Connectedness of cotensors

Peter J. Haine

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1 Overview

The purpose of this note is to generalize the following observation: given an $(n+1)$-connected map of spaces $f\colon X \to Y$, the induced morphism $\mathrm{L}f\colon \mathrm{L}X \to \mathrm{L}Y$ on free loop spaces is n-connected. To cast this result in a more general context, note that $\mathrm{L}X$ is the cotensor X^{S^1} of X by the circle S^1. The circle S^1 only has cells in dimensions ≤ 1, and the observation generalizes by saying that in an ∞-topos, cotensoring with a finite space with cells in dimensions $\leq m$ decreases connectedness of morphisms by m.

1.1 Proposition. *Let $\boldsymbol{X}$ be an ∞-topos, let $m, n \geq 0$ be integers, and let K be a space that can be written as a retract of a* finite *space with cells in dimensions $\leq m$. If $f\colon X \to Y$ is an $(m+n)$-connected morphism of $\boldsymbol{X}$, then the cotensor $f^K\colon X^K \to Y^K$ is n-connected.*

As long as ℓ-connected morphisms are stable under inifinite products in $\boldsymbol{X}$ (e.g., $\boldsymbol{X} = \mathbf{Spc}$) the conclusion of Proposition 1.1 is also valid without the finiteness hypothesis on K. See Proposition 3.4.

Outline. Included for the unfamiliar reader, Section 2 reviews a bit of background on cotensors and finite spaces. Section 3 is dedicated to the proof of Proposition 1.1.

2 Background on cotensors & finite spaces

We begin by recalling the cotensoring of an ∞-category with all limits over the ∞-category of spaces.

2.1 Notation. We write $\mathbf{Spc}$ for the ∞-category of spaces.

2.2 Recollection (cotensoring over $\mathbf{Spc}$)**.** Every ∞-category C with all limits is naturally *cotensored over* the ∞-category $\mathbf{Spc}$ of spaces [HTT, Remark 5.5.2.6]. That is, there is a functor

$$\begin{aligned} \mathbf{Spc}^{\mathrm{op}} \times C &\to C \\ (K, X) &\mapsto X^K , \end{aligned}$$

along with natural equivalences

$$\mathrm{Map}_C(X', X^K) \simeq \mathrm{Map}_{\mathbf{Spc}}(K, \mathrm{Map}_C(X', X)) .$$

Note that for each object $X \in C$, the functor $X^{(-)} \colon \mathbf{Spc}^{\mathrm{op}} \to C$ preserves limits.

The cotensor can be defined explicitly as follows. Recall that **Spc** is obtained by the terminal category $*$ by freely adjoining colimits. Under the equivalence

$$\mathrm{Fun}(\mathbf{Spc}^{\mathrm{op}} \times C, C) \simeq \mathrm{Fun}(\mathbf{Spc}^{\mathrm{op}}, \mathrm{Fun}(C, C)) \ ,$$

the cotensor corresponds to the functor

$$\mathbf{Spc}^{\mathrm{op}} \to \mathrm{Fun}(C, C)$$

obtained by extending the functor $\mathrm{id}_C \colon *^{\mathrm{op}} \to \mathrm{Fun}(C, C)$ along limits.

In this note, we are particularly interested in the case that C is an ∞-topos. In this case, the cotensor has a very explicit description.

2.3 Observation. Let $\boldsymbol{X}$ be an ∞-topos and write $\Gamma^* \colon \mathbf{Spc} \to \boldsymbol{X}$ for the constant sheaf functor (the left adjoint to the global sections functor). Write

$$\mathrm{Hom}_{\boldsymbol{X}} \colon \boldsymbol{X}^{\mathrm{op}} \times \boldsymbol{X} \to \boldsymbol{X}$$

for the internal-Hom in $\boldsymbol{X}$. The cotensor of an object $X \in \boldsymbol{X}$ by a space K is given by the internal-Hom

$$X^K \simeq \mathrm{Hom}_{\boldsymbol{X}}(\Gamma^*(K), X) \ .$$

In the statement of Proposition 1.1, we need to restrict to cotensoring by (retracts of) *finite spaces*. One way of identifying finite spaces is as the underlying homotopy types of CW complexes with finitely many cells; here is an invariant way.

2.4 Definition. The subcategory $\mathbf{Spc}^{\mathrm{fin}} \subset \mathbf{Spc}$ of *finite spaces* is the smallest full subcategory containing the terminal object and closed under finite colimits.

Similar to Recollection 2.2, ∞-categories with finite limits are cotensored over $\mathbf{Spc}^{\mathrm{fin}}$.

