The Bégueri Resolution

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Let S be a scheme and G a commutative, finite locally free S-group scheme. The purpose of this note is to explain how to resolve G by smooth affine S-group schemes. Specifically, we give an exposition of Bégueri's result [1, Proposition 2.2.1] that there exists a (functorial) short exact sequence of fppf sheaves

$$0 \longrightarrow G \hookrightarrow \mathrm{R}(G) \twoheadrightarrow \mathrm{Q}(G) \longrightarrow 0\,,$$

where $\mathrm{R}(G)$ and $\mathrm{Q}(G)$ are commutative, smooth affine S-group schemes. This resolution is a key technical ingredient used in Česnavičius and Scholze's recent work on purity for flat cohomology [3, §§5.2–5.3]. The reason why the Bégueri resolution is so useful is that it allows one to transfer many questions about fppf cohomology with coefficients in a commutative, finite locally free S-group into questions about fppf cohomology with coefficients in a commutative, smooth affine S-group. Since fppf cohomology with smooth coefficients agrees with étale chomology, one can often deduce the desired results about fppf cohomology from existing results about étale cohomology.

To explain Bégueri's construction, it will be easier to work with the Cartier dual $G^\vee = \mathrm{Hom}_S(G, \mathbf{G}_{\mathrm{m},S})$ of G. The reason for this is that there is a natural choice of a larger group $\mathrm{R}(G^\vee)$ to embed $\mathrm{Hom}_S(G, \mathbf{G}_{\mathrm{m},S})$ into: the sheaf $\mathrm{Mor}_S(G, \mathbf{G}_{\mathrm{m},S})$ with the pointwise group structure. This is actually a reasonable choice of $\mathrm{R}(G^\vee)$ because the sheaf $\mathrm{Mor}_S(G, \mathbf{G}_{\mathrm{m},S})$ is the *Weil restriction* of $\mathbf{G}_{\mathrm{m},G}$ and there are standard existence and smoothness results for Weil restrictions. We fix some notation that we'll use throughout this note and then recall the facts about Weil restrictions that we need.

1 Notation. Let $\boldsymbol{X}$ be a topos, U and V objects of $\boldsymbol{X}$, and A and B abelian group objects of $\boldsymbol{X}$.

(1) We write $V_U = V \times U$ for the pullback of V to $\boldsymbol{X}_{/U}$.

(2) We write $\mathrm{Mor}_{\boldsymbol{X}}(U, V) \in \boldsymbol{X}$ for the sheaf of morphisms $U \to V$ in $\boldsymbol{X}$. If V is an abelian group object of $\boldsymbol{X}$, then we regard $\mathrm{Mor}_{\boldsymbol{X}}(A, B)$ as an abelian group object of $\boldsymbol{X}$ with the pointwise group structure coming from the group structure on B.

(3) We write $\mathrm{Hom}_{\boldsymbol{X}}(A, B) \subset \mathrm{Mor}_{\boldsymbol{X}}(A, B)$ for the subsheaf of group homomorphisms $A \to B$.

(4) For each integer n, we write $\mathrm{Ext}^n_{\boldsymbol{X}}(A, B) \in \mathrm{Ab}(\boldsymbol{X})$ for the n^{th} sheaf of extensions of A by B.

(5) For each integer n, we write $\mathrm{H}^n_{\mathbf{X}}(U; A) \in \mathrm{Ab}(\boldsymbol{X})$ for the n^{th} cohomology sheaf of U with coefficients in A.

Let S be a scheme. When $\boldsymbol{X} = S_{\text{fppf}}$ is the topos of sheaves on S with respect to the fppf topology, we simply use the subsectipt 'S' rather than a subscript 'S_{fppf}'.

2 Recollection (Weil restriction)**.** Let $f : T \to S$ be a morphism of schemes and Y a scheme over T. If the image of Y under the pushforward functor

$$\begin{aligned} f_\star \colon \mathrm{PSh}(\mathbf{Sch}_T) &\to \mathrm{PSh}(\mathbf{Sch}_S) \\ F &\mapsto [U \mapsto F(U \times_S T)] \end{aligned}$$

is representable by an S-scheme, we call the scheme $f_\star(Y)$ the *Weil restriction of Y along f*, and write $\mathrm{Res}_{T/S}(Y) \coloneqq f_\star(Y)$.

