Copenhagen Masterclass

Exodromy

Beyond Conicality

Goal of the lecture series. Explain the exodromy equivalence beyond the setting of conically stratified spaces along with applications

> E.g., representability of derived moduli of constructible and perverse sheaves.

Joint with. Mauro Porta $ Jean-Baptiste Teyssier

Lecture 1

Exodromic Stratified Spaces & the Stability Theorem

Remark. In this lecture, we'll state some results with **hypersheaves** in order to remove some point-set topology assumptions.

- If you don't know what these are, this won't affect the key ideas.
- In a later lecture, we'll explain the meaning.

Exit-path + monodromy

Exodromy Thm, conical case (MacPherson, Treumann, Lurie, Porta-Teyssier).

Let (X,P) be a stratified space with locally weakly contractible strata. If the stratification of X is conical, then:

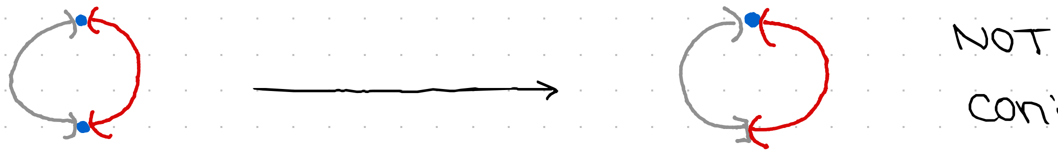
E.g., Whitney

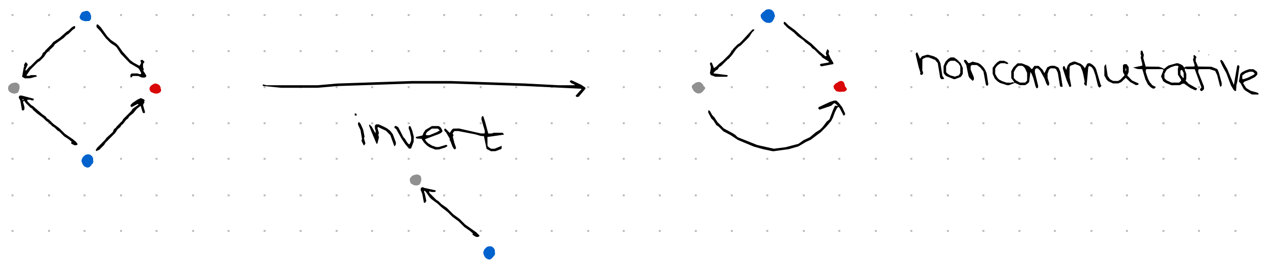
(1) There exists an ∞-category $\mathrm{Exit}(X,P)$ and an equivalence
$$\mathrm{Cons}_P^{hyp}(X) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Exit}(X,P),\mathrm{Spc}).$$

(2) $\mathrm{Cons}_P^{hyp}(X) \subset \mathrm{Sh}^{hyp}(X)$ is closed under limits & colimits

(3) $s^*: \underset{\mathrm{Fun}(P,\mathrm{Spc})}{\underset{\wr\!\!\mid}{\mathrm{Sh}^{hyp}(P)}} \longrightarrow \mathrm{Sh}^{hyp}(X)$ preserves limits.

Issue. Many naturally-arising stratifications are not conical (!) but should satisfy exodromy





Rem. There are non-conically stratified spaces of interest in mirror symmetry — see Favero–Huang.

Ayala–Francis–Rozenblyum, Clausen–Ørsnes Jansen

Idea. Make the conclusion of the exodromy theorem into a definition

Atomic Generation

Def. $\mathcal{C}$ presentable ∞-category.

(1) $c \in \mathcal{C}$ is **atomic** if $\mathrm{Map}_{\mathcal{C}}(c,-) : \mathcal{C} \longrightarrow \mathrm{Spc}$ preserves colimits.

- Write $\mathcal{C}^{at} \subset \mathcal{C}$ for the full subcategory spanned by the atomic objects.

(2) A small full subcat $\mathcal{C}_0 \subset \mathcal{C}$ **atomically generates** $\mathcal{C}$ if the colimit-preserving extension

$$\mathrm{PSh}(\mathcal{C}_0) \longrightarrow \mathcal{C}$$

is an equivalence.

- In this case, we say that $\mathcal{C}$ is **atomically generated**.

Notes.

(1) $\mathcal{C}^{at} \subset \mathcal{C}$ is idempotent complete.

(2) $\mathcal{C}$ atomically generated $\Rightarrow$ $\mathrm{PSh}(\mathcal{C}^{at}) \xrightarrow{\sim} \mathcal{C}$

(3) $\mathcal{C}_0$ atomically generates $\mathcal{C} \Rightarrow \mathcal{C}_0 \hookrightarrow \mathcal{C}^{at}$ is idem. completion

Obs. Let $L: \mathcal{C} \to \mathcal{D}$ be a left adjoint between pres ∞-cats

(1) If the right adj $R: \mathcal{D} \to \mathcal{C}$ pres. colims, then L preserves atomic objects.

(2) If $\mathcal{C}$ and $\mathcal{D}$ are atomically gen. and L preserves atomic objects, then R preserves colims.

Prop. The functors

$$\mathcal{C}_0 \longmapsto \mathrm{PSh}(\mathcal{C}_0), \text{ LKE}$$

$$\mathrm{Cat}_\infty^{\mathrm{idem}} \rightleftarrows \left\{ \begin{array}{l} \text{atomically gen. presentable } \infty\text{-cats} \\ \text{\& left adjoints} \\ \text{whose right adj} \\ \text{admits a right adj} \end{array} \right\} =: \mathrm{Pr}^L_{\mathrm{at}}$$

$$\mathcal{C}^{\mathrm{at}} \longleftarrow\!\shortmid \mathcal{C}$$

are inverse equivalences of ∞-cats.

> Hence also $(\mathrm{Cat}_\infty^{\mathrm{idem}})^{op} \xrightarrow[\mathrm{PSh},\text{ pullback}]{\sim} \mathrm{Pr}^R_{\mathrm{at}} \simeq (\mathrm{Pr}^L_{\mathrm{at}})^{op}$

Obs. Given a poset P and functor $s: \mathcal{C} \longrightarrow P$, TFAE:

(1) s is conservative.

(2) The fibers of s are ∞-groupoids.

Lem. TFAE for an ∞-category $\mathcal{C}$:

(1) There exists a conservative functor $s: \mathcal{C} \longrightarrow P$ to a poset.

(2) $\mathcal{C}$ is **layered** (or an **EI ∞-category**): for all $c \in \mathcal{C}$, every endomorphism $c \longrightarrow c$ is an equivalence.

Proof.

(1) $\Rightarrow$ (2) Clear

(2) $\Rightarrow$ (1) Take P to be the poset $h_0(\mathcal{C})$ of iso classes of objeces of $\mathcal{C}$ with $[x] \leq [y]$ iff there exists a morphism $x \longrightarrow y$ in $\mathcal{C}$. The hypothesis that $\mathcal{C}$ is layered guarentees this is a poset and $\mathcal{C} \longrightarrow h_0(\mathcal{C})$ is conservative. $\square$

Lem. Every layered ∞-category is idempotent complete.

Ex. (1) Posets are idempotent complete

(2) If (X,P) is a conically stratified space, then $\mathrm{Exit}(X,P)$ is idempotent complete.

Upshot. In the setting of exodromy,

$$\begin{array}{c} \mathrm{Exit}(X,P) \simeq \left(\mathrm{Cons}_P(X)^{at}\right)^{op}. \\ \parallel \\ \mathrm{Sing}_P(X) \end{array}$$

Exodromic Stratified Spaces

Reflection Thm (Adámek-Rosický for 1-cats, Ragimov-Schlank for ∞-cats).
If $\mathcal{C}$ is a presentable ∞-cat and $\mathcal{D} \subset \mathcal{C}$ is closed under limits and κ-filtered colimits for some regular cardinal κ, then $\mathcal{D}$ is presentable and $\mathcal{D} \hookrightarrow \mathcal{C}$ admits a left adj.

