References for the equivalence of different approaches to ∞-categorical enhancements of derived categories

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In this document we compile some reference material on the different approaches to working with ∞-categorical derived enhancements of dervied categories. There are three approaches: one using model categories, one using dg categories, and one using ∞-categorical localizations. These approaches all produce the same answer: this is explained in various subsections of [HA, §1.3], and we've tried to provide some useful references to all statements throughout. Lurie's Kerodon project also provides a very nice explaination [Ker, Tags 00JN & 00ND]. We use the term '∞-category' to mean 'quasicategory'.

1 The bounded-above derived ∞-category $\mathrm{D}^+(\mathbf{A})$

Throughout, let $\mathbf{A}$ be an abelian category with enough injectives[1] (e.g., for a scheme X the category of $\mathcal{O}_X$-modules or quasicoherent sheaves of $\mathcal{O}_X$-modules). Then there are number of different candidates for a bounded-above derived ∞-category of $\mathbf{A}$:

Notation. Let $\mathbf{A}^{\mathrm{inj}} \subset \mathbf{A}$ denote the full subcategory spanned by the injective objects, and write $\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}}) \subset \mathrm{Ch}(\mathbf{A}^{\mathrm{inj}})$ for those chain complexes of injectives M_* for which $M_i = 0$ for $i \gg 0$.

1.1 The differential graded nerve

The category $\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}})$ has the structure of a differential graded (dg) category. Hence we can consider the differential graded nerve $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}}))$ of $\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}})$ (see [HA, Construction 1.3.1.6] and [Ker, Tag 00PK]; the sign conventions in these two sources differ, but the one in Kerodon seems to be the correct one as Lurie actually checks it). The dg nerve $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}}))$ is an ∞-category ([HA, Proposition 1.3.1.10] or [Ker, Tag 00PW]). The dg structure on chain complexes and what the dg nerve is explicitly in this

[1] In [HA, §1.3], Lurie writes things mostly for abelian categories with enough projectives, and remarks that the story works exactly the same when there are enough injectives, which is the setting we're in for $\mathcal{O}_X$-modules.

case isn't explained until [HA, §1.3.2], in particular [HA, Definition 1.3.2.1 & Remark 1.3.2.2]. See also [Ker, Tags 00PE & 00PP].

1.2 The homotopy coherent nerve

By the Dold–Kan correspondence, every differential graded category C determines a simplicial category C^{Δ} with the same objects of C, and the mapping simplicial set in C^{Δ} from X to Y is the underlying simplicial set of the simplicial abelian group corresponding to the nonnegative part of the mapping chain complex $\mathrm{Map}_C(X,Y)$ in C under the Dold–Kan correspondence. There is a natural functor

$$\mathrm{N}^{\mathrm{hc}}(C^{\Delta}) \to \mathrm{N}^{\mathrm{dg}}(C)$$

from the *homotopy coherent* (or *simplicial*) *nerve* of C^{Δ} to the dg nerve of C, and this is an equivalence of ∞-categories [HA, Proposition 1.3.17; Ker, Tag 00SV].

1.3 Inverting quasi-isomorphisms in the ∞-categorical sense

Now consider the category $\mathrm{Ch}^+(\mathbf{A}) \subset \mathrm{Ch}(\mathbf{A})$ of bounded-above chain complexes in $\mathbf{A}$ (not just bounded above complexes of injectives). Let qis denote the collection of morphisms in $\mathrm{Ch}^+(\mathbf{A})$ that are quasi-isomorphisms of chain complexes. Then there is a canonical equivalence of ∞-categories

$$\mathrm{Ch}^+(\mathbf{A})[\mathrm{qis}^{-1}] \xrightarrow{\sim} \mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}}))$$

between the universal ∞-category obtained from the (ordinary) category $\mathrm{Ch}^+(\mathbf{A})$ by inverting the quasi-isomorphisms, and the ∞-category $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}}))$ [HA, Theorem 1.3.4.4]. Cisinski's book provides a very nice exposition of how to formally invert a class of morphisms in an ∞-category [1, §7.1].

In summary, there are equivalences of ∞-categories

$$\mathrm{Ch}^+(\mathbf{A})[\mathrm{qis}^{-1}] \xrightarrow{\sim} \mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}})) \xleftarrow{\sim} \mathrm{N}^{\mathrm{hc}}(\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}})^{\Delta}) .$$

This common ∞-category is what is called the *positive* or *left-bounded* or *bounded-above derived ∞-category* of $\mathbf{A}$.

