Results on Martin's Conjecture

by

Patrick Lutz

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Theodore Slaman, Chair
Professor Antonio Montalbán
Professor John Steel

Spring 2021

Results on Martin's Conjecture

Copyright 2021
by
Patrick Lutz

Abstract

Results on Martin's Conjecture

by

Patrick Lutz

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Theodore Slaman, Chair

Martin's conjecture is an attempt to classify the behavior of all definable functions on the Turing degrees under strong set theoretic hypotheses. Very roughly it says that every such function is either eventually constant, eventually equal to the identity function or eventually equal to a transfinite iterate of the Turing jump. It is typically divided into two parts: the first part states that every function is either eventually constant or eventually above the identity function and the second part states that every function which is above the identity is eventually equal to a transfinite iterate of the jump. If true, it would provide an explanation for the unique role of the Turing jump in computability theory and rule out many types of constructions on the Turing degrees.

In this thesis we will introduce a few tools which we use to prove several cases of Martin's conjecture. It turns out that both these tools and these results on Martin's conjecture have some interesting consequences both for Martin's conjecture and for a few related topics.

The main tool that we introduce is a basis theorem for perfect sets, improving a theorem due to Groszek and Slaman [GS98]. We also introduce a general framework for proving certain special cases of Martin's conjecture which unifies a few pre-existing proofs. We will use these tools to prove three main results about Martin's conjecture: that it holds for regressive functions on the hyperarithmetic degrees (answering a question of Slaman and Steel), that part 1 holds for order preserving functions on the Turing degrees, and that part 1 holds for a class of functions that we introduce, called measure preserving functions.

This last result has several interesting consequences for the study of Martin's conjecture. In particular, it shows that part 1 of Martin's conjecture is equivalent to a statement about the Rudin-Keisler order on ultrafilters on the Turing degrees. This suggests several possible strategies for working on part 1 of Martin's conjecture, which we will discuss.

The basis theorem that we use to prove these results also has some applications outside of Martin's conjecture. We will use it to prove a few theorems related to Sacks' question about

whether it is provable in ZFC that every locally countable partial order of size continuum embeds into the Turing degrees. We will show that in a certain extension of ZF (which is incompatible with ZFC), this holds for all partial orders of height two, but *not* for all partial orders of height three. We will also present an obstacle to embedding height three partial orders into the Turing degrees in ZFC which shows that one of the most natural ways of trying to do so cannot work.

We will end the thesis with a list of open questions related to Martin's conjecture, which we hope will stimulate further research.

Contents

Acknowledgments

I have always been a fan of expansive acknowledgments sections. I like the brief glimpse it gives you into the author's personal life and the sense it gives of the network of human relationships in which research typically takes place. I also like the idea that some of the people thanked in the acknowledgments may stumble onto the work months or years later and feel happy to see their name.

Still, a long acknowledgments section carries a risk. Paradoxically, the more people you thank, the greater the risk that you have left someone out. When you thank only your advisor and your parents, there is no chance that you have forgotten anyone. Not so when you thank dozens of people.

So though I will thank many people in this section, I am sure I will forget many others. If you come across this thesis months or years from now and feel that your name deserves to be included in these acknowledgments, you are probably right and I'm sorry for leaving you out.

I believe that the two most important factors in the graduate school experience are one's advisor and one's fellow graduate students. I was lucky on both accounts. My advisor, Ted Slaman, has always been incredibly kind to me. He has also been incredibly patient with me (maybe too patient) as I took time in grad school to pursue all my intellectual interests, both mathematical and non-mathematical. He has also given me many valuable insights into computability theory and mathematical logic.

I have learned a tremendous amount, both about mathematics and about life, from my fellow grad students. I am grateful for Benny Siskind for taking me to a coffee shop on my first day of grad school and telling me about the existence of research seminars and for being a good friend and research collaborator throughout grad school. Izaak Meckler and German Stefanich have both greatly influenced my perspective on mathematics and I have been inspired by their enthusiasm and fearlessness when pursuing mathematical ideas. Chris and Wendy have been dear friends and great office-mates. James Walsh was a good friend and a great research collaborator. Rahul and Ravi have always been sources of thought-provoking conversations. I enjoyed working on stuff in Lean with Thomas and Jordan and I am sorry I had to take a break from it this semester (in part to write this thesis). I am proud to be in the same cohort as Milind, Ritvik, Mario, Yanhe, Mariel, Joe, Leon, Nic, Mariana and too many others to mention. I am also grateful to all the other logic students at Berkeley: James, Reid, Kentaro, Benny, Yifeng, Ben, Mariana, Adele, David Casey, Jacopo, Matthew, Diego, Guillaume, Scott, David Gonzalez, Ahmee, Jordan, Ronan and Shoshana. They have been good friends, teachers and sources of interesting ideas and discussions. I would also like to thank all the grad students from outside Berkeley who befriended me and taught me many things about math and logic, especially Jiachen, Wei Quan, Chieu-Minh and Jun Le.

This thesis wouldn't exist without my research collaborators—James, Benny, Vittorio and Kojiro. A few other people have also played a key role in shaping the research in this thesis. Ted, of course, but also especially Gabe Goldberg, Takayuki Kihara, Andrew Marks and Steffen Lempp. I would like to thank Gabe and Benny especially for putting up with dozens of hours of phone calls from me throughout the pandemic.

I have been lucky to have had many good teachers and mentors throughout my life. In high school, Linda Dewees and Renee Goin were great teachers and Katharine Ott gave me my first introduction to rigorous mathematical reasoning. In college, I benefited from the teaching and advice of Paul Hilfinger, John Steel, Michael Christ and Itay Neeman. In grad school, I have also learned a lot from Antonio Montalban and Leo Harrington. Also thanks to Vicky Lee for being a wonderful person and a truly great graduate advisor.

Finally, I would like to thank all the other people who were good friends to me during my time in college and grad school and who helped me in various ways, including Summer, Griffin, Ran, Jesus, Derek, William, Hunter, Greg, Emily, Brendan, Meha, Sarah, Jimmy, Leo, Munim, Kunal, Kayon, Teddy, Govind, Nat and Chenyang and of course my parents and siblings.

Chapter 1

Introduction

Martin's conjecture is a proposed classification of the behavior of functions on the Turing degrees. Very roughly, it states that every definable function on the Turing degrees is either eventually constant, eventually equal to the identity function, or eventually equal to a transfinite iterate of the Turing jump.

The conjecture was proposed by Martin in the 1970s. In the 70s and 80s, Martin, Lachlan, Steel and Slaman proved several special cases [Mar68; Lac75; Ste82; SS88]. Since that time, there has been considerable conceptual progress, connecting Martin's conjecture to the theory of countable Borel equivalence relations and descriptive set theory more generally and also refining our understanding of the past results ([Bec88; DK00; MSS16; KM18; Bar20]. However, there has been no direct progress on the conjecture itself. We will present proofs of the first cases of Martin's conjecture that have been proved since the results of Slaman and Steel in the 1980s.

We have three main goals in this thesis. First, to present our proofs of a few special cases of Martin's conjecture, and explore some of the consequences of these results. Second, to collect and compare results about Martin's conjecture in different degree structures. Third, to compile a list of open questions about Martin's conjecture, which we hope will stimulate further research. In particular, we will only occasionally comment on applications of Martin's conjecture to set theory or its connections with other parts of set theory and computability theory—our focus is on proving instances of Martin's conjecture.

Special Cases of Martin's Conjecture

We will prove two special cases of Martin's conjecture: in Chapter 6 we will prove part 1 of Martin's conjecture for order preserving functions and in Chapter 5 we will prove part 1 of Martin's conjecture for a class of functions which we call measure preserving functions. The result on measure preserving functions yields, as a corollary, a connection between Martin's conjecture and the Rudin-Keisler order on ultrafilters.

To prove these theorems, we introduce a couple of technical tools. The first tool is a basis theorem for perfect sets, presented in section 2.3. We have also found applications of this tool to Sacks' question about which partial orders embed into the Turing degrees and to related questions about the role of the Turing degrees in the theory of locally countable Borel quasi orders (which parallels the theory of countable Borel equivalence relations). The second tool is a somewhat general method for converting instances of Martin's conjecture which involve potentially very complicated functions into instances which only involve continuous functions. This method is surprisingly simple, but has proved useful in several of the results of this thesis. We present the basic ideas behind this in the first section of Chapter 2.

Comparing Martin's Conjecture on Different Degree Structures

Martin's conjecture is a statement about functions on the Turing degrees, but it is possible to formulate versions of Martin's conjecture for most computability-theoretic degree structures—and in particular for functions on the arithmetic degrees and the hyperarithmetic degrees. We will provide an overview of what is known about Martin's conjecture for functions on the Turing degrees, arithmetic degrees, and hyperarithmetic degrees. We will also prove one new case of Martin's conjecture for functions on the hyperarithmetic degrees (section 4.1).

Open Questions

We will end the thesis with a list of open questions related to Martin's conjecture. I cannot claim that this list is a complete inventory of every question about Martin's conjecture that has been asked, but I can promise that I find each one compelling and would find an answer to any one of them quite interesting. It is my hope that these questions will help spur further research on Martin's conjecture by providing some tractable but appealing intermediate goals.

Note: Joint Work

Some of the results in this thesis are the result of collaboration. In particular, the results of Chapter 3 are joint work with Vittorio Bard, Chapters 5 and 6 are joint work with Benjamin Siskind, and Chapter 7 is joint work with Kojiro Higuchi. Also, the main idea of the third proof in section 6.2 is due to Takayuki Kihara.

1.1 Motivation: Natural Functions on the Turing Degrees

Since the dawn of time (or at least since the 1940s), computability theorists have noted the unique role that the Turing jump seems to play in computability theory. It has a simple and philosophically appealing definition and is even definable from the partial order on the Turing degrees [SS99]. It has many different equivalent definitions, some of which have fairly different motivations. The halting problem was the first uncomputable problem discovered and every time an uncomputable problem shows up in mathematics outside of computability theory, it seems to be equivalent to the jump or, occasionally, some higher version of the jump like the ω-jump or the hyperjump. It seems to be the one really fundamental operation on the Turing degrees and all other operations are either built out of it (or higher versions of it, like the hyperjump) or are produced by ad-hoc constructions[1].

This observation cries out for an explanation. Are there other fundamental operations on the Turing degrees that we've just been missing this whole time? And if not, can we prove it? Can we even state the question in a mathematically precise way?

Attempts to address this question have driven much of the development of computability theory. For example, they led Post to define the many-one degrees and the Turing degrees and to pose Post's problem, which famously asked whether there might be no r.e. degree strictly between the Turing degree of the computable sets and the Turing degree of the halting problem [Pos44]. And though this hope proved false, the techniques discovered to disprove it led to the development of the priority argument as a sophisticated tool in computability theory and the discovery of a rich structure in the Turing degrees (see, for example, section 5 of Odifreddi's book [Odi89]). However, despite all the ensuing progress in computability theory, there is still no completely satisfactory explanation for the unique role of the Turing jump.

Martin's conjecture is one more attempt to explain this phenomenon. The key idea is to switch focus from individual Turing degrees to operations defined on all Turing degrees. It is not clear whether Martin's conjecture is true (and there is evidence pointing in both directions) but it seems likely that whichever way it is resolved will contribute to computability theory: either we will gain new insight into the special role of the jump or we will discover powerful new constructions for building functions on the Turing degrees.

Before getting into the technical details of how Martin's conjecture is stated, which we will do in the next few sections, let's set the stage by exploring how one might try to come up with a mathematically precise formulation of the idea that every natural operation on the Turing degrees is built out of the jump. A natural starting point is to focus on functions F on the Turing degrees such that $F(\mathbf{x})$ is always in-between $\mathbf{x}$ and $\mathbf{x}'$. An excessively naive question to ask here is whether every such function is either equal to the identity function or the jump. However, this is obviously false. Here are two counterexamples.

Example 1.1. By the relativized version of the Kleene-Post theorem, for every Turing degree $\mathbf{x}$ there is some degree $\mathbf{y}$ strictly between $\mathbf{x}$ and $\mathbf{x}'$. Pick one such $\mathbf{y}$ for every $\mathbf{x}$ to get a function that is always strictly in-between the identity and the jump.

[1] There is arguably a natural computability theoretic operation on 2^ω that is not equivalent to the jump—namely the relativization of Chaitin's Ω to an arbitrary real (see [Dow+05])—but notably this operation does not induce a well-defined operation on the Turing degrees.

Example 1.2. Fix some Turing degree $\mathbf{z}$ and define a function by

$$f(\mathbf{x}) = \begin{cases} \mathbf{x}', & \text{if } \mathbf{x} \geq_T \mathbf{z} \\ \mathbf{x}, & \text{if } \mathbf{x} \not\geq_T \mathbf{z} \end{cases}$$

This function is sometimes equal to the identity and sometimes equal to the jump, but it is clearly not equal to either one on all degrees.

