstract. These notes outline the "fundamental theorems" of étale cohomology, following [4, Ch. vi], as well as briefly discuss the Weil conjectures.

Contents

1. Cohomological Dimension

1.1. **Definition.** Let X be a scheme, τ a topology on X, and ℓ a prime. A τ-sheaf $\mathcal{F}$ of abelian groups on X is ***ℓ-torsion*** if for every object $U \to X$ of the site X_τ such that U is quasicompact, the abelian group $\mathcal{F}(U)$ is ℓ-torsion.

Torsion sheaves are defined in the obvious way.

1.2. **Definition.** Let X be a scheme, τ a topology on X, and ℓ a prime. The ***ℓ-cohomological dimension*** $\mathrm{cd}_\ell(X_\tau)$ of X_τ is defined to be the smallest integer so that $\mathrm{H}^r_E(X; \mathcal{F}) = 0$ for all $r > \mathrm{cd}_\ell(X_\tau)$ and ℓ-torsion sheaves $\mathcal{F}$ on X_τ, and ∞ if no such integer exists.

The ***cohomological dimension*** of X_τ is defined by

$$\mathrm{cd}(X_\tau) \coloneqq \sup_{\ell \text{ prime}} \mathrm{cd}_\ell(X_\tau)$$

1.3. **Warning.** This definition of cohomological dimension is highly nonstandard and it is not obvious whether or not it agrees with the standard one from topos theory.

Date: October 19, 2017.

Acknowledgments: Ng Hoi Hei Janson.

1.4. **Theorem** (Tate [3, Thm. 15.2]). *If $k \subset K$ is a field extension, then*

$$\mathrm{cd}_\ell(K) \leq \mathrm{cd}_\ell(k) + \mathrm{trdeg}_k(K)\,,$$

where cd_ℓ refers to the étale ℓ-cohomological dimension of the field.

1.5. **Theorem** ([4, Ch. VI Thm. 1.1]). *If X is a scheme of finite type over a separably closed field, then $\mathrm{cd}_\ell(X_{\acute{e}t}) \leq 2\dim(X)$.*

1.6. **Corollary** ([4, Ch. VI Cor. 1.4]). *Let X be a scheme of finite type over a field k. For all primes $\ell \neq \mathrm{char}(k)$ we have*

$$\mathrm{cd}_\ell(X_{\acute{e}t}) \leq \mathrm{cd}_\ell(k) + 2\dim(X)\,.$$

2. The Proper Base Change Theorem

2.1. **Theorem** ([4, Ch. VI Thm. 2.1]). *If $\pi\colon Y \to X$ is a proper morphism of schemes and $\mathcal{F} \in \mathbf{Cnstr}(X)$, then the sheaves $R^i\pi_\star(\mathcal{F})$ are constructible for all $i \geq 0$.*

2.2. **Corollary** (proper base change [4, Ch. VI Cor. 2.3]). *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{\bar{f}} & Y \\ {\scriptstyle\bar{\pi}}\downarrow & \lrcorner & \downarrow{\scriptstyle\pi} \\ X' & \xrightarrow[f]{} & X \end{array}$$

be a pullback square of schemes. If π is proper and $\mathcal{F}$ is a torsion étale sheaf on Y, then the base change morphism

$$(f^{-1} \circ R^i\pi_\star)(\mathcal{F}) \to (R^i\bar{\pi}_\star \circ \bar{f}^{-1})(\mathcal{F})$$

is an isomorphism for all $i \geq 0$.

2.3. **Corollary** ([4, Ch. VI Cor. 2.5]). *Let $\pi\colon Y \to X$ be a proper morphism of schemes and $\bar{x} \to X$ a geometric point. If $\mathcal{F} \in \mathbf{Shv}_{\acute{e}t}(Y)$ is torsion, then there is a canonical isomorphism*

$$R^i\pi_\star(\mathcal{F})_{\bar{x}} \xrightarrow{\sim} \mathrm{H}^i_{\acute{e}t}(Y_{\bar{x}}; \mathcal{F}|_{\bar{x}})$$

for all $i \geq 0$.

If, moreover, all fibers of π have dimension at most n, then $R^i\pi_\star(\mathcal{F}) = 0$ for $i > 2n$. If all fibers of π have dimension at most n, X is a characteristic p scheme, and $\mathcal{F}$ is p-torsion, then $R^i\pi_\star(\mathcal{F}) = 0$ for $i > n$.

2.4. **Corollary** ([4, Ch. VI Cor. 2.6]). *Let $k \subset K$ be separably closed fields and X a proper k-scheme. Write $f\colon X_K \to X$ for the basechange projection. If $\mathcal{F} \in \mathbf{Shv}_{\acute{e}t}(X)$ is torsion, then the natural map*

$$\mathrm{H}^i_{\acute{e}t}(X; \mathcal{F}) \to \mathrm{H}^i_{\acute{e}t}(X_K; f^{-1}\mathcal{F})$$

is an isomorphism for all $i \geq 0$.

2.5. **Corollary** ([4, Ch. VI Cor. 2.7]). *Let A be a Henselian ring and $s_0 \in \mathrm{Spec}(A)$ the closed point. Let $\pi\colon X \to \mathrm{Spec}(A)$ be a proper morphism and $X_0 \coloneqq X_{s_0}$ the closed fiber. If $\mathcal{F} \in \mathbf{Shv}_{\acute{e}t}(X)$ is torsion, then the natural map*

$$\mathrm{H}^i_{\acute{e}t}(X; \mathcal{F}) \to \mathrm{H}^i_{\acute{e}t}(X_0; \mathcal{F}|_{X_0})$$

is an isomorphism for all $i \geq 0$.

3. Higher Direct Images with Compact Support

3.1. **Definition.** A morphism of schemes $\pi\colon X \to S$ is ***compactifiable*** if there exists a factorization

$$\begin{array}{ccc} X & \xhookrightarrow{j} & \overline{X} \\ & \underset{\pi}{\searrow} & \downarrow{\scriptstyle \overline{\pi}} \\ & & S\,, \end{array}$$

where j is an open immersion and $\overline{\pi}$ is proper.

A ***compactification*** of a compactifiable morphism π is a choice of a factorization $\pi = \overline{\pi} j$, where j is an open immersion and π is proper.

3.2. **Definition.** Let $\pi\colon X \to S$ be a compactifiable morphism and $\pi = \overline{\pi} j$ a compactification of π. Define $\mathbf{R}_c\pi_\star \coloneqq \mathbf{R}(\overline{\pi}_\star) \circ j_!$, and for $r \geq 0$ define

$$R^r_c\pi_\star \coloneqq R^r(\overline{\pi}_\star) \circ j_!\ .$$

3.3. **Warning.** The notations of Definition 3.2 are abusive as they depend on the choice of compactification and it is not generally possible to make a functorial choice of compactifications. In general $\mathbf{R}_c$ is not even functorial in proper maps of compactifiable S-schemes.

