Interactions Between Homotopy Theory & Algebraic Geometry

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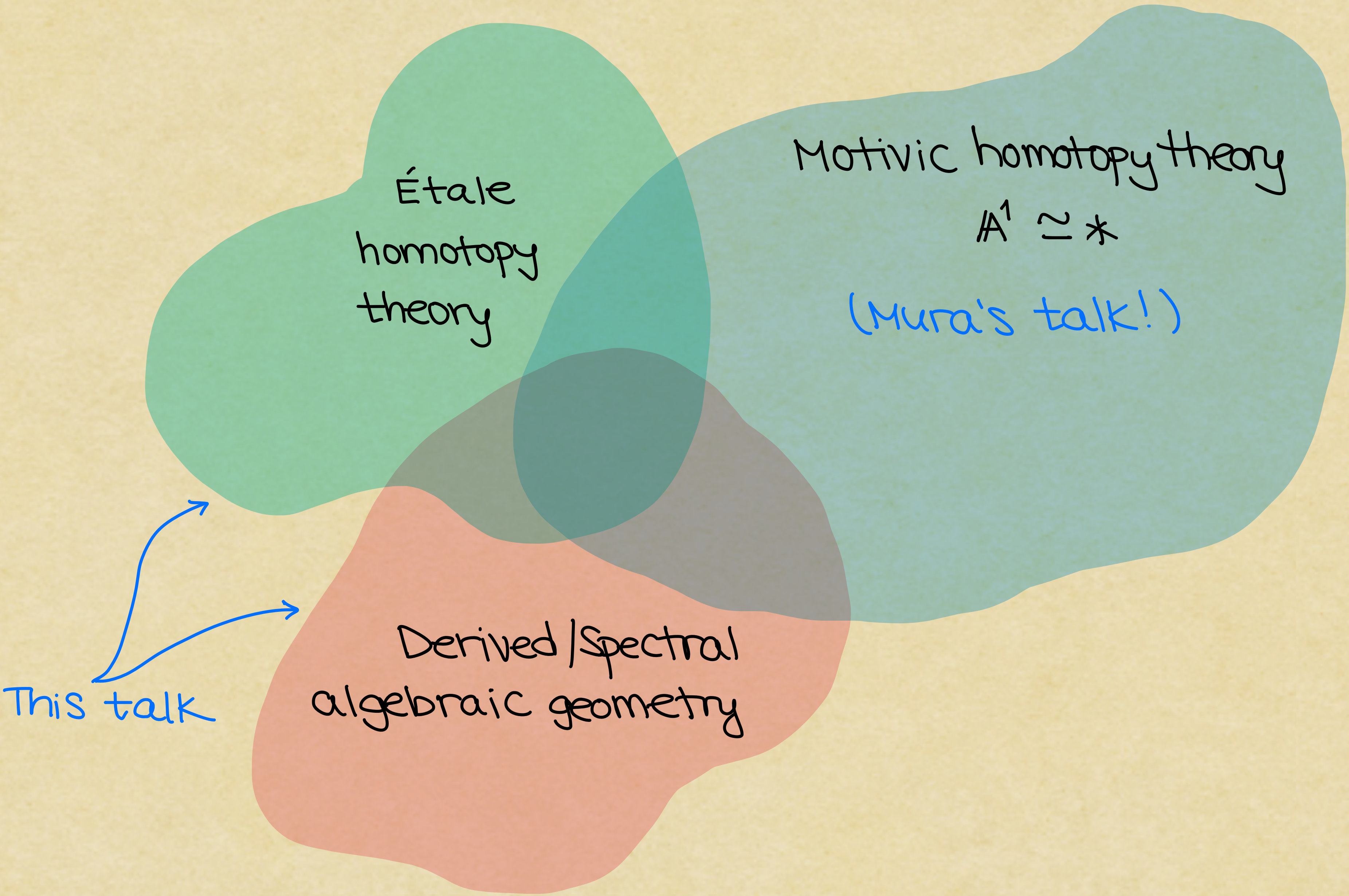
Three main intersections



Why inject homotopy theory into algebraic geometry?