But though these examples do answer our naive question, they do not seem to really be counterexamples to our initial intuition about the uniqueness of the Turing jump. The first example requires using the Axiom of Choice to define the function and the second example is just built out of the identity and the jump in a simple way and is just equal to the jump on all large enough degrees. The idea of Martin's conjecture is that these are essentially the only possible types of counterexamples. We can eliminate the first type of counterexample by either working in a setting without the Axiom of Choice or by restricting our attention to some class of definable functions (Borel functions, for example). The usual formulation of Martin's conjecture replaces the Axiom of Choice with the Axiom of Determinacy (an axiom of set theory which contradicts Choice), though there are also versions of Martin's conjecture which instead restrict the class of functions considered. And we can eliminate the second type of counterexample by only considering the behavior of functions "in the limit," i.e. their behavior on all sufficiently large degrees.

A few things about this description of Martin's conjecture require more explanation. First, what do we mean by "only consider the behavior of functions in the limit"? And second, what is the Axiom of Determinacy, and why should we want to prove a theorem that assumes it rather than just prove a consistency result over ZF? In the next two sections, we will address these questions, at which point we will be able to state Martin's conjecture and explain the sense in which it explains the phenomenon that we started with.

1.2 The Martin Order and the Martin Measure

In this section we will explain what it means to talk about the behavior of functions on the Turing degrees "in the limit." The basic idea is that we will consider two functions equivalent as long as they are equal on every large enough Turing degree. More generally, we will think of a property of Turing degrees as being "eventually true" if it holds for every large enough Turing degree. The key notion to making this precise is that of a **cone** of Turing degrees—this is a set which contains all degrees above some fixed degree (in other words, all "large enough" degrees).

Definition 1.3. *A **cone** of Turing degrees is a set of Turing degrees of the form* $\{\mathbf{x} \mid \mathbf{x} \geq_T \mathbf{z}\}$ *for some fixed degree* $\mathbf{z}$*. This is sometimes also referred to as the **cone above*** $\mathbf{z}$ *and* $\mathbf{z}$ *is called the **base** of the cone.*

Notation 1.4. We will sometimes use the notation $\mathrm{Cone}(\mathbf{z})$ to refer to the cone above a degree $\mathbf{z}$.

One of the core ideas of Martin's conjecture is that if we switch from asking for some property to hold for every Turing degree to asking for it to hold on a cone of Turing degrees then we get much more robust statements. This "on-a-cone" way of thinking has also found applications well outside of Martin's conjecture. For example, it has been used by Montalbán in [Mon17] to get cleaner versions of theorems in computable structure theory, such as the following: a countable structure is computably categorical on a cone if and only if it is $\exists$-atomic over a finite set of parameters (which is easier to state than the version that just talks about computable categoricity instead of computable categoricity on a cone).

The Martin Order

We can now define what it means for two functions on the Turing degrees to be "eventually equal" or for one function to be "eventually above" another.

Notation 1.5. We will use $\mathcal{D}_T$ to refer to the Turing degrees.

Definition 1.6. *If $F, G\colon \mathcal{D}_T \to \mathcal{D}_T$ are both functions on the Turing degrees then F is* ***Martin equivalent*** *to G, written $F \equiv_M G$, if they are equal on a cone. In other words, if there is some $\mathbf{z}$ such that for all $\mathbf{x} \geq_T \mathbf{z}$, $F(\mathbf{x}) = G(\mathbf{x})$.*

Definition 1.7. *If $F, G\colon \mathcal{D}_T \to \mathcal{D}_T$ are two functions on the Turing degrees then F is* ***Martin below*** *G, written $F \leq_M G$, if F is Turing below G on a cone. In other words, if there is some $\mathbf{z}$ such that for all $\mathbf{x} \geq_T \mathbf{z}$, $F(\mathbf{x}) \leq_T G(\mathbf{x})$.*

The relation $\leq_M$ is called the **Martin order**. Note that the Martin order is a quasi-order on functions on the Turing degrees and if we quotient out by Martin equivalence then it forms a partial order.

The Martin Measure

There is a natural way to think about Martin equivalence and the Martin order using concepts from measure theory. We wish to emphasize this now because this perspective will be useful for understanding our results later in this thesis.

Here's the idea. Call a set of Turing degrees "measure 1" if it contains a cone and "measure 0" if it is disjoint from a cone. This forms a $\{0, 1\}$-valued measure on the Turing degrees (i.e. a countably complete filter), called the Martin measure. Martin equivalence is just equivalence almost everywhere with respect to the Martin measure, and likewise for the Martin order. For the sake of completeness, we will now state all of this formally.

Definition 1.8. *The **Martin measure**, denoted U_M, is the $\{0,1\}$-valued measure on the Turing degrees defined by*

$$U_M(A) = \begin{cases} 1, & \text{if } A \text{ contains a cone} \\ 0, & \text{if } A \text{ is disjoint from a cone.} \end{cases}$$

Note that it is easy to show that the Axiom of Choice implies that not every set of Turing degrees is measurable. The next proposition reassures us about our use of the word "measure" in the term "Martin measure" and is straightforward to prove using standard facts from computability theory.

Proposition 1.9. *The Martin measure is, in fact, a measure—i.e. a countably complete filter.*

We can now restate Martin equivalence and Martin order in terms of the Martin measure. The proof of the following proposition is simply a matter of comparing definitions and seeing that they are identical.

Proposition 1.10. *Let F and G be functions on the Turing degrees.*

- *$F \equiv_M G$ if and only if F and G agree U_M-almost everywhere*
- *$F \leq_M G$ if and only if $F(\mathbf{x}) \leq_T G(\mathbf{x})$ for U_M-almost every $\mathbf{x}$.*

Turing Invariant Functions on the Reals

It is most common to state Martin's conjecture not for all functions on the Turing degrees, but instead only for those functions on the Turing degrees which are induced by Turing invariant functions on the reals. The main reason for this is that it allows for access to more technical tools and because if we assume a limited form of the Axiom of Choice (which is consistent with the Axiom of Determinacy), then this includes all functions on the Turing degrees (we will comment on this again later).

Another benefit of framing things in terms of Turing invariant functions on the reals rather than functions on the Turing degrees is that it gives a better perspective on why the Axiom of Choice was required in Example 1.1. The proof that Kleene and Post gave to construct a real whose Turing degree is in-between that of a real, x, and its jump, x', builds a real by finite extensions, meeting an infinite list of requirements along the way to make sure that the real is not computed by x and does not compute x'. The Turing degree of the real produced by this construction depends in a very sensitive way on x—if x is changed even in one position, it may change the way in which the construction tries to satisfy the requirements and this can easily change the Turing degree of the real produced[2]. Thus the

[2]And priority arguments are even worse: the *order* in which requirements are satisfied can be changed by flipping a single bit of x.

Kleene-Post construction should really be viewed as a function not from Turing degrees to Turing degrees, but from reals to reals. The way that Martin's conjecture excludes things like the Kleene-Post construction, then, is by restricting our attention from all functions on the reals to just the Turing invariant functions. This also agrees with our intuition—the Turing jump is special in part because even though it is naturally defined in terms of reals, it *does* induce a well-defined function on the Turing degrees.

We will now give a formal definition of a **Turing invariant function** on the reals and of a **Turing invariant set** of reals and briefly state how to rephrase the definitions from earlier in this section to talk about Turing invariant functions on the reals rather than functions on the Turing degrees.

Definition 1.11. *A function $f\colon 2^\omega \to 2^\omega$ is **Turing invariant** if for all x and y in 2^ω,*

$$x \equiv_T y \implies f(x) \equiv_T f(y).$$

The point is that a Turing invariant function on the reals induces a function on the Turing degrees in a well-defined way. Likewise, we can define Turing invariant subsets of 2^ω.

Definition 1.12. *A set $A \subseteq 2^\omega$ is **Turing invariant** if for all x and y in 2^ω,*

$$x \equiv_T y \implies (x \in A \iff y \in A).$$

Again, the point is that a Turing invariant subset of 2^ω gives a subset of the Turing degrees in a well-defined way.

Just as we defined cones, Martin equivalence and the Martin order for sets of Turing degrees and functions on the Turing degrees, we can make analogous definitions for Turing invariant sets and functions. Note that doing so means we are overloading the terms involved. For instance, we will sometimes use "cone of Turing degrees" to mean a set of Turing degrees and sometimes to mean the corresponding Turing invariant set of reals. We will also engage in some abuse of terminology by saying things like **"f is constant on a cone"** to mean that a Turing invariant function f is Martin equivalent to a constant function (even though it is only the function on the Turing degrees induced by f which is actually constant on a cone and f itself may not be). Likewise, we will occasionally say things like **"f is equal to the identity on a cone"** to mean f is Martin equivalent to the identity function.

Definition 1.13. *A **cone** of Turing degrees is a set of reals of the form $\{x \mid x \geq_T z\}$ for some real z. As before, we will sometimes refer to this as **the cone above** z, use the notation $\mathrm{Cone}(z)$ to refer to it, and refer to z as the **base** of the cone.*

Definition 1.14. *If $f, g\colon 2^\omega \to 2^\omega$ are Turing invariant functions then f is **Martin equivalent** to g, written $f \equiv_M g$, if they are Turing equivalent on a cone. In other words, if there is some z such that for all $x \geq_T z$, $f(x) \equiv_T g(x)$.*

Definition 1.15. *If $f, g\colon 2^\omega \to 2^\omega$ are two Turing invariant functions then f is **Martin below** g, written $f \leq_M g$, if f is Turing below g on a cone. In other words, if there is some z such that for all $x \geq_T z$, $f(x) \leq_T g(x)$.*

Notation 1.16. At a few times during this thesis, we will need to shift back and forth between Turing invariant functions on the reals and functions on the Turing degrees, as well as between reals and the Turing degrees of those reals. To help keep things straight, we will use the following notation.

- Lowercase letters like f and g will be used for functions on the reals.
- Uppercase letters like F and G will be used for functions on the Turing degrees.
- If f is a Turing invariant function on the reals, we will use F to denote the function on the Turing degrees induced by F.
- Lowercase, italic letters like x and y will be used for reals.
- Lowercase, boldface letters like $\mathbf{x}$ and $\mathbf{y}$ will be used for Turing degrees.
- If x is a real then we will use $\deg_T(x)$ to denote its Turing degree.

1.3 The Axiom of Determinacy

In this section, we will introduce the Axiom of Determinacy and answer three questions about it: what is it, why do we want to use it, and how will we use it. We will also mention some weak forms of choice that are consistent with the Axiom of Determinacy and which it is often convenient to assume.

What is the Axiom of Determinacy?

The Axiom of Determinacy, usually shortened to AD, is a strong axiom of set theory which is inconsistent with the Axiom of Choice, but which is provably consistent with ZF, assuming the existence of sufficiently large cardinals[3] (in fact, assuming the existence of sufficiently large cardinals, $L(\mathbb{R})$ is a model of $\mathsf{ZF} + \mathsf{AD}$). Historically, people began to study the Axiom of Determinacy because it implies that sets of reals are very well-behaved, satisfying many "regularity" properties. For descriptive set theorists, AD is a kind of paradise: every set of reals is Lebesgue measurable, satisfies the property of Baire and the perfect set property and there is a very nice and orderly pattern of which of the projective pointclasses satisfy the separation and uniformization properties, just to name a few of the consequences (see chapter 6 of [Mos09] and chapter 33 of [Jec03] for proofs of some of these results). Eventually, connections between AD and other parts of set theory were discovered, especially connections to large cardinals.

We will now explain briefly what the Axiom of Determinacy actually says. We will never need to use this definition, so the benefit of explaining it is purely psychological. The Axiom

[3] For example, the consistency of $\mathsf{ZF} + \mathsf{AD}$ follows from the consistency of the existence of a supercompact cardinal [Woo88].

of Determinacy states that for every two player game of a certain sort, it is always the case that one of the players has a winning strategy. The games we consider are the following kind. Two players alternate playing natural numbers for infinitely many rounds. The game is perfect information so on each turn the current player can see all moves made so far. After infinitely many rounds have passed, the players have together formed a sequence $x \in \omega^\omega$. There is some fixed set $A \subseteq \omega^\omega$, called the payoff set for the game, and player 1 wins if x is in A. Otherwise, player 2 wins. The Axiom of Determinacy says that for every payoff set A, either player 1 or player 2 has a winning strategy in the corresponding game.

Why Should We Assume the Axiom of Determinacy?

It may seem a bit odd that Martin's conjecture is stated assuming AD. Sure, we said we wanted to avoid the Axiom of Choice to get rid of artificial counterexamples, but why replace it with this weird new axiom? There are a few reasons.

Reason 1: Using AD is a way of establishing a consistency result. One way of thinking about our use of AD is that it is simply a way of establishing a consistency result over ZF. Just as it is common to prove that some statement is consistent with ZFC by showing that it is implied by the continuum hypothesis or Martin's Axiom, if we prove Martin's conjecture in $\mathsf{ZF} + \mathsf{AD}$ then it establishes that Martin's conjecture is consistent with ZF, which is already enough to tell us that it is impossible to construct weird Turing invariant functions (such as functions which are always strictly in-between the identity and the jump) without using the Axiom of Choice.