3.4. **Theorem** ([4, Ch. VI Thm. 3.1]). *If $\pi\colon X \to S$ is a compactifiable morphism and $\mathcal{F} \in \mathbf{Shv}_{\acute{e}t}(X)$ is torsion, then the sheaves $R^r_c\pi_\star(\mathcal{F})$ are independent of the choice of compactification of π.*

4. The Smooth Base Change Theorem

4.1. **Definition.** Let X be a scheme. The ***characteristic*** of X is the set

$$\operatorname{char}(X) \coloneqq |\operatorname{im}(X \to \operatorname{Spec}(\mathbf{Z}))| = \{\operatorname{char}(\kappa(x)) \mid x \in X\}\ ,$$

where $|{-}|$ denotes the underlying space of a scheme.

4.2. **Definition.** Let X be a scheme and $\mathcal{F} \in \mathbf{Shv}_{\acute{e}t}(X)$. We say that $\mathcal{F}$ ***has torsion prime to*** $\operatorname{char}(X)$ if for all *primes* $p \in \operatorname{char}(X)$, the multiplication by p map $\mathcal{F} \to \mathcal{F}$ is a monomorphism.

4.3. **Corollary** (smooth base change [4, Ch. VI Thm. 4.1]). *Let*

$$\begin{array}{ccc} Y' & \xrightarrow{\bar{f}} & Y \\ {\scriptstyle \bar{\pi}}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ X' & \xrightarrow[f]{} & X \end{array}$$

be a pullback square of schemes. If π is quasicompact, f is smooth, and $\mathcal{F} \in \mathbf{Shv}_{\acute{e}t}(Y)$ has torsion prime to $\operatorname{char}(X)$*, then the base change morphism*

$$(f^{-1} \circ R^i\pi_\star)(\mathcal{F}) \to (R^i\bar{\pi}_\star \circ \bar{f}^{-1})(\mathcal{F})$$

is an isomorphism for all $i \geq 0$.

4.4. **Corollary** ([4, Ch. VI Cor. 4.2]). *Let $\pi\colon Y \to X$ be a smooth proper morphism and $\mathcal{F} \in \mathbf{Shv}_{\acute{e}t}(Y)$ a constructible locally constant sheaf with torsion prime to* $\operatorname{char}(X)$*. Then for all $i \geq 0$, the sheaf $R^i\pi_\star(\mathcal{F})$ is constructible and locally constant.*

If, in addition, X is connected, then the groups $\mathrm{H}^i_{\acute{e}t}(Y_{\bar{x}}; \mathcal{F}|_{Y_{\bar{x}}})$ are isomorphic for all geometric points $\bar{x} \to X$.

4.5. **Corollary** ([4, Ch. VI Cor. 4.3]). *Let $k \subset K$ be separably closed fields and X a k-scheme. Write $f\colon X_K \to X$ for the basechange projection. If $\mathcal{F} \in \mathbf{Shv}_{ét}(X)$ is torsion with torsion prime to* $\mathrm{char}(X)$, *then the natural map*

$$\mathrm{H}^i_{ét}(X; \mathcal{F}) \to \mathrm{H}^i_{ét}(X_K; f^{-1}\mathcal{F})$$

is an isomorphism for all $i \geq 0$.

4.6. **Corollary** ([4, Ch. VI Cor. 4.5]). *Let $\pi\colon X \to S$ be a morphism of finite type schemes that are locally of finite type over a field k, and let $\mathcal{F} \in \mathbf{Cnstr}(X)$. If resolution of singularities holds for X (e.g., if* $\mathrm{char}(k) = 0$ *or* $\dim(X) \leq 2$*) and the torsion of $\mathcal{F}$ is prime to* $\mathrm{char}(X)$, *then the sheaves $R^i\pi_\star(\mathcal{F})$ are constructible for all $i \geq 0$.*

5. Purity

5.1. **Definition.** Let S be a scheme. A ***smooth S-pair*** (Z, X) is a closed immersion $i\colon Z \hookrightarrow X$ of S-schemes.

A smooth S-pair (Z, X) ***has codimension*** c if for all $s \in S$, the fiber Z_s has pure codimension c in X_s.

5.2. **Theorem** (cohomological purity [4, Ch. VI Thm. 5.1]). *Let S be a scheme and $i\colon Z \hookrightarrow X$ a smooth S-pair of codimension c. Let $\mathcal{F}$ be a locally constant torsion sheaf of torsion prime to* $\mathrm{char}(\mathcal{F})$. *Then the following equivalent statements hold.*

(5.2.a) *We have that $R^{2c}i^!(\mathcal{F})$ is locally isomorphic to $i^{-1}\mathcal{F}(-c)$ (as a sheaf on Z), and for $q \neq 2c$ we have that $R^q i^!(\mathcal{F}) = 0$.*

(5.2.b) *The unit $\mathcal{F} \to j_\star j^{-1}(\mathcal{F})$ is an isomorphism, $R^{2c-1}j_\star(j^{-1}\mathcal{F})$ is locally isomorphic to $i_\star i^{-1}\mathcal{F}$, and $R^q j_\star(j^{-1}\mathcal{F}) = 0$ for $q \neq 0, 2c-1$.*

5.3. **Observation.** Assume the hypotheses of Theorem 5.2, that $n \in \mathbf{Z}$ is prime to $\mathrm{char}(X)$, and the multiplication by n map $\mathcal{F} \to \mathcal{F}$ is zero. If $V \to X$ is a finite étale map and $s \in \Gamma(V; \mathcal{F})$, the map $\mathbf{Z}/n \to \mathcal{F}|_V$ defined by s defines a map

$$\mathrm{H}^{2c}_{Z \times V}(V; \mathbf{Z}/n) \to \mathrm{H}^{2c}_{Z \times V}(V; \mathcal{F}|_V)\ .$$

By varying V we get a morphism of étale sheaves on X

$$\mathcal{F} \to \mathrm{Hom}_X(i_\star R^{2c}i^!(\mathbf{Z}/n), i_\star R^{2c}i^!(\mathcal{F}))\ . \tag{5.4}$$

Since $R^{2c}i^!(\mathbf{Z}/n)$ is locally isomorphic to $\mathbf{Z}/n$ by Theorem 5.2, we see that $R^{2c}i^!(\mathbf{Z}/n)$ is locally free of finite type. Hence the proof of Theorem 5.2 shows that the morphism (5.4) is an isomorphism. Thus we may write the morphism (5.4) as an isomorphism

$$i^{-1}(\mathcal{F}) \otimes R^{2c}i^!(\mathbf{Z}/n) \xrightarrow{\sim} R^{2c}i^!(\mathcal{F})\ .$$

5.5. **Notation.** Let S be a scheme, $i\colon Z \hookrightarrow X$ a smooth S-pair of codimension c, and n an integer prime to $\mathrm{char}(X)$. We (abusively) write $T_{Z/X} \coloneqq R^{2c}i^!(\mathbf{Z}/n)$.