Better Descent Results

Thm

(1) $X \longmapsto D(X;\mathbb{Z})$ is a sheaf of ∞-cats

top space — derived ∞-cat of sheaves

(2) $X \longmapsto D_{qc}(X)$ is an fpqc sheaf of ∞-cats

scheme — derived ∞-cat of quasicoherent sheaves

(3) $X \longmapsto C^*(X;\mathbb{Z})$ is a sheaf valued in $D(\mathbb{Z})$

top space — complex of singular cochains

Note

Descent for Sheaf ∞-categories $\Longrightarrow$ Descent for $R\Gamma$ $\Longrightarrow$ "cohomological descent" spectral seq. for H^*

> However, it turns out to often be easier to prove descent for the sheaf ∞-categories

Case Study Bhatt–Mathew, "The arc-topology"

> Get more conceptual proofs with fewer assumptions, e.g.,

Almost immediate from (3)

Thm If X is a locally weakly contractible top space, then

$$R\Gamma_{\text{sheaf}}(X;\mathbb{Z}) \simeq C^*(X;\mathbb{Z})$$

in $D(\mathbb{Z})$.

Intersection Theory

Bézout's Thm $C, C' \subset \mathbb{P}^2_{\mathbb{C}}$ smooth curves w/ transverse intersection and fundamental classes $[C], [C'] \in H^2(\mathbb{P}^2_{\mathbb{C}}; \mathbb{Z})$,

$$[C] \smile [C'] = [C \cap C'] \quad \text{in } H^4(\mathbb{P}^2_{\mathbb{C}}; \mathbb{Z}) \cong \mathbb{Z}$$

> if nontransverse but $C \cap C'$ is a finite # of points

Idea To deal with non-transverse $\cap$, replace pullback by a derived enhancement locally modelled on derived $\otimes$

> Since so much has been said elsewhere, we won't say more

Hidden Smoothness

Philosophy (Beilinson, Deligne, Drinfeld, Konsevich ...)

Many classical moduli stacks are $\underbrace{\text{poorly behaved}}_{\text{not smooth / lci, wrong dim}}$

but have derived refinements that are better-behaved

Thm X/k smooth proper, $\mathcal{M}$ the derived moduli of either:

(1) vector bundles on X

(2) coherent sheaves on X

(3) principal G-bundles on X (for G an algebraic group)

Then $\mathbb{L}_{\mathcal{M}}$ is perfect and

$$\dim(X) \leq d \implies \text{Tor amplitude}(\mathbb{L}_{\mathcal{M}}) \leq d-1 .$$

$d=1$ $\mathcal{M}$ smooth

$d=2$ $\mathcal{M}$ quasi-smooth

However If $d \geq 2$, then the underlying classical stack $\mathcal{M}_{cl} \hookrightarrow \mathcal{M}$ is singular with unbounded cotangent complex

"Underived" statements with proofs naturally using derived tools

Absolute purity for flat cohomology (Česnavičius–Scholze)

(R, m) noetherian local ring that's a complete intersection

G finite flat commutative R-group.

Then

$$H^i_m(R; G) = 0 \text{ for } i < \dim(R)$$

fppf cohomology

$\leq \dim(R)$ if R is regular and not a field

A preview of the Setup: Animation

Key point If we take the functor of points perspective seriously, the main obstruction to the setup of DAG is what the ∞-cat of "rings" is

- Simplicial commutative rings: OK for homotopy theorists, but maybe difficult for others

- There's an alternative perspective:

Ntn R ring, $\mathrm{Poly}_R \subset \mathrm{CAlg}_R$ full subcat on fg polynomial R-algs $R[x_1, \dots, x_n]$

Thm The 1-category CAlg_R is obtained from Poly_R by freely adjoining filtered colims + reflexive coequalizers

$$\bullet \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \bullet$$

If D is a 1-category with these colimits, then there is an equivalence of categories

$$\mathrm{Fun}^{\mathrm{filt},\mathrm{refcoeq}}(\mathrm{CAlg}_R, D) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Poly}_R, D)$$

$$\overline{F} \longmapsto \overline{F}|_{\mathrm{Poly}_R}$$

Explicitly

$$\otimes \longmapsto \times$$

$$\mathrm{CAlg}_R \simeq \mathrm{Fun}^{\times}(\mathrm{Poly}_R^{op}, \mathrm{Set})$$

$$S \longmapsto \mathrm{Hom}_{\mathrm{CAlg}_R}(-, S)$$

Def The ∞-category of

animated R-algebras

OR

simplicial commutative R-algs

$\mathrm{CAlg}_R^{\mathrm{ani}}$

is the universal ∞-category obtained from the 1-category Poly_R by freely adjoining filtered colimits and colimits of simplicial objects

$$\mathrm{Fun}^{\mathrm{filt},\Delta^{\mathrm{op}}}(\mathrm{CAlg}_R^{\mathrm{ani}}, D) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{Poly}_R, D)$$

$$F \longmapsto F|_{\mathrm{Poly}_R}$$

Explicitly

$$\mathrm{CAlg}_R^{\mathrm{ani}} := \mathrm{Fun}^{\times}(\mathrm{Poly}_R^{\mathrm{op}}, \mathrm{Spc})$$