2.5 Recollection (cotensoring over $\mathbf{Spc}^{\mathrm{fin}}$)**.** Every ∞-category C with finite limits is naturally cotensored over the ∞-category $\mathbf{Spc}^{\mathrm{fin}}$ of finite spaces. The cotensor can be defined explicitly as follows. Note that $\mathbf{Spc}^{\mathrm{fin}}$ is obtained by the terminal category $*$ by freely adjoining finite colimits. Under the equivalence

$$\mathrm{Fun}(\mathbf{Spc}^{\mathrm{fin,op}} \times C, C) \simeq \mathrm{Fun}(\mathbf{Spc}^{\mathrm{fin,op}}, \mathrm{Fun}(C, C)) \ ,$$

the cotensor corresponds to the functor

$$\mathbf{Spc}^{\mathrm{fin,op}} \to \mathrm{Fun}(C, C)$$

obtained by extending the functor $\mathrm{id}_C \colon *^{\mathrm{op}} \to \mathrm{Fun}(C, C)$ along finite limits.

2.6 Observation. Let $F \colon C \to D$ be a functor between ∞-categories with finite limits, let $X \in C$, and let K be a finite space. Functoriality provides a natural comparison morphism

$$c_{X,K} \colon F(X^K) \simeq F(\lim_K X) \to \lim_K F(X) \simeq F(X)^K \ .$$

Note that if the functor F is left exact, then for every $X \in C$ and finite space K, the morphism $c_{X,K}$ is an equivalence.

The last thing we need to explain in the statement of Proposition 1.1 is an invariant description of the underlying homotopy types of a CW complexes with cells in dimensions $\leq m$.

2.7 Notation. We write $\mathrm{S}^0 \in \mathbf{Spc}$ for the coproduct $* \sqcup *$, and for each integer $m \geq 1$, we write $\mathrm{S}^m \in \mathbf{Spc}$ for the pushout

$$\begin{array}{ccc} \mathrm{S}^{m-1} & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{S}^m \end{array} .$$

2.8 Definition. For each integer $m \geq 0$, we define *spaces with cells in dimensions $\leq m$* recursively as follows.

- A space K *has cells in dimensions ≤ 0* if and only if K is discrete.
- For $m \geq 1$, a space K *has cells in dimensions $\leq m$* if there exists a space $K_{(m-1)}$ with cells in dimensions $\leq m-1$ and a pushout diagram

$$\begin{array}{ccc} \coprod_{i \in I} \mathrm{S}^{m-1} & \longrightarrow & K_{(m-1)} \\ \downarrow & & \downarrow \\ \coprod_{i \in I} * & \longrightarrow & K \end{array}$$

  in $\mathbf{Spc}$.

3 The proof

The idea is prove Proposition 1.1 by induction on m. We express a finite space with cells in dimensions $\leq m$ as a pushout by attaching cells of dimension m to a finite space of cells of dimension $\leq m-1$, and use the following result to prove the claim.

3.1 Proposition ([1, Proposition 4.13])**.** *Let $\mathbf{X}$ be an ∞-topos, $\ell \geq -2$ be an integer, and*

$$\begin{array}{ccccc} A & \xrightarrow{f} & C & \xleftarrow{g} & B \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ A' & \xrightarrow[f']{} & C' & \xleftarrow[g']{} & B' \end{array}$$

be a commutative diagram in $\mathbf{X}$. If a and b are ℓ-connected and c is $(\ell+1)$-connected, then the induced morphism on pullbacks $a \times_c b \colon A \times_C B \to A' \times_{C'} B'$ is ℓ-connected.

We first treat the special case of cotensoring by a sphere.

3.2 Lemma. *Let $\mathbf{X}$ be an ∞-topos and let $m, n \geq 0$ be integers. If $f \colon X \to Y$ is an $(m+n)$-connected morphism of $\mathbf{X}$, then the cotensor*

$$f^{S^m} \colon X^{S^m} \to Y^{S^m}$$

is n-connected.

Proof. We prove the claim by induction on the integer m. For the base case $m = 0$, note that the cotensor f^{S^0} is the product map

$$f \times f \colon X \times X \to Y \times Y .$$

The claim follows from the fact that the class of n-connected morphisms of an ∞-topos is stable under finite products.

For the induction step, assume that we have shown that for each integer $n \geq 0$, and $(m+n)$-connected morphism g in $\mathbf{X}$, the cotensor g^{S^m} is n-connected. Let $f \colon X \to Y$ be an $(m+1+n)$-connected morphism of $\mathbf{X}$. To see that $f^{S^{m+1}}$ is n-connected, use the fact that $S^{m+1} \simeq * \sqcup^{S^m} *$ to express $X^{S^{m+1}}$ and $Y^{S^{m+1}}$ as the pullbacks of the top and bottom rows of the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_X} & X^{S^m} & \xleftarrow{\Delta_X} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{S^m}} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\Delta_Y]{} & Y^{S^m} & \xleftarrow[\Delta_Y]{} & Y . \end{array} \tag{3.3}$$

By assumption f is $(m+1+n)$-connected, and by the induction hypothesis f^{S^m} is $(n+1)$-connected. Hence applying Proposition 3.1 to the diagram (3.3) shows that $f^{S^{m+1}}$ is n-connected. □

Under the hypothesis that ℓ-connected morphisms are closed under arbitrary products, we now prove a variant of Proposition 1.1 with stronger conclusion. This hypothesis holds in any presheaf ∞-topos, and we deduce Proposition 1.1 by noting that the conclusion of Proposition 1.1 is preserved by passage to a left exact localization.