5.6. **Corollary** (Gysin sequence [4, Ch. VI Cor. 5.3]). *Let S be a scheme, $i\colon Z \hookrightarrow X$ a smooth S-pair of codimension c, n an integer prime to* $\mathrm{char}(X)$, *and $\mathcal{F} \in \mathbf{LC}_{ét}(X)$ be killed by n. Write $f\colon X \to S$ for the structure morphism, $j\colon U \coloneqq X \smallsetminus Z \hookrightarrow X$ for the open complement of Z, and write $i' \coloneqq fi$ and $j' \coloneqq fj$. Then for $0 \leq r \leq 2c-2$ the comparison map*

$$R^r f_\star(\mathcal{F}) \to R^r j'_\star(\mathcal{F}|_U)$$

is an isomorphism and there is a long exact sequence

$$0 \longrightarrow R^{2c-1} f_\star(\mathcal{F}) \longrightarrow R^{2c-1} j'_\star(\mathcal{F}|_U) \longrightarrow i'_\star(i^{-1}(\mathcal{F}) \otimes T_{Z/X}) \xrightarrow{i_\star} R^{2c} f_\star(\mathcal{F}) \longrightarrow \cdots$$

$$\cdots \longrightarrow R^{r-1} j'_\star(\mathcal{F}|_U) \to R^{r-2c} i'_\star(i^{-1}\mathcal{F} \otimes T_{Z/X}) \xrightarrow{i_\star} R^r f_\star(\mathcal{F}) \longrightarrow \cdots .$$

The maps labeled $i_\star$ are called ***Gysin maps.***

5.7. **Example** ([4, Ch. VI Ex. 5.6]). Let k be a separably closed field and $n \in \mathbf{Z}$ prime to $\mathrm{char}(k)$. Then

$$\mathrm{H}^r_{\acute{e}t}(\mathbf{P}^m_k; \mathbf{Z}/n) \cong \begin{cases} (\mathbf{Z}/n)(-r/2), & r \text{ even } 0 \le r \le 2m \\ 0, & \text{otherwise}. \end{cases}$$

With a few more results it is possible to show that if X is a smooth projective k-variety, for a generic curve $i\colon C \hookrightarrow X$, the map

$$i^\star\colon \mathrm{H}^1_{\acute{e}t}(X; (\mathbf{Z}/n)(1)) \to \mathrm{H}^1_{\acute{e}t}(C; (\mathbf{Z}/n)(1))$$

is an injection. Using Poincaré duality one can show that this result extends to smooth complete intersections.

5.8. **Theorem** (Deligne [4, Ch. VI Rem. 5.7]). *Let S be a quasicompact scheme. Then for any morphism $\pi\colon Y \to X$ of finite type S-schemes and $\mathcal{F} \in$ **Cnstr**$(Y; \mathbf{Z}/n)$, there exists a dense open subscheme $U_\pi \subset S$ such that:*

(5.8.a) *Over U_π, the sheaves $R^r\pi_\star(\mathcal{F})$ are constructible for all but finitely many $r \ge 0$.*

(5.8.b) *The formation of the sheaves $R^r\pi_\star(\mathcal{F})$ is compatible with all base changes along morphisms $T \to S$ with image in U_π.*

Moreover, if S is regular and 0 or 1-dimensional, the sheaves $R^r\pi_\star(\mathcal{F})$ are constructible (over S) for all $r \ge 0$.

6. Finiteness Theorems

6.1. **Corollary** ([4, Ch. VI Cor. 2.8]). *If X is a proper k-scheme and $\mathcal{F} \in$ **Cnstr**(X), then $\mathrm{H}^i_{\acute{e}t}(X; \mathcal{F})$ is finite for all $i \ge 0$.*

6.2. **Corollary** ([4, Ch. VI Cor. 5.5]). *Let X be a smooth k-variety and $\mathcal{F}$ a finite locally constant étale sheaf on X with torsion prime to* $\mathrm{char}(k)$. *Then $\mathrm{H}^i_{\acute{e}t}(X; \mathcal{F})$ is finite for all $i \ge 0$*

7. Fundamental Classes

7.1. **Notation.** Throughout this section k denotes a separably closed field, n a integer prime to $\mathrm{char}(k)$, and $\Lambda := \mathbf{Z}/n$.

7.2. **Conventions.** In this section all schemes are smooth k-varieties and all sheaves are Λ-modules.

7.3. **Observation.** If $i\colon Z \hookrightarrow X$ is a smooth k-pair of codimension c, then the spectral sequence

$$\mathrm{H}^p_{\acute{e}t}(Z; i^{-1}R^q i^!(\mathcal{F})) \Longrightarrow \mathrm{H}^{p+q}_Z(X; \mathcal{F})$$

gives canonical isomorphisms

$$\mathrm{H}^r_{\acute{e}t}(Z; i^{-1}R^{2c} i^!(\mathcal{F})) \xrightarrow{\sim} \mathrm{H}^{2c+r}_Z(X; \mathcal{F}),$$

for any locally free sheaf $\mathcal{F}$ of finite rank. In particular,

$$\Gamma(Z; i^{-1}R^{2c} i^!(\mathcal{F})) \xrightarrow{\sim} \mathrm{H}^{2c}_Z(X; \mathcal{F}).$$

7.4. **Observation.** If $i\colon Z \hookrightarrow X$ is a smooth divisor of codimension 1 with open complement $j\colon U \coloneqq X \smallsetminus Z \hookrightarrow X$, we have an isomorphism of exact sequences

$$\begin{array}{ccccccc}
\mathrm{H}^0_{\acute{e}t}(U;\mathbf{G}_m) & \xrightarrow{\partial} & \mathrm{H}^1_Z(X;\mathbf{G}_m) & \longrightarrow & \mathrm{H}^1_{\acute{e}t}(X;\mathbf{G}_m) & \xrightarrow{j^\star} & \mathrm{H}^1_{\acute{e}t}(U;\mathbf{G}_m) \\
\wr\downarrow & & \wr\downarrow & & \downarrow\wr & & \downarrow\wr \\
\Gamma(U;\mathcal{O}_U^\times) & \xrightarrow[\mathrm{ord}_Z]{} & \mathbf{Z} & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow[j^\star]{} & \mathrm{Pic}(U)
\end{array}$$

Moreover, the Kummer sequence gives an exact sequence

$$\begin{array}{ccccc}
\mathrm{H}^1_Z(X;\mathbf{G}_m) & \xrightarrow{\cdot n} & \mathrm{H}^1_Z(X;\mathbf{G}_m) & \xrightarrow{\partial} & \mathrm{H}^2_{\acute{e}t}(X;\boldsymbol{\mu}_n) \\
\wr\downarrow & & \wr\downarrow & & \Vert \\
\mathbf{Z} & \xrightarrow[\cdot n]{} & \mathbf{Z} & \xrightarrow[\partial]{} & \mathrm{H}^2_Z(X;\Lambda(1))\,.
\end{array} \tag{7.5}$$

7.6. **Definition.** If $Z \hookrightarrow X$ is a smooth divisor of codimension 1, then define the ***fundamental class*** $s_{Z/X} \in \mathrm{H}^2_Z(X;\Lambda(1))$ by setting $s_{Z/X} \coloneqq \partial(1)$, where ∂ is the boundary morphism in the sequence (7.5).

7.7. **Observations.**

(7.7.a) By the exactness of the sequence (7.5), $s_{Z/X}$ is n-torsion.

(7.7.b) The fundamental class $s_{Z/X}$ generates $R^2 i^!(\Lambda(1))$.

