# On coherent topoi & coherent 1-localic ∞-topoi

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**Abstract**

In this note we prove the following useful fact that seems to be missing from the literature: the ∞-category of coherent ordinary topoi is equivalent to the ∞-category of coherent 1-localic ∞-topoi. We also collect a number of examples of coherent geometric morphisms between ∞-topoi coming from algebraic geometry.

## Contents



## Overview

Let $f\colon X \to Y$ be a morphism between quasicompact quasiseparated schemes. It follows from [11, Example 7.1.7] that the induced geometric morphism

$$f_*\colon \mathbf{Sh}_{\mathit{pro\acute{e}t}}(X;\mathbf{Set}) \to \mathbf{Sh}_{\mathit{pro\acute{e}t}}(Y;\mathbf{Set})$$

on proétale topoi is a coherent geometric morphism between coherent topoi in the sense of [SGA $4_{\mathrm{II}}$, Exposé VI]. It is often helpful to be able to apply methods of homotopy theory to topos theory, especially if one needs to work with stacks. To do this, one works with the 1-localic ∞-topos associated to an ordinary topos, obtained by taking sheaves of *spaces* rather than sheaves of sets. There is again an induced geometric morphism

$$f_*\colon \mathbf{Sh}_{\mathit{pro\acute{e}t}}(X;\mathbf{Spc}) \to \mathbf{Sh}_{\mathit{pro\acute{e}t}}(Y;\mathbf{Spc})\ ,$$

and these $\infty$-topoi are coherent in the sense of [SAG, Appendix A]. One naturally expects this geometric morphism to satisfy the same kinds of good finiteness conditions as the morphism of ordinary topoi does, i.e., be *coherent* in the sense of [SAG, Appendix A]. However, a proof of this fact is not currently in the literature. This claim is not completely obvious either: from the perspective of higher topos theory, the pullback in a coherent geometric morphisms of ordinary topoi is only required to preserve 0-truncated coherent objects, rather than *all* coherent objects.

In this note we fill this small gap in the literature. We show that the theories of coherent ordinary topoi and coherent geometric morphisms (in the sense of [SGA $4_{\text{II}}$, Exposé VI]) and of coherent 1-localic $\infty$-topoi and coherent geometric morphisms (in the sense of [SAG, Appendix A]) are equivalent (Proposition 2.11). This point is surely known to experts, but does not seem to be explicitly addressed in [SAG, Appendix A] or elsewhere. Our main aim in proving this equivalence is to make the $\infty$-categorical version of sheaf theory more accessible to (non-derived) algebraic geometers who are interested in applying results from [SAG, Appendix A] to ordinary coherent topoi.

The proof of this equivalence reduces to showing that a coherent geometric morphism of ordinary coherent topoi induces a coherent geometric morphism of corresponding 1-localic $\infty$-topoi. This follows from the more general fact that a morphism of finitary $\infty$-sites induces a coherent gometric morphism on corresponding $\infty$-topoi (Corollary 2.9). In ordinary topos theory this is well-known [SGA $4_{\text{II}}$, Exposé VI, Corollaire 3.3], but the $\infty$-toposic version seems to be missing from the literature.

Our original motivation for proving Proposition 2.11 was the following. In recent work with Barwick and Glasman [2] we proved a basechange theorem for oriented fiber product squares of bounded coherent $\infty$-topoi [2, Theorem 8.1.4]. In the original version of [2], we claimed [2, Corollary 8.1.6] that this implies the basechange theorem for oriented fiber products of coherent topoi of Moerdijk and Vermeulen [12, Theorem 2(i)] (which is the nonabelian refinement of a result of Gabber [6, Exposé XI, Théorème 2.4]). While this is true, our original proof implicitly used that a coherent geometric morphism of ordinary topoi induces a coherent geometric morphism on corresponding 1-localic $\infty$-topoi.

In § 1 we review the classification of coherent topoi in terms of pretopoi as well as the classification of bounded coherent $\infty$-topoi in terms of bounded $\infty$-pretopoi. This review is aimed at readers familiar with [SGA $4_{\text{II}}$, Exposé VI], but not necessarily with pretopoi or coherent $\infty$-topoi; the familiar reader should skip straight to § 2. At the end of § 2 we collect a number of examples of coherent geometric morphisms between $\infty$-topoi coming from algebraic geometry.

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## Terminology & notations

- We write $\boldsymbol{N}$ for the poset of *nonnegative integers*, and $\boldsymbol{N}^{\triangleright} \coloneqq \boldsymbol{N} \cup \{\infty\}$.
- We write $\mathbf{Cat}_{\infty}$ for the $\infty$-category of $\infty$-categories.

- We write $\mathbf{Top}_\infty \subset \mathbf{Cat}_\infty$ for the $\infty$-category of $\infty$-topoi and geometric morphisms. We typically write $f_* \colon \boldsymbol{X} \to \boldsymbol{Y}$ to denote a geometric morphism from an $\infty$-topos $\boldsymbol{X}$ to an $\infty$-topos $\boldsymbol{Y}$ and write $f^*$ for the left exact left adjoint of $f_*$.
- We write $\mathbf{Cat}$ for the $(2,1)$-category of (ordinary) categories, functors, and natural isomorphisms, which we tacitly regard as an $\infty$-category (via the Duskin nerve [Ker, Tag 009P]). We write $\mathbf{Top} \subset \mathbf{Cat}$ for the subcategory of topoi and geometric morphisms.