Def. A stratified space $s: X \longrightarrow P$ is **exodromic** if:

(1) $\mathrm{Cons}_P^{hyp}(X) \subset \mathrm{Sh}^{hyp}(X)$ is closed under limits and colimits.

Reflection $\Rightarrow$ $\mathrm{Cons}_P^{hyp}(X)$ is presentable and the inclusion admits both adjoints

(2) $\mathrm{Cons}_P^{hyp}(X)$ is atomically generated

(3) $s^*: \mathrm{Sh}^{hyp}(P) \simeq \mathrm{Fun}(P, \mathrm{Spc}) \longrightarrow \mathrm{Cons}_P^{hyp}(X) \subset \mathrm{Sh}(X)$ preserves limits $\Leftrightarrow$ induced by a functor $\Pi_\infty(X,P) \longrightarrow P$

> The **exit-path ∞-category** of (X,P) is

$$\Pi_\infty(X,P) := (\mathrm{Cons}_P(X)^{at})^{op}$$

> By definition,

$$\mathrm{Cons}_P^{hyp}(X) \simeq \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Spc}).$$

Ex. If (X,P) is conically stratified with lwc strata, then (X,P) is exodromic and

$$\Pi_\infty(X,P) \simeq \mathrm{Exit}(X,P).$$

Def. (X,P), (Y,Q) exodromic stratified spaces. A stratified map $f:(X,P) \longrightarrow (Y,Q)$ is **exodromic** if

$$f^*: \mathrm{Cons}_Q^{hyp}(Y) \longrightarrow \mathrm{Cons}_P^{hyp}(X)$$

preserves limits. $\Leftrightarrow$ induced by a functor $\Pi_\infty(X,P) \to \Pi_\infty(Y,Q)$

> For the following Theorem, we need the following:

Def. Write $\mathrm{Cat}_\infty^{fin} \subset \mathrm{Cat}_\infty$ for the smallest full subcategory containing $\emptyset$, Δ^0, and Δ^1 and closed under pushouts. We say that an ∞-category $\mathcal{C}$ is **finite** if $\mathcal{C} \in \mathrm{Cat}_\infty^{fin}$. $\leftarrow$ ∞-categorical version of a space having a finite CW structure

Ex. A finite poset is a finite ∞-category.

Lem. TFAE for an ∞-category $\mathcal{C}$:

(1) $\mathcal{C}$ is a compact object of Cat_∞ (i.e., $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, -)$ preserves filtered colimits).

(2) $\mathcal{C}$ is a retract of a finite ∞-category.

Stability Thm for Exodromic Stratified Spaces (HPT).

(1) **Stability under pulling back to locally closed subposets:**
If (X,P) is exodromic and $S \subset P$ is locally closed, then (X_S, S) is exodromic and

$$\Pi_\infty(X \times_P S, S) \xrightarrow{\sim} \Pi_\infty(X,P) \times_P S.$$

- In particular, taking $S = \{p\}$, we deduce that $\Pi_\infty(X,P) \longrightarrow P$ is conservative
[will explain this later]

(2) **Functoriality:** If (X,P) and (Y,Q) are exodromic, then every map of stratified spaces $f:(X,P) \to (Y,Q)$ is exodromic.

(3) Stability under coarsening: If (X,R) is exodromic and $\phi : R \to P$ is any map of posets, then (X,P) is exodromic and the natural functor $\Pi_\infty(X,R) \to \Pi_\infty(X,P)$ induces an equivalence

$$\Pi_\infty(X,R)[W_P^{-1}] \xrightarrow{\sim} \Pi_\infty(X,P).$$

Here W_P is the collection of morphisms sent to identities by

$$\Pi_\infty(X,R) \longrightarrow R \xrightarrow{\phi} P.$$

- If X is locally weakly contractible, this implies that

$$\underbrace{B}_{\substack{\text{classifying space} \\ \text{left adj to } Spc \hookrightarrow Cat_\infty}}(\Pi_\infty(X,R)) \xrightarrow{\sim} \underbrace{\Pi_\infty(X)}_{\substack{\text{underlying} \\ \text{homotopy type}}}$$

(4) van Kampen: Given a cocone diagram $(X_\bullet, P_\bullet): I^{\triangleright} \to \mathrm{StrTop}$ with (X_i, P_i) exodromic for $i \in I$, if

$$\mathrm{Sh}^{hyp}(X_\infty) \xrightarrow{\sim} \lim_{i \in I^{op}} \mathrm{Sh}^{hyp}(X_i)$$ Mikala gave examples

and

$$\underbrace{\mathrm{Cons}^{hyp}_{P_\infty}(X_\infty) \xrightarrow{\sim} \lim_{i \in I^{op}} \mathrm{Cons}^{hyp}_{P_i}(X_i)}_{\text{allong } * \text{- pullbacks}},$$

then (X_∞, P_∞) is exodromic. Moreover,

$$\underbrace{\operatorname*{Colim}_{i \in I} \Pi_\infty(X_i, P_i)}_{\text{formed in } \mathrm{Cat}_\infty^{idem}} \xrightarrow{\sim} \Pi_\infty(X_\infty, P_\infty)$$

- Hypotheses are satisfied if we have a stratified space $(X_\infty, P_\infty) = (X,P)$ and open cover $\{U_\alpha\}_{\alpha\in A}$ such that all

$$(U_{\alpha_0}\cap\cdots\cap U_{\alpha_n}, P)$$

are exodromic, and take the Čech diagram

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; \coprod_{\alpha_0,\alpha_1\in A} U_{\alpha_0}\cap U_{\alpha_1} \;\substack{\longrightarrow\\\longrightarrow}\; \coprod_{\alpha_0\in A} U_{\alpha_0} \longrightarrow X$$

(5) **Homotopy-invariance**: (X,P) stratified space, Y lwc space. Then

$$pr^*: Sh^{hyp}(X) \longrightarrow Sh^{hyp}(X\times Y)$$

admits a left adjoint $pr_\#$. Moreover, if Y is wcontr.:

(a) $pr_\#$ preserves constructibility

(b) pr^* is fully faithful and $Cons_P^{hyp}(X) \xrightarrow[pr^*]{\sim} Cons_P^{hyp}(X\times Y)$

(c) (X,P) exodromic $\Rightarrow$ $(X\times Y, P)$ exodromic and

$$\Pi_\infty(X\times Y, P) \xrightarrow{\sim} \Pi_\infty(X,P).$$

(6) **Stability of fineness/compactness**: the property of an exit-path ∞-category being finite/compact:

(a) is stable under pulling back to locally closed subposets

(b) is stable under coarsening

(c) can be checked on a finite open cover (including intersections).

(7) **Change of coefficients**: If (X,P) is an exodromic stratified space and $\mathcal{E}$ is a presentable ∞-cat, then there is a natural fully faithful functor

$$\begin{array}{ccc} \mathrm{Cons}_P^{\mathrm{hyp}}(X)\otimes\mathcal{E} & \xhookrightarrow{\boxtimes} & \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathcal{E}) \\ \wr\| & & \\ \mathrm{Fun}(\Pi_\infty(X,P),\mathcal{E}) & & \end{array}$$

If $\mathcal{E}$ is compactly assembled, then $\boxtimes$ is an equivalence.

Lecture 2

Examples of exodromic stratified spaces

Functoriality & Objects of $\Pi_\infty(X,P)$

Observe. Let $f: (X,P) \to (Y,Q)$ be a stratified map between exodromic stratified spaces. Then (2) in the Stability Thm shows that

$$f^*: \mathrm{Cons}_Q^{hyp}(Y) \longrightarrow \mathrm{Cons}_P^{hyp}(X)$$

admits both a left adjoint $f^c_\#$ and a right adjoint f^c_*.

⚠ The right adjoint f^c_* is not generally the restriction of $f_*: \mathrm{Sh}^{hyp}(X) \longrightarrow \mathrm{Sh}^{hyp}(Y)$ to constructible sheaves. In general, f_* doesn't preserve constructibility (even the pushforward from an open stratum). However, if f_* preserves constructibility, then f^c_* is the restriction of f_*.