We only recall two facts about Weil restrictions that we need here, and refer the reader to [2, §7.6; 4; 6, Chapter 4] for basic properties of Weil restriction as well as existence criteria.

3 Remark. Let $f : T \to S$ be a morphism of schemes and let X be an S-scheme. If the Weil restriction of $T \times_S X$ along f exists, then by the universal property we have that

$$\mathrm{Res}_{T/S}(T \times_S X) \cong \mathrm{Mor}_S(T, X) \,.$$

The following is the standard existence result for Weil restrictions that we need:

4 Proposition ([2, §7.6, Theorem 4; 6, Proposition 4.4])**.** *Let $f : T \to S$ be a finite locally free morphism of schemes. Then for every smooth affine T-scheme Y, the Weil restriction $\mathrm{Res}_{T/S}(Y)$ exists and is a smooth affine S-scheme.*

5 Example. Since $\mathbf{G}_{\mathrm{m},G}$ is a smooth affine G-group scheme, Remark 3 and Proposition 4 show that

$$\mathrm{Mor}_S(G, \mathbf{G}_{\mathrm{m},S}) \cong \mathrm{Res}_{G/S}(\mathbf{G}_{\mathrm{m},G})$$

is a commutative, smooth affine S-group scheme. Thus in order to construct the desired resolution of $G^\vee$, it suffices to show that the cokernel of the inclusion

$$\mathrm{Hom}_S(G, \mathbf{G}_{\mathrm{m},S}) \hookrightarrow \mathrm{Mor}_S(G, \mathbf{G}_{\mathrm{m},S})$$

is a smooth affine S-scheme.

If we can show that this cokernel is representable by an affine S-group, the smoothness comes for free from the fact that if the quotient of a smooth group by a flat subgroup is representable, then it is also smooth:

6 Proposition ([SGA 3_{I}, Exposé VI_{B}, Proposition 9.2 (xii)])**.** *Let S be a scheme and let*

$$0 \longrightarrow K \hookrightarrow H \overset{p}{\twoheadrightarrow} Q \longrightarrow 0$$

be a short exact sequence of commutative S-group schemes. Then:

(1) *The S-group K is flat and locally of finite presentation if and only if the morphism $p \colon H \to Q$ is faithfully flat and locally of finite presentation.*

(2) *If K is flat and locally of finite presentation over S and H is smooth over S, then the quotient Q is smooth over S.*

Identification of the quotient

By Example 5 and Proposition 6, in order to construct the Bégueri resolution of $G^\vee$, we're reduced to showing that the cokernel of the inclusion $\mathrm{Hom}_S(G, \mathbf{G}_\mathrm{m}) \hookrightarrow \mathrm{Mor}_S(G, \mathbf{G}_\mathrm{m})$ is representable by an affine S-scheme. The first thing to notice is that we can identify the cokernel in the more general setup where we replace the fppf topos of S with an arbitrary topos, and G and $\mathbf{G}_\mathrm{m}$ by arbitrary abelian group objects. The correct term is a variant of an Ext group where the short exact sequences admit local sections on the level of sheaves of sets.

Let $\boldsymbol{X}$ be a topos. In the following we identify the category $\mathrm{Ab}(\boldsymbol{X})$ of abelian group objects of $\boldsymbol{X}$ with the category of limit-preserving functors $\boldsymbol{X}^{\mathrm{op}} \to \mathbf{Ab}$. If one prefers, the following construction can be presented in terms of sheaves of abelian groups on a site; here we have chosen present the invariant formulation that does not make use of sites.

7 Construction. Let $\boldsymbol{X}$ be a topos and A and B abelian group objects of $\boldsymbol{X}$. Define an abelian group object $\mathrm{Ext}^1_{\boldsymbol{X}}(A,B)^{\mathrm{sec}}$ of $\boldsymbol{X}$ as follows. For each object $U \in \boldsymbol{X}$, write $\mathrm{E}(A,B)(U)$ for the set of of isomorphism classes of short exact sequences

$$0 \longrightarrow B_U \longrightarrow C \xrightarrow{p} A_U \longrightarrow 0 \tag{8}$$

in $\mathrm{Ab}(\boldsymbol{X}_{/U})$ equipped with a morphism $s\colon A_U \to C$ in $\boldsymbol{X}_{/U}$ that is a section of p. We simply write $(C,s) \in \mathrm{E}(A,B)(U)$ for the isomorphism class of the short exact sequence (8) equipped with the section s.