7.8. **Theorem** ([4, Ch. VI Thm. 6.1]). *There is a unique function $(Z,X) \mapsto s_{Z/X}$ sending a smooth k-pair of codimension c to a **fundamental class** $s_{Z/X} \in \mathrm{H}^{2c}_Z(X;\Lambda(c))$ satisfying the following:*

(7.8.a) *$s_{Z/X}$ has order n.*

(7.8.b) *If $c = 1$ and Z is connected, then $s_{Z/X}$ is the class of Definition 7.6.*

(7.8.c) *If $\phi\colon (Z',X') \to (Z,X)$ is a morphism of smooth k-pairs of codimension c, then $\phi^\star(s_{Z/X}) = s_{Z'/X'}$.*

(7.8.d) *If*

$$\begin{array}{ccc}
Z & \overset{v}{\hookrightarrow} & Y \\
& {\scriptstyle i}\searrow \quad \swarrow{\scriptstyle u} & \\
& X &
\end{array}$$

is a commutative triangle where (Z,Y), (Y,X), and (Z,X) are smooth k-pairs of codimensions a, b, and c, respectively, then $s_{Z/Y} \otimes s_{Y/X} = s_{Z/X}$, once we have made the canonical identifications

$$\mathrm{H}^{2a}_Z(Y;R^{2b}u^!(\Lambda(c))) \xrightarrow{\sim} \mathrm{H}^{2c}_Z(X;\Lambda(c))\,,$$

$$\mathrm{H}^{2a}_Z(Y;R^{2b}u^!(\Lambda(c))) \xrightarrow{\sim} \mathrm{H}^{2c}_Z(Y;\Lambda(a)) \otimes \mathrm{H}^{2b}_Y(X;\Lambda(b))\,.$$

7.9. **Corollary** ([4, Ch. VI Cor. 6.4]). *Let (Z,X) be a smooth k-pair of codimension c. Then $T_{Z/X}$ is canonically isomorphic to $\Lambda(-c)$.*

7.10. **Proposition** ([4, Ch. VI Prop. 6.5]).

(7.10.a) Projection formulas: *Let $i\colon Z \hookrightarrow X$ be a smooth k-pair of codimension c, write $i_\star\colon \mathrm{H}^r_{\acute{e}t}(Z;\Lambda) \to \mathrm{H}^{r+2c}_{\acute{e}t}(X;\Lambda(c))$ for the Gysin map, and let $1_Z \in \mathrm{H}^0_{\acute{e}t}(Z;\Lambda) \cong \Lambda$ denote the identity. Then*

— *$i_\star(1_Z)$ is the image of $s_{Z/X}$ under the map $\mathrm{H}^{2c}_Z(X;\Lambda(c)) \to \mathrm{H}^{2c}_{\acute{e}t}(X;\Lambda(c))$.*

— *For all* $x \in \mathrm{H}^{r+2c}_{ét}(X;\Lambda)$ *and* $z \in \mathrm{H}^{s}_{ét}(X;\Lambda)$ *we have* $i_\star(i^\star(x) \smile z) = x \smile i_\star(z)$. *In particular,* $i_\star i^\star(x) = x \smile i_\star(1_Z)$.

(7.10.b) Gysin maps compose: *If* $i_1 \colon Z \hookrightarrow Y$ *and* $i_2 Y \hookrightarrow X$ *are both smooth k-pairs, then* $(i_2 i_1)_\star = i_{2,\star} i_{1,\star}$.

8. The Weak Lefschetz Theorem

8.1. **Theorem** ([4, Ch. VI Thm. 7.1]). *Let X be an m-dimensional projective variety over a separably closed field and* $i \colon Z \hookrightarrow X$ *the inclusion of a hyperplane section.*

(8.1.a) *If X and Z are smooth,* $n \in \mathbf{Z}$ *is prime to* $\mathrm{char}(k)$, *and* $\mathcal{F} \in \mathbf{LC}_{ét}(X;\mathbf{Z}/n)$, *then the Gysin map* $i_\star \colon \mathrm{H}^r_{ét}(Z; i^{-1}\mathcal{F}) \to \mathrm{H}^{r+2}_{ét}(X;\mathcal{F}(1))$ *is an isomorphism for* $r \geq m$ *and a surjection for* $r = m-1$.

(8.1.b) *If* $\mathcal{F} \in \mathbf{Shv}_{ét}(X)$ *is torsion, then the map* $\mathrm{H}^r_Z(X;\mathcal{F}) \to \mathrm{H}^r_{ét}(X;\mathcal{F})$ *is an isomorphism for* $r \geq m+2$ *and a surjection for* $r = m+1$.

8.2. **Theorem** ([4, Ch. VI Thm. 7.2]). *If X is a scheme affine and of finite type over a separably closed field, then* $\mathrm{cd}(X) = \dim(X)$.

9. The Künneth Formula

9.1. **Convention.** Throughout this section Λ is a finite ring and unless otherwise stated all sheaves are Λ-modules.

9.2. **Theorem** (Künneth formula [4, Ch. VI Thm. 8.5]). *Consider a pullback square of schemes*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\bar{f}} & Y \\ {\scriptstyle \bar{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S\,, \end{array}$$

and set $h \coloneqq g\bar{f} = f\bar{g}$. *Assume the following:*

(9.2.a) *S is quasicompact,*

(9.2.b) *f and g are compactifiable,*

(9.2.c) $\mathcal{F} \in \mathbf{Shv}_{ét}(X;\Lambda)$, $\mathcal{G} \in \mathbf{Shv}_{ét}(Y;\Lambda)$, *and* $\mathcal{F}$ *is flat as a* Λ*-module.*

Then there is a canonical quasi-isomorphism

$$\mathbf{R}_c f_\star(\mathcal{F}) \otimes^{\mathbf{L}}_\Lambda \mathbf{R}_c g_\star(\mathcal{G}) \simeq \mathbf{R}_c h_\star(\mathcal{F} \boxtimes^{\mathbf{L}}_\Lambda \mathcal{G})$$

If, moreover, $R^r_c f_\star(\mathcal{F})$ *is flat for all* $r \geq 0$, *then for all* $m \geq 0$ *we have a canonical isomorphism*

$$\bigoplus_{r+s=m} R^r_c f_\star(\mathcal{F}) \otimes_\Lambda R^s_c g_\star(\mathcal{G}) \simeq R^m_c h_\star(\mathcal{F} \boxtimes_\Lambda \mathcal{G})\,.$$

9.3. **Lemma** (proper base change [4, Ch. VI Lem. 8.9]). *Consider a pullback square of schemes*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\bar{f}} & Y \\ {\scriptstyle \bar{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & S\,, \end{array}$$

where g is proper, and let $\mathcal{G} \in \mathbf{Shv}_{ét}(Y;\Lambda)$. *Then there is a canonical quasi-isomorphism* $f^\star \mathbf{R} g_\star(\mathcal{G}) \simeq \mathbf{R}\bar{g}_\star(\bar{f}^\star \mathcal{G})$.