Upshot. $(X,P) \longmapsto \Pi_\infty(X,P)$ assembles into a functor as

$$
\begin{array}{ccccccc}
\left\{\begin{matrix}\text{exodromic} \\ \text{stratified} \\ \text{spaces}\end{matrix}\right\} & \longrightarrow & \left(\mathrm{Pr}^R_{at}\right)^{op} & \overset{\text{pass to left adj}}{\simeq} & \mathrm{Pr}^L_{at} & \xrightarrow{\sim} & \mathrm{Cat}^{idem}_\infty \\
(X,P) & \longmapsto & \mathrm{Cons}^{hyp}_P(X) & \longmapsto & \mathrm{Cons}^{hyp}_P(X) & & \\
 & & *\text{-pullback} & & \#\text{-pushforward} & & \\
 & & & & \mathcal{E} & \longmapsto & (\mathcal{E}^{at})^{op}
\end{array}
$$

> In the conical setting for exodromy we said that the functor $\Pi_\infty(X,P)$ corresponding to a constructible sheaf F is given on objects by $x \longmapsto F_x$. We'd like to see this more generally.

Lem. (X,P) exodromic stratified space. Then:

(1) $\Pi_\infty(X,P)^{\simeq} \simeq \coprod_{p\in P} \Pi_\infty(X_p,\{p\})$.

maximal sub-∞-groupoid

(2) If the strata of X are locally weakly contractible, then

$$\Pi_\infty(X,P)^{\simeq} \simeq \coprod_{p\in P} \left(\begin{array}{c}\text{underlying homotopy}\\ \text{type of } X_p\end{array}\right)$$

Proof.

(1) Note that if $\mathcal{C} \longrightarrow P$ is a conservative functor from an ∞-cat to a poset, then

$$\mathcal{C}^{\simeq} \simeq \coprod_{p\in P} \mathcal{C} \times_P \{p\}.$$

By Stability (1), $\Pi_\infty(X,P)$ is conservative and

$$\Pi_\infty(X,P) \times_P \{p\} \simeq \Pi_\infty(X_p,\{p\}).$$

(2) Since each X_p is locally weakly contractible, by monodromy for lwc spaces,

$$\underbrace{\Pi_\infty(X_p,\{p\})}_{\substack{\text{by def classifies} \\ \text{locally constant sheaves} \\ \text{on } X_p}} \simeq \begin{pmatrix} \text{underlying homotopy} \\ \text{type of } X_p \end{pmatrix}.$$

$\square$

Lem. If (X,P) is exodromic with lwc strata, then the equivalence

$$\mathrm{Cons}_P^{hyp}(X,P) \overset{Ex}{\simeq} \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Spc})$$

sends a constructible sheaf F to a functor $\Pi_\infty(X,P) \longrightarrow \mathrm{Spc}$ given on objects by $x \mapsto F_x$.

Proof.

Let $x \in X$ and write $i_x : \{x\} \hookrightarrow X$ for the inclusion. By functoriality, we have a commutative diagram

$$
\begin{tikzcd}
\mathrm{Cons}_P^{hyp}(X) \arrow[r, "\simeq", "\mathrm{Ex}"'] \arrow[d, "i_x^*"'] & \mathrm{Fun}(\Pi_\infty(X,P),\mathrm{Spc}) \arrow[d, "\text{precomp w/ } \{x\}\to\Pi_\infty(X,P)"] \arrow[dr, "\mathrm{eval}_x"] & \\
\mathrm{Cons}_P^{hyp}(\{x\}) \arrow[r, "\simeq", "\mathrm{Ex}"'] & \mathrm{Fun}(\{x\},\mathrm{Spc}) \arrow[r, "\sim", "\mathrm{eval}"'] & \mathrm{Spc}
\end{tikzcd}
$$

$$F \mapsto \mathrm{Ex}(F) \qquad F \mapsto F_x \qquad \mathrm{Ex}(F) \mapsto \mathrm{Ex}(F)(x)$$

$\square$

Examples

Def. $s: X \longrightarrow P$ stratified space.

(1) A **conical refinement** of $s: X \longrightarrow P$ is a conical stratification $t: X \longrightarrow R$ with locally weakly contractible strata and a map of posets $\phi: R \longrightarrow P$ such that $s = \phi t$.

(2) (X, P) is **locally conically refineable** if there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of X such that each (U_α, P) is conically refineable.

Note. If (X, P) is conically stratified and $U \subset X$ is open, then (U, P) is conically stratified.

> Showing this uses the fact that we give cones the "teardrop topology"!

Cor. If (X,P) is locally conically refineable, then (X,P) is exodromic. Moreover, X is locally weakly contractible.

Proof.

Since open subspaces of conically stratified spaces are conically stratified, the first statement is immediate from van Kampen and stability under coarsening. Since the final statement is local, we can reduce to the case where X is conically refineable. Then this is a general fact about conically stratified spaces with locally weakly contractible strata, see [HPT, Lemma 5.2.6]. $\square$

Ex. Let $(V, S \subset \mathrm{Sub}_{\mathrm{fin}}(V))$ be a simplicial complex.
nonempty + downward closed

Write

$$\Delta^{(V,S)} := \left\{ (t_v)_{v \in V} \in [0,1]^V \;\middle|\; \begin{array}{l} \{v \in V \mid t_v > 0\} \in S \\ \sum_{v \in V} t_v = 1 \end{array} \right\}$$

Give $[0,1]^V$ the topology induced by

$$[0,1]^V = \operatorname{colim}^{\mathrm{set}}_{V' \subset V \text{ finite}} [0,1]^{V'} \text{ w/ prod topology}$$

for its geometric realization. There is a natural stratification

$$\begin{aligned} \Delta^{(V,S)} &\longrightarrow S \quad \text{ordered by inclusion} \\ (t_v)_{v \in V} &\longmapsto \{v \in V \mid t_v > 0\} \end{aligned}$$

If (V,S) is locally finite, then this stratification is conical and

$$\Pi_\infty(\Delta^{(V,S)}, S) \xrightarrow{\sim} S.$$

See [HA, Prop. A.6.8 & Thm. A.6.10].

Upshot. If (X,P) locally admits a refinement by a locally finite triangulation, then (X,P) is exodromic.

Ex. The **tree stratification** of an abstract simplicial complex considered by Favero-Huang in the context of Mirror Symmetry is conically refineable, but not usually conical.

Def. An exodromic stratified space (X,P) is **categorically finite/compact** if $\Pi_\infty(X,P)$ is a finite/compact ∞-category.

Examples from Subanalytic geometry

Def (see [Bierstone - Milman]). Let M be a real analytic manifold.

(1) A subset $B \subset M$ is **basic semianalytic** if

$$B = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n} Y_{i,j}$$

Smallest family containing $\{f(x)>0\}$ and stable under complement, finite $\cap$, finite $\cup$

where each $Y_{i,j}$ is either of the form

$$\{f_{i,j}(x)=0\} \quad \text{or} \quad \{f_{i,j}(x)>0\}$$

for $f_{i,j}: M \longrightarrow \mathbb{R}$ real analytic

(2) A subset $S \subset M$ is **semianalytic** if for each point $x \in S$, there is an open $x \in U \subset M$ such that $S \cap U$ is **basic semianalytic** in U.

(3) $S \subset M$ is **subanalytic** if for each $x \in S$, there is an open $x \in U \subset M$ such that $S \cap U$ is a projection of a relatively compact semianalytic set, i.e., there is a real analytic manifold N such that $S \cap U$ is the image of a relatively compact semianalytic subset $T \subset U \times N$ under $pr_1 : U \times N \longrightarrow U$.

Ex. Closed and half-open intervals in $\mathbb{R}$ and S^1 are semianalytic.

Def. A **subanalytic stratified space** is a triple (M, X, P) where M is a real analytic mfld, $X \subset M$ is a locally closed subanalytic subset, and $X \longrightarrow P$ is a locally finite stratification by subanalytic subsets of M.

Ex. Both stratifications of S^1 from Lecture 1 are subanalytic stratified spaces (with $M = X = S^1$).

Thm. Let (M, X, P) be a subanalytic stratified space. Then:
(1) (X, P) admits a refinement by a locally finite triangulation. Hence (X, P) is exodromic.