We give the set $\mathrm{E}(A,B)(U)$ the structure of an abelian group as follows. Given elements $(C,s),(C',s') \in \mathrm{E}(A,B)(U)$, the sum of (C,s) and (C',s') is given by forming the Baer sum of the short exact sequences with middle terms C and C', and noting that the map $(s,s')\colon A_U \to C \oplus C'$ factors through the pullback $C \times_{A_U} C' \subset C \oplus C'$, hence induces a section of the Baer sum of C and C'.

Finally, we write $\mathrm{Ext}^1_{\boldsymbol{X}}(A,B)^{\mathrm{sec}}$ for the sheafification of the presheaf of abelian groups given by the assignment

$$U \mapsto \mathrm{E}(A,B)(U)\,.$$

Note that there is a natural morphism of abelian group objects

$$\mathrm{Ext}^1_{\boldsymbol{X}}(A,B)^{\mathrm{sec}} \to \mathrm{Ext}^1_{\boldsymbol{X}}(A,B)$$

that simply forgets the additional data of the section.

Define a group homomorphism $\mathrm{d}\colon \mathrm{Mor}_{\boldsymbol{X}}(A,B) \to \mathrm{E}(A,B)$ as follows: for each object $U \in \boldsymbol{X}$, the map d sends a map $f\colon A_U \to B_U$ to the split short exact sequence

$$0 \longrightarrow B_U \longrightarrow A_U \oplus B_U \xrightarrow{\mathrm{pr}_1} A_U \longrightarrow 0$$

equipped with the section $(\mathrm{id}, f)\colon A_U \to A_U \oplus B_U$. We also simply write d for the composite of the morphism $\mathrm{d}\colon \mathrm{Mor}_{\boldsymbol{X}}(A,B) \to \mathrm{E}(A,B)$ with the sheafification map $\mathrm{E}(A,B) \to \mathrm{Ext}^1_{\boldsymbol{X}}(A,B)^{\mathrm{sec}}$.

With the somewhat involved definition of $\operatorname{Ext}^1_{\mathbf{X}}(A,B)^{\mathrm{sec}}$ in place, the fact that the following sequence is exact amounts to an exercise in the definitions.

9 Lemma ([1, Lemme 2.1.1]). *Let* $\mathbf{X}$ *be a topos and* A *and* B *abelian group objects of* $\mathbf{X}$*. Then there is a functorial exact sequence*

$$0 \longrightarrow \operatorname{Hom}_{\mathbf{X}}(A,B) \hookrightarrow \operatorname{Mor}_{\mathbf{X}}(A,B) \xrightarrow{\mathrm{d}} \operatorname{Ext}^1_{\mathbf{X}}(A,B)^{\mathrm{sec}} \longrightarrow \operatorname{Ext}^1_{\mathbf{X}}(A,B) \longrightarrow \mathrm{H}^1_{\mathbf{X}}(A;B)\,.$$

To use Lemma 9 to construct the Bégueri resolution, we now make use of the following standard vanishing result of Ext groups.

10 Lemma ([5, Lemme 6.2.2]). *Let* S *be a scheme and* G *a commutative, finite locally free* S*-group. Then* $\operatorname{Ext}^1_S(G,\mathbf{G}_{\mathrm{m}}) = 0$.

Thus we have a short exact sequence of fppf sheaves

$$0 \longrightarrow G^{\vee} \hookrightarrow \operatorname{Res}_{G/S}(\mathbf{G}_{\mathrm{m}}) \twoheadrightarrow \operatorname{Ext}^1_S(G,\mathbf{G}_{\mathrm{m}})^{\mathrm{sec}} \longrightarrow 0\,.$$

Hence we will have succeeded if we can show that $\operatorname{Ext}^1_S(G,\mathbf{G}_{\mathrm{m}})^{\mathrm{sec}}$ is representable by a smooth affine S-group. To do this, we provide a *cocycle* description of $\operatorname{Ext}^1_S(G,\mathbf{G}_{\mathrm{m}})^{\mathrm{sec}}$.