There is one potential worry about this, however. We mentioned above that the proof that $\mathsf{ZF} + \mathsf{AD}$ is consistent uses fairly strong large cardinal hypotheses. If we are simply using AD as a tool to establish a consistency result over ZF, wouldn't it be more reasonable to use a tamer hypothesis with lower consistency strength? One answer to this is that it is easy to prove that Martin's conjecture implies a weak form of the Axiom of Determinacy, known as Turing Determinacy (or TD for short). So it seems that it is unavoidable to use some determinacy when trying to prove Martin's conjecture. It is unknown whether Martin's conjecture (or even TD) implies AD. It is also unknown whether TD is sufficient to prove all currently-known cases of Martin's conjecture, though some researchers have investigated this question and proved some partial results [CWY10; Bar20].

Reason 2: Uses of determinacy often smoothly restrict to smaller classes of sets. When we introduced the idea behind Martin's conjecture earlier in this chapter, we mentioned that there is a version of it where instead of replacing the Axiom of Choice with the Axiom of Determinacy, you simply restrict the class of functions being considered to some nice "definable" class of functions, such as Borel functions. It turns out that it is possible to restrict the Axiom of Determinacy to only talk about games whose payoff sets are Borel and this version—often called "Borel determinacy"—is outright provable in ZF (due to a remarkable proof by Martin [Mar85]). It is often assumed that if Martin's conjecture is proved in $\mathsf{ZF} + \mathsf{AD}$ then the same proof will also show that Martin's conjecture restricted to

Borel functions is provable in ZF by using Borel determinacy. In other words, it is assumed that the uses of determinacy in the proof will all be for games whose payoff sets are of the same complexity as the function being considered.

This assumption—that the amount of determinacy used in the proof is directly tied to the complexity of the function being considered—is often referred to by saying that the proof uses determinacy in a "local" way. If this assumption pans out, then it will be possible to simultaneously prove many different versions of Martin's conjecture where the level of determinacy is adjusted depending on the large cardinal assumptions that you are comfortable with and the class of functions that you would like Martin's conjecture to hold for. Borel functions aren't enough for you, but you're willing to accept the existence of a measurable cardinal? Then you can get Martin's conjecture for functions with analytic graphs using analytic determinacy (which is provable from the existence of a measurable cardinal). Analytic functions aren't enough and you're fine with any large cardinals but don't want to give up the Axiom of Choice just yet? Then you can get Martin's conjecture for all functions in $L(\mathbb{R})$ using the fact that $L(\mathbb{R})$ is a model of AD (provable from the existence of infinitely many Woodin cardinals). And in the other direction, if you think ZF is too strong then you can prove forms of Martin's conjecture for smaller classes of functions than Borel using weaker forms of determinacy provable in fragments of ZF.

This seems like an appealing vision and is a large part of the reason to use determinacy in formulating Martin's conjecture. Unfortunately some of the results of this thesis provide at least a little bit of evidence that determinacy may not be used in a strictly "local" way in a proof of Martin's conjecture. For example, our proof of Martin's conjecture for regressive functions on the hyperarithmetic degrees does work both in $\mathsf{ZF} + \mathsf{AD}$ and for Borel functions in ZF but modifying the AD proof to work for Borel functions and use only Borel determinacy requires some nontrivial changes to the proof. And we currently do not know how to prove part 1 of Martin's conjecture for measure preserving Borel functions using only Borel determinacy—instead we need to assume analytic determinacy. We will discuss this issue more when we come to these two cases of Martin's conjecture later in this thesis.

Reason 3: Assuming AD allows us to actually prove stuff. The biggest reason to assume AD in the statement of Martin's conjecture is purely pragmatic. The Axiom of Determinacy provides powerful tools to a computability theorist and some special cases of Martin's conjecture have already been proven by using these tools. Mathematicians, above all, want to prove things and if accepting AD lets us do that then it seems like a worthwhile trade-off.

In fact, part of Martin's motivation for stating his conjecture was simply his recognition that the tools provided by determinacy might allow such a theorem to be proved (he was partly inspired by the at-the-time-recent success in thoroughly understanding the structure of Wadge reducibility under AD). We will next get some sense of what tools determinacy provides for us.

How is Determinacy Used in Computability Theory?

The single most important consequence of AD for computability theory is the following fact, known as "Martin's cone theorem."

Theorem 1.17 (ZF + AD; Martin [Mar68])**.** *Every set of Turing degrees either contains a cone or is disjoint from a cone.*

This has a few important corollaries. First, it tells us that the Martin measure is actually an *ultrafilter* on the Turing degrees, which will be fairly conceptually important later in this thesis. Second, it tells us that to show that a set of Turing degrees contains a cone, it is enough to show that it is not disjoint from a cone, which is often a significantly easier task. So much easier, in fact, that Martin's cone theorem often feels a bit miraculous. We will now state a definition that allows us to repackage this second consequence in a convenient way.

Definition 1.18. *A set of Turing degrees A is* ***cofinal*** *if for every Turing degree $\mathbf{x}$, there is a degree $\mathbf{y} \geq_T \mathbf{x}$ which is in A.*

It is easy to see that A set of Turing degrees is cofinal if and only if it is not disjoint from any cone. Thus we can think of Martin's cone theorem as saying that if a set of Turing degrees is cofinal then it contains a cone. This gives rise to what can be thought of as "the first principle of using determinacy in computability theory."

> **The First Principle of Using Determinacy in Computability Theory.** To show that something happens on a cone, just describe what you want, show it happens cofinally, and let determinacy do the rest.

Let's see a simple example of this principle in action. We will use it to prove a version of the jump inversion theorem.

Example 1.19 (Jump inversion via nuclear flyswatter)**.** We will show that there is a cone of Turing degrees $\mathbf{x}$ such that $\mathbf{x}$ is the jump of some other degree—i.e. such that there is some $\mathbf{y}$ such that $\mathbf{y}' = \mathbf{x}$. To do so, we will follow the first principle of using determinacy in computability theory.

The first step is to describe the set which we want to contain a cone. For us, this set is simply the set of degrees which are the jump of some other degree, i.e. $A = \{\mathbf{x} \mid \exists \mathbf{y}\,(\mathbf{y}' = \mathbf{x})\}$. The next step is to show that this set is cofinal. In other words, we have to start with some arbitrary degree $\mathbf{z}$ and find a degree above it which is in A. That's easy: just use $\mathbf{z}'$, which is obviously both above $\mathbf{z}$ and is the jump of some other degree. And now we apply determinacy and we are done!

Note, by the way, that we did not need the full strength of AD here. The set A that we are applying the Martin cone theorem to is easily seen to be Borel and so it is sufficient to use Borel determinacy, which is provable in ZF.

This example may seem a little silly since we were only able to prove that there is *some* cone on which every degree is the jump of something and the Friedberg jump inversion theorem already tells us that this actually happens on *the cone above* $\mathbf{0}'$. But it illustrates the way that we will often use determinacy.

By the way, we will often need to use a fancier version of Martin's cone theorem adapted to work for sets of reals which are not Turing invariant (and thus do not correspond to sets of Turing degrees). But stating this version requires defining the notion of a "pointed perfect tree" and so we will delay it until section 2.1.

We will now describe a second important way that Martin's cone theorem can be used, which can be thought of as "the second principle of using determinacy in computability theory." This principle is really just a restatement of the fact that Martin measure is a countably complete ultrafilter on the Turing degrees.

> **The Second Principle of Using Determinacy in Computability Theory.** If a countable union of sets of Turing degrees contains a cone then one of those sets contains a cone.

Let's now see an example of this second principle in action, being used to prove what we will later recognize as a very simple special case of Martin's conjecture.

Example 1.20 (Martin's conjecture for bounded functions)**.** Suppose $F\colon \mathcal{D}_T \to \mathcal{D}_T$ is a *bounded* function on the Turing degrees—i.e. there is some degree $\mathbf{a}$ such that for all degrees $\mathbf{x}$, $F(\mathbf{x}) \leq_T \mathbf{a}$. Then F is actually constant on a cone.

To prove this, we want to apply the second principle of using determinacy in computability theory. The main idea of the proof is that since $\mathbf{a}$ can only compute countably many things, the range of F must be countable. Let $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \ldots$ enumerate the range and for each n, let A_n be the preimage of $\mathbf{b}_n$ under F. Since the union of the A_n's is all of $\mathcal{D}_T$, which is itself a cone (the cone above $\mathbf{0}$), the second principle tells us that there is some n such that A_n contains a cone. And therefore F is constant on a cone, with constant value $\mathbf{b}_n$.

The Axiom of Determinacy and Weak Forms of Choice

Even though AD contradicts the full Axiom of Choice, it is common to use it in conjunction with weak forms of choice which it is consistent with. We will now briefly discuss some forms of choice which are compatible with determinacy.

There is one weak form of choice which is actually just provable using AD. This is the axiom of choice for countable collections of sets of real numbers, often denoted $\mathsf{CC}_{\mathbb{R}}$. We will not give a proof here (though it is not hard to prove), but the basic idea is that a way to choose one element from each set can be thought of as a winning strategy in a certain game. This form of choice is used frequently throughout this thesis (for example, it is needed to prove that Martin measure is countably complete) and since it is provable from AD, we will not comment on when it is being used[4].

Another form of choice which is often useful but which is not provable from AD (though it is compatible with AD) is the Axiom of Dependent Choice, usually abbreviated to DC. This axiom says that we can make countably many choices in a row, where our options for each choice may depend on the choices we have already made. In the context of Martin's conjecture, we never really need the full Axiom of Dependent Choice, but only a weak form of it which just talks about sets of real numbers (which is usually written $\mathsf{DC}_{\mathbb{R}}$).

Definition 1.21 (The Axiom of Dependent Choice)**.** *The Axiom of Dependent Choice,* DC *for short, states that for every set A and every binary relation R on A, there is a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ of elements of A such that for each n, $R(x_n, x_{n+1})$ holds. The weaker form of this axiom where we require that the set A is just the set of real numbers is known as* $\mathsf{DC}_{\mathbb{R}}$.

This axiom is not provable from AD, but it is consistent with it and, in fact, ZF + AD + DC is equiconsistent with ZF + AD. Thus it is common when using AD to assume DC as well. In this thesis, we will usually work in ZF + AD, but we will occasionally assume $\mathsf{DC}_{\mathbb{R}}$.

Finally, we will sometimes need a form of choice which, when added to AD, actually has greater consistency strength than AD alone. This form of choice is just choice for sets of real numbers indexed by real numbers, also known as uniformization for all binary relations on the real numbers. We will refer to this form of choice as $\mathsf{Uniformization}_{\mathbb{R}}$.

Definition 1.22. *The Axiom of Uniformization for reals, denoted* $\mathsf{Uniformization}_{\mathbb{R}}$*, is a form of choice which states that if R is a binary relation on 2^ω then there is a function $f\colon 2^\omega \to 2^\omega$ such that for all $x \in 2^\omega$,*

$$\exists y\, R(x,y) \iff R(x, f(x)).$$

It is provable from sufficiently large cardinals that $\mathsf{ZF}+\mathsf{AD}+\mathsf{Uniformization}_{\mathbb{R}}$ is consistent, but, as we mentioned above, it has greater consistency strength than ZF + AD. In fact, Woodin and Martin have shown that $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}+\mathsf{Uniformization}_{\mathbb{R}}$ is equivalent to a certain strengthening of AD, known as $\mathsf{AD}_{\mathbb{R}}$ (as in logically equivalent, not just equiconsistent)[5].

For a few of the results in this thesis, and in particular for our proof of part 1 of Martin's conjecture for measure preserving functions, we will need to use $\mathsf{Uniformization}_{\mathbb{R}}$. Since this is implied by $\mathsf{AD}_{\mathbb{R}}$, we will usually just state these results as consequences of $\mathsf{AD}_{\mathbb{R}}$. We will also see that we can get away without assuming $\mathsf{Uniformization}_{\mathbb{R}}$ by using the axiom AD^+, which is a different strengthening of AD. One reason to prefer AD^+ to $\mathsf{AD}_{\mathbb{R}}$, by the way, is

[4] There is a potential reason to keep track of its use: if you want to know whether various cases of Martin's conjecture require the full Axiom of Determinacy or if they are provable from weaker forms like TD. However, Peng and Yu recently showed (by a really clever proof) that TD implies $\mathsf{CC}_{\mathbb{R}}$ [PY20] so this reason may not be so good after all.

[5] For the curious reader, $\mathsf{AD}_{\mathbb{R}}$, which was first introduced by Solovay in [Sol78], is like AD but the class of games considered is expanded to include games where the players play real numbers on each turn instead of just natural numbers.

that while $\mathsf{AD}_{\mathbb{R}}$ cannot hold in $L(\mathbb{R})$, sufficiently large cardinals imply that AD^+ does hold in $L(\mathbb{R})$.

One use of $\mathsf{Uniformization}_{\mathbb{R}}$, which we have already touched on, is to prove that every function on the Turing degrees comes from a Turing invariant function on the reals.

Example 1.23. It is provable from $\mathsf{Uniformization}_{\mathbb{R}}$ (and thus from $\mathsf{AD}_{\mathbb{R}}$) that every function $F\colon \mathcal{D}_T \to \mathcal{D}_T$ is induced by a Turing invariant function on 2^ω. This is because we can obtain such a function as any uniformization of the binary relation R on 2^ω defined by

$$R(x, y) \iff y \in F(\deg_T(x)).$$

1.4 Statement of Martin's Conjecture

We are now ready to state Martin's conjecture. It essentially says that every Turing invariant function is eventually constant, eventually equal to the identity, or eventually equal to a transfinite iterate of the Turing jump. It is traditionally divided into two parts. The first part deals with functions which are not above the identity (and says that they are all constant) and the second part deals with functions that are above the identity (and says more or less that they are all transfinite iterates of the jump).