(2) If X is compact, then (X, P) admits a refinement by a **finite** triangulation. Hence (X, P) is categorically finite.

Proof.
The refinement statements go back to Verdier. In these forms, they can be found in §1.7 of Goresky & MacPherson's book "Stratified Morse Theory".
The rest of the statements follow from van Kampen and stability under localization. $\square$

Examples from real algebraic geometry

Def. An **algebraic stratified space** is a stratified space (X,P) where X is (the real points of) an $\mathbb{R}$-variety and the stratification is by subvarieties.

Thm. Let (X,P) be an algebraic stratified space. Then:

(1) If X is affine, then (X,P) admits a conical refinement (X,R) with R finite and $\Pi_\infty(X,R)$ finite.
  - Hence (X,P) is exodromic and categorically finite.

(2) (X,P) is locally conically refineable and categorically finite.

Lem [Porta–Teyssier, Lemma 2.10]. If (X, P) is conically stratified and $S \subset P$ is locally closed, then (X_S, S) is also conically stratified.

Proof of Thm.

(1) Write $s: X \longrightarrow P$ for the stratification. Since X is affine, we can choose a closed immersion $X \hookrightarrow \mathbb{A}^n$. We further embed $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$. Write $\overline{X}$ for the closure of X in $\mathbb{P}^n$. Note that since $X \subset \mathbb{P}^n$ is locally closed, $\overline{X} \setminus X$ is closed. Moreover, since $X \subset \mathbb{P}^n$ is semianalytic, $\overline{X}$ is semianalytic. Extend the stratification of X to $\mathbb{P}^n$ by

$$\begin{aligned} \mathbb{P}^n &\longrightarrow \{-\infty\} \cup P \cup \{\infty\} =: Q \\ x &\longmapsto \begin{cases} s(x), & x \in X \\ \infty, & x \in \mathbb{P}^n \setminus \overline{X} \\ -\infty, & x \in \overline{X} \setminus X \end{cases} \end{aligned}$$

Then $(\mathbb{P}^n, Q)$ is a compact subanalytic stratified space, hence admits a refinement $\mathbb{P}^n \longrightarrow Q' \longrightarrow Q$ by a finite triangulation. By stability under pullback $(X, Q' \times_Q P)$ is conically stratified and categorically finite. By stability under coarsening, (X, P) is also categorically finite.

(2) Note that X admits a finite cover by affine varieties and all intersections are also affine. So by (1) and van Kampen, (X,P) is exodromic and $\Pi_\infty(X,P)$ is a finite colimit of finite ∞-cats. Hence (X,P) is categorically finite. $\square$

Another Example

Ex[Lejay, after Lurie]. If M is a (Fréchet) mfld, then there are two commonly-considered topologies on the set $\mathrm{Ran}(M)$ of finite subsets of M: one coming from a metric on M, and another that makes sense for arbitrary top. spaces.

- In both topologies, there is a natural stratification

$$\begin{aligned} \mathrm{Ran}(M) &\longrightarrow \mathbb{Z}_{\geq 0} \\ S &\longmapsto \# S \end{aligned}$$

- The metric one is conical, but the "general" one is not. However, both are exodromic and have equivalent ∞-cats of constructible sheaves.

Note. Constructible sheaves on $\mathrm{Ran}(M)$ are used in the theory of factorization algebras.

Lecture 3

Topos-theoretic background

Motivations for toposic generalization.

(1) Want exodromy for sheaves & hypersheaves to be on the same footing [See next page]
- both ∞-topoi.

(2) There are natural examples of stratified topological stacks that satisfy exodromy ($\overline{\mathcal{M}}_{g,n}$ and the stack Broken of broken lines).
- Sheaves on these are not sheaves on a space. But they are ∞-topoi.

Observe. The conditions of being exodromic only depend on the ∞-cats $Sh^{hyp}(X)$, $Cons_P^{hyp}(X)$, and the functor s^*.
$\rightsquigarrow$ We can generalize these even more.

Sheaves vs. hypersheaves

Issue. For a space X, the stalk functors $Sh(X) \to An$ aren't usually jointly conservative.

Def. $Sh^{hyp}(X) := Sh(X)[(\text{stalkwise equivs})^{-1}]$

Fact: There's a localization $Sh^{hyp}(X) \hookrightarrow Sh(X)$ and the left adjoint is left exact.

Fact. If P is a poset, then:

(1) If P satisfies the ascending chain condition,

$$Sh(P) \simeq Fun(P, Spc)$$

(2) In general,

$$Sh^{hyp}(P) \simeq Fun(P, Spc).$$

⚠ For the poset

$$P = \begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \bullet \arrow[r] \arrow[dr] & \bullet \arrow[r] \arrow[dr] & \cdots \\ \bullet \arrow[r] \arrow[ur] & \bullet \arrow[r] \arrow[ur] & \bullet \arrow[r] \arrow[ur] & \cdots \end{tikzcd}$$

See Aoki's paper "Tensor triangular..." Example A.13

$$\mathrm{Sh}(P) \not\simeq \mathrm{Fun}(P, \mathrm{Spc}).$$

Thm. If X satisfies one of the following conditions, then $\mathrm{Sh}^{hyp}(X) \hookrightarrow \mathrm{Sh}(X)$ is an equivalence:

(1) X is paracompact of finite covering dimension.

(2) X admits a CW structure [MO:168852G]

(3) X is a spectral space of finite Krull dimension [Clausen-Mathew, Thm. 3.12]

Crash course on ∞-topoi

Thm. Let $\mathcal{X}$ be an ∞-category. TFAE:

(1) $\exists$ a small ∞-cat $\mathcal{C}$ and a left exact accessible localization

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, shift right, "\text{acces.}"'] \arrow[r, phantom, "\perp"] & \mathrm{PSh}(\mathcal{C}) \arrow[l, shift right, "\text{lex}"'] \end{tikzcd}$$

(2) Giraud's axioms: $\mathcal{X}$ is presentable and

(a) Coproducts are disjoint: $\forall\, U, V \in \mathcal{X}$, the square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \sqcup V \end{tikzcd}$$

is a pullback

(b) Colimits are universal: for every diagram $U_\bullet : A \to \mathcal{X}$, objects $V, W \in \mathcal{X}$ and morphisms

$$\operatorname*{colim}_{\alpha \in A} U_\alpha \longrightarrow W \longleftarrow V$$

$$\operatorname*{colim}_{\alpha\in A}\left(U_\alpha \times_W V\right) \xrightarrow{\sim} \left(\operatorname*{colim}_{\alpha\in A} U_\alpha\right)\times_W V.$$

(c) **Groupoid objects are effective:** if $U_\bullet : \Delta^{op} \to \mathcal{X}$ is a simplicial object so that for every partition $[n] = S \cup T$ with $S \cap T = \{i\}$ a single point, then

$$\begin{array}{ccc} U_n \simeq U_\bullet([n]) & \longrightarrow & U_\bullet(T) \\ \downarrow & & \downarrow \\ U_\bullet(S) & \longrightarrow & U_\bullet(\{i\}) \simeq U_0 \end{array}$$

is a pullback,

then $U_\bullet \simeq \underbrace{\text{Čech nerve of } U_0 \longrightarrow |U_\bullet|}$

$f: X \longrightarrow Y$ Čech nerve is

$$\cdots \; \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} \; X \times_Y X \;\begin{smallmatrix}\xrightarrow{\mathrm{pr}_1}\\ \longleftarrow\\ \xrightarrow[\mathrm{pr}_2]{}\end{smallmatrix}\; X$$

(3) $\mathcal{X}$ is presentable and colimits in $\mathcal{X}$ are **van Kampen**: the functor

$$
\begin{tikzcd}
\mathcal{X}^{op} \arrow[r] & \mathrm{Cat}_\infty \\
U \arrow[r, mapsto] \arrow[d] & \mathcal{X}_{/U} \\
V \arrow[r, mapsto] & \mathcal{X}_{/V} \arrow[u, "U \times_V (-)"']
\end{tikzcd}
$$

preserves limits, i.e., $\mathcal{X}_{/\operatorname{colim}_{\alpha \in A} U_\alpha} \xrightarrow{\sim} \lim_{\alpha \in A^{op}} \mathcal{X}_{/U_\alpha}$.