11 Construction (symmetric 2-cocycles). Let S be a scheme and G a commutative, finite locally free S-group. We define the commutative, affine S-group scheme $\mathrm{Z}^2_S(G;\mathbf{G}_{\mathrm{m}})^{\mathrm{sym}}$ of *symmetric 2-cocycles on* G *with values in* $\mathbf{G}_{\mathrm{m}}$ as follows. First notice by Remark 3, the fppf sheaf

$$\operatorname{Mor}_S(G\times_S G,\mathbf{G}_{\mathrm{m}}) \cong \operatorname{Res}_{(G\times_S G)/S}(\mathbf{G}_{\mathrm{m}})$$

is a commutative, smooth affine S-group scheme. The S-group scheme $\mathrm{Z}^2_S(G;\mathbf{G}_{\mathrm{m}})^{\mathrm{sym}}$ is the closed scheme of $\operatorname{Mor}_S(G\times_S G,\mathbf{G}_{\mathrm{m}})$ consisting of those morphisms f satisfying the *symmetry condition*

$$f(x,y) = f(x,y)$$

and the *cocycle condition*

$$f(x,y)f(xy,z) = f(y,z)f(x,yz)\,.$$

Note that since closed immersions are affine, $\mathrm{Z}^2_S(G;\mathbf{G}_{\mathrm{m}})^{\mathrm{sym}}$ is also affine over S.

12 Proposition ([1, Proposition 2.2.1]). *Let* S *be a scheme and* G *a commutative, finite locally free* S*-group scheme. Then the fppf sheaf* $\operatorname{Ext}^1_S(G,\mathbf{G}_{\mathrm{m}})^{\mathrm{sec}}$ *is representable by the commutative, affine* S*-group scheme* $\mathrm{Z}^2_S(G;\mathbf{G}_{\mathrm{m}})^{\mathrm{sym}}$.

Proof sketch. We define a morphism presheaves of abelian groups

$$\theta\colon \mathrm{E}(G,\mathbf{G}_{\mathrm{m},S}) \rightsquigarrow \mathrm{Z}^2_S(G;\mathbf{G}_{\mathrm{m}})^{\mathrm{sym}}$$

as follows. For each S-scheme U and element $(C,s) \in \mathrm{E}(G,\mathbf{G}_{\mathrm{m},S})(U)$, write $\theta(U)(C,s)$ for the morphism (informally) defined by the assignment

$$\begin{aligned} G_U \times_U G_U &\to \mathbf{G}_{\mathrm{m},U} \\ (x,y) &\mapsto s(x)s(y)(s(xy))^{-1}\,. \end{aligned}$$

It is not difficult to check that θ is an isomorphism with inverse given by sending a symmetric 2-cocycle $c \in Z^2_S(G; \mathbf{G}_m)^{\mathrm{sym}}(U)$ to the class of the short exact sequence

$$0 \longrightarrow \mathbf{G}_{m,U} \longrightarrow G_U \times_U \mathbf{G}_{m,U} \xrightarrow{\mathrm{pr}_1} G_U \longrightarrow 0$$

with multiplication on $G_U \times_U \mathbf{G}_{m,U}$ defined (informally) by

$$(x,a)\cdot(y,b) \coloneqq (xy, c(x,y)ab) \,,$$

and section $s\colon G_U \to G_U \times_U \mathbf{G}_{m,U}$ defined (informally) by $s(x) \coloneqq (x,1)$. □

Combining Proposition 6, Lemma 9, Lemma 10, and Proposition 12 provides us with the Bégueri resolution:

13 Corollary (Bégueri resolution)**.** *Let S be a scheme and G a commutative, finite locally free S-group scheme. Then there are functorial short exact sequences of commutative S-group schemes*

$$0 \longrightarrow G^\vee \hookrightarrow \mathrm{Res}_{G/S}(\mathbf{G}_m) \xrightarrow{\mathrm{d}} Z^2_S(G; \mathbf{G}_m)^{\mathrm{sym}} \longrightarrow 0$$

and

$$0 \longrightarrow G \hookrightarrow \mathrm{Res}_{G^\vee/S}(\mathbf{G}_m) \xrightarrow{\mathrm{d}} Z^2_S(G^\vee; \mathbf{G}_m)^{\mathrm{sym}} \longrightarrow 0\,.$$

Moreover, the second and hence the third terms in each sequence are smooth and affine over S.

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