9.4. **Corollary** ([4, Ch. VI Cor. 8.13]). *Let X and Y be schemes over a separably closed field k, $\mathcal{F} \in \mathbf{Shv}_{\acute{e}t}(X;\Lambda)$, $\mathcal{G} \in \mathbf{Shv}_{\acute{e}t}(Y;\Lambda)$. If X and Y are proper over k, $\mathcal{F}$ is flat, and $\mathrm{H}^r_{\acute{e}t}(X;\mathcal{F})$ is flat for all $r \geq 0$, then for all $m \geq 0$ we have a canonical isomorphism*

$$\bigoplus_{r+s=m} \mathrm{H}^r_{\acute{e}t}(X;\mathcal{F}) \otimes_\Lambda \mathrm{H}^s_{\acute{e}t}(Y;\mathcal{G}) \xrightarrow{\sim} \mathrm{H}^m_{\acute{e}t}(X \times Y;\mathcal{F} \boxtimes_\Lambda \mathcal{G})$$

induced by the cup product pairings.

9.5. **Theorem** (ℓ-adic Künneth formula [4, Ch. VI Cor. 8.21]). *Let k be a separably closed field and ℓ a prime different from* $\mathrm{char}(k)$. *Let X and Y be compactifiable k-schemes and A the integral closure of $\mathbf{Z}_\ell$ in a finite extension of $\mathbf{Q}_\ell$. If $\mathcal{F} \in \mathbf{Cnstr}(X;A)$, $\mathcal{G} \in \mathbf{Cnstr}(Y;A)$, and $\mathcal{F}$ and $\mathcal{G}$ are flat A-modules, then:*

(9.5.a) *We have a canonical isomorphism $\mathbf{H}^{\bullet}_c(X;\mathcal{F}) \otimes_A \mathbf{H}^{\bullet}_c(Y;G) \xrightarrow{\sim} \mathbf{H}^{\bullet}_c(X \times Y;\mathcal{F} \boxtimes \mathcal{G})$.*

(9.5.b) *For all $m \geq 0$ there is a short exact sequence*

$$0 \longrightarrow \bigoplus_{r+s=m} \mathrm{H}^r_c(X;\mathcal{F}) \otimes_A \mathrm{H}^s_c(Y;\mathcal{G}) \longrightarrow \mathrm{H}^m_c(X \times Y;\mathcal{F} \boxtimes_A \mathcal{G}) \longrightarrow \bigoplus_{r+s=m} \mathrm{Tor}^A_1(\mathrm{H}^r_c(X;\mathcal{F}), \mathrm{H}^s_c(Y;\mathcal{G})) \longrightarrow 0\,.$$

10. The Cycle Class Map

10.1. **Convention.** Fix an algebraically closed field k. All schemes in this section are smooth k-varieties.

10.2. **Notation.** Fix an integer n prime to $\mathrm{char}(k)$, and write $\Lambda \coloneqq \mathbf{Z}/n$. For a smooth k-variety X, write $\mathrm{H}^*_{\acute{e}t}(X;\Lambda) \coloneqq \bigoplus_{r\geq 0} \mathrm{H}^{2r}_{\acute{e}t}(X;\Lambda(r))$.

10.3. **Remark.** As a graded group, the degree r piece of $\mathrm{H}^*_{\acute{e}t}(X;\Lambda)$ is $\mathrm{H}^{2r}_{\acute{e}t}(X;\Lambda(r))$. Moreover, the cup product makes $\mathrm{H}^*_{\acute{e}t}(X;\Lambda)$ a graded-commutative ring.

10.4. **Recollection.** Let X be a smooth k-variety. An ***elementary r-cycle*** on X is a closed integral subscheme $Z \subset X$ of codimension r. The ***group of algebraic r-cycles*** $C^r(X)$ is the free abelian group on the set of elementary r-cycles. Write

$$C^*(X) \coloneqq \bigoplus_{r\geq 0} C^r(X)\,.$$

Elements of $C^*(X)$ are called ***algebraic cycles*** on X.

10.5. **Definition.** Let X be a smooth k-variety. An elementary r-cycle $Z \subset X$ and an elementary r'-cycle $Z' \subset X$ ***intersect properly*** if each irreducible component of $Z \cap Z'$ has codimension $r + r'$. In this case $Z \cdot Z'$ is defined and belongs to $C^{r+r'}(X)$.

Algebraic cycles $Z, Z' \in C^*(X)$ ***intersect properly*** if every elementary cycle of Z intersects every elementary cycle of Z' properly, in which case $Z \cdot Z'$ is defined in the obvious manner.

10.6. **Observation.** For a morphism $\pi\colon Y \to X$ of smooth k-varieties we can sometimes define a morphism $\pi^\star : C^*(X) \to C^*(Y)$ by setting

$$\pi^\star(Z) \coloneqq \Gamma_\pi \cdot (Y \times Z)$$

for an elementary cycle $Z \subset X$, where $\Gamma_\pi \subset Y \times X$ is the graph of π, and extending linearly to algebraic cycles. The map $\pi^\star$ makes sense as long as all of the intersection products $\Gamma_\pi \cdot (Y \times Z)$ are well-defined.

10.7. **Remark.** According to [1, Prop. 1.3], $\pi^\star$ is well-defined if π is a flat map of algebraic k-schemes (though we only consider *smooth* k-schemes), or in the case that both the source and target are smooth (which is the case that we are interested in).

10.8. **Observation.** If $\pi\colon Y \to X$ is a proper morphism of smooth k-varieties, there is a morphism $\pi_\star : C^*(Y) \to C^*(X)$ defined on elementary cycles $Z \subset Y$ by

$$\pi_\star(Z) \coloneqq \begin{cases} 0, & \dim(\pi(Z)) < \dim(Z) \\ \deg(\pi|_Z)\pi(Z), & \text{otherwise}, \end{cases}$$

and extended linearly to $C^*(Y)$.

10.9. **Lemma** (projection formula for cycles). *Let $\pi\colon Y \to X$ be a proper morphism of smooth k-varieties, $Z \in C^*(Y)$, and $Z' \in C^*(X)$. Then*

$$\pi_\star(\pi^\star(Z') \cdot Z) = Z' \cdot \pi_\star(Z),$$

if the intersection products $\pi^\star(Z') \cdot Z$ and $Z' \cdot \pi_\star(Z)$ are defined.

10.10. **Definition.** Let X be a smooth k-variety. The ***cycle class map*** $\mathrm{cl}_X : C^*(X) \to \mathrm{H}^*_{\acute{e}t}(X;\Lambda)$ is the homomorphism of graded groups defined as follows.

(10.10.a) If $i\colon Z \hookrightarrow X$ is a *smooth* elementary r-cycle, then define $\mathrm{cl}_X(Z) \coloneqq i_\star(1_Z)$, where

$$i_\star : \mathrm{H}^0_{\acute{e}t}(Z;\Lambda) \to \mathrm{H}^{2r}_{\acute{e}t}(X;\Lambda(r))$$

is the Gysin map and $1_Z \in \mathrm{H}^0_{\acute{e}t}(Z;\Lambda)$ is the identity. Equivalently, $\mathrm{cl}_X(Z)$ is the image of the fundamental class $s_{Z/X}$ under the map $\mathrm{H}^{2r}_Z(X;\Lambda(r)) \to \mathrm{H}^{2r}_{\acute{e}t}(X;\Lambda(r))$.