# 1 Premilinaries on (higher) coherent topoi & pretopoi

In this section we review the classification of coherent topoi in terms of pretopoi, as well as the theory of coherent $\infty$-topoi and the classification of *bounded* coherent $\infty$-topoi in terms of *bounded* $\infty$-pretopoi.

## Classification of coherent topoi

We assume that the reader is familiar with coherent topoi in the sense of [SGA $4_{\text{II}}$, Exposé VI]. Excellent accounts of coherent topoi can also be found in [8; 11, §§C.5 & C.6]. The classification of coherent topoi in terms of pretopoi is sketched in [SGA $4_{\text{II}}$, Exposé VI, Exercise 3.11]; a self-contained account can be found in [9].

**1.1 Definition.** Let $\boldsymbol{X}$ be a topos.

(1.1.1) An object $U \in \boldsymbol{X}$ is ***quasicompact*** if every covering of $U$ has a finite subcovering.

(1.1.2) An object $U \in \boldsymbol{X}$ is ***quasiseparated*** if for every pair of morphisms $U' \to U$ and $U'' \to U$ where $U'$ and $U''$ are quasicompact, the fiber product $U' \times_U U''$ is quasicompact.

(1.1.3) An object $U \in \boldsymbol{X}$ is ***coherent*** if $U$ is quasicompact and quasiseparated.

(1.1.4) The topos $\boldsymbol{X}$ is ***coherent*** if the terminal object $1_{\boldsymbol{X}} \in \boldsymbol{X}$ is coherent, every object of $\boldsymbol{X}$ admits a cover by coherent objects, and the coherent objects of $\boldsymbol{X}$ are closed under finite products.

We write $\boldsymbol{X}^{coh} \subset \boldsymbol{X}$ for the full subcategory spanned by the coherent objects.

A geometric morphism of topoi $f_* \colon \boldsymbol{X} \to \boldsymbol{Y}$ is ***coherent*** if and only if, for every coherent object $F \in \boldsymbol{Y}$, the object $f^*(F) \in \boldsymbol{X}$ is coherent. We write $\mathbf{Top}^{coh}$ for the subcategory of $\mathbf{Top}$ whose objects are coherent topoi and whose morphisms are coherent geometric morphisms.

**1.2 Definition** ([11, Definition A.4.1])**.** A category $X$ is a ***pretopos*** if $X$ satisfies the following conditions:

(1.2.1) The category $X$ admits finite limits.

(1.2.2) The category $X$ admits finite coproducts, which are universal and disjoint.

(1.2.3) Equivalence relations in $X$ are effective.

(1.2.4) Effective epimorphisms in $X$ are stable under pullback

If $X$ and $Y$ are pretopoi, we say that a functor $f^*: Y \to X$ is a ***morphism of pretopoi*** if $f^*$ preserves finite limits, finite coproducts, and effective epimorphisms. Write *pre***Top** $\subset$ **Cat** for the subcategory consisting of *essentially small* pretopoi and morphisms of pretopoi.

**1.3 Example** ([11, Corollary C.5.14]). Let $\boldsymbol{X}$ be a coherent topos. Then the full subcategory $\boldsymbol{X}^{coh} \subset \boldsymbol{X}$ of coherent objects is an essentially small pretopos. If $f_*: \boldsymbol{X} \to \boldsymbol{Y}$ is a coherent geometric morphism of coherent topoi, then the functor $f^*: \boldsymbol{Y}^{coh} \to \boldsymbol{X}^{coh}$ is a morphism of pretopoi.

If $\boldsymbol{X}$ is the étale topos of a quasicompact quasiseparated scheme $X$, then $\boldsymbol{X}$ is coherent and $\boldsymbol{X}^{coh}$ is the category of constructible étale sheaves of sets on $X$.

**1.4 Definition** ([11, Definition B.5.3]). Let $X$ be a pretopos. The ***effective epimorphism topology*** on $X$ is the Grothendieck topology *eff* on $X$ where a collection of morphisms $\{U_i \to U\}_{i\in I}$ is a covering if and only if there exists a finite subset $I_0 \subset I$ such that the induced morphism $\coprod_{i\in I_0} U_i \to U$ is an effective epimorphism in $X$.

The effective epimorphism topology is subcanonical [11, Corollary B.5.6].

**1.5 Theorem** ([9, Corollary 7; 11, Proposition C.6.3]). *The constructions* $\boldsymbol{X} \mapsto \boldsymbol{X}^{coh}$ *and* $X \mapsto \mathbf{Sh}_{eff}(X; \mathbf{Set})$ *are mutually inverse equivalences of* $(2,1)$*-categories*

$$\mathbf{Top}^{coh} \simeq pre\mathbf{Top}^{op}.$$

**1.6 Remark.** The equivalence of Theorem 1.5 is really an equivalence of $(2,2)$-categories, but we do not need noninvertible 2-morphisms in this note.