∞-cat of spaces / ∞-groupoids / Kan complexes / ...

Étale homotopy theory

What is the étale homotopy type?

- Defined by Artin-Mazur in the late '60s, refined by Friedlander in the early '80s

  ⇝ Difficult-to-use definition, no universal proproperty

- Used by Sullivan ('74) and Friedlander (following Quillen) to prove the Adams Conjecture in topology

- 2015: Hoyois gives a new definition using Lurie's shape theory

  ⇝ uses ∞-topoi, has a universal property

- X scheme (or stack) $\rightsquigarrow$ profinite space $\Pi_\infty^{\text{ét}}(X) \in \mathrm{Pro}(\underbrace{\mathrm{Spc}_\pi}_{})$

  Spaces w/ finite π_0, all π_i are finite and $\pi_i = 0$ for $i \gg 0$

  - $\pi_1 \Pi_\infty^{\text{ét}}(X) \cong \pi_1^{\text{ét}}(X)$
  - For a finite ring R

$$H^i(\Pi_\infty^{\text{ét}}(X); R) \cong H^i_{\text{ét}}(X; R)$$

$$D_{\text{lisse}}(X_{\text{ét}}; R) \simeq \mathrm{Fun}^{\text{cts}}(\Pi_\infty^{\text{ét}}(X), \mathrm{Perf}(R)).$$

locally constant w/ perfect stalks

  - $\Pi_\infty^{\text{ét}}(\mathrm{Spec}(k)) \simeq BG_k$

- Work of Barwick-Glasman-H. gives a new description of $\Pi_\infty^{\text{ét}}(X)$ as the classifying prospace of an explicit procategory

Basic properties

TLDR "All basic results about étale cohomology are true at the level of the étale homotopy type"

- Work of many: Artin-Mazur, Friedlander, Lurie, Hoyois, Carchedi, Chough, Orgogozo, H.-Holzschuh-Wolf, ...

Fundamental Fiber Sequence k field w/ sep closure $\bar{k} \supset k$, X/k qcqs, then there is a fiber sequence

$$\Pi_\infty^{\text{ét}}(X_{\bar{k}}) \longrightarrow \Pi_\infty^{\text{ét}}(X) \longrightarrow BG_k$$

$\Longleftrightarrow \pi_n^{\text{ét}}(X_{\bar{k}}) \cong \pi_n^{\text{ét}}(X)$ for $n \geq 2$ & usual π_1 SES

Riemann Existence If $X/\mathbb{C}$ is finite type, then there is a natural map

$$\Pi_\infty(X(\mathbb{C})) \longrightarrow \Pi_\infty^{\text{ét}}(X)$$

that becomes an equivalence after profinite completion

Künneth Formulas $k=\bar{k}$ of char $p \geq 0$, X, Y qcqs k-schemes

(1) If Y/k is proper, then

$$\Pi_\infty^{\text{ét}}(X \times_k Y) \xrightarrow{\sim} \Pi_\infty^{\text{ét}}(X) \times \Pi_\infty^{\text{ét}}(Y).$$

(2) With no additional assumptions

$$\Pi_\infty^{\text{ét}}(X \times_k Y)^\wedge_{p'} \xrightarrow{\sim} \Pi_\infty^{\text{ét}}(X)^\wedge_{p'} \times \Pi_\infty^{\text{ét}}(Y)^\wedge_{p'}$$

$(-)^\wedge_{p'}$ completion away from p

(3) Formulas for symmetric products

Affine analogue of proper basechange If (A, I) is a henselian pair, then

$$\Pi_\infty^{\text{ét}}(\mathrm{Spec}(A/I)) \xrightarrow{\sim} \Pi_\infty^{\text{ét}}(\mathrm{Spec}(A))$$

> This is just a sample... many more results!