(10.10.b) This definition extends to singular elementary cycles by virtue of the following lemma, and extends to each graded piece $C^r(X)$ by linearity.

10.11. **Lemma** ([4, Ch. VI Lem. 9.1]). *Let X be a smooth k-variety. For any reduced closed subscheme $Z \subset X$ of codimension r, we have $\mathrm{H}^s_Z(X;\Lambda) = 0$ for $s < 2r$.*

10.12. **Proposition** ([4, Ch. VI Prop. 9.3]). *Let $i\colon Z \hookrightarrow X$ be a closed immersion of smooth k-varieties and $c \coloneqq \mathrm{codim}_X(Z)$. Then for all $r \geq 0$, the square*

$$\begin{array}{ccc} C^r(Z) & \hookrightarrow & C^{r+c}(X) \\ \downarrow{\scriptstyle \mathrm{cl}_Z} & & \downarrow{\scriptstyle \mathrm{cl}_X} \\ \mathrm{H}^{2r}_{\acute{e}t}(Z;\Lambda(r)) & \xrightarrow[i_\star]{} & \mathrm{H}^{2(r+c)}_{\acute{e}t}(X;\Lambda(r+c)) \end{array}$$

commutes.

10.13. **Proposition** ([4, Ch. VI Prop. 9.4]). *Let X and Y be smooth k-varieties. The square*

$$\begin{array}{ccc} C^*(X) \times C^*(Y) & \xrightarrow{\times} & C^*(X \times Y) \\ \downarrow{\scriptstyle \mathrm{cl}_X \times \mathrm{cl}_Y} & & \\ \mathrm{H}^*_{\acute{e}t}(X;\Lambda) \times \mathrm{H}^*_{\acute{e}t}(Y;\Lambda) & & \downarrow{\scriptstyle \mathrm{cl}_{X\times Y}} \\ \downarrow{\scriptstyle \mathrm{pr}_X^\star \times \mathrm{pr}_Y^\star} & & \\ \mathrm{H}^*_{\acute{e}t}(X \times Y;\Lambda)^{\times 2} & \xrightarrow[\smile]{} & \mathrm{H}^*_{\acute{e}t}(X \times Y;\Lambda) \end{array}$$

commutes.

10.14. **Proposition** ([4, Ch. VI Prop. 9.4]). *Let X be a smooth k-variety and Z and Z' algebraic cycles on X. If Z and Z' intersect transversally, then*

$$\mathrm{cl}_X(Z \cdot Z') = \mathrm{cl}_X(Z) \smile \mathrm{cl}_X(Z') \,.$$

10.15. **Remark** ([4, Ch. VI Rem. 9.6]). The map $\mathrm{cl}_X^1 \colon C^1(X) \to \mathrm{H}^2_{\acute{e}t}(X; \Lambda(1))$ is the composite of the canonical maps $C^1(X) \to \mathrm{Pic}(X)$ given by $Z \mapsto O_X(Z)$ and $\mathrm{Pic}(X) \to \mathrm{H}^2_{\acute{e}t}(X; \Lambda(1))$ coming from the identifications $\mathrm{Pic}(X) \cong \mathrm{H}^1_{\acute{e}t}(X; \mathbf{G}_m)$ and $\Lambda(1) \cong \boldsymbol{\mu}_n$ along with the Kummer sequence.

10.16. **Example** ([4, Ch. VI Ex. 9.7]). If $L^r \subset \mathbf{P}^m_k$ is a complete intersection of codimension r, then the Gysin map $\Lambda \to \mathrm{H}^{2r}_{\acute{e}t}(\mathbf{P}^m_k; \Lambda(r))$ is an isomorphism. Hence $\mathrm{H}^{2r}_{\acute{e}t}(\mathbf{P}^m_k; \Lambda(r))$ is generated by $\mathrm{cl}^r_{\mathbf{P}^m_k}(L^r)$.

(10.16.a) Since $\mathrm{Pic}(\mathbf{P}^m_k) \cong \mathbf{Z}$ is generated by the class of any hypersurface L^1, $\mathrm{cl}_{\mathbf{P}^m_k}(L^1)$ is independent of the hypersurface L^1.

(10.16.b) Thus for a codimension r complete intersection L^r we have $\mathrm{cl}_{\mathbf{P}^m_k}(L^r) = \mathrm{cl}_{\mathbf{P}^m_k}(L^1)^{\smile r}$. So the cycle class of a codimension r complete intersection is also independent of L^r.

(10.16.c) Thus the map $\Lambda[t]/\langle t^{m+1}\rangle \to \mathrm{H}^*_{\acute{e}t}(\mathbf{P}^m_k; \Lambda)$ defined by sending $t^r \mapsto \mathrm{cl}_{\mathbf{P}^m_k}(L^1)^{\smile r}$ is an isomorphism of graded rings.

11. Chern Classes

11.1. **Convention.** Let k be an algebraically closed field. In this section all schemes are smooth quasiprojective k-varieties.

11.2. **Notation.** Let $n \in \mathbf{Z}$ be prime to $\mathrm{char}(k)$. We write $\Lambda \coloneqq \mathbf{Z}/n$.

11.3. **Proposition** (projective bundle formula [4, Ch. VI Prop. 10.1]). *Let X be a smooth quasiprojective k-variety and $\mathcal{E}$ a rank m vector bundle on X. Let $\xi \in \mathrm{H}^2_{\acute{e}t}(\mathbf{P}(\mathcal{E}); \Lambda(1))$ be the image of $O_{\mathbf{P}(\mathcal{E})}(1)$ under the map*

$$\mathrm{Pic}(\mathbf{P}(\mathcal{E})) \to \mathrm{H}^2_{\acute{e}t}(\mathbf{P}(\mathcal{E}); \Lambda(1))$$

coming from the Kummer sequence. Let $\pi \colon \mathbf{P}(\mathcal{E}) \to X$ be the structure morphism. Then the map

$$\mathrm{H}^*_{\acute{e}t}(X; \Lambda)[t]/\langle t^m\rangle \to \mathrm{H}^*_{\acute{e}t}(\mathbf{P}(\mathcal{E}); \Lambda)$$

given by $\pi^\star$ on $\mathrm{H}^*_{\acute{e}t}(X; \Lambda)$ and by sending $t \mapsto \xi$ is an isomorphism of graded $\mathrm{H}^*_{\acute{e}t}(X; \Lambda)$-modules.

11.4. **Corollary.** *Let X be a smooth quasiprojective k-variety and $\mathcal{E}$ a rank m vector bundle on X. There are unique elements $c_r(\mathcal{E}) \in \mathrm{H}^{2r}_{\acute{e}t}(X; \Lambda)$ such that*

$$\begin{cases} c_0(\mathcal{E}) = 1 & \\ c_r(\mathcal{E}) = 0, & \textit{for } r > m \\ \sum_{r=0}^m c_r(\mathcal{E})\xi^{m-r} = 0 \,. & \end{cases}$$

*The element $c_r(\mathcal{E})$ is called the r^{th} **Chern class** of $\mathcal{E}$.*

11.5. **Definition.** Let X be a smooth quasiprojective k-variety and $\mathcal{E}$ a vector bundle on X. The ***total Chern class*** of X is the element

$$c(\mathcal{E}) \coloneqq \sum_{r \geq 0} c_r(\mathcal{E}) \in \mathrm{H}^*_{\acute{e}t}(X; \Lambda) \,.$$

11.6. **Definition.** Let X be a smooth quasiprojective k-variety and $\mathcal{E}$ a vector bundle on X. The ***Chern polynomial*** of X is the element

$$\mathrm{ch}_t(\mathcal{E}) \coloneqq \sum_{r\geq 0} c_r(\mathcal{E})t^r \in \bigoplus_{r\geq 0} \mathrm{H}^{2r}_{ét}(X;\Lambda(r))t^r \,.$$

11.7. **Notation.** Write $\mathbf{Var}_k$ for the category of smooth quasiprojective k-varieties.