Conjecture 1.24 (Martin's conjecture)**.** *Assume* $\mathsf{ZF} + \mathsf{AD}$*. Then*

(1) Every Turing invariant function $f\colon 2^\omega \to 2^\omega$ is either Martin equivalent to a constant function or Martin above the identity.

(2) The Martin order prewellorders the Turing invariant functions which are Martin above the identity. Moreover, the successor of f in this well order is the jump of f—i.e. the function $x \mapsto f(x)'$.

Let us comment on exactly how—and to what extent—Martin's conjecture really captures the idea that every Turing invariant function is eventually constant, equal to the identity, or equal to a transfinite iterate of the jump.

In order to only discuss "eventual behavior" of functions, we will mod out by Martin equivalence. If we do this then we can see that the first part of the conjecture says that every Turing invariant function is either constant or above the identity. So if we ignore the constant functions then there is a least element in the Martin order, namely the identity.

$$x \mapsto x.$$

The second part of the conjecture then implies that there is a least function which is Martin above the identity and we can get this function by taking the jump of the identity. This gives us

$$x \mapsto x',$$

which is just the Turing jump itself. And the conjecture implies that there is also a least function Martin above this one. Namely

$$x \mapsto x'',$$

which is also known as the double jump. And obviously we can keep going. If we do, we will see that the first ω Turing invariant functions above the identity are just the finite iterates of the Turing jump.

The second part of the conjecture also implies that the finite iterates of the Turing jump have a least upper bound in the Martin order. So what is this least upper bound? Intuitively, it should just be the ω-jump, $x \mapsto x^{(\omega)}$ (where $x^{(\omega)}$ denotes the infinite join of all the finite jumps of x). But curiously, Martin's conjecture does not seem to obviously imply that the ω-jump actually is the least upper bound of the finite jumps. In particular, it seems difficult to rule out the possibility that there could be "pseudo ω-jumps"—functions which are above each finite jump, but strictly below the ω-jump.

It would be interesting to know whether it is possible to use Martin's conjecture to rule out the existence of such functions. Bizarrely, it is not too hard to prove that there can be at most *two* of them. This can be proved using the following theorem due to Enderton and Putnam [EP70].

Theorem 1.25 (Enderton-Putnam). *If $\mathbf{x}$ is any Turing degree and $\mathbf{y}$ is an upper bound for all the finite jumps of $\mathbf{x}$ (i.e. $\mathbf{y} \geq_T \mathbf{x}^{(n)}$ for each $n \in \mathbb{N}$) then $\mathbf{y}'' \geq_T \mathbf{x}^{(\omega)}$.*

Proposition 1.26 ($\mathsf{ZF} + \mathsf{AD}$). *Assume that Martin's conjecture holds. Then there can be at most two "pseudo ω-jumps" (up to Martin equivalence).*

Proof. Let $f\colon 2^\omega \to 2^\omega$ be a Turing invariant function which is a member of the Martin equivalence class which is the least upper bound of all the finite iterates of the jump. Since f is Martin above all the finite iterates of the jump, there is a cone on which $f(x)$ is always an upper bound for all the finite jumps of x. So by the Enderton-Putnam theorem, f'' is above the ω-jump on this cone. By Martin's conjecture, f'' is the successor of the successor of f. So either f, f', or f'' is Martin equivalent to the ω-jump. □

By the way, Enderton and Putnam's theorem works for any countable ordinal, not just ω. So for each countable α, Martin's conjecture implies there are at most two "pseudo α-jumps" (the situation at limit ordinals of uncountable cofinality seems to be more complicated).

But in spite of this potential oddity at limit ordinals, it does seem reasonable to say that if a function is part of a well-ordered hierarchy of functions where the successor in that hierarchy is the jump then that function is a transfinite iterate of the Turing jump. Thus Martin's conjecture does seem to do a reasonable job at capturing our intuitions about the special role of the jump.

1.5 Special Cases of Martin's Conjecture

At the beginning of this chapter, we said that while Martin's conjecture is still open, it has been proved for several special classes of functions. In this section we will introduce some of these classes and review what is known about them.

Uniformly Invariant Functions

The best understood special case of Martin's conjecture is that of **uniformly Turing invariant functions**. These are Turing invariant functions $f\colon 2^\omega \to 2^\omega$ such that if $x \equiv_T y$ then to find programs witnessing the Turing equivalence of $f(x)$ and $f(y)$, it is enough to know which programs witness the Turing equivalence of x and y (i.e. you do not need to know what x and y are themselves). This condition is satisfied by the jump because if you know how to compute x from y then there is a uniform procedure to turn a program with an oracle for x into a program with an oracle for y and thus a uniform procedure to compute x' from y'.

Here's the precise definition of a uniformly Turing invariant function.

Definition 1.27. *Suppose x and y are reals and $i, j \in \mathbb{N}$. We say $x \equiv_T y$ **via** (i, j) if $\Phi_i(x) = y$ and $\Phi_j(y) = x$ (in other words if i and j are indices for programs witnessing the Turing equivalence of x and y).*

Definition 1.28. *A function $f\colon 2^\omega \to 2^\omega$ is called **uniformly Turing invariant** if there is a function $u\colon \mathbb{N}^2 \to \mathbb{N}^2$ such that for all $x, y \in 2^\omega$ and $i, j \in \mathbb{N}$, if $x \equiv_T y$ via (i, j) then $f(x) \equiv_T f(y)$ via $u(i, j)$. The function u is called a **uniformity function** for f.*

A series of papers by Lachlan, Steel, and Slaman and Steel proved that Martin's conjecture holds when restricted to uniformly Turing invariant functions. The first step was taken in 1975 by Lachlan [Lac75], who showed that part 2 of Martin's conjecture holds for all uniformly Turing invariant r.e. operators—in other words that there is no uniformly Turing invariant solution to Post's problem which works relative to any oracle.

Theorem 1.29 (Lachlan [Lac75])**.** *Suppose W is an r.e. operator such that $W^x \geq_T x$ for all x and $x \mapsto W^x$ is a uniformly Turing invariant function. Then either $W^x \equiv_T x$ on a cone or $W^x \equiv_T x'$ on a cone.*

Next, Steel extended Lachlan's result by showing that part 2 of Martin's conjecture holds for all uniformly invariant functions, not just the ones given by r.e. operators [Ste82].

Theorem 1.30 ($\mathsf{ZF}+\mathsf{AD}$; Steel [Ste82])**.** *Part 2 of Martin's conjecture holds when restricted to the uniformly Turing invariant functions—i.e. uniformly Turing invariant functions which are above the identity on a cone are prewellordered by the Martin order and the successor in this prewellorder is the jump.*

Finally, Slaman and Steel proved part 1 of Martin's conjecture for uniformly invariant functions [SS88], thus finishing the proof of the full conjecture for the case of uniformly invariant functions.

Theorem 1.31 ($\mathsf{ZF} + \mathsf{AD}$; Slaman-Steel [SS88])**.** *Part 1 of Martin's conjecture holds when restricted to the uniformly Turing invariant functions—i.e. if $f\colon 2^\omega \to 2^\omega$ is a uniformly Turing invariant function then either f is constant on a cone or f is above the identity on a cone.*

A subsequent paper by Becker gave a thorough analysis of all uniformly Turing invariant functions, showing that they all arise as the universal set of some fairly well-behaved lightface pointclass [Bec88]. Kihara and Montalbán have also done follow-up work analyzing the case of uniformly invariant functions from the Turing degrees to the many-one degrees [KM18].

Steel has conjectured that every Turing invariant function is Martin equivalent to a uniformly invariant function. In light of the results stated above, this would imply Martin's conjecture.

Regressive Functions

Another case of Martin's conjecture which has been completely proved is that of **regressive functions**, where "regressive" here just means "Martin below the identity function."

Definition 1.32. *A Turing invariant function $f\colon 2^\omega \to 2^\omega$ is called **regressive** if for all $x \in 2^\omega$, $f(x) \leq_T x$.*

Of course, only part 1 of Martin's conjecture is relevant to regressive functions and it implies that every regressive function is constant on a cone or equal to the identity on a cone. This was proved by Slaman and Steel in [SS88].

Theorem 1.33 ($\mathsf{ZF}+\mathsf{AD}$; Slaman-Steel [SS88])**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is regressive on a cone then it is either constant on a cone or equal to the identity on a cone.*

Order Preserving Functions

Another special case of Martin's conjecture that was examined by Slaman and Steel is that of **order preserving functions** on the Turing degrees. These are Turing invariant functions which preserve Turing reducibility in addition to Turing equivalence.

Definition 1.34. *A Turing invariant function $f\colon 2^\omega \to 2^\omega$ is called **order preserving** if for all x and y in 2^ω,*

$$x \leq_T y \implies f(x) \leq_T f(y).$$

Slaman and Steel have proved a portion of part 2 of Martin's conjecture for order preserving functions [SS88]. In particular, they have shown that part 2 of Martin's conjecture holds for order preserving functions which are not above the hyperjump, and thus for all order preserving Borel functions[6]. In fact, their proof of this fact yields something a bit stronger than just part 2 of Martin's conjecture for such functions. It actually shows that every order preserving function which is above the identity but not above the hyperjump must be Martin equivalent to the α-jump for some countable ordinal α. Thus for order preserving functions at least, it is not possible to have the kind of "pseudo α-jumps" that we discussed earlier.

Theorem 1.35 (ZF+AD; Slaman-Steel [SS88])**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is order preserving, above the identity on a cone and not above the hyperjump on any cone, then there is some countable ordinal α such that, on a cone, $f(x) = x^{(\alpha)}$ (note that this only makes sense if x is large enough so that $\alpha < \omega_1^x$).*

In chapter 6, we will prove a complementary result to Slaman and Steel's theorem: part 1 of Martin's conjecture holds for all order preserving functions.

Theorem 1.36 (ZF + AD)**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is order preserving then either f is constant on a cone or f is above the identity on a cone.*

Measure Preserving Functions

In this thesis, we will introduce a class of Turing invariant functions which we call **measure preserving functions**, and then prove part 1 of Martin's conjecture for this class. We do this not out of a love for generating new special cases of Martin's conjecture to try to prove, but rather because identifying this class of functions gave greater insight into our proof of part 1 of Martin's conjecture for order preserving functions and allowed us to find a connection between Martin's conjecture and the Rudin-Keisler order on ultrafilters on the Turing degrees.

A measure preserving function can be thought of as a function on the Turing degrees which "goes to infinity in the limit." What we mean by this is that it eventually gets above every fixed degree. This is made precise in the following definition.

Definition 1.37. *A Turing invariant function $f\colon 2^\omega \to 2^\omega$ is called **measure preserving** if for every $z \in 2^\omega$, there is some $y \in 2^\omega$ such that*

$$x \geq_T y \implies f(x) \geq_T z.$$

In other words, for every z, f is above z on a cone.

In section 5.1 we will prove several alternative characterizations of measure preserving functions and explain why the name "measure preserving" was chosen. As we mentioned above, we will also prove that part 1 of Martin's conjecture holds for measure preserving

[6] Thus this is a case of Martin's conjecture where the Borel version is known but the AD version is not.

functions. It turns out that this theorem has a historical precedent: in what was possibly the first known case of Martin's conjecture, Martin himself showed that it holds for all *regressive* measure preserving functions (though he didn't use the term "measure preserving").

Theorem 1.38 ($\mathsf{ZF} + \mathsf{AD}$; Martin)**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is regressive and measure preserving then f is equal to the identity on a cone.*

Our proof of part 1 of Martin's conjecture for all measure preserving functions (not just the regressive ones) can actually be seen as an evolution of Martin's proof. We will discuss this more in chapter 5.

Theorem 1.39 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$)**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is measure preserving then f is above the identity on a cone.*

We will also show that every order preserving function is either constant on a cone or measure preserving (in fact, this is a key part of our proof of part 1 of Martin's conjecture for order preserving functions). So the theorem above can be seen as a generalization of the theorem on order preserving functions.

Note, by the way, that if a function is Martin above the identity then it is automatically measure preserving, so part 2 of Martin's conjecture for measure preserving functions is identical to the full part 2 of Martin's conjecture.

1.6 Martin's Conjecture on Other Degree Structures

The Turing degrees are not the only computability-theoretic degree structure which have a concept of a jump operator. For example, the arithmetic degrees have the ω-jump and the hyperarithmetic degrees have the hyperjump. This suggests that there should be an analogue of Martin's conjecture for these degree structures.

It turns out that it is possible to do this—to state versions of Martin's conjecture for a number of degree structures other than the Turing degrees and, in particular, for the arithmetic degrees and the hyperarithmetic degrees. The main point is that the definition of cone makes sense for these degree structures and that Martin's cone theorem still holds[7].