Anabelian Geometry

Conj (Grothendieck, 1983) $K \supset \mathbb{Q}$ finitely generated extension,

$X, Y / K$ 'anabelian' varieties

Then the natural map

mod conjugation by $\pi_1^{\text{ét}}(Y)$

$$\mathrm{Isom}_K(X,Y) \longrightarrow \mathrm{Isom}^{\text{out}}_{G_K}(\pi_1^{\text{ét}}(X), \pi_1^{\text{ét}}(Y))$$

is bijective.

compatible with homomorphism

$\pi_1^{\text{ét}}(X) \longrightarrow \pi_1^{\text{ét}}(\mathrm{Spec}(K)) \cong G_K$

Thm (Tamagawa, Mochizuki, late 90s) Grothendieck's conj. true if X and Y are smooth hyperbolic curves.

Issue We don't really know what 'anabelian' means

- Clearly a restrictive condition
- Only reasonable if X and Y are étale $K(\pi,1)$s

$\rightsquigarrow$ Grothendieck's conjecture should hold for more general varieties if we replace $\pi_1^{\text{ét}}$ by $\Pi_\infty^{\text{ét}}$

Thm (A. Schmidt – Stix) $k \supset \mathbb{Q}$ finitely generated extension, X, Y smooth k-varieties st $\exists$ embeddings $X, Y \underset{\text{loc closed}}{\hookrightarrow} \prod$ hyperbolic curves

Then the natural map

$$\mathrm{Isom}_k(X,Y) \longrightarrow \pi_0 \, \mathrm{Equiv}_{BG_k}(\Pi_\infty^{\text{ét}}(X), \Pi_\infty^{\text{ét}}(Y))$$

is a split injection with a natural retraction.

Cor If $\Pi_\infty^{\text{ét}}(X) \simeq \Pi_\infty^{\text{ét}}(Y)$ over BG_k, then $X \cong Y$ as k-schemes.

Thm (Holschbach - J. Schmidt - Stix)

k alg. closed field, $\mathrm{char}(k) > 0$

X/k smooth variety

Then $\Pi_\infty^{\text{ét}}(X) \simeq *$ if and only if $X \cong \mathrm{Spec}(k)$.

Thm (Achinger) If X is a connected affine $\mathbb{F}_p$-scheme, then

$$\Pi_\infty^{\text{ét}}(X) \text{ is 1-truncated}$$

$$\text{i.e., } \simeq B\pi_1^{\text{ét}}(X)$$

Conj (Deligne–Morava) For any $g, n \geq 0$, the étale homotopy type $\Pi_\infty^{\text{ét}}(\mathcal{M}_{g,n})$ is 1-truncated.

$\iff$ the mapping class group $\Gamma_{g,n}$ is good in the sense of Serre

for a finite $\hat{G}$-module M,
$H^*(G;M) \xrightarrow{\sim} H^*(\hat{G};M)$

> Known for $g \leq 2$ and all n by Birman–Hilden

The Artin-Tate Pairing

Setup $X/\mathbb{F}_q$ smooth geometrically connected surface

$\ell \nmid q$ prime $\qquad \mathrm{Br}_{nd}(X) := \dfrac{\mathrm{Br}(X)}{\text{divisible elts.}}$

> M. Artin and Tate defined a pairing

$$\mathrm{Br}_{nd}(X)[\ell^\infty] \times \mathrm{Br}_{nd}(X)[\ell^\infty] \longrightarrow \mathbb{Q}/\mathbb{Z}$$

Conj (Tate, 1966) The Artin-Tate pairing is alternating

History

1967 & 86 Manin gave examples disproving Tate's conj.

1996 Urabe found mistakes in Manin's work

Thm (Feng, 2017) Tate's conjecture is true.

- uses analogies from arithmetic topology

Idea Because of the fiber sequence

$$\Pi_\infty^{\text{ét}}(X_{\overline{\mathbb{F}}_q}) \longrightarrow \Pi_\infty^{\text{ét}}(X) \longrightarrow B\hat{\mathbb{Z}} \simeq (S^1)^\wedge_\pi$$

"dim 4" "dim 5" "dim 1"

$\Pi_\infty^{\text{ét}}(X)$ behaves like a real 5-manifold

Artin-Tate pairing $\longleftrightarrow$ linking form on an orientable $(4d+1)$-manifold

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