11.8. **Theorem** ([4, Ch. VI Thm. 10.3]). *The Chern classes of smooth quasiprojective k-varieties are uniquely characterized by the following properties.*

(11.8.a) Functoriality: *If $\pi\colon Y \to X$ is a morphism in $\mathbf{Var}_k$ and $\mathcal{E}$ is a vector bundle on X, then for all $r \geq 0$ we have $c_r(\pi^{-1}(\mathcal{E})) = \pi^\star(c_r(\mathcal{E}))$.*

(11.8.b) Normalization: *If $\mathcal{E}$ is a vector bundle on X, then $\mathrm{ch}_t(\mathcal{E}) = 1 + p_X(\mathcal{E})t$, where*

$$p_X\colon \mathrm{Pic}(X) \to \mathrm{H}^2_{ét}(X;\Lambda)$$

is the natural morphism from the Kummer sequence.

(11.8.c) Additivity: *If $X \in \mathbf{Var}_k$ and*

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0\,.$$

is a short exact sequence of vector bundles on X, then $\mathrm{ch}_t(\mathcal{E}) = \mathrm{ch}_t(\mathcal{E}')\,\mathrm{ch}_t(\mathcal{E}'')$.

11.9. **Remark.** By (11.8.c) the total Chern class defines a homomorphism of abelian groups $c\colon K_0(X) \to \mathrm{H}^*_{ét}(X;\Lambda)$. Moreover, this homomorphism respects the gradings.

11.10. **Corollary.** *Let $X \in \mathbf{Var}_k$. If $(\dim(X)-1)!$ is invertible in Λ, then the cycle class map cl_X defines a homomorphism $\mathrm{cl}_X\colon \mathrm{CH}^\star(X) \to \mathrm{H}^*_{ét}(X;\Lambda)$ which is natural in X.*

11.11. **Remark.** Corollary 11.10 is true with Λ replaced by $\mathbf{Q}_\ell$, where $\ell \neq \mathrm{char}(k)$. In this case, the invertibility condition on $(\dim(X)-1)!$ is vacuous.

12. Poincaré Duality

12.1. **Notation.** We write $\Lambda \coloneqq \mathbf{Z}/n$, and n is always taken to be prime to $\mathrm{char}(X)$ for any scheme X that we take cohomology of with coefficients in a Λ-module.

12.2. **Notation.** If X is a seperated variety of dimension d and $x \in X$ is a closed point, we abusively write $\mathrm{cl}_X(x)$ for the image of the fundamental class $s_{x/X}$ under the natural map

$$\mathrm{H}^{2d}_x(X;\Lambda(d)) \to \mathrm{H}^{2d}_c(X;\Lambda(d))\,.$$

12.3. **Theorem** (Poincaré duality for smooth varieties [4, Ch. VI Thm. 11.1]). *Let k be a separably closed field and X a dimension d smooth separated k-variety. Then:*

(12.3.a) *There is a unique **trace map** $\eta_X\colon \mathrm{H}^{2d}_c(X;\Lambda(d)) \to \Lambda$ such that $\eta_X(\mathrm{cl}_X(x)) = 1$ for all closed points $x \in X$. Moreover, η_X is an isomorphism.*

(12.3.b) *If $\mathcal{F} \in \mathbf{Cnstr}(X;\Lambda)$, the natural pairings*

$$\mathrm{H}^r_c(X;\mathcal{F}) \times \mathrm{Ext}^{2d-r}_X(\mathcal{F},\Lambda(d)) \longrightarrow \mathrm{H}^{2d}_c(X;\Lambda(d)) \xrightarrow[\eta_X]{\sim} \Lambda$$

are perfect for $0 \leq r \leq 2d$.

Proof Idea. To construct the trace map:

(12.3.a.i) If X is a dimension d separated variety over a separably closed field, then

$$\mathrm{H}^{2d}_c(X;\Lambda(d)) \simeq \Lambda\,.$$

(12.3.a.ii) If $\pi\colon Y \to X$ is a separated étale morphism, where X is a dimension d separated variety over a separably closed field, then for every closed point $y \in Y$,

$$\pi_\star\colon \mathrm{H}_c^{2d}(Y;\Lambda(d)) \to \mathrm{H}_c^{2d}(X;\Lambda(d))$$

sends $\mathrm{cl}_Y(y)$ to $\mathrm{cl}_X(\pi(y))$.

To prove that the pairing is perfect:

(12.3.b.i) If $\pi\colon X' \to X$ is finite étale, the pairings are perfect for $(X', \mathcal{F}')$ if and only if they are perfect for $(X, \pi_\star \mathcal{F}')$.

(12.3.b.ii) If $U \subset X$ is an open subvariety, the pairings are perfect for $(X, \mathcal{F})$ are perfect for $(X, \mathcal{F})$ if and only if they are perfect for $(U, \mathcal{F}|_U)$. To do this, we show that the pairings are prefect for sheaves with support in a smooth closed subvariety $Z \subsetneq X$.

(12.3.b.iii) If $\pi\colon X \to S$ is a smooth projective morphism with 1-dimensional fibers, the pairings are perfect for locally constant constructible sheaves. To do this, we show that if X is a variety for which the pairings are perfect, then the appropriate derived versions of the pairings are perfect.

(12.3.b.iv) The pairings are perfect for constant sheaves.

(12.3.b.v) The pairings are perfect for locally constant sheaves.

(12.3.b.vi) By pulling back to an open subvariety on which a constructible sheaf is locally constant, we conclude the theorem. □

12.4. **Corollary** ("classical" Poincaré duality [4, Ch. VI Cor. 11.2]). *Let X be a dimension d separated variety over a separably closed field. Then for any locally constant constructible Λ-module, the cup product pairing*

$$\mathrm{H}_c^r(X;\mathcal{F}) \times \mathrm{H}_c^{2d-r}(X;\mathcal{F}(d)) \xrightarrow{\smile} \mathrm{H}_c^{2d}(X;\Lambda(d)) \xrightarrow[\eta_X]{\sim} \Lambda$$

is perfect for $0 \le r \le 2d$.