Surprisingly, these different versions of Martin's conjecture have turned out to work somewhat differently from each other. Some of the special cases of Martin's conjecture which are known to hold in the Turing degrees actually fail to hold in the arithmetic degrees. And there are other instances of Martin's conjecture which are known to hold for the Turing degrees, but whose status is open for the arithmetic degrees and the hyperarithmetic degrees.

Aside from the intrinsic interest in classifying the behavior of functions on the various degree structures that are studied in computability theory, there are a couple of other reasons to study these versions of Martin's conjecture.

[7] It turns out there actually *are* degree structures where the cone theorem doesn't hold, typically degree structures where the type of reducibility considered involves subexponential time computation. For example, Marks has shown that the cone theorem does not hold for the polynomial time degrees, see [Mar18].

One reason is that the differences between how Martin's conjecture works in different degree structures yield interesting insights about the differences between different types of computability-theoretic reducibilities. For example, the reason that some cases of Martin's conjecture fail in the arithmetic degrees but hold in the Turing degrees seems to be that it is possible to "amalgamate" uniformly computable sequences of Turing reductions but not uniformly computable (much less arithmetic) sequences of arithmetic reductions (we will say a bit more about this after introducing the arithmetic degrees). In the hyperarithmetic degrees, it is again possible to amalgamate uniform sequences of reductions, but proving this requires the Σ^1_1-bounding theorem (which we will discuss in the next chapter) and the resulting amalgamation is less well-behaved than for the Turing degrees. This explains why some cases of Martin's conjecture which hold in the Turing degrees also hold in the hyperarithmetic degrees, but with different or more complicated proofs. Thus the extent to which sequences of reductions can be amalgamated is revealed to be an important structural feature of computability-theoretic reducibilities.

There is also a reason to study versions of Martin's conjecture on other degree structures that could appeal to someone who has no interest in those other structures for their own sake. You can think of Martin's conjecture on degree structures other than the Turing degrees as a kind of parallel universe version of Martin's conjecture where everything works slightly differently—some new techniques are available but also there are some new obstacles. These "parallel universes" can be used as a laboratory for trying out new approaches to Martin's conjecture and for testing the limits of old approaches. For example, we know there is a counterexample to Martin's conjecture on the arithmetic degrees. How far can this counterexample be pushed? Can it be made to work in the hyperarithmetic degrees? If so, perhaps that could be the first step to getting it to work in the Turing degrees as well. And if not, perhaps that would tell us something about the limitations of the technique.

We will now introduce the arithmetic degrees and the hyperarithmetic degrees more formally and discuss what is known about Martin's conjecture for each of them. Note that for most of the classes of functions we defined in the previous section, it is obvious how to modify the definitions to make sense for the arithmetic degrees or the hyperarithmetic degrees. The only case in which this is not quite clear is for uniformly invariant functions.

Arithmetic Degrees

One definition of arithmetic reducibility is that $y \leq_A x$ if and only if y is definable using an arithmetic formula with x as a parameter. An equivalent definition is that there is some natural number n such that y is computable from $x^{(n)}$. It is not hard to check that this notion of reducibility is transitive and reflexive and thus it can be used to define an equivalence relation—arithmetic equivalence—and the structure of the arithmetic degrees (the quotient by this relation).

The ω-jump plays the role of the jump on the arithmetic degrees. For any real x, $x^{(\omega)}$ is not arithmetic in x and if x and y are arithmetically equivalent then $x^{(\omega)}$ and $y^{(\omega)}$ are always also arithmetically equivalent (actually, Turing equivalent).

An important structural difference between Turing reducibility and arithmetic reducibility is that there is a natural hierarchy of arithmetic reductions, just given by how many jumps of x are needed to compute y. This hierarchy is related to the failure of the ability to "amalgamate" sequences of arithmetic reductions that we mentioned earlier. The idea is that if x is a real and $y_0, y_1, y_2, \ldots$ is a sequence of reals which are all arithmetic in x then even if the sequence of arithmetic formulas witnessing those reductions is computable, the join of the y_n's might not be arithmetic in x because there might be no bound on the number of jumps that it takes to compute the y_n's from x (i.e. each y_n could be computable from $x^{(n+1)}$ but not $x^{(n)}$ so no finite jump of x can compute all of them).

This structural feature of arithmetic reducibility was exploited by Slaman and Steel to construct a counterexample to part 1 of Martin's conjecture on the arithmetic degrees. Moreover, they were able to show that this counterexample is uniformly invariant.

Theorem 1.40 (ZF; Slaman-Steel [MSS16])**.** *There is a uniformly arithmetically invariant function $f\colon 2^\omega \to 2^\omega$ such that f is neither constant on any cone of arithmetic degrees, nor equal to the identity on any cone of arithmetic degrees.*

With some more care, this example can also be made order preserving.

Theorem 1.41 (ZF; Slaman-Steel)**.** *There is an arithmetically invariant function $f\colon 2^\omega \to 2^\omega$ such that f is order preserving on the arithmetic degrees and neither constant on any cone of arithmetic degrees, nor equal to the identity on any cone of arithmetic degrees.*

However, there actually *are* a few cases of Martin's conjecture that hold on the arithmetic degrees. In particular, Slaman and Steel's result about part 2 of Martin's conjecture for order preserving functions also works in the arithmetic degrees, as do the results in this thesis about part 1 of Martin's conjecture for measure preserving functions.

Theorem 1.42 (ZF + AD; Slaman-Steel)**.** *If $f\colon 2^\omega \to 2^\omega$ is an arithmetically invariant function which is order preserving, above the identity on a cone of arithmetic degrees, and not above the hyperjump on any cone of arithmetic degrees then there is some countable ordinal α such that $f(x) = x^{(\omega\times\alpha)}$ on a cone of arithmetic degrees.*

Theorem 1.43 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$)**.** *If $f\colon 2^\omega \to 2^\omega$ is an arithmetically invariant function which is measure preserving then either f is constant on a cone of arithmetic degrees or f is above the identity on a cone of arithmetic degrees.*

Most other special cases of Martin's conjecture on the arithmetic degrees remain open. In particular, it is unknown whether it holds for regressive functions and whether there is a counterexample to part 2 in general (even for uniformly invariant functions).

Hyperarithmetic Degrees

The easiest definition of hyperarithmetic reducibility is that $y \leq_H x$ if and only if y is $\Delta^1_1(x)$ definable (in which case we will often say that y is hyperarithmetic in x). It is not very hard to see that this relation is transitive and reflexive and thus deserves the title "reducibility." As usual, we can then define hyperarithmetic equivalence and the structure of the hyperarithmetic degrees.

But there is another characterization of hyperarithmetic reducibility which is often useful and which we will now explain. Letting ω_1^x denote the least countable ordinal with no presentation computable from x, Davis observed that there is a well-defined (up to many-one degree) notion of iterating the jump of x up to any ordinal below ω_1^x. If $\alpha < \omega_1^x$ then we denote this α^{th} jump of x by $x^{(\alpha)}$. It is possible to show that y is hyperarithmetic in x if and only if y is computable from $x^{(\alpha)}$ for some $\alpha < \omega_1^x$. In fact, this was Davis's original definition of hyperarithmetic reducibility—its equivalence to the definition involving Δ_1^1 definability was only shown several years later by Kleene (and by the way, this equivalence is not at all obvious—it is maybe surprising that the definition in terms of transfinite iterates of the jump is even transitive since it is not obvious that $x \leq_H y$ implies that $\omega_1^x \leq \omega_1^y$).

It will be helpful later on if we make some of this more precise. Suppose r is a linear order on $\mathbb{N}$ and 0 is the minimum element according to r. If x is any real, then a jump hierarchy on r which starts with x is a set $H \subset \mathbb{N}^2$ such that the 0^{th} column of H is x and for each $n \neq 0$, the n^{th} column of H is equal to the jump of the smaller columns of H (smaller according to the ordering given by r). In other words, if we define

$$H_n = \{i \mid \langle n, i\rangle \in H\}$$
$$H_{<n} = \{\langle m, i\rangle \mid m <_r n \text{ and } \langle m, i\rangle \in H\}$$

then we have $H_0 = x$ and $H_n = (H_{<n})'$ for all $n \neq 0$.

If r is a presentation of a well-order then there is always a unique H satisfying the conditions above. Moreover, if $\alpha < \omega_1^x$ and r is a presentation of α computable from x then the many-one degree of the unique jump hierarchy on r starting from x is independent of the choice of r. Such a jump hierarchy is considered to be the α^{th} jump of x (which is only well-defined up to many-one degree). This makes precise the alternative characterization of hyperarithmetic reducibility mentioned above.

It is also worth mentioning here that hyperarithmetic reducibility is closely connected to Borel measurability. Just as continuous functions correspond to computable functions (relative to some oracle), Borel functions correspond to hyperarithmetic functions (relative to some oracle). More precisely, if f is Borel then there is some countable ordinal α, some presentation r of α, some real y and some Turing functional Φ such that for all x, $f(x) = \Phi((x \oplus y)^{(\alpha)})$, where $(x \oplus y)^{(\alpha)}$ is taken to mean the unique jump hierarchy on r starting from $x \oplus y$.

We have also already mentioned that there is an operation on hyperarithmetic degrees that acts like the jump—namely the hyperjump. This is the operation that sends a real x to a $\Pi_1^1(x)$-complete set, usually denoted $\mathcal{O}^x$ (defined formally as the set of indices for programs which, using x as an oracle, compute a presentation of a well-order). For any real x, $\mathcal{O}^x$ is not hyperarithmetic in x and if two reals, x and y, are hyperarithmetically equivalent then their hyperjumps, $\mathcal{O}^x$ and $\mathcal{O}^y$, are also hyperarithmetically equivalent (actually, Turing equivalent).

In terms of at least some structural features, the hyperarithmetic degrees act like an intermediate between the arithmetic degrees and the Turing degrees. If x is a fixed real, then like the arithmetic degrees there is a natural hierarchy of reals which are hyperarithmetic in x: this is the hierarchy indexed by ω_1^x which simply counts how many jumps of x it takes to compute y. But unlike the arithmetic degrees, this hierarchy depends on which real x we consider (since it depends on ω_1^x) and it is not clear how to construct a single hierarchy of hyperarithmetic reductions that is independent of the oracle x. In this respect, the hyperarithmetic degrees are more like the Turing degrees.

This structural intermediate-ness is also reflected in the status of Martin's conjecture on the hyperarithmetic degrees. Unlike with the arithmetic degrees, there are no known counterexamples to Martin's conjecture on the hyperarithmetic degrees. And some of the cases of Martin's conjecture that are known to hold for the Turing degrees are also known to hold for the hyperarithmetic degrees. For example, Slaman and Steel observed that the uniform case of Martin's conjecture holds for the hyperarithmetic degrees.

Theorem 1.44 ($\mathsf{ZF}+\mathsf{AD}$; Slaman-Steel)**.** *Martin's conjecture for the hyperarithmetic degrees holds when restricted to the uniformly hyp-invariant*[8] *functions.*

Slaman and Steel's proof of part 2 of Martin's conjecture for order preserving functions on the Turing degrees which are not above the hyperjump also applies to order preserving functions on the hyperarithmetic degrees, although in that case it takes a simpler form.

Theorem 1.45 ($\mathsf{ZF}+\mathsf{AD}$; Slaman-Steel)**.** *If $f\colon 2^\omega \to 2^\omega$ is a hyp-invariant function which is above the identity on a cone of hyperarithmetic degrees then either f is equal to the identity on a cone of hyperarithmetic degrees of f is above the hyperjump on a cone of hyperarithmetic degrees.*

We will also prove in this thesis that the regressive case of Martin's conjecture also holds for the hyperarithmetic degrees.

Theorem 1.46 ($\mathsf{ZF}+\mathsf{AD}$)**.** *If $f\colon 2^\omega \to 2^\omega$ is a hyp-invariant function such that $f(x) \leq_H x$ on a cone of hyperarithmetic degrees then f is either constant on a cone of hyperarithmetic degrees or equal to the identity on a cone of hyperarithmetic degrees.*

On the other hand, the theorem above came only thirty years after the analogous theorem for the Turing degrees (and was stated as an open problem in Slaman and Steel's paper [SS88] where they proved it for the Turing degrees) and there are also cases of Martin's conjecture which are known to hold for the Turing degrees but which are open for the hyperarithmetic degrees. For example, it is unknown whether part 1 of Martin's conjecture holds for all order preserving functions on the hyperarithmetic degrees and in my opinion, resolving this question could yield progress on Martin's conjecture for the Turing degrees as well.

[8] Note that we will often abbreviate "hyperarithmetically invariant" to "hyp-invariant."

We will finish by mentioning that just like with both the Turing degrees and the arithmetic degrees, part 1 of Martin's conjecture holds for measure preserving functions on the hyperarithmetic degrees.

The Landscape of Martin's Conjecture

There has been a decent amount of research on how Martin's conjecture looks for a few specific degree structures (besides the ones we have mentioned here, it has also been studied in some form for the polynomial time degrees and the many-one degrees), but there has been little focus so far on the "landscape" of Martin's conjecture across different degree structures. That is, there has been little effort to understand what behaviors are possible for Martin's conjecture across many different degree structures. Is there a reasonable degree structure on which Martin's conjecture holds for all order preserving functions but does not hold in general? On which it holds for all uniformly invariant functions but not in general? Is there a degree structure on which Martin's conjecture can be proved outright (even one which may look quite different from the Turing degrees)? This thesis will not answer any of these questions but I think they are all interesting and answering them could lead to progress on Martin's conjecture.