⚠ Note that we're not saying that there is an ∞-site $(\mathcal{C}, \tau)$ and an equivalence $\mathcal{X} \simeq Sh_\tau(\mathcal{C})$. This is expected to be false!

Ex. For a topological space X, $Sh^{hyp}(X)$ is an ∞-topos.

Cor (of (3)). If $\mathcal{X}$ is an ∞-topos, then for any $U \in \mathcal{X}$, the ∞-cat $\mathcal{X}_{/U}$ is an ∞-topos.

Ex. X top space, $U \subset X$ open, $よ(U) \in Sh(X)$ sheaf represented by U. There is a natural equivalence

$$Sh(X)_{/よ(U)} \simeq Sh(U)$$

Geometric morphisms

Def. Given ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, a **geometric morphism** $f_*: \mathcal{X} \longrightarrow \mathcal{Y}$ is a right adjoint whose left adjoint $f^*: \mathcal{Y} \longrightarrow \mathcal{X}$ is left exact (= preserves finite limits).

- $\mathrm{RTop}_\infty :=$ ∞-topoi and geometric morphisms
- $\mathrm{LTop}_\infty :=$ ∞-topoi and left exact left adjoints

$$\begin{array}{c} \mathrm{LTop}_\infty \\ \cong \\ (\mathrm{RTop}_\infty)^{op} \end{array}$$

Ex. If $\mathcal{X}$ is an ∞-topos with terminal object $1_{\mathcal{X}}$, then

$$\Gamma_* := \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, -): \mathcal{X} \longrightarrow \mathrm{Spc}$$

is a geometric morphism. We call Γ_* the **global sections functor** and Γ^* the **constant sheaf functor**.

Ex. Given $U \in \mathcal{X}$, by universality of colimits and the adjoint functor theorem,

$$f^* := U \times (-) : \mathcal{X} \longrightarrow \mathcal{X}_{/U}$$

admits both a left adjoint

$$f_\# := \text{forget} : \mathcal{X}_{/U} \longrightarrow \mathcal{X}$$

and a right adjoint

$$f_* : \mathcal{X}_{/U} \longrightarrow \mathcal{X},$$

which is harder to describe explicitly.

Thm [HTT, Prop. 6.3.2.3 & Cor. 6.3.4.7]. LTop_∞ has all limits and colimits. Moreover, the forgetful functor $\mathrm{LTop}_\infty \longrightarrow \mathrm{Cat}_\infty^{\mathrm{large}}$ preserves limits.

Ex. If X is a topological stack, then $\mathrm{Sh}(X)$ is an ∞-topos.

> By definition, $\mathrm{Sh}(X)$ is a limit of ∞-topoi along left exact left adjoints.

To generalize to ∞-topoi.

(1) We need to generalize stratifications.

(2) We need to generalize local constancy and constructibility.

The first is easy:

Def. X ∞-topos, P poset. A **P-stratification** of X is a geometric morphism $S_*: X \longrightarrow \mathrm{Fun}(P, \mathrm{Spc})$.

The second is also easy if we understand local constancy properly

Lecture 4

More on ∞-topoi

&

Outline of the proof

Effective epimorphisms

Def. $\mathcal{C}$ ∞-cat with pullbacks. A morphism $f: X \to Y$ is an **effective epimorphism** if the natural diagram

$$\underbrace{\cdots \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X\times_Y X\times_Y X \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X\times_Y X \;\substack{\xrightarrow{pr_1}\\ \xleftarrow{\Delta}\\ \xrightarrow[pr_2]{}}\; X}_{\text{Čech nerve of } f.} \xrightarrow{f} Y$$

is a colimit diagram.

Ex. X top space, for $U \subset X$ open, write $\text{よ}(U) \in \mathrm{Sh}(X)$ for the sheaf represented by U. For an open cover $\{U_\alpha\}_{\alpha\in A}$,

$$\coprod_{\alpha\in A} \text{よ}(U_\alpha) \longrightarrow \text{よ}(X) = 1_{\mathrm{Sh}(X)}$$

is an effective epi in $\mathrm{Sh}(X)$.

Idea. Effective epis are "covers".

Shape theory & monodromy

After Lurie. See [HA, § 1].

Def. $\mathcal{X}$ an ∞-topos. An object $L \in \mathcal{X}$ is **locally constant** if there exists an effective epimorphism $\coprod_{\alpha \in A} U_\alpha \twoheadrightarrow 1_{\mathcal{X}}$ such that for each $\alpha \in A$, the pullback

$$L \times U_\alpha \in \mathcal{X}_{/U_\alpha}$$

is a constant object of the ∞-topos $\mathcal{X}_{/U_\alpha}$.

> $LC(\mathcal{X}) \subset \mathcal{X}$ full subcategory of locally constant objects.

Lem. For a topological space X,

$$LC(Sh(X)) = LC(X).$$

$$LC(Sh^{hyp}(X)) = LC^{hyp}(X).$$

Question. When is there an ∞-groupoid $\Pi_\infty(X)$ and an equivalence

$$LC(X) \simeq \mathrm{Fun}(\Pi_\infty(X), \mathrm{Spc})?$$

Def. An ∞-topos X is **locally of contractible shape** (or **locally of constant shape**) if the constant sheaf functor $\Gamma^*: \mathrm{Spc} \longrightarrow X$ admits a left adjoint $\Gamma_\#: X \longrightarrow \mathrm{Spc}$. We write

$$\Pi_\infty(X) := \Gamma_\#(1_X)$$

and call $\Pi_\infty(X)$ the **shape** of X.

> Then $\Gamma_\#$ factors as

$$X \xrightarrow[\text{left adj}]{\Gamma_\#} \mathrm{Spc}_{/\Pi_\infty(X)} \xrightarrow{\text{forget}} \mathrm{Spc}$$

Q. Why this definition? What are examples?

Thm (Lurie). TFAE for an ∞-topos $\mathcal{X}$:

(1) $\mathcal{X}$ is locally of constant shape

(2) $LC(\mathcal{X}) \subset \mathcal{X}$ is closed under limits and colimits and $LC(\mathcal{X})$ is atomically generated

"$\mathcal{X}$ is monodromic"

In this case

$$\Gamma_{\#} : LC(\mathcal{X}) \longrightarrow Spc_{/\Pi_\infty(\mathcal{X})} \simeq Fun(\Pi_\infty(\mathcal{X}), Spc)$$

is an equivalence of ∞-cats.

straightening/unstraightening

Remark. Given an ∞-topos $\mathcal{X}$, there is a left exact accessible localization $\mathcal{X}^{hyp} \hookrightarrow \mathcal{X}$ called the **hypercompletion** of $\mathcal{X}$. This generalizes hypersheaves on a top space. One can show that if $\mathcal{X}$ is locally of constant shape, then so is $\mathcal{X}^{hyp}$ and $LC(\mathcal{X}^{hyp}) = LC(\mathcal{X})$.

Fundamental Example

Claim. If X is a lwc top. space, then $Sh^{hyp}(X)$ is locally of contractible shape.

van Kampen Thm. For any top. space X,

$$\mathrm{Open}(X) \xrightarrow{\mathrm{Sing}} \mathrm{Spc}$$

is a hypercomplete cosheaf.

Construction. For any top space X, define a left adjoint

$$\mathrm{Sing} : \mathrm{PSh}(X) \longrightarrow \mathrm{Spc}$$

as the colimit-preserving extension of

$$\mathrm{Sing} : \mathrm{Open}(X) \longrightarrow \mathrm{Spc}.$$

Then Sing has a right adjoint $R : \mathrm{Spc} \longrightarrow \mathrm{PSh}(X)$

given by

$$R\colon \quad K \longmapsto [U \longmapsto \mathrm{Map}(\mathrm{Sing}(U), K)].$$
$$K \in \mathrm{Spc}$$

By van Kampen, R factors through $\mathrm{Sh}^{hyp}(X)$.

More refined claim. If X is lwc, then $R \simeq \Gamma^*$.