12.5. **Corollary** ([4, Ch. VI Cor. 11.5]). *Let X be a dimension d separated variety over a separably closed field. If X is not complete, then for any locally constant torsion sheaf $\mathcal{F}$ on X we have $\mathrm{H}_{ét}^{2d}(X;\mathcal{F}) = 0$.*

12.6. **Theorem** ([4, Ch. VI Thm. 11.7]). *Let X be a dimension d projective variety over a separably closed field. Then for all $r \ge 0$, the group $N^r(X)$ of r-cycles modulo numerical equivalence is finitely generated.*

12.7. **Proposition** (Poincaré duality for compactifiable morphisms [4, Ch. VI Prop. 11.8]). *Let $\pi\colon X \to S$ be a smooth compactifiable morphism of schemes with d-dimensional fibers. There exists a unique* ***relative trace map*** *$\eta_{X/S}\colon R_c^{2d}\pi_\star(\Lambda(d)) \to \Lambda$ such that for any geometric point $\bar{s} \to S$ and any closed point $x \in X_{\bar{s}}$, the trace map*

$$\eta_{X_{\bar{s}}/\bar{s}}\colon \mathrm{H}_c^{2d}(X_{\bar{s}};\Lambda(d)) \to \Lambda$$

sends $\mathrm{cl}_{X_{\bar{s}}}(x)$ to 1. If, moreover, the fibers of π are connected, then $\eta_{X/S}$ is an isomorphism.

Moreover, $\eta_{X/S}$ is compatible with composition of compactifiable morphisms and base change along arbitrary morphisms.

Proof Idea. Work fiberwise, using Poincaré duality for varieties and the assumptions that the fibers are equidimensional. The assumption that π is compactifiable allows us to globalize the result on fibers. □

13. The Zeta Function & the Weil Conjectures

13.1. **Notation.** In this section all schemes are $\mathbf{F}_q$-varieties. Given an $\mathbf{F}_q$-variety X, we write $\overline{X} \coloneqq X \times_{\mathrm{Spec}(\mathbf{F}_q)} \mathrm{Spec}(\overline{\mathbf{F}}_q)$.

13.2. **Definition.** Let X be an $\mathbf{F}_q$-variety. The ***Zeta function*** of X is the formal power series with $\mathbf{Q}$-coefficients defined by

$$Z(X,t) \coloneqq \exp\left(\sum_{n\geq 1} \frac{\#X(\mathbf{F}_{q^n})}{n} t^n\right) .$$

13.3. **Conjectures** (Weil). *Let X be a smooth projective $\mathbf{F}_q$-variety of dimension d.*

(13.3.a) Rationality of the Zeta function: *we have that*

$$Z(X,t) = \frac{P_1(X,t)\cdots P_{2d-1}(X,t)}{P_0(X,t)\cdots P_{2d}(X,t)} ,$$

where each $P_r(X,t)$ is a polynomial with coefficients in a field of characteristic 0.

(13.3.b) Integrality: $P_0(X,t) = 1-t$, $P_{2d}(X,t) = 1 - q^d t$, *each each $P_r(X,t)$ has roots whose inverses are algebraic integers.*

(13.3.c) Functional equation: $Z(X, q^{-d}t^{-1}) = \pm q^{d\chi/2} t^{\chi} Z(X,t)$, *where χ is the Euler characteristic of X.*

(13.3.d) Riemann Hypothesis: *the inverses of the roots of $P_r(X,t)$ and all of their conjugates have complex absolute value $q^{r/2}$.*

(13.3.e) Specialization: *If X is the specialization of a smooth projective variety $\widetilde{X}$ over a number field, then*

$$\deg(P_r(X,t)) = \beta_r(\widetilde{X}(\mathbf{C})^{an}) ,$$

where $\beta_r(Y)$ denotes the r^{th} Betti number of a space Y.

13.4. **Theorem** (Lefschetz trace formula [4, Ch. VI Thm. 12.3]). *Let $\phi\colon X \to X$ be an endomorphism of a smooth projective variety over an algebraically closed field k such that the intersection $\Gamma_\phi \cdot \Delta_X$ is defined. Then for any prime $\ell \neq \mathrm{char}(k)$ we have*

$$(\Gamma_\phi \cdot \Delta_X) = \sum_{r=0}^{2\dim(X)} (-1)^r \operatorname{tr}(\phi^\star \mid \mathrm{H}^r_{\acute{e}t}(X;\mathbf{Q}_\ell)) .$$

13.5. **Theorem** ([2, Ch. IV Thm. 1.2]). *Let X be a smooth projective $\mathbf{F}_q$-variety. For all $r \geq 0$, and primes $\ell \neq \mathrm{char}(\mathbf{F}_q)$, the polynomials*

$$P_{r,\ell}(X,t) \coloneqq \det(\mathrm{id} - t\,\mathrm{Fr}^\star \mid \mathrm{H}^r_{\acute{e}t}(\overline{X};\mathbf{Q}_\ell)) .$$

in $\mathbf{Q}_\ell[t]$ have coefficients in $\mathbf{Z}$. Moreover the polynomials $P_{r,\ell}(X,t)$ are independent of the prime ℓ.

13.6. **Notation.** Let X be a smooth projective $\mathbf{F}_q$-variety. Write $P_r(X,t)$ for the common polynomial $P_{r,\ell}(X,t) \in \mathbf{Z}[t]$.

13.7. **Theorem** (Riemann hypothesis for varieties [2, Ch. IV Thm. 1.2]). *Let X be a smooth projective $\mathbf{F}_q$-variety and $\ell \neq \mathrm{char}(\mathbf{F}_q)$ a prime. Then the eigenvalues of* $\mathrm{Fr}^\star$ *on* $\mathrm{H}^r_{\acute{e}t}(\overline{X};\mathbf{Q}_\ell)$ *and all of their conjugates have complex absolute value $q^{r/2}$, and the inverse roots of $P_r(X,t)$ are algebraic integers of absolute value $q^{r/2}$.*

13.8. **Theorem** (rationality of the Zeta function [4, Ch. VI Thm. 12.4]). *Let X be a smooth projective $\mathbf{F}_q$-variety. Then*

$$Z(X,t) = \frac{P_1(X,t)\cdots P_{2d-1}(X,t)}{P_0(X,t)\cdots P_{2d}(X,t)} .$$

13.9. **Theorem** (functional equation for the Zeta function [2, Ch. IV Thm. 1.2; 4, Ch. VI Thm. 12.6]). *Let X be a smooth projective $\mathbf{F}_q$-variety of dimension d. Then*

$$Z(X, q^{-d}t^{-1}) = (-1)^{(d-1)N} q^{d\chi/2} t^{\chi} Z(X,t) ,$$

where χ is the Euler characteristic of X and N is the multiplicity of the eigenvalue $q^{d/2}$ of $\mathrm{Fr}^\star$ on $\mathrm{H}^d_{ét}(\overline{X};\mathbf{Q}_\ell)$ (for any prime $\ell \neq \mathrm{char}(\mathbf{F}_q)$).

13.10. **Theorem** ([4, Ch. VI 13.1]). *Let X be a separated $\mathbf{F}_q$-variety. Then for any prime $\ell \neq \mathrm{char}(\mathbf{F}_q)$ we have*

$$Z(X,t) = \prod_{r\geq 0} \det(\mathrm{id} - t\,\mathrm{Fr}^\star \mid \mathrm{H}^r_{ét}(\overline{X};\mathbf{Q}_\ell))^{(-1)^{r+1}} .$$

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