1.7 Summary of What's Known

The table below summarizes the current status of Martin's conjecture for each of the special classes of functions and degree structures that we introduced in the last two sections. The check-marks indicate statements that have been proved, the X's indicate statements which have known counterexamples and the question marks indicate statements which are open. Blue text indicates a new result of this thesis and green text indicates a result that is known, but unpublished. Also we have abbreviated "order preserving" to "OP" and "measure preserving" to "MP." Note that in the part of the table on part 2 of Martin's conjecture, we have omitted the case of regressive functions (because it's trivial) and the case of measure preserving functions (because it's equivalent to the full part 2 of Martin's conjecture).

Structure	**Part 1**				**Part 2**	
	Uniform	Regressive	OP	MP	Uniform	OP
Turing Degrees	✓	✓	✓ (blue)	✓ (blue)	✓	✓*
Arithmetic Degrees	✗	?	✗ (green)	✓ (blue)	?	✓* (green)
Hyperarithmetic Degrees	✓ (green)	✓ (blue)	?	✓ (blue)	✓ (green)	✓* (green)

✓ = True ✗ = False ? = Open ■ (blue) = New result ■ (green) = Unpublished

*Only proved for functions which are not above the hyperjump

Chapter 2

Tools

In this chapter, we will introduce some tools that we will use in the rest of this thesis. Some of these tools are well-known facts and concepts in computability theory and some of them are original to this thesis. Here are the main things we will prove.

A general framework for proving instances of part 1 of Martin's conjecture. In section 2.1 we will lay out a general framework for proving that functions are above the identity on a cone. This framework has essentially two components: inverting computable functions on perfect trees and using determinacy to find computable functions on pointed perfect trees. One of the key lemmas is something that we have dubbed the "Computable Uniformization Lemma," which is a simple consequence of determinacy but surprisingly useful in the context of Martin's conjecture. This general framework will be used later in the paper to prove part 1 of Martin's conjecture for measure preserving functions and order preserving functions on the Turing degrees and for regressive functions on the hyperarithmetic degrees.

Ordinal invariants. In section 2.2 we will introduce the idea of "ordinal invariants of Turing degrees"—essentially just functions from the Turing degrees to the ordinals—and use determinacy to establish a few facts about them. We will use these later to help implement the general framework described in section 2.1.

A basis theorem for perfect sets. In section 2.3 we will prove a basis theorem for perfect sets[1] This lemma is the key to proving part 1 of Martin's conjecture for order preserving functions and is also used in the application to locally countable Borel quasi orders and Sacks' question in Chapters 7 and 8.

The Solecki Dichotomy. In section 2.4 we will explain a useful theorem from descriptive set theory called "the Solecki dichotomy." Later, using an idea due to Kihara, we will apply this theorem to improve the hypotheses used in our first proof of part 1 of Martin's conjecture for order preserving functions (specifically to improve $\boldsymbol{\Pi^1_1}$ determinacy to Borel determinacy in the Borel case).

[1] Arguably it's not exactly a basis theorem, but I couldn't come up with a snappier name for it.

The Σ^1_1-bounding theorem. In section 2.5 we will review the Σ^1_1-bounding theorem, and a few other things that we will need to work with hyperarithmetic reducibility.

Upper bounds in the Turing degrees. In section 2.6 we will review some classic theorems of computability theory about finding minimal upper bounds in the Turing degrees.

2.1 A Framework for Proving Instances of Part 1 of Martin's Conjecture

In this section we will explain a basic strategy to prove that a function is above the identity on a cone. Versions of this strategy have been used before, first by Martin to prove Martin's conjecture for regressive measure preserving functions and later by Slaman and Steel to prove Martin's conjecture for all regressive functions [SS88]. We hope that explicitly identifying this common proof strategy may prove useful in future research on Martin's conjecture. In this thesis, we will use this strategy to prove Martin's conjecture for regressive functions on the hyperarithmetic degrees (section 4.1) and part 1 of Martin's conjecture for measure preserving functions (section 5.3) and to give an alternate proof of part 1 of Martin's conjecture for order preserving functions (the second proof in section 6.2).

The main idea underlying the strategy is actually just the computability theory version of a simple topological fact.

> **Basic topological fact:** If $f : X \to X$ is a continuous, injective function on a compact, Hausdorff space, then f has a continuous inverse $\text{range}(f) \to X$.
>
> **Computability theory version:** If $f : 2^\omega \to 2^\omega$ is a computable, injective function, then for each x, $f(x)$ can compute x.

The point is that if a computable function on 2^ω is injective then it is automatically above the identity. Hence, one way to try to prove that a function f is above the identity is to find a computable injective function g such that f is above g.

In practice, it is not always possible to find a computable function which is injective on all of 2^ω and below f. So we will instead usually use a refined version of this strategy where we find a computable function below f which is injective not on all of 2^ω, but only on a sufficiently large subset of 2^ω. For us, "sufficiently large subset of 2^ω" will mean the set of paths through a perfect tree (which, in practice, will pretty much always be a pointed perfect tree).

The strategy can thus be seen as consisting of two parts: a method for trying to find a pointed perfect tree and a computable function which is below f on that tree, and a method for inverting computable injective functions on pointed perfect trees. We will explain in more detail how to implement all of this: how to define perfect trees and pointed perfect trees, how to prove the key lemmas that we will need about them, how to put it all together, and what the obstacles to getting it to work in practice usually are.

Inverting Computable Functions on Perfect Trees

We will now explain the first component of the strategy: inverting computable functions on perfect trees. We will start with the definition of **perfect tree**.

Definition 2.1. *A* ***perfect tree*** *is a nonempty binary tree T such that every node in T has incomparable descendants in T—i.e. for each σ in T there is a descendant τ of σ such that $\tau^\frown\langle 0\rangle$ and $\tau^\frown\langle 1\rangle$ are both in T.*

The name "perfect tree" is chosen because the set of infinite paths through T is a perfect subset of 2^ω. You can picture a perfect tree T as a kind of warped version of 2^ω: there are no dead ends and if you follow any path long enough you will eventually come to a place where you can choose between going left or right while remaining in the tree. In 2^ω you can make this decision after every step, but in a perfect tree you may have to take many steps before being able to choose.

Notation 2.2. If T is a binary tree then $[T]$ denotes the set of infinite paths through T.

We will now prove a refined version of the basic computability theory fact mentioned above. It is essentially just an application of the compactness of Cantor space[2]. This lemma is also one of the main ideas in constructions of minimal degrees and minimal covers.

Lemma 2.3. *If T is a perfect tree and Φ is a Turing functional which is total on each x in $[T]$ and injective on $[T]$ then for each $x \in [T]$,*

$$\Phi(x) \oplus T \geq_T x.$$

Proof. The main idea of the proof is just a routine application of compactness.

First, let's explain the algorithm to compute x given $\Phi(x)$ and T. Say we want to compute $x \upharpoonright n$. For each σ in level n of T we do the following search (and we do all of these searches in parallel):

> Look for an $m > n$ such that for all descendants τ of σ on level m of T, $\Phi(\tau)[m]$ disagrees with $\Phi(x) \upharpoonright m$.

Once all but one of these searches have terminated, we output the remaining element of level n of T as our guess for $x \upharpoonright n$.

Hopefully it is clear that the search we describe above will never terminate for $x \upharpoonright n$ (since on every level above $m > n$ there is a descendant of $x \upharpoonright n$ in T, namely $x \upharpoonright m$, which will not make Φ disagree with $\Phi(x)$). So all we really need to do is show that the search will terminate for every $\sigma \in 2^n \cap T$ which is not equal to $x \upharpoonright n$.

[2]As a totally irrelevant aside, you can use this same trick with compactness to exhibit programs that seem to "perform exhaustive search" over all of 2^ω in finite time (as long as what you are searching for is a real that satisfies a computable property), a fact which constructive mathematicians seem to have rediscovered a few times—see [Esc07] for example.

Suppose this is not the case and let σ be such a node in T. Then by König's lemma we can find some $y \in [T]$ extending σ such that for all m, $\Phi(y)[m]$ does not disagree with $\Phi(x)$. However, we know that Φ is total on y and injective on $[T]$, hence $\Phi(y)$ and $\Phi(x)$ must disagree somewhere, so this is a contradiction. □

We can already note a couple of problems with trying to use this lemma in the strategy we outlined above. First, instead of getting that the computable injective function is above the identity, we only got that it is above the identity after being joined with T. In other words, instead of showing that $\Phi(x)$ computes x, the lemma above only showed that $\Phi(x) \oplus T$ computes x. Essentially, this is where some additional facts about the original function, f, must be invoked; we will return to this issue later. But second, even if we ignore the first problem, there is another issue. If T is an arbitrary perfect tree then the lemma doesn't tell us that the computable injective function is above the identity on a cone, only that it is above the identity on the elements of $[T]$. The solution is to restrict our attention to a special class of perfect trees, called "pointed perfect trees." So we will now explain what those are and how to work with them.

Using Determinacy to Find Pointed Perfect Trees

We will start with the definition of a **pointed perfect tree**.

Definition 2.4. *A **pointed perfect tree** is a perfect tree T such that every element of $[T]$ computes T.*

The idea of pointed perfect trees, made precise by the next lemma, is that if T is a pointed perfect tree then $[T]$ contains a representative from every Turing degree on the cone above T and thus acts like a "non Turing invariant" version of a cone.

Lemma 2.5. *If T is a pointed perfect tree and $x \geq_T T$ then there is some $\widetilde{x} \in [T]$ such that $\widetilde{x} \equiv_T x$.*

Proof. Think of x as a series of decisions about whether to go left or right. We get $\widetilde{x}$ by following these decisions to build a path through T—each time we need to make a decision about whether to go left or right (i.e. each time we can choose either and stay in T) we choose according to the first bit of x that we haven't already used.

It should be clear that if we form $\widetilde{x}$ in this way then $x \oplus T$ computes $\widetilde{x}$ and $\widetilde{x} \oplus T$ computes x. Since x and $\widetilde{x}$ both compute T (x by assumption and $\widetilde{x}$ because T is pointed), this implies that $x \equiv_T \widetilde{x}$. □

The following corollary demonstrates how this lemma is typically used.

Corollary 2.6. *Suppose $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function and T is a pointed perfect tree. If for all $x \in [T]$, $f(x)$ computes x, then f is above the identity in the Martin order.*

Next, we will prove some lemmas that we will use as tools to find pointed perfect trees with desirable properties. Our first lemma tells us that if f is a computable function then in many situations we can find a pointed perfect tree on which f is injective. The content of the lemma is essentially that if f is a computable function on a pointed perfect tree T then we can either refine the tree to make f constant or refine the tree to make f injective.

Lemma 2.7 (Tree thinning lemma)**.** *If T is a pointed perfect tree and Φ is a Turing functional which is total on every x in $[T]$ then one of the following must hold:*

- *We can "thin out" T to make Φ injective: there is a pointed perfect tree S such that $S \subseteq T$ and Φ is injective on T.*
- *Φ is constant on a large set: there is a node σ in T such that Φ is constant on $[T_\sigma]$.*

In particular, this implies that Φ is either constant or injective on a pointed perfect subtree of T.

Proof. The idea is basically the same as in Spector's construction of a minimal degree (i.e. Sacks forcing). Suppose that the second condition does not hold—i.e. that Φ is not constant on T_σ for any σ in T—and we will show how to find a pointed perfect tree $S \subseteq T$ on which Φ is injective.

We will define S in a series of stages. In stage 0, we let S_0 consist of just the empty sequence (i.e. the root node of T). In stage $n+1$ we have a finite tree $S_n \subseteq T$ which we want to extend to S_{n+1} in a way that makes sure that every leaf in S_n has two incompatible extensions in S_{n+1} and Φ is injective on the leaves of S_{n+1}. This is actually pretty straightforward to do: for each leaf σ of S_n we know that Φ is not constant on $[T_\sigma]$ so we can find descendants τ_1 and τ_2 of σ in T and an m such that

$$\Phi(\tau_1)[m] \text{ disagrees with } \Phi(\tau_2)[m]$$

(i.e. there is a place where they both converge and are not equal). Put these τ_1 and τ_2, along with all their ancestors, into S_{n+1}.

Now define S as the union of all the S_n's. It is clear that if we construct S in this way then S is a perfect tree, $S \subseteq T$, and Φ is injective on S. To see that S is pointed, just note that the above process was computable in T and so $S \leq_T T$. Since $[S] \subseteq [T]$ and each element of $[T]$ computes T, we have that each element of $[S]$ computes T and hence also computes S. □

Our next lemma tells us that we can use determinacy to find suitable pointed perfect trees—more precisely, that if some property of reals is not false for every real on a cone then it is true for all paths through some pointed perfect tree. Before we can state this lemma formally, we need to give one more definition (which is just a version of 1.18 for sets which are not Turing invariant).