Proof.

crucially uses hypersheaves!

The ∞-categorical version of the ["basis theorem"] shows that

$$\mathrm{Sh}^{hyp}(X) \xrightarrow[\text{restrict}]{\sim} \mathrm{Sh}^{hyp}(\underbrace{\mathrm{Open}_{wc}(X)}_{\text{weakly contr opens}})$$

Consider the composite

$$\mathrm{Spc} \xrightarrow{R} \mathrm{Sh}^{hyp}(X) \xrightarrow{\sim} \mathrm{Sh}^{hyp}(\mathrm{Open}_{wc}(X))$$
$$K \longmapsto [U \mapsto \mathrm{Map}(\underbrace{\mathrm{Sing}(U)}_{*}, K)]$$

This is the constant presheaf functor, and also a sheaf! Hence the constant sheaf functor. □

Cor. If X is lwc, then

$$\underbrace{\Pi_\infty}_{\text{Shape}}(\mathrm{Sh}^{\mathrm{hyp}}(X)) \simeq \mathrm{Sing}(X).$$

Remark. This argument generalizes to coefficients in any presentable ∞-cat. $\mathcal{E}$. When $\mathcal{E} = D(R)$, this implies that singular and sheaf cohomology agree for lwc spaces.

> Given van Kampen, this proof is easier than the classical one! Also much more general.

Exodromic Stratified ∞-topoi

Def. Given a stratified ∞-topos $s: \mathcal{X} \longrightarrow \mathrm{Fun}(P, \mathrm{Spc})$ and locally closed subposet $S \subset P$, write $\mathcal{X}_S$ for the pullback in RTop_∞

$$\mathcal{X}_S := \mathcal{X} \times_{\mathrm{Fun}(P,\mathrm{Spc})} \mathrm{Fun}(S, \mathrm{Spc}).$$

$i_{S,*}: \mathcal{X}_S \hookrightarrow \mathcal{X}$ induced (fully faithful) geometric mor.

> p-th stratum $\mathcal{X}_p = \mathcal{X}_{\{p\}}$.

Prop. Given a pullback square of spaces

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ T \arrow[r, hook, "i"'] & Y \end{tikzcd}$$

where i is a locally closed immersion, the induced square

$$\begin{tikzcd} Sh(S) \arrow[r, hook] \arrow[d] & Sh(X) \arrow[d] \\ Sh(T) \arrow[r, hook, "i_*"'] & Sh(Y) \end{tikzcd}$$

is a pullback square in $RTop_\infty$. The same is true for hypersheaves.

Upshot. For a stratified space (X, P)

$$Sh(X)_S \simeq Sh(X_S) \quad [P \text{ noetherian}]$$

$$Sh^{hyp}(X)_S \simeq Sh^{hyp}(X_S)$$

Def. $(\mathcal{X}, P)$ stratified ∞-topos. $F \in \mathcal{X}$ is **constructible** if $\forall p \in P$, the restriction $i_p^*(F) \in \mathcal{X}_p$ is a locally constant object of $\mathcal{X}_p$.

> $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ full subcat of constructible sheaves.

Ex. For stratified spaces, this notion of constructibility agrees with the usual one.

Def. A stratified ∞-topos $S_*: \mathcal{X} \longrightarrow \mathrm{Fun}(P, \mathrm{Spc})$ is **exodromic** if:

(1) $\mathrm{Cons}_P(\mathcal{X}) \subset \mathcal{X}$ is closed under limits & colimits

(2) $\mathrm{Cons}_P(\mathcal{X})$ is atomically generated.

(3) $S^*: \mathrm{Fun}(P, \mathrm{Spc}) \longrightarrow \mathcal{X}$ preserves limits.

In this case, the **exit-path ∞-category** is

$$\Pi_\infty(\mathcal{X}, P) := \left(\mathrm{Cons}_P(\mathcal{X})^{at}\right)^{op}.$$

Cor. A stratified space $s: X \longrightarrow P$ is exodromic in the sense of Lecture 1 iff the stratified ∞-topos $s_*: Sh^{hyp}(X) \longrightarrow Fun(P, Spc)$ is exodromic.

Ex.

(1) [Ørsnes Jansen]: $Sh(\overline{\mathcal{M}}_{g,n})$ is exodromic with exit-path ∞-cat the Charney-Lee 1-cat!

(2) [Lurie-Tanaka]: $Sh(Broken) \simeq Fun(\mathbb{\Delta}_{surj}, Spc)$ is exodromic.

verbatim the same, just replace "stratified space" by "stratified ∞-topos"

Stability Thm.

(1) Stability under pulling back to locally closed subposets:
If (X,P) is exodromic and $S \subset P$ is locally closed, then (X_S, S) is exodromic and

$$\Pi_\infty(X_S, S) \xrightarrow{\sim} \Pi_\infty(X,P) \times_P S.$$

- Taking $S = \{p\}$ shows $\Pi_\infty(X,P) \longrightarrow P$ is conservative.

(2) Functoriality: If (X,P) and (Y,Q) are exodromic, and $f_*: (X,P) \longrightarrow (Y,Q)$ is a stratified morphism, then

$$f^*: \mathrm{Cons}_Q(Y) \longrightarrow \mathrm{Cons}_P(X)$$

preserves limits and colimits.

(3) Stability under coarsening: If (X,R) is exodromic and $\phi: R \longrightarrow P$ is any map of posets, then (X,P) is exodromic and

$$\Pi_\infty(X,P) \xleftarrow{\sim} \Pi_\infty(X,R)\left[\begin{pmatrix} \text{maps } \Pi_\infty(X,R) \to R \xrightarrow{\phi} P \\ \text{sends to identities} \end{pmatrix}^{-1}\right]$$

- Taking $P = *$, we deduce that

$$B(\Pi_\infty(X,R)) \xrightarrow{\sim} \underbrace{\Pi_\infty(X)}_{\text{Shape}}.$$

(4) van Kampen: Given a diagram $(X_\bullet, P_\bullet): I^{\triangleright} \to \mathrm{StrTop}_\infty$ if for each $i \in I$, (X_i, P_i) is exodromic and

$$X_\infty \xrightarrow{\sim} \lim_{i \in I^{op}}{}^{*} X_i$$

and

$$\mathrm{Cons}_{P_\infty}(X_\infty) \xrightarrow{\sim} \lim_{i \in I^{op}}{}^{*} \mathrm{Cons}_{P_i}(X_i),$$

then (X_∞, P_∞) is exodromic and

$$\operatorname*{colim}_{i \in I} \Pi_\infty(X_i, P_i) \xrightarrow{\sim} \Pi_\infty(X_\infty, P_\infty).$$

(6) Stability of finiteness/compactness

(7) Change of coefficients: If (X, P) is exodromic and $\mathcal{E}$ is compactly assembled, then

$$\begin{array}{c} \mathrm{Cons}_P(X) \otimes \mathcal{E} \xrightarrow{\sim} \mathrm{Cons}_P(X; \mathcal{E}) \subset X \otimes \mathcal{E}. \\ \wr\mid \\ \mathrm{Fun}(\Pi_\infty(X,P), \mathcal{E}) \end{array}$$

Outline of the Proof

Stability under pulling back to locally closed subposets

- We'll prove the more refined statement:

Claim. If (X,P) is exodromic and $S \subset P$ is locally closed, then (X_S, S) is exodromic and $i_{S,*} : (X_S, S) \hookrightarrow (X,P)$ is exodromic.

Observe. By factoring $S \subset P$ as a closed immersion followed by an open immersion, it suffices to treat the cases where S is open or closed.

Setup. $Z \subset P$ closed with open complement $U \subset P$.