3.4 Proposition. *Let $\mathbf{X}$ be an ∞-topos, and assume that for each integer $\ell \geq 0$, the class of ℓ-connected morphisms in $\mathbf{X}$ is stable under arbitrary products (e.g., $\mathbf{X} = \mathbf{Spc}$). Let $m, n \geq 0$ be integers, and let K be a space that can be written as a retract of a space with cells in dimensions $\leq m$. If $f \colon X \to Y$ is an $(m+n)$-connected morphism of $\mathbf{X}$, then the cotensor $f^K \colon X^K \to Y^K$ is n-connected.*

Proof. First notice that since the class of ℓ-connected morphisms in an ∞-topos is closed under retracts, it suffices to prove:

(∗) Let $m, n \geq 0$ be integers, and let K a space with cells in dimensions $\leq m$. If $f \colon X \to Y$ is an $(m+n)$-connected morphism of $\mathbf{X}$, then the cotensor $f^K \colon X^K \to Y^K$ is n-connected.

We prove ($*$) by induction on m. For $m = 0$, notice that since n-connected morphisms are stable under products in $\boldsymbol{X}$, if K is a discrete space, then

$$f^K \simeq \prod_{i \in K} f : \prod_{i \in K} X \to \prod_{i \in K} Y$$

is n-connected.

For the induction step, assume that we have proven the claim for spaces with cells in dimensions $\leq m$, and let K be a space with cells in dimensions $\leq m+1$. Express K as a pushout

$$\begin{array}{ccc} \coprod_{i \in I} S^m & \longrightarrow & K_{(m)} \\ \downarrow & & \downarrow \\ \coprod_{i \in I} * & \longrightarrow & K \, , \end{array}$$

where $K_{(m)}$ is a space with cells in dimensions $\leq m$. Then f^K is the morphism on pullbacks induced by the diagram of cospans

$$\begin{array}{ccccc} X^I & \longrightarrow & X^{\coprod_{i \in I} S^m} & \longleftarrow & X^{K_{(m)}} \\ {\scriptstyle f^I}\downarrow & & \downarrow {\scriptstyle f^{\coprod_{i \in I} S^m}} & & \downarrow {\scriptstyle f^{K_{(m)}}} \\ Y^I & \longrightarrow & Y^{\coprod_{i \in I} S^m} & \longleftarrow & Y^{K_{(m)}} \, . \end{array} \tag{3.5}$$

Since f is $(m+1+n)$-connected by assumption:

- By the base case, f^I is $(m+1+n)$-connected.
- By the inductive hypothesis, $f^{K_{(m)}}$ is $(n+1)$-connected.
- By Lemma 3.2 and the assumption that $(n+1)$-connected morphisms in $\boldsymbol{X}$ are closed under products, the morphism

  $$f^{\coprod_{i \in I} S^m} \simeq \prod_{i \in I} f^{S^m} : \prod_{i \in I} X^{S^m} \to \prod_{i \in I} Y^{S^m}$$

  is $(n+1)$-connected.

Applying Proposition 3.1 to the diagram (3.5) concludes the proof of the induction step. □

The observation necessary to deduce Proposition 1.1 from Proposition 3.4 is that left exact functors commute with cotensors by finite spaces (Observation 2.6).

Proof of Proposition 1.1. Since the class of ℓ-connected morphisms in $\boldsymbol{X}$ is closed under retracts, it suffices to prove the claim for finite spaces K. Choose a small ∞-category C and left exact localization $L \colon \mathrm{PSh}(C) \to \boldsymbol{X}$. Since L commutes with cotensors by finite spaces (Observation 2.6), the claim now follows from Proposition 3.4 applied to $\mathrm{PSh}(C)$ and the fact that a morphism g in $\boldsymbol{X}$ is ℓ-connected if and only if there exists a ℓ-connected morphism g' in $\mathrm{PSh}(C)$ and an equivalence $g \simeq L(g')$ [HTT, Remark 6.5.1.15]. □

3.6 Remark. An alternative way to prove Proposition 1.1 is to repeat the proof Proposition 3.4, where we only allow the set I to be finite.

References

HTT J. Lurie, *Higher topos theory*, Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 2009, vol. 170, pp. xviii+925, ISBN: 978-0-691-14049-0; 0-691-14049-9.

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