Definition 2.8. *A subset A of 2^ω is **cofinal** if for all $x \in 2^\omega$ there is some $y \geq_T x$ such that $y \in A$. Note that A is* not *required to be Turing invariant.*

Lemma 2.9 (AD; Martin; [MSS16] lemma 3.5)**.** *Suppose $A \subseteq 2^\omega$ is cofinal in the Turing degrees. Then there is a pointed perfect tree, T, such that $[T] \subseteq A$.*

We will often need the following souped-up version of this lemma.

Lemma 2.10 (AD; Martin; [MSS16] lemma 3.5)**.** *Suppose $A \subseteq 2^\omega$ is cofinal in the Turing degrees and h is a function on A with countable range. Then there is a pointed perfect tree, T, such that $[T] \subseteq A$ and h is constant on $[T]$.*

The above two lemmas are widely used in works on Martin's conjecture as a replacement for Martin's cone theorem when dealing with non-Turing-invariant sets. During the research presented in this thesis, we discovered an easy, but surprisingly useful, consequence of Lemma 2.10. To the best of our knowledge, it has not been explicitly stated anywhere before. We should point out that the argument in this lemma is not new, but rather it is a distillation of an argument that appeared in an ad-hoc way in a number of proofs in prior work on Martin's conjecture and in our own early drafts of this work.

Lemma 2.11 (AD; Computable uniformization lemma)**.** *Suppose R is a binary relation on 2^ω such that both of the following hold.*

- *The domain of R is cofinal: for all z there is some $x \geq_T z$ and y such that $(x, y) \in R$*
- *R is a subset of Turing reducibility: for every $(x, y) \in R$, $x \geq_T y$.*

Then there is a pointed perfect tree T and a Turing functional Φ such that for every $x \in [T]$, $\Phi(x)$ is total and $(x, \Phi(x)) \in R$. In other words, Φ is a computable choice function for R on $[T]$.

Proof. For each x in the domain of R there is some e such that $\Phi_e(x)$ is total and $R(x, \Phi_e(x))$ holds. Let e_x denote the smallest such e. By determinacy (in the form of lemma 2.10), there is a pointed perfect tree T on which e_x is constant. Let e be this constant value. Then T and Φ_e satisfy the conclusion of the lemma. □

Later, we will need the following corollary of this lemma, which also shows how it is typically used. The corollary says that any increasing function can be inverted by a computable function on a pointed perfect tree.

Corollary 2.12. *If $f \colon 2^\omega \to 2^\omega$ is a function such that $f(x) \geq_T x$ for all x then there is a pointed perfect tree, T, and a Turing functional, Φ, such that Φ is a right inverse for f on $[T]$. That is, for each $x \in [T]$, $\Phi(x)$ is total and $f(\Phi(x)) = x$. Note that f is not required to be Turing invariant.*

Proof. Let R be the binary function defined as follows.

$$R(x, y) \iff x = f(y).$$

Applying Lemma 2.11 to this relation gives us what we want. To show that we can apply the lemma, we need to check that R is a subset of Turing reducibility and that its domain is cofinal. The former is just a consequence of the fact that $f(y) \geq_T y$ for all y. For the latter, consider any $z \in 2^\omega$. We need to show that there is some $x \geq_T z$ which is in the domain of R. For this, we can just take $x = f(z)$. □

Overview of the Framework and Main Obstacles to Using It

Now that we have provided all the major pieces of the general strategy outlined at the beginning of this section, let's provide a brief recap of how they fit together. A typical proof using this strategy looks something like this: We start with a Turing invariant function, f. We first use the computable uniformization lemma to find a computable function g which is below f, at least on a pointed perfect tree. We then use the tree thinning lemma to refine this pointed perfect tree to one on which g is injective. By Lemma 2.3, this implies that up to joining with a constant (namely the pointed perfect tree itself), g is above the identity on a cone and hence so is f. We then finish by showing that f is eventually above this constant.

There are usually two obstacles to getting this to work. The first is the problem of how to ensure that we can use the tree thinning lemma—i.e. how to ensure that g is not constant on a pointed perfect tree. This is typically dealt with by enforcing some kind of connection between g and f. For example, in the case of regressive functions on the hyperarithmetic degrees, f and g are required to be hyp-equivalent. In our second proof of part 1 of Martin's conjecture for measure preserving functions, g is required to preserve a certain ordinal invariant for Turing degrees that is built using f (we will discuss ordinal invariants in the next section). And sometimes if we choose g in a clever enough way, it is already guaranteed to be injective and we can skip the tree thinning lemma altogether.

The second obstacle is something we have already mentioned: how can we ensure that f eventually gets above every constant? The solution is typically to use some additional facts about f. Sometimes this is easy: in the case of measure preserving functions, it's basically just the definition of measure preserving. And other times this is quite hard and constitutes a large chunk of the proof: in the case of regressive functions on the Turing degrees or hyperarithmetic degrees, it requires a complicated coding argument and in the case of order preserving functions on the Turing degrees it requires invoking the basis theorem which we present in section 2.3 (which itself is an evolved version of the coding argument used for regressive functions on the hyperarithmetic degrees).

2.2 Ordinal Invariants of Turing Degrees

In the previous section we introduced a general strategy for proving instances of part 1 of Martin's conjecture. A key part of this strategy consisted of the following: given a function

f that you want to show is above the identity, somehow produce a computable function g which is below f and which can be made injective on a pointed perfect tree. We also saw that we can always find a pointed perfect tree on which g is injective as long as g is not constant on a cone and noted that making sure g is not constant on a cone is one of the main difficulties of implementing the strategy. In this section, we will introduce a tool to help with this difficulty. The tool is the notion of an **ordinal invariant** of Turing degrees, which is a function that assigns an ordinal to every Turing degree. The idea is that if α is an ordinal invariant and g is a computable function such that $\alpha(\deg_T(g(x))) = \alpha(\deg_T(x))$ for all x, then as long as α is not constant on a cone, neither is g.

This will be used in Chapter 5 to give an alternate proof of part 1 of Martin's conjecture for measure preserving functions and in Chapter 6 to give one of several proofs of part 1 of Martin's conjecture for order preserving functions. We also believe that the concept of ordinal invariants of Turing degrees may find further use in work on Martin's conjecture. One piece of evidence in favor of this is that a similar notion was used by Peng and Yu in their paper on proving countable choice for sets of reals from Turing determinacy [PY20].

So what is an ordinal invariant? As we mentioned, it's basically just a function from Turing degrees to ordinals. However, we will officially define them as Turing invariant functions from reals to ordinals. This is just a matter of notational convenience[3] and makes no material difference anywhere.

Definition 2.13. *An ordinal invariant is a Turing-invariant function $\alpha\colon 2^\omega \to \mathit{Ord}$ (where Ord denotes the class of ordinals).*

The prototypical example of an ordinal invariant is $x \mapsto \omega_1^x$, i.e. the function which maps a real to the least ordinal which the real does not compute any presentation of. In fact, this example is actually part of a much larger class of examples from descriptive set theory. For many pointclasses, $\mathbf{\Gamma}$, the function that maps x to the prewellordering number of $\Gamma(x)$ is an ordinal invariant. The function $x \mapsto \omega_1^x$ is simply an instance of this example where the pointclass $\mathbf{\Gamma}$ is $\mathbf{\Delta}^1_1$.

At first it may seem that the notion of an ordinal invariant is much too general to say anything meaningful about or to be very useful, but we will now see that using determinacy there are several nontrivial things that we can prove about them. None of these facts will be used anywhere in the rest of this thesis, but we include them because they give a clearer picture of what ordinal invariants can look like and how determinacy can used to reason about them.

The first of these facts is due to Martin and is an application of Σ^1_1-bounding. The rest are essentially folklore.

Theorem 2.14 ($\mathsf{ZF} + \mathsf{AD}$; Martin)**.** *If α is an ordinal invariant such that $\alpha(x) < \omega_1^x$ then α is constant on a cone.*

[3] We will be using them in situations where we are working with functions from reals to reals and it is easier to say things like $\alpha(x) = \alpha(f(x))$ than $\alpha(\deg_T(x)) = \alpha(\deg_T(f(x)))$.

Proposition 2.15 (ZF + AD)**.** *If α is an ordinal invariant, then α is order preserving on a cone—i.e. for all x and y in some cone*

$$x \geq_T y \implies \alpha(x) \geq \alpha(y).$$

Proof. Define $\alpha_{\min}\colon 2^\omega \to \mathrm{Ord}$ by

$$\alpha_{\min}(x) = \min\{\alpha(y) \mid y \geq_T x\}.$$

The claim that α is order preserving on a cone is equivalent to the claim that $\alpha(x) = \alpha_{\min}(x)$ on a cone.

For each x, there is some $y \geq_T x$ such that $\alpha(y) = \alpha_{\min}(x) = \alpha_{\min}(y)$. In other words, $\alpha(x) = \alpha_{\min}(x)$ holds cofinally. By determinacy, this means $\alpha(x) = \alpha_{\min}(x)$ on a cone. □

Proposition 2.16 (ZF + AD)**.** *The relation "$\alpha(x) \leq \beta(x)$ on a cone" prewellorders the ordinal invariants (i.e. it well orders them after quotienting by the relation "$\alpha(x) = \beta(x)$ on a cone").*

Proof. Let $\preceq$ be the binary relation on ordinal invariants defined by

$$\alpha \preceq \beta \iff \alpha(x) \leq \beta(x) \text{ on a cone.}$$

It is easy to check that $\preceq$ is transitive and reflexive. To see why $\preceq$ is linear, note that for any α and β, determinacy implies that either $\alpha(x) \leq \beta(x)$ on a cone or $\beta(x) < \alpha(x)$ on a cone.

To see why there are no strictly descending chains, suppose for contradiction that $\alpha_0 \succ \alpha_1 \succ \alpha_2 \succ \ldots$ is a strictly descending chain. Thus there are reals $x_0, x_1, x_2, \ldots$ such that $\alpha_n(x) > \alpha_{n+1}(x)$ on the cone above x_n. Let x be any upper bound for the x_n's. So we have

$$\alpha_0(x) > \alpha_1(x) > \alpha_2(x) > \ldots$$

In other words, a descending sequence of ordinals, which is a contradiction. □

We will see later in this thesis that this last proposition can be reinterpreted as saying that the ultrapower of the ordinals by the Martin measure is wellfounded (which is a consequence of the countable completeness of the Martin measure).

2.3 A Basis Theorem for Perfect Sets

In this section, we will prove a basis theorem for perfect sets, which we will use to prove that order preserving functions on the Turing degrees are either constant on a cone or measure preserving. We will also use it to prove a result about embedding partial orders into the Turing degrees. The theorem was inspired by, and is a strengthening of, a theorem proved by Groszek and Slaman in [GS98], which we will state next.

Definition 2.17. *Suppose that A is a perfect subset of 2^ω and $x \in A$. Say that x is* ***eventually constant*** *in A if there is some n such that*

$$\forall y \in A\,(x \upharpoonright n = y \upharpoonright n \implies x < y) \quad \text{or} \quad \forall y \in A\,(x \upharpoonright n = y \upharpoonright n \implies y < x)$$

where the ordering is the usual lexicographic ordering on 2^ω.

If you think of A is the set of branches through a perfect tree, this is saying that x eventually either always goes to the left or always goes to the right in the tree.

Theorem 2.18 (Groszek and Slaman [GS98] lemma 2.2)**.** *Suppose that A is a perfect subset of 2^ω, B is a countable dense subset of A which contains no element which is eventually constant in A, and $\langle c_i \rangle_{i\in\mathbb{N}}$ is a countable sequence which contains every element of B. Then for every x there are some y_0 and y_1 in A such that*

$$\left(\bigoplus_{i\in\mathbb{N}} c_i\right) \oplus y_0 \oplus y_1 \geq_T x.$$

The main shortcoming of Groszek and Slaman's theorem for our purposes is that to compute x you need to be able to compute the countable sequence $\langle c_i \rangle_{i\in\omega}$, but in the situation where we would like to use the theorem we can only be assured of having some real which computes every element of the sequence, but does not necessarily compute the sequence itself (i.e. it may compute the sequence in a non-uniform way). The basis theorem we prove below was formulated to fix this problem.

To prove our strengthened version of the basis theorem, we had to find a proof that is somewhat different from Groszek and Slaman's (and which is essentially a souped-up version of the coding argument we use for Martin's conjecture for regressive functions on the hyperarithmetic degrees in section 4.1). This proof also allows us to get rid of the requirement that no element of $\langle c_i \rangle$ is eventually constant in A (though this is mostly just a cosmetic improvement). Roughly speaking, here's what's different about our proof. In Groszek and Slaman's proof, they start with a real x which they want to code using two elements of A. To do so, they code the bits of A into the sequence of decisions about whether to turn left or right in the tree whose branches are A. In our proof, we instead essentially code the bits of x into the Kolmogorov complexity of initial segments of the elements of A.

Theorem 2.19. *Suppose that A is a perfect subset of 2^ω, B is a countable dense subset of A, and c is a real such that $b \leq_T c$ for each $b \in B$. Then for every x there are some y_0, y_1, y_2, y_3 in A such that*

$$c \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3 \geq_T x.$$

Remark. The proof of this theorem is the kind of thing that is not that hard to explain on a blackboard in an in-person conversation, but which looks quite complicated when all the details are written down. In the proof below, we have tried as best we can to explain the idea of the construction without getting lost in the messy details. If we have succeeded, then the construction should not actually seem so complicated. If we have failed then we hope the reader will forgive us.