## Classification of bounded coherent $\infty$-topoi

Coherent $\infty$-topoi admit a classification in terms of a higher-categorical analogue of pretopoi, as long as they can be recovered from the collection of their $n$-topoi of $(n-1)$-truncated objects. This subsection is a breif summary of [SAG, §§A.2, A.3, A.6, & A.7].

**1.7 Notation.** We use here the theory of *n-topoi* for $n \in \boldsymbol{N}^{\triangleright}$; see [HTT, Chapter 6]. We write $\mathbf{Top}_n \subset \mathbf{Cat}_\infty$ for the subcategory of $n$-topoi and geometric morphisms.

**1.8 Example.** Recall that 1-topoi are topoi in the classical sense [HTT, Remark 6.4.1.3].

**1.9 Example.** Let $m, n \in \boldsymbol{N}^{\triangleright}$ with $m \leq n$. An $m$***-site*** is a small $m$-category[1] $X$ equipped with a Grothendieck topology $\tau$. Attached to this $m$-site is the $n$-topos $\mathbf{Sh}_{\tau,\leq(n-1)}(X)$ of sheaves of $(n-1)$-truncated spaces on $X$. We simply write $\mathbf{Sh}_\tau(X)$ for the $\infty$-topos of sheaves of spaces on $X$.

Not all $\infty$-topoi are of the form $\mathbf{Sh}_\tau(X)$ for some $\infty$-site $X$; however, if $n \in \boldsymbol{N}$, then every $n$-topos is of the form $\mathbf{Sh}_{\tau,\leq(n-1)}(X)$ for some $n$-site $(X, \tau)$ [HTT, Theorem 6.4.1.5(1)].

[1] By an $m$***-category*** we mean an $\infty$-category whose mapping spaces are $(m-1)$-truncated.

**1.10 Definition** ([HTT, §6.4.5]). For any integer $n \geq 0$, passage to $(n-1)$-truncated objects defines a functor $\tau_{\leq n-1} : \mathbf{Top}_\infty \to \mathbf{Top}_n$. The functor $\tau_{\leq n-1}$ admits admits a fully faithful right adjoint $\mathbf{Top}_n \hookrightarrow \mathbf{Top}_\infty$ whose essential image we denote by $\mathbf{Top}^n_\infty \subset \mathbf{Top}_\infty$. The $\infty$-category $\mathbf{Top}^n_\infty$ is the $\infty$-category of ***n-localic*** $\infty$-topoi.

**1.11 Example.** For any topological space $T$, the $\infty$-topos $\mathbf{Sh}(T)$ of sheaves on $T$ is 0-localic.

**1.12 Example.** If $\boldsymbol{X}$ is a topos presented as sheaves of sets on a site $(X, \tau)$ *with finite limits*, then the 1-localic $\infty$-topos associated to $\boldsymbol{X}$ is the $\infty$-topos $\mathbf{Sh}_\tau(X)$ of sheaves of *spaces* on $(X, \tau)$.

**1.13.** Let $n \in \boldsymbol{N}$. The proof of [HTT, Proposition 6.4.5.9] demonstrates that an $\infty$-topos $\boldsymbol{X}$ is $n$-localic if and only if $\boldsymbol{X} \simeq \mathbf{Sh}_\tau(X)$ for some $n$-site $(X, \tau)$ *with finite limits*.

**1.14 Warning.** If $(X, \tau)$ is an $n$-site and the $n$-category $X$ does *not* have finite limits, then the $\infty$-topos $\mathbf{Sh}_\tau(X)$ is not generally $N$-localic for any integer $N \geq 0$. See [SAG, Counterexample 20.4.0.1] for a basis $B$ for the topology on the Hilbert cube $\prod_{i \in \boldsymbol{Z}} [0, 1]$ for which the $\infty$-topos of sheaves on $B$ is not $N$-localic for any $N \geq 0$.

**1.15 Definition** ([SAG, Definition A.7.1.2]). An $\infty$-topos $\boldsymbol{X}$ is ***bounded*** if $\boldsymbol{X}$ can be written as the limit of a diagram $\boldsymbol{Y} : I \to \mathbf{Top}_\infty$ where $I^{op}$ is a filtered $\infty$-category and for each $i \in I$ the $\infty$-topos $\boldsymbol{Y}_i$ is $n_i$ localic for some $n_i \in \boldsymbol{N}$.

**1.16 Definition** ([SAG, Definition A.2.0.12]). Let $\boldsymbol{X}$ be an $\infty$-topos. We say that $\boldsymbol{X}$ is ***0-coherent*** or ***quasicompact*** if and only if every cover $\{U_i \to 1_{\boldsymbol{X}}\}_{i \in I}$ of the terminal object $1_{\boldsymbol{X}} \in \boldsymbol{X}$ admits a finite subcover. Let $n \geq 1$ be an integer, and define $n$-coherence of $\infty$-topoi and their objects recursively as follows:

(1.16.1) An object $U \in \boldsymbol{X}$ is ***n-coherent*** if and only if the $\infty$-topos $\boldsymbol{X}_{/U}$ is $n$-coherent.