Lem. Given a small ∞-cat $\mathcal{C}$ and functor $s : \mathcal{C} \to P$, write $\mathcal{C}_S := \mathcal{C} \times_P S$. Write $i : \mathcal{C}_Z \hookrightarrow \mathcal{C}$ and $j : \mathcal{C}_U \hookrightarrow \mathcal{C}$ for the inclusions. Then

$$\mathrm{Fun}(\mathcal{C}_Z, \mathrm{Spc}) \xleftarrow{i^*} \mathrm{Fun}(\mathcal{C}, \mathrm{Spc}) \xrightarrow{j^*} \mathrm{Fun}(\mathcal{C}_U, \mathrm{Spc})$$

exhibit $\mathrm{Fun}(\mathcal{C}, \mathrm{Spc})$ as the **recollement** of $\mathrm{Fun}(\mathcal{C}_Z, \mathrm{Spc})$ and $\mathrm{Fun}(\mathcal{C}_U, \mathrm{Spc})$. That is:

(1) i^* and j^* admit fully faithful right adjoints

(2) i^* and j^* are left exact.

(3) i^* and j^* are jointly conservative.

(4) $j^* i_* \simeq *$ [constant functor @ terminal object]

Proof.
Exercise! Just calculate Kan extensions. $\square$

Prop. If (X, P) is exodromic, then

$$\mathrm{Cons}_Z(X_Z) \xleftarrow{\ i_Z^*\ } \mathrm{Cons}_P(X) \xrightarrow{\ i_U^*\ } \mathrm{Cons}_U(X_U)$$

are a recollement.

Idea for rest of proof. We know the middle term is equivalent to $\mathrm{Fun}(\Pi_\infty(X,P),\mathrm{Spc})$ and we have a recollement into

$$\mathrm{Fun}(\Pi_\infty(X,P)_Z,\mathrm{Spc}) \text{ and } \mathrm{Fun}(\Pi_\infty(X,P)_U,\mathrm{Spc})$$

By general yoga of recollements, if we can compare one of the side terms, we get the other.
> It is easy to compare the open side.

From the comparison. Get i_Z^* and i_U^* are restriction along $\Pi_\infty(X,P)_Z \hookrightarrow \Pi_\infty(X,P)$ and $\Pi_\infty(X,P)_U \hookrightarrow \Pi_\infty(X,P)$, hence preserve limits.

Cor. If (X,P) is exodromic, then

$$\{i_p^*: \mathrm{Cons}_P(X) \longrightarrow LC(X_p)\}_{p\in P}$$

are jointly conservative and preserve limits and colimits.

Functoriality

> First, let's treat the unstratified setting.

[HTT, Cor. 6.3.5.9]

Cancellation for étale geometric mors. $\mathcal{X}$ ∞-topos $U, V \in \mathcal{X}$. Given a left exact left adjoint

$$f^* : \mathcal{X}_{/V} \longrightarrow \mathcal{X}_{/U},$$

there is a unique morphism $f : U \longrightarrow V$ in $\mathcal{X}$ such that $f^* \simeq U \times_V (-)$.

> Hence f^* admits a left adjoint $f_\#$.

Cor. Given $K, L \in \mathrm{Spc}$, every left exact left adjoint

$$f^* : \mathrm{Fun}(L, \mathrm{Spc}) \longrightarrow \mathrm{Fun}(K, \mathrm{Spc})$$

preserves limits.

Cor. Given a geometric morphism $f_*: \mathcal{X} \longrightarrow \mathcal{Y}$ with $\mathcal{X}$ and $\mathcal{Y}$ locally of contractible shape,

$$f^*: LC(\mathcal{Y}) \longrightarrow LC(\mathcal{X})$$

preserves limits.

Lem (exodromy $\Rightarrow$ monodromy). If $(\mathcal{X}, P)$ is exodromic, then $\mathcal{X}$ is locally of constant shape.

Proof.

$\Gamma^*: \mathrm{Spc} \longrightarrow \mathcal{X}$ is the composite

$$\mathrm{Spc} \xrightarrow{\text{const}} \mathrm{Fun}(P, \mathrm{Spc}) \xrightarrow{s^*} \mathcal{X}$$

has left adj colim_P

has left adj by assumption

$\square$

Lem. Let $f_*: (\mathcal{X}, P) \longrightarrow (\mathcal{Y}, *)$ be a morphism between exodromic stratified ∞-topoi (where the target has trivial stratification). Then f_* is exodromic.

Proof.
Factor f_* as

$$(\mathcal{X}, P) \xrightarrow[\substack{\text{exodrom} \\ \text{by prev lem}}]{} (\mathcal{X}, *) \xrightarrow[\substack{\text{exodrom} \\ \text{by prev}^2 \\ \text{lem}}]{f_*} (\mathcal{Y}, *).$$

$\square$

Thm. If $f_*: (\mathcal{X}, P) \longrightarrow (\mathcal{Y}, Q)$ is a morphism between exodromic stratified ∞-topoi, then f_* is exodromic.

Proof.
Write $\phi: P \longrightarrow Q$ for the map of posets and $P_q := \phi^{-1}(q)$. Then P_q is locally closed. Moreover, by stability under pulling back to loc closed subposets,

$$\{ i_{P_q}^* : \mathrm{Cons}_P(X) \longrightarrow \mathrm{Cons}_{P_q}(X_q) \}_{q \in Q}$$

are jointly conservative and preserve limits.

Upshot. Suffices to show each $i_{P_q}^* f^*$ preserves limits. I.e., $f_* i_{P_q,*} : (X_{P_q}, P_q) \longrightarrow (Y, Q)$ is exodromic.

We have a commutative square

$$\begin{tikzcd}
(X_{P_q}, P_q) \arrow[r, "f_{q,*}"] \arrow[d, hook, "i_{P_q,*}"'] & (Y_q, \{q\}) \arrow[d, hook, "i_{q,*}"] \\
(X, P) \arrow[r, "f_*"'] & (Y, Q)
\end{tikzcd}$$

exodromic by prev lem

exodromic by stability under PB

□

Stability under Coarsening

> The key technical result is:

Prop. Let $\mathcal{D}$ be an atomically generated presentable ∞-cat and $i: \mathcal{E} \hookrightarrow \mathcal{D}$ a full subcat closed under limits and colimits. Then:

(1) i admits adjoints $L \dashv i \dashv R$.

(2) $\mathcal{E}$ is atomically generated by $L(\mathcal{D}^{at})$.

(3) Let $W_L \subset Mor(\mathcal{D})$ be the L-equivalences and $W \subset W_L \cap Mor(\mathcal{D}^{at})$ a collection of mors st $\mathcal{E} = \{W\text{-local objs of } \mathcal{D}\}$. Then

$$\left(\mathcal{D}^{at}[W^{-1}]\right)^{idem} \xrightarrow[L]{\sim} \mathcal{E}^{at}.$$

Then.

(1) First prove for (X,R) exodromic $\phi: R \longrightarrow *$.

"Exodromy $\Rightarrow$ monodromy" shows that $(X,*)$ is exodromic and $LC(X) \subset Cons_R(X)$ is closed under limits and colimits.

- The previous Prop. shows that

$$\Pi_\infty(X,R)[all^{-1}] \xrightarrow{\sim} \Pi_\infty(X).$$

(2) Use stability under pullback to reduce to (1).

van Kampen

This is actually immediate from the definitions, the equivalence $\mathrm{Cat}_\infty^{\mathrm{idem}} \simeq \mathrm{Pr}^{L,\mathrm{at}}$, and the fact that the forgetful functors

$$\mathrm{Pr}^L \longrightarrow \mathrm{Cat}_\infty^{\mathrm{large}} \longleftarrow \mathrm{Pr}^R$$

both preserve limits.

Lecture 5
Applications

Finiteness of cohomology

Observe. (X,P) exodromic stratified space, A $\mathbb{E}_1$-ring. Then by functoriality, under exodromy the global sections functor

$$R\Gamma(X;-) : \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A) \longrightarrow \mathrm{Mod}_A$$

is identified with the limit functor

$$\lim : \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Mod}_A) \longrightarrow \mathrm{Mod}_A.$$

Ntn. If (X,P) has lwc strata, write

$$\mathrm{Cons}_P^{\omega}(X;A) \subset \mathrm{Cons}_P^{\mathrm{hyp}}(X;\mathrm{Mod}_A)$$

for those objects with perfect stalks.