Notation. Let $\mathbf{A}$ be an abelian category with enough injectives. We write $\mathrm{D}^+(\mathbf{A})$ for the bounded-above derived ∞-category of $\mathbf{A}$.

2 Another universal property of $\mathrm{D}^+(\mathbf{A})$

The ∞-category $\mathrm{D}^+(\mathbf{A})$ also has another universal property that doesn't make reference to chain complexes. To state this, first recall that if C is a stable ∞-category with t-structure $(C_{\geq 0}, C_{\leq 0})$, the *heart* of the t-structure on C is the intersection $C_{\geq 0} \cap C_{\leq 0}$ of the positive and negative parts of the t-structure. The heart is always an abelian category [HA, Remark 1.2.1.12]. Informally, $\mathrm{D}^+(\mathbf{A})$ is the universal stable ∞-category equipped with a right-complete t-structure with heart the abelian category $\mathbf{A}$. See [HA, Theorem 1.3.3.2] for a precise statement of the universal property of $\mathrm{D}^+(\mathbf{A})$.

3 The unbounded derived ∞-category $\mathrm{D}(\mathbf{A})$

Now assume that $\mathbf{A}$ is a *Grothendieck abelian category*, that is, $\mathbf{A}$ is presentable and the collection of monomorphism is closed under filtered colimits. Examples include $\mathcal{O}_X$-modules and quasicoherent sheaves of $\mathcal{O}_X$-modules (the latter is a result of Gabber [2, Lemma 2.1.7; 3, §3]; see Akhil Mathew's blog post [5] for a really nice discussion). By Grothendieck's Tôhoku paper [4], Grothendieck abelian categories have enough injectives (see also [HA, Corollary 1.3.5.7]). In this setting, the injective model structure on $\mathrm{Ch}(\mathbf{A})$ is combinatorial [HA, Proposition 1.3.5.3]. As before, there is a canonical equivalence between the ∞-category $\mathrm{Ch}(\mathbf{A})[\mathrm{qis}^{-1}]$ obtained from the category $\mathrm{Ch}(\mathbf{A})$ by inverting the quasi-isomorphisms (i.e., the underlying ∞-category of $\mathrm{Ch}(\mathbf{A})$ in the injective model structure) and the dg nerve $\mathrm{N}^{\mathrm{dg}}(\mathrm{Ch}(\mathbf{A})^\circ)$ of the full subcategory $\mathrm{Ch}(\mathbf{A})^\circ \subset \mathrm{Ch}(\mathbf{A})$ spanned by the fibrant-cofibrant objects in the injective model structure. This common ∞-category is called the *(unbounded) derived ∞-category* of $\mathbf{A}$, denoted by $\mathrm{D}(\mathbf{A})$.

The category $\mathrm{Ch}(\mathbf{A})^\circ$ in particular contains all bounded-above chain complexes of injectives [HA, Proposition 1.3.5.6], so applying the dg nerve to the inclusion $\mathrm{Ch}^+(\mathbf{A}^{\mathrm{inj}}) \subset \mathrm{Ch}(\mathbf{A})^\circ$ defines a fully faithful functor of ∞-categories $\mathrm{D}^+(\mathbf{A}) \subset \mathrm{D}(\mathbf{A})$. Alternatively, we have an inclusion $\mathrm{Ch}^+(\mathbf{A}) \subset \mathrm{Ch}(\mathbf{A})$ and can invert quasi-isomorphisms on both sides. The ∞-category $\mathrm{D}(\mathbf{A})$ has a t-structure with $\mathrm{D}(\mathbf{A})_{\geq 0}$ those chain complexes with homology vanishing in negative degrees and $\mathrm{D}(\mathbf{A})_{\leq 0}$ with homology vanishing in positive degrees [HA, Proposition 1.3.5.21]. The ∞-category $\mathrm{D}^+(\mathbf{A})$ can be identified with the bounded-above objects of $\mathrm{D}(\mathbf{A})$ with respect to this t-structure. The bounded derived ∞-category $\mathrm{D}^b(\mathbf{A})$ can then be defined to be the full subcategory of $\mathrm{D}(\mathbf{A})$ (or $\mathrm{D}^+(\mathbf{A})$) spanned by those objects that are bounded with respect to the t-structure (i.e., have homology concentrated in a finite range).

References

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