(1.16.2) The $\infty$-topos $\boldsymbol{X}$ is ***locally n-coherent*** if and only if every object $U \in \boldsymbol{X}$ admits a cover $\{U_i \to U\}_{i \in I}$ where each $U_i$ is $n$-coherent.

(1.16.3) The $\infty$-topos $\boldsymbol{X}$ is ***(n + 1)-coherent*** if and only if $\boldsymbol{X}$ is locally $n$-coherent, and the $n$-coherent objects of $\boldsymbol{X}$ are closed under finite products.

An $\infty$-topos $\boldsymbol{X}$ is ***coherent*** if and only if $\boldsymbol{X}$ is $n$-coherent for every $n \geq 0$. An object $U$ of an $\infty$-topos $\boldsymbol{X}$ is ***coherent*** if and only if $\boldsymbol{X}_{/U}$ is a coherent $\infty$-topos. Finally, an $\infty$-topos $\boldsymbol{X}$ is ***locally coherent*** if and only if every object $U \in \boldsymbol{X}$ admits a cover $\{U_i \to U\}_{i \in I}$ where each $U_i$ is coherent.

**1.17 Definition.** A geometric morphism of $\infty$-topoi $f_* : \boldsymbol{X} \to \boldsymbol{Y}$ is ***coherent*** if and only if, for every coherent object $F \in \boldsymbol{Y}$, the object $f^*(F) \in \boldsymbol{X}$ is coherent. We write $\mathbf{Top}^{coh}_\infty$ for the subcategory of $\mathbf{Top}_\infty$ whose objects are coherent $\infty$-topoi and whose morphisms are coherent geometric morphisms.

Write $\mathbf{Top}^{bc}_\infty \subset \mathbf{Top}^{coh}_\infty$ for the full subcategory spanned by those coherent $\infty$-topoi that are also bounded, that is, the ***bounded coherent*** $\infty$-topoi

**1.18 Notation.** If $\boldsymbol{X}$ is an $\infty$-topos, then write $\boldsymbol{X}^{coh} \subset \boldsymbol{X}$ for the full subcategory of $\boldsymbol{X}$ spanned by the coherent objects and $\boldsymbol{X}^{coh}_{<\infty} \subset \boldsymbol{X}$ for the full subcategory of $\boldsymbol{X}$ spanned by the truncated coherent objects.

**1.19 Example.** The $\infty$-topos **Spc** of spaces is coherent. An object $U \in \mathbf{Spc}$ is truncated coherent if and only if $U$ is a *π-finite* space, i.e., $U$ is truncated, has finitely many connected components, and all of the homotopy groups of $U$ are finite.

**1.20 Definition** ([SAG, Definition A.3.1.1])**.** An $\infty$-site $(X, \tau)$ is ***finitary*** if and only if $X$ admits all fiber products, and, for every object $U \in X$ and every covering sieve $S \subset X_{/U}$, there is a finite subset $\{U_i\}_{i \in I} \subset S$ that generates a covering sieve.

Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be finitary $\infty$-sites. A morphism of $\infty$-sites $f^* : (Y, \tau_Y) \to (X, \tau_X)$ is a ***morphism of finitary*** $\infty$***-sites*** if $f^*$ is preserves fiber products.

**1.21 Proposition** ([SAG, Proposition A.3.1.3])**.** *Let $(X, \tau)$ be a finitary $\infty$-site. Then the $\infty$-topos $\mathbf{Sh}_\tau(X)$ locally coherent, and for every object $x \in X$, the sheaf よ$(x)$ is a coherent object of $\mathbf{Sh}_\tau(X)$, where よ$: X \to \mathbf{Sh}_\tau(X)$ is the sheafified Yoneda embedding. If, in addition, $X$ admits a terminal object, then $\mathbf{Sh}_\tau(X)$ is coherent.*

**1.22 Definition** ([SAG, Definition A.6.1.1])**.** An $\infty$-category $X$ is an $\infty$-***pretopos*** if $X$ satisfies the following conditions:

(1.22.1) The category $X$ admits finite limits.

(1.22.2) The category $X$ admits finite coproducts, which are universal and disjoint.

(1.22.3) Groupoid objects in $X$ are effective, and their geometric realizations are universal.

If $X$ and $Y$ are $\infty$-pretopoi, we say that a functor $f^* : Y \to X$ is a ***morphism of*** $\infty$-***pretopoi*** if $f^*$ preserves finite limits, finite coproducts, and effective epimorphisms. We write $pre\mathbf{Top}_\infty \subset \mathbf{Cat}_\infty$ for the subcategory consisting of $\infty$-pretopoi and morphisms of $\infty$-pretopoi.

**1.23 Example** ([SAG, Corollary A.6.1.7])**.** If $\boldsymbol{X}$ is a coherent $\infty$-topos, then the full subcategory $\boldsymbol{X}^{coh} \subset \boldsymbol{X}$ spanned by the coherent objects is an $\infty$-pretopos.