Obs. Exodromy restricts to an equivalence

$$\mathrm{Cons}_P^{\omega}(X;A) \xrightarrow[\sim]{\mathrm{Ex}} \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Perf}_A).$$

Cor. If (X,P) is an exodromic strat space with locally weakly contr. strata and $\Pi_\infty(X,P)$ is finite, then for any $F \in \mathrm{Cons}_P^\omega(X;A)$, then

$$R\Gamma(X;F) \in \mathrm{Perf}_A$$

Proof

$R\Gamma(X;F) \simeq \lim\left(\Pi_\infty(X,P) \xrightarrow{Ex(F)} \mathrm{Perf}_A\right)$ and $\mathrm{Perf}_A \subset \mathrm{Mod}_A$ is closed under finite limits. $\square$

Localizing invariants of constructible sheaves

With. Qingyuan Bai

Observation. $\mathcal{C} \longrightarrow P$ functor from a small ∞-cat to a poset, $Z \subset P$ closed, $U := P \setminus Z$. $i : \mathcal{C}_Z \hookrightarrow \mathcal{C}$ and $j : \mathcal{C}_U \hookrightarrow \mathcal{C}$
$\mathcal{E}$ stable ∞-cat with limits + colimits.
Then we have adjoints

$$\begin{tikzcd}[column sep=huge]
\mathrm{Fun}(\mathcal{C}_Z, \mathcal{E}) \arrow[r, bend left=40, "i_\#"] \arrow[r, bend right=40, "i_*"'] & \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \arrow[l, "i^*"] \arrow[l, bend left=70, "i^!"] \arrow[r, "j^*"] \arrow[r, bend left=70, "j^L_\#"] & \mathrm{Fun}(\mathcal{C}_U, \mathcal{E}) \arrow[l, bend right=40, "j_\#"'] \arrow[l, bend left=40, "j_*"]
\end{tikzcd}$$

- $i_\#, j_\#$ LKE
- i^*, j^* restriction
- i_*, j_* RKE

- $i^!$ exists by the i^*, j^* recollement
  $i^! \xrightarrow{\sim} \mathrm{fib}\left(i^* \xrightarrow{i^* \mathrm{unit}} i^* j_* j^*\right)$

- $j_\#^L$ also exists by recollement and is given by

$$j_\#^L \simeq \mathrm{cofib}\left(j^* i_\# i^* \xrightarrow{j^*\mathrm{counit}} j^*\right).$$

- Moreover, $j_\#^L$ and i^* also form a recollement! (with the closed & open roles switched)

Upshot. If $(\mathcal{X}, P)$ is an exodromic stratified ∞-topos and $\mathcal{E}$ is a compactly assembled presentable ∞-cat, then we also get these adjoints and relations.

Recall. Let $\mathcal{H}$ be a stable presentable ∞-Cat. A localizing invariant is a functor

$$L: \underbrace{\Pr^L_{st,cg}}_{\substack{\text{compactly gen} \\ \text{stable } \infty\text{-cats} \\ \text{+ left adjs whose} \\ \text{right adj pres.} \\ \text{colimits}}} \underset{(-)^\omega}{\overset{\text{Ind}}{\underset{\sim}{\leftrightarrows}}} \underbrace{\mathrm{Cat}^{perf}_\infty}_{\substack{\text{idem complete} \\ \text{small stable } \infty\text{-cats} \\ \text{+ exact functors}}} \longrightarrow \mathcal{H}$$

such that $L(0)=0$ and L sends sequences that are both fiber and cofiber sequences to fiber seqs.

> Efimov has extended the theory of localizing invariants to [**dualizable**] objects of $\Pr^L_{st}$:

(equivalent to compactly assembled

$\mathrm{Ind}(\mathcal{C}_0)$ has dual $\mathrm{Ind}(\mathcal{C}_0^{op})$

$$L \text{ extends to } L^{cont}: \underbrace{\Pr^{dual}_{st}}_{\substack{\text{dualizable pres stable } \infty\text{-cats} \\ \text{+ left adjs whose rt adj preserves colims}}} \longrightarrow \mathcal{H}$$

Ex. K-theory, THH, ...

- Using that continuous localizing invariants split "continuous" semiorthogonal decompositions (with enough adjoints to occur in $\mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}}$), we deduce:

Thm. $\mathcal{C} \to P$ functor from a small ∞-cat to a poset, $\mathcal{E} \in \mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}}$. If P is finite or L is finitary, then:

$$L^{\mathrm{cont}}(\mathrm{Fun}(\mathcal{C},\mathcal{E})) \simeq \bigoplus_{p \in P} L^{\mathrm{cont}}(\mathrm{Fun}(\mathcal{C}_p,\mathcal{E})).$$

Cor. (X,P) exodromic stratified ∞-topos, $\mathcal{E} \in \mathrm{Pr}^{\mathrm{dual}}_{\mathrm{st}}$. If P is finite or L is finitary, then:

$$L^{\mathrm{cont}}(\mathrm{Cons}_P(X;\mathcal{E})) \simeq \bigoplus_{p \in P} L^{\mathrm{cont}}(\mathrm{Fun}(\Pi_\infty(X_p),\mathcal{E}))$$

Ex. If (X,P) is exodromic and all $\Pi_\infty(X_p)$ are compact spaces (e.g., in the real algebraic or subanalytic settings), then

$$K^{\mathrm{cont}}(\mathrm{Cons}_P(X;\mathrm{Sp})) \simeq \bigoplus_{p\in P} \underbrace{A(\Pi_\infty(X_p))}_{\substack{\text{Waldhausen}\\ A\text{-theory}}}$$

Ex. $$\mathrm{THH}^{\mathrm{cont}}(\mathrm{Cons}_P(X;\mathrm{Sp})) \simeq \bigoplus_{p\in P} \mathbb{S}[\underbrace{\mathcal{L}\Pi_\infty(X_p)}_{\substack{\mathcal{L}Y = \mathrm{Map}(S^1, Y)\\ \text{free loop object}}}]$$

Also. Results for constructible sheaves w/ compact stalks, but more involved to state.

Moduli of constructible sheaves

Throughout. (X,P) exodromic stratified space with lwc strata, A animated commutative ring

Ntn. $\underline{\mathrm{Cons}}_P(X) : \mathrm{dAff}_A^{op} \longrightarrow \mathrm{Spc}$

$$\begin{tikzcd}
\mathrm{Spec}(C) \arrow[r, mapsto] \arrow[d] & \mathrm{Cons}_P^\omega(X;C)^\simeq \\
\mathrm{Spec}(B) \arrow[r, mapsto] & \mathrm{Cons}_P^\omega(X;B)^\simeq \arrow[u, "C\otimes_B(-)"']
\end{tikzcd}$$

Lem. $\underline{\mathrm{Cons}}_P(X)$ satisfies flat hyperdescent, in particular a derived stack.

Proof.

Since $B \mapsto \mathrm{Perf}_B$ satisfies flat hyperdescent,

$$B \mapsto \mathrm{Fun}(\Pi_\infty(X,P), \mathrm{Perf}_B) \text{ does too.} \qquad \square$$

Thm. If $\Pi_\infty(X,P)$ is compact, then:

(1) $\underline{\mathrm{Cons}}_P(X)$ is locally geometric and locally of finite presentation.

(2) For a point $x : \mathrm{Spec}(B) \longrightarrow \underline{\mathrm{Cons}}_P(X)$ corresponding to a constructible sheaf of B-mods F with perfect stalks,

$$x^* \mathbb{T}_{\underline{\mathrm{Cons}}_P(X)} \simeq \mathrm{map}(F,F)[1].$$

Thm. If (X,R) has compact exit-path ∞-cat and $\phi : R \longrightarrow P$ is any map of posets, then

$$\underline{\mathrm{Cons}}_P(X) \hookrightarrow \underline{\mathrm{Cons}}_R(X)$$

is a representable open immersion.

Similar results. For moduli of perverse sheaves.

Stokes Data. Porta–Teyssier use these results to prove representability of derived moduli of Stokes data for subanalytic stratified spaces of any dimension.

- Previously only in dim ≤ 2 by hard results of Kedlaya and Takuro Mochizuki
- Also generalize many hard results of Sabbah.
- They crucially need the finiteness of exit-paths for subanalytic stratified spaces.