Proof. The basic idea here is to build up $y_0, \ldots, y_3$ by finite extensions and on each step code one more bit of x. To use c together with $y_0, \ldots, y_3$ to compute x, we then have to try to decode the results of this coding process: figure out what happened on each step and recover the bits of x as a consequence.

Perhaps you could imagine that we have two rooms which are completely separated from each other. In the first room, someone—let's call them **the coder**—is given the real x (and whatever other information they need) and tasked with building the y_i's one bit at a time. In the other room, the coder's friend—let's call them **the decoder**—is given c and then receives the bits of the y_i's one at a time, and needs to reconstruct what the coder was doing in the first room. So what the coder needs to accomplish is to give the decoder enough information to reconstruct what they did. Note, by the way, that the process followed by the decoder needs to be computable, but there is no such requirement on the coder.

There is also one more constraint: the coder needs to end up building elements of the set A. They can accomplish this by making sure that on each step, the portions of the y_i's that they have built so far are each consistent with some element of B, since that will ensure that each y_i eventually formed is the limit of a sequence of elements of A.

It is the most natural to describe how these two processes work together. That is, describe what the coder is doing on a single step of the process, and at the same time, describe what the decoder is doing on the same step. In particular, we will assume that the decoder has so far reconstructed all the steps correctly and see how they can also reconstruct the current step correctly. We will actually see, by the way, that the decoder doesn't need to be completely correct in their reconstruction—as long as they are mostly correct, the coder can give them enough information to correct their mistakes and keep going. We will now try to describe what happens in a single step in both processes.

Suppose the coder has just finished the n^{th} step of the coding process. In other words, they have formed finite initial segments $y_0^n, y_1^n, y_2^n, y_3^n$ of y_0, y_1, y_2, y_3, respectively. And to make sure that the reals being built will end up in A, they have also picked elements $b_0^n, b_1^n, b_2^n, b_3^n$ of B such that each y_i^n is actually an initial segment of b_i^n. They now want to code the $(n+1)^{\text{th}}$ bit of x by extending each of $y_0^n, y_1^n, y_2^n, y_3^n$ by a finite amount, making sure each one is an initial segment of some element of B (though perhaps not the same element as at the end of the n^{th} step), while also giving the decoder enough information to recover what happened on this step.

Let's now consider what things look like for the decoder. After the end of the n^{th} step, they have a guess about the current status of the coding process. In other words, they have finite strings $\widetilde{y_0^n}, \widetilde{y_1^n}, \widetilde{y_2^n}, \widetilde{y_3^n}$ which constitute their guess about $y_0^n, y_1^n, y_2^n, y_3^n$, respectively.

They also know that these finite strings is each the initial segment of *some* element of B. But since the decoding process is supposed to be computable, we should not imagine that the decoder can actually store elements of B in their head—we should think of their memory as being finite (though unbounded) and thus unable to hold real numbers. However, they *do* know that each element of B is computable from c. So in some sense, the decoder *can* make guesses about this part of the coding process also, as long as those guesses take the form of programs.

Concretely, that means we will assume that Person 2 has indices for programs $e_0^n, e_1^n, e_2^n, e_3^n$, which you should think of as their guesses for the reals $b_0^n, b_1^n, b_2^n, b_3^n$ in the following way: they are guessing that b_0^n is $\Phi_{e_0^n}(c)$, that b_1^n is $\Phi_{e_1^n}(c)$, and so on. The key to the proof is explaining how the coder can give the decoder enough information at each step to determine the correct guesses for that step.

Now we are finally ready to start describing what actually happens on step $n+1$. We will make the assumption that at the beginning of this step, the decoder's guesses $\widetilde{y_0^n}, \widetilde{y_1^n}, \widetilde{y_2^n}, \widetilde{y_3^n}$ are all correct and so are at least two of their guesses $e_0^n, e_1^n, e_2^n, e_3^n$—and further, that they know which two are correct (it won't matter if the other two are correct or not, as long as there are two they know they can trust). For convenience, let's assume that they know that e_0^n and e_1^n are correct. What I mean is that they know that $\Phi_{e_0^n}(c)$ and $\Phi_{e_1^n}(c)$ are both total and are equal to b_0^n and b_1^n, respectively.

Let's describe what happens on the decoding side first. The decoder will begin to look at the bits of y_0 until they find a place where y_0 disagrees with $\Phi_{e_0^n}(c)$—in other words, a place where y_0 disagrees with their guess for b_0^n. Let l_0 be the first position where this occurs. They next do the same thing for y_1 and $\Phi_{e_1^n}(c)$. Let l_1 be the first place where these disagree.

The decoder will now try to determine the correct guesses e_2^{n+1} and e_3^{n+1}. They know there is a chance that b_2^{n+1} and b_3^{n+1} are not the same as b_2^n and b_3^n. They want to determine what they are. And they will assume that the coder (who is their good friend) has given them enough information to do so, in the form of l_0 and l_1.

Here's how the decoder uses l_0 and l_1. Recall that they are trying to find programs e_2^{n+1} and e_3^{n+1} such that $b_2^{n+1} = \Phi_{e_2^{n+1}}(c)$ and $b_3^{n+1} = \Phi_{e_3^{n+1}}(c)$. Let's describe how they try to figure out e_2^{n+1}. The idea is that they will look for the first program whose output when run with oracle c matches the first l_0 bits of y_2. Of course, some of these programs may take a very long time to run and some may even run forever[4]. So the decoder will only allow each program to run for l_1 steps. To recap: the decoder sets e_2^{n+1} to be the least index e such that $\Phi_e(c)[l_1]$ converges on all inputs less than l_0, and also agrees with y_2 on all of these inputs. And e_3^{n+1} is found in exactly the same way.

How does the decoder extract the $(n+1)^{\text{th}}$ bit of x? After all, that was the ultimate point of this whole thing. It's actually very easy: they simply compare e_2^{n+1} and e_3^{n+1}. If e_2^{n+1} is smaller then the $(n+1)^{\text{th}}$ bit of x was a 0. Otherwise, it was a 1.

And how does the decoder determine their other guesses, namely $\widetilde{y_0^{n+1}}, \ldots, \widetilde{y_3^{n+1}}$ and

[4]This is the fundamental difference between our theorem and the theorem of Groszek and Slaman, by the way: our theorem needs to deal with programs that do not converge.

e_0^{n+1}, e_1^{n+1}? That's quite simple as well. To find the $\widetilde{y_i^{n+1}}$'s, the decoder can just set:

$$\begin{aligned}
\widetilde{y_0^{n+1}} &:= y_0 \restriction l_0 \\
\widetilde{y_1^{n+1}} &:= y_1 \restriction l_1 \\
\widetilde{y_2^{n+1}} &:= y_2 \restriction l_0 \\
\widetilde{y_3^{n+1}} &:= y_3 \restriction l_0.
\end{aligned}$$

As for e_0^{n+1} and e_1^{n+1}, it doesn't really matter what the decoder does. On the next step of the decoding process, they will assume that e_2^{n+1} and e_3^{n+1} are correct. And if you have been following along closely, you will see that the decoder never had to use the two guesses that they were not confident about, so it doesn't matter what they are.

That completes our description of the decoding process. If we have done a good job, then the reader should be able to fill in all the details about the coding process for themselves. But we will describe them here anyways for the sake of completeness.

At the beginning of this step, the coder knows exactly what the decoder's guesses are (after all, they have access to all the same information as the decoder, so they can figure out exactly what the decoder did on each previous step). The coder also knows the next bit of x that they need to code.

Suppose for convenience that the next bit is a 0. The coder will begin by choosing some value which they will make sure is the decoder's guess for e_2^{n+1} on the next step. One choice that works well enough is to simply let e_2^{n+1} be the first e such that $\Phi_e(c)$ is total and equal to b_2^n. In this case, the coder will set b_2^{n+1} to be b_2^n (i.e. they will not change b_2^n on this step).

Next, the coder wants to choose some value and ensure that this value is the decoder's guess for e_3^{n+1} on the next step. In particular, since bit $n+1$ of x is a 0, the coder wants to make sure that e_3^{n+1} is larger than e_2^{n+1}. It is also easy enough to accomplish this. The coder simply looks for the least e greater than e_2^{n+1} such that $\Phi_e(c)$ is total and equal to some element of B which has y_3^n as an initial segment. Since B has infinitely many elements which have y_3^n as an initial segment, it is no problem to find such an e and such an element of B. The coder then sets b_3^{n+1} to be whatever element of B they used.

Now the coder needs to make sure that the values l_0 and l_1 that the decoder recovers are large enough to make the correct guess about e_2^{n+1} and e_3^{n+1}. To do this, the coder defines l_0 to be large enough such that e_2^{n+1} and e_3^{n+1} are the first indices e and e' for which $\Phi_e(c) \restriction l_0$ and $\Phi_{e'}(c) \restriction l_0$ agree with b_2^{n+1} and b_3^{n+1}, respectively. The coder can then choose the second number, l_1, to be large enough to give $\Phi_{e_2^{n+1}}(c)$ and $\Phi_{e_3^{n+1}}(c)$ to both converge on all of the first l_0 inputs.

Next, the coder can choose b_0^{n+1} to be an element of B which has y_0^n as an initial segment and agrees with b_0^n on the first l_0 bits, but which eventually disagrees with b_0^n. They should retroactively increase the value of l_0 to the first position of disagreement between b_0^n and b_0^{n+1}, which may necessitate increasing l_1 as well (so that the programs e_2^{n+1} and e_3^{n+1} are given enough time to converge on the first l_0 inputs).

The coder can now choose b_1^{n+1} in a similar manner, to be some element of B which has y_1^n as an initial segment and agrees with b_1^n on the first n bits, but eventually disagrees. They should then retroactively increase l_1 to this first position of disagreement. Notice that increasing l_0 and l_1 in this way is harmless.

Now the coder defines

$$\begin{aligned} y_0^{n+1} &:= b_0^{n+1} \upharpoonright l_0 \\ y_1^{n+1} &:= b_1^{n+1} \upharpoonright l_1 \\ y_2^{n+1} &:= b_2^{n+1} \upharpoonright l_0 \\ y_3^{n+1} &:= b_3^{n+1} \upharpoonright l_0. \end{aligned}$$

to make sure they match the decoder's guesses.

That completes our description of the coding process. The main thing to notice is that we have managed to describe actions for the coder which guarantee that if the decoder follows the decoding process described above, then all of their guesses at the end of step $n+1$ are correct—or at least all of those guesses which we were assuming inductively to be correct. And moreover, the decoder has correctly extracted the $(n+1)^{\text{th}}$ bit of x. □

A Useful Corollary

We now prove a corollary of theorem 2.19 that will allow us to give a more streamlined proof that every order preserving function is either constant on a cone or measure preserving.

Definition 2.20. *A subset A of 2^ω is **countably directed for Turing reducibility** if for all countable subsets $B \subset A$, A contains an upper bound for the Turing degrees of the elements of B—i.e. there is some $x \in A$ such that for all $y \in B$, $y \leq_T x$.*

Corollary 2.21. *If A is a subset of 2^ω which is countably directed for Turing reducibility and contains a perfect set then A is cofinal (see definition 2.8).*

Proof. We start with a set A which contains a perfect set and which is countably directed. Let P be a perfect set contained in A and let B be a countable dense subset of P. Since A is countably directed, we can find some c in A which is an upper bound for B.

Our goal is to show that A is cofinal. So we start with an arbitrary x and we want to find some y in A that computes x. By the basis theorem we just proved, we can find reals y_0, y_1, y_2, y_3 in P (and therefore also in A) such that

$$c \oplus y_0 \oplus y_1 \oplus y_2 \oplus y_3 \geq_T x.$$

Since A is countably directed, we can find an upper bound for c, y_0, y_1, y_2, y_3 in A. This upper bound obviously computes x, so it is the y we are after. □

2.4 The Solecki Dichotomy

In this section, we will introduce the Solecki dichotomy. This is a powerful dichotomy theorem in descriptive set theory originally due to Solecki and later expanded by Zapletal, Sabok and others. We will eventually use it to give an alternative proof of part 1 of Martin's conjecture for order preserving functions (following an idea due to Kihara, who was only missing the fact that order preserving functions are measure preserving to complete the proof).

The basic idea of the Solecki dichotomy is that it tells us that every function which cannot be written as a countable union of partial continuous functions is at least as complicated as the Turing jump. Since the Solecki dichotomy comes from a different tradition than computability theory, it is typically stated in terms of something called the Pawlikowski function rather than in terms of the Turing jump. We will first give the traditional statement of the Solecki dichotomy in terms of the Pawlikowski function and then show that it implies a similar statement about the Turing jump, which is what we will actually use later in this thesis. We will start with a few definitions.

Definition 2.22. *Suppose X and Y are topological spaces and $f \colon X \to Y$ is any function. Then f is **σ-continuous** if it can be written as a countable union of continuous functions. In other words, if there are sets $A_0, A_1, A_2, \dots$ such that $\bigcup_{n \in \mathbb{N}} A_n = X$ and for each n