**1.24 Definition** ([SAG, Definition A.6.2.4])**.** Let $X$ be an $\infty$-pretopos. The ***effective epimorphism topology*** on $X$ is the Grothendieck topology *eff* where a collection of morphisms $\{U_i \to U\}_{i \in I}$ is a covering if and only if there exists a finite subset $I_0 \subset I$ such that the induced morphism $\coprod_{i \in I_0} U_i \to U$ is an effective epimorphism in $X$.

The effective epimorphism topology is finitary and subcanonical [SAG, Corollary A.6.2.6].

**1.25 Definition** ([SAG, Definition A.7.4.1])**.** An $\infty$-pretopos $X$ is ***bounded*** if and only if $X$ is essentially small and every object of $X$ is truncated. We write $pre\mathbf{Top}^b_\infty \subset pre\mathbf{Top}_\infty$ for the full subcategory spanned by the bounded $\infty$-pretopoi.

**1.26 Theorem** ([SAG, Theorem A.7.5.3])**.** *The constructions $\boldsymbol{X} \mapsto \boldsymbol{X}^{coh}_{<\infty}$ and $X \mapsto \mathbf{Sh}_{eff}(X)$ are mutually inverse equivalences of $\infty$-categories*

$$\mathbf{Top}^{bc}_\infty \simeq pre\mathbf{Top}^{b,op}_\infty .$$

## 2 Coherence for 1-localic $\infty$-topoi

In this section we show that the $\infty$-category of coherent ordinary topoi is equivalent to the $\infty$-category of coherent 1-localic $\infty$-topoi (Proposition 2.11). This follows from the fact that morphisms of finitary $\infty$-sites induce coherent geometric morphisms (Corollary 2.9). First we'll have to give $\infty$-toposic versions of a number of points from [SGA $4_{\text{II}}$, Exposé VI, §§1–3], which follow easily from [SAG, §A.2.1].

**2.1 Definition.** Let $n \in \boldsymbol{N}$ and let $\boldsymbol{X}$ be a locally $n$-coherent $\infty$-topos. A morphism $U \to V$ in $\boldsymbol{X}$ is ***relatively*** $\boldsymbol{n}$***-coherent*** if for every $n$-coherent object $V' \in \boldsymbol{X}$ and every morphism $V' \to V$, the fiber product $U \times_V V'$ is also $n$-coherent.

**2.2 Example** ([SAG, Example A.2.1.2])**.** Let $\boldsymbol{X}$ be a locally $n$-coherent $\infty$-topos and $f : U \to V$ a morphism in $\boldsymbol{X}$. If $U$ is $n$-coherent and $V$ is $(n+1)$-coherent, then $f$ is relatively $n$-coherent.

**2.3 Lemma.** *Let $\boldsymbol{X}$ be an $\infty$-topos. If $e : U \twoheadrightarrow V$ is an effective epimorphism in $\boldsymbol{X}$ and $U$ is quasicompact, then $V$ is quasicompact.*

*Proof.* This is a special case of [SAG, Proposition A.2.1.3]. □

**2.4 Lemma.** *Let $n \geq 1$ be an integer and $\boldsymbol{X}$ a locally $(n-1)$-coherent $\infty$-topos. Let $U \in \boldsymbol{X}$ and let $e : \coprod_{i \in I} U_i \twoheadrightarrow U$ be a cover of $U$ where $I$ is finite and $U_i$ is $n$-coherent for each $i \in I$. The following are equivalent:*

(2.4.1) *The effective epimorphism $e$ is relatively $(n-1)$-coherent.*

(2.4.2) *For all $i, j \in I$, the object $U_i \times_U U_j$ is $(n-1)$-coherent.*

(2.4.3) *The object $U$ is $n$-coherent.*

*Proof.* If $e$ is relatively $(n-1)$-coherent, then since coproducts in $\boldsymbol{X}$ are universal, the fiber product

$$\left(\textstyle\coprod_{i \in I} U_i\right) \times_U \left(\textstyle\coprod_{j \in I} U_j\right) \simeq \coprod_{i,j \in I} U_i \times_U U_j$$

is $(n-1)$-coherent. Thus $U_i \times_U U_j$ is $(n-1)$-coherent for all $i, j \in I$ [SAG, Remark A.2.0.16].

If each $U_i \times_U U_j$ is $(n-1)$-coherent, then since each $U_i$ is $n$-coherent the pullback of $e$ along itself

$$\coprod_{i,j \in I} U_i \times_U U_j \twoheadrightarrow \coprod_{i \in I} U_i$$

is relatively $(n-1)$-coherent (Example 2.2). Applying [SAG, Corollary A.2.1.5] we deduce that $e : \coprod_{i \in I} U_i \twoheadrightarrow U$ is relatively $(n-1)$-coherent.

To conclude, note that if $e : \coprod_{i \in I} U_i \twoheadrightarrow U$ is relatively $(n-1)$-coherent, then [SAG, Proposition A.2.1.3] shows that $U$ is $n$-coherent. On the other hand, if $U$ is $n$-coherent, then $e$ is $(n-1)$-coherent by Example 2.2. □

**2.5 Proposition.** *Let $f_* : \boldsymbol{X} \to \boldsymbol{Y}$ be a geometric morphism of $\infty$-topoi and $n \in \boldsymbol{N}$. Assume that:*

(2.5.1) *There exists a collection of $n$-coherent objects $\boldsymbol{Y}_0 \subset \mathrm{Obj}(\boldsymbol{Y})$ of $\boldsymbol{Y}$ such that for every $n$-coherent object $U \in \boldsymbol{Y}$ there exists a cover $\coprod_{i\in I} U_i \twoheadrightarrow U$ where $U_i \in \boldsymbol{Y}_0$ for each $i \in I$.*

(2.5.2) *The pullback functor $f^* : \boldsymbol{Y} \to \boldsymbol{X}$ takes objects of $\boldsymbol{Y}_0$ to $n$-coherent objects of $\boldsymbol{X}$.*

(2.5.3) *If $n \geq 1$, the $\infty$-topoi $\boldsymbol{X}$ and $\boldsymbol{Y}$ are locally $(n-1)$-coherent and $f^* : \boldsymbol{Y} \to \boldsymbol{X}$ takes $(n-1)$-coherent objects of $\boldsymbol{Y}$ to $(n-1)$-coherent objects of $\boldsymbol{X}$.*

*Then $f^*$ takes $n$-coherent objects of $\boldsymbol{Y}$ to $n$-coherent objects of $\boldsymbol{X}$.*

*Proof.* Let $U \in \boldsymbol{Y}$ be an $n$-coherent object; we show that $f^*(U)$ is $n$-coherent. By assumption there exists a cover

$$e\colon \coprod_{i\in I} U_i \twoheadrightarrow U$$

where $U_i \in \boldsymbol{Y}_0$ for each $i \in I$ and $I$ is finite (since $U$ is, in particular, 0-coherent). For all $i \in I$ the object $f^*(U_i)$ is $n$-coherent by assumption, so since $n$-coherent objects are closed under finite coproducts [SAG, Remark A.2.0.16], the object

$$f^*\left(\textstyle\coprod_{i\in I} U_i\right) \simeq \coprod_{i\in I} f^*(U_i)$$

is $n$-coherent.

Note that

$$f^*(e)\colon \coprod_{i\in I} f^*(U_i) \twoheadrightarrow f^*(U)$$

is an effective epimorphism in $\boldsymbol{X}$. If $n = 0$, this proves the claim (Lemma 2.3). If $n \geq 1$, then Lemma 2.4 shows that it suffices to show that for all $i, j \in I$, the object

$$f^*(U_i) \times_{f^*(U)} f^*(U_j) \simeq f^*(U_i \times_U U_j)$$

is $(n-1)$-coherent. This follows from the fact that $U_i \times_U U_j$ is $(n-1)$-coherent (by Lemma 2.4) and the assumption that $f^*$ sends $(n-1)$-coherent objects of $\boldsymbol{Y}$ to $(n-1)$-coherent objects of $\boldsymbol{X}$. □

Proposition 2.5 shows that coherence of a geometric morphism between locally coherent $\infty$-topoi (Definition 1.17) is equivalent to the *a priori* stronger condition that the pullback functor preserve $n$-coherent objects for all $n \geq 0$:[2]

**2.6 Corollary.** *Let $f_* : \boldsymbol{X} \to \boldsymbol{Y}$ be a geometric morphism between locally coherent $\infty$-topoi. Then $f_*$ is coherent if and only if $f^*$ takes $n$-coherent objects of $\boldsymbol{Y}$ to $n$-coherent objects of $\boldsymbol{X}$ for all $n \geq 0$.*

Proposition 2.5 also shows that coherence of a geometric morphism can be checked on a generating set of coherent objects.

[2] This second notion is how Grothendieck and Verdier originally defined coherence for ordinary topoi [SGA $4_{\mathrm{II}}$, Exposé VI, Définition 3.1].

**2.7 Corollary.** *Let $f_* \colon \boldsymbol{X} \to \boldsymbol{Y}$ be a geometric morphism between locally coherent $\infty$-topoi. Let $\boldsymbol{Y}_0 \subset \mathrm{Obj}(\boldsymbol{Y}^{coh})$ be a collection of coherent objects such that for every object $U \in \boldsymbol{Y}$ there exists a cover $\coprod_{i \in I} U_i \twoheadrightarrow U$ where $U_i \in \boldsymbol{Y}_0$ for each $i \in I$. If for all $U \in \boldsymbol{Y}_0$ the object $f^*(U)$ is coherent, the geometric morphism $f_* \colon \boldsymbol{X} \to \boldsymbol{Y}$ is coherent.*

For the next result, we need the following lemma.

**2.8 Lemma.** *Let $f^* \colon (Y, \tau_Y) \to (X, \tau_X)$ be a morphism of $\infty$-sites, and write $\text{よ}_Y \colon Y \to \mathbf{Sh}_{\tau_Y}(Y)$ for the sheafified Yoneda embedding. If the topology $\tau_X$ is finitary, then*

$$f^* \text{よ}_Y \colon Y \to \mathbf{Sh}_{\tau_X}(X)$$

*factors through $\mathbf{Sh}_{\tau_X}(X)^{coh} \subset \mathbf{Sh}_{\tau_X}(X)$.*

*Proof.* We have a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{p^*} & X \\ {\scriptstyle \text{よ}_Y}\big\downarrow & & \big\downarrow{\scriptstyle \text{よ}_X} \\ \mathbf{Sh}_{\tau_Y}(Y) & \xrightarrow[p^*]{} & \mathbf{Sh}_{\tau_X}(X) \end{array}$$

where the vertical functors are sheafified Yoneda embeddings. The claim now follows from the fact that $\text{よ}_X \colon X \to \mathbf{Sh}_{\tau_X}(X)$ factors through $\mathbf{Sh}_{\tau_X}(X)^{coh}$, since the topology $\tau_X$ is finitary (Proposition 1.21). □

**2.9 Corollary.** *Let $f^* \colon (Y, \tau_Y) \to (X, \tau_X)$ be a morphism of finitary $\infty$-sites. Then the geometric morphism*

$$f_* \colon \mathbf{Sh}_{\tau_X}(X) \to \mathbf{Sh}_{\tau_Y}(Y)$$

*is coherent.*

*Proof.* By Proposition 1.21, both $\mathbf{Sh}_{\tau_X}(X)$ and $\mathbf{Sh}_{\tau_Y}(Y)$ are locally coherent. The image $\text{よ}_Y(Y)$ of $Y$ under the sheafified Yoneda embedding generates $\mathbf{Sh}_{\tau_Y}(Y)$ under colimits, so by Corollary 2.7 it suffices to check that $f^*$ carries objects in $\text{よ}_Y(Y)$ to coherent objects of $\boldsymbol{X}$; this the content of Lemma 2.8. □

**2.10 Notation.** Write $\mathbf{Top}_\infty^{1,coh} \subset \mathbf{Top}_\infty^{coh}$ for the full subcategory spanned by the 1-localic coherent $\infty$-topoi.

Corollary 2.9 and the definitions immediately imply the following:

**2.11 Proposition.** *The equivalence of $\infty$-categories $\tau_{\leq 0} \colon \mathbf{Top}_\infty^1 \xrightarrow{\sim} \mathbf{Top}$ (Definition 1.10) restricts to an equivalence*

$$\tau_{\leq 0} \colon \mathbf{Top}_\infty^{1,coh} \xrightarrow{\sim} \mathbf{Top}^{coh}$$

**2.12 Corollary.** *The following are equivalent for a geometric morphism $f_* \colon \boldsymbol{X} \to \boldsymbol{Y}$ between 1-localic coherent $\infty$-topoi:*

(2.12.1) *The geometric morphism $f_* \colon \boldsymbol{X} \to \boldsymbol{Y}$ is coherent.*

(2.12.2) *The pullback functor* $f^*\colon \boldsymbol{Y} \to \boldsymbol{X}$ *carries* 0*-truncated* 1*-coherent objects of* $\boldsymbol{Y}$ *to* 1*-coherent objects of* $\boldsymbol{X}$.

**2.13 Remark.** If $n \geq 2$, there doesn't already exist a notion of 'coherent $n$-topos' (other than saying that the corresponding $n$-localic $\infty$-topos is coherent). However, if one declares that an $n$-topos $\boldsymbol{X}$ is 'coherent' if $\boldsymbol{X}$ is '$(n+1)$-coherent', then Corollary 2.9 allows one to immediately deduce variants of Proposition 2.11 and Corollary 2.12 for coherent $n$-topoi. Sections 5.4 through 5.6 of the newest version of [2] address this more general point.

## The $\infty$-pretopos associated to an ordinary pretopos

In this subsection we exploit the equivalence of Proposition 2.11 to show how to associate a bounded $\infty$-pretopos to an essentially small pretopos. Lurie briefly touches upon this point (without details) in [10].

**2.14.** If $\boldsymbol{X}$ is a bounded coherent $\infty$-topos, then the associated ordinary topos $\tau_{\leq 0}\boldsymbol{X}$ is coherent. Moreover, if $f_*\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a coherent geometric morphism of bounded coherent $\infty$-topoi, then the induced geometric morphsim $f_*\colon \tau_{\leq 0}\boldsymbol{X} \to \tau_{\leq 0}\boldsymbol{Y}$ is a coherent geometric morphism of ordinary topoi. Hence the adjunction $\mathbf{Top}_\infty \rightleftarrows \mathbf{Top}$ restricts to an adjunction

$$\mathbf{Top}_\infty^{bc} \underset{}{\overset{\tau_{\leq 0}}{\rightleftarrows}} \mathbf{Top}^{coh} . \tag{2.15}$$

**2.16.** Transporting the adjunction (2.15) across the equivalences

$$(-)^{coh}\colon \mathbf{Top}^{coh} \xrightarrow{\sim} pre\mathbf{Top}^{op} \qquad \text{and} \qquad (-)^{coh}_{<\infty}\colon \mathbf{Top}_\infty^{bc} \xrightarrow{\sim} pre\mathbf{Top}_\infty^{b,op}$$

of Theorems 1.5 and 1.26 we see that the functor $\tau_{\leq 0}\colon pre\mathbf{Top}_\infty^b \to pre\mathbf{Top}$ admits a fully faithful *right* adjoint

$$(-)^+\colon pre\mathbf{Top} \hookrightarrow pre\mathbf{Top}_\infty^b$$

given by $X^+ \coloneqq \mathbf{Sh}_{eff}(X)^{coh}_{<\infty}$.

**2.17 Example.** The bounded $\infty$-pretopos $\mathbf{Fin}^+$ associated to the pretopos $\mathbf{Fin}$ of finite sets is the $\infty$-pretopos $\mathbf{Spc}_\pi$ of $\pi$-finite spaces.

## Examples from algebraic geometry

We conclude with a few examples from algebraic geometry that Corollary 2.9 puts on the same footing.

**2.18 Example.** For a spectral topological space[3] $S$, write $\mathrm{Open}^{qc}(S) \subset \mathrm{Open}(S)$ for the locale of quasicompact opens in $S$. Since the quasicompact opens of $S$ form a basis for the

[3] A topological space $S$ is ***spectral*** if and only if $S$ is homeomorphic to the underlying topological space of a quasicompact quasiseparated scheme.

topology on $S$ that is closed under finite intersections, the $\infty$-topos $\mathbf{Sh}(\mathrm{Open}^{qc}(S))$ is 0-localic. Applying [11, Proposition B.6.4] we see that the inclusion $\mathrm{Open}^{qc}(S) \subset \mathrm{Open}(S)$ induces an equivalence of 0-localic $\infty$-topoi

$$\mathbf{Sh}(S) \simeq \mathbf{Sh}(\mathrm{Open}^{qc}(S)) .$$

The Grothendieck topology on $\mathrm{Open}^{qc}(S)$ is finitary, so the $\infty$-topos $\mathbf{Sh}(S)$ of sheaves on $S$ is a coherent $\infty$-topos. (Cf. [SAG, Lemma 2.3.4.1]).

If $f : S \to T$ is a quasicompact continuous map of spectral topological spaces, the inverse image map $f^{-1} : \mathrm{Open}(T) \to \mathrm{Open}(S)$ restricts to a map

$$f^{-1} : \mathrm{Open}^{qc}(T) \to \mathrm{Open}^{qc}(S) .$$

Corollary 2.9 shows that the induced geometric morphism $f_* : \mathbf{Sh}(S) \to \mathbf{Sh}(T)$ is coherent. Since spectral topological spaces are sober, a continuous map $f : S \to T$ of spectral topological spaces induces a coherent geometric morphism on the level of $\infty$-topoi if and only if $f$ is quasicompact.

**2.19.** If $\boldsymbol{X}$ is a coherent $\infty$-topos, then the underlying topological space of $\boldsymbol{X}$ is spectral [7, Chapter II, §§3.3–3.4].

Combining the fact that the Zariski, Nisnevich[4], étale, and proétale[5] topoi of a scheme all have the same underlying topological space with the fact that if a scheme $X$ is quasicompact and quasiseparated, then the topoi of sheaves on $X$ in each of these topologies is coherent [SAG, Proposition 2.3.4.2 & Remark 3.7.4.2; 1, Appendix A; 11, Example 7.1.7], we deduce the following:

**2.20 Proposition.** *The following are equivalent for a scheme $X$:*

(2.20.1) *The scheme $X$ is quasicompact and quasiseparated.*

(2.20.2) *The Zariski $\infty$-topos $X_{zar}$ of $X$ is a coherent $\infty$-topos.*

(2.20.3) *The Nisnevich $\infty$-topos $X_{nis}$ of $X$ is a coherent $\infty$-topos.*

(2.20.4) *The étale $\infty$-topos $X_{ét}$ of $X$ is a coherent $\infty$-topos.*

(2.20.5) *The proétale $\infty$-topos $X_{proét}$ of $X$ is a coherent $\infty$-topos.*

**2.21 Example** ([2, Example 10.4.13])**.** Let $X$ be a quasicompact quasiseparated scheme. Then the bounded $\infty$-pretopos of truncated coherent objects of the coherent $\infty$-topos $X_{ét}$ is the $\infty$-category of constructible étale sheaves of spaces on $X$.

**2.22 Example.** Let $f : X \to Y$ be a morphism of quasicompact quasiseparated schemes and let $\tau \in \{zar, nis, ét, proét\}$. Then the induced geometric morphism $f_* : X_\tau \to Y_\tau$ on $\infty$-topoi of $\tau$-sheaves is a coherent geometric morphism of coherent $\infty$-topoi.

**2.23 Example.** Let $X$ be a quasicompact quasiseparated scheme. Then the natural geometric morphisms

$$X_{proét} \to X_{ét} , \quad X_{ét} \to X_{nis} , \quad \text{and} \quad X_{nis} \to X_{zar}$$

are all coherent geometric morphisms of coherent $\infty$-topoi.

[4] For background on the Nisnevich topology, see [SAG, §3.7; 5; 4; 13].

[5] For background on the proétale topology, see [STK, Tags 0988 & 099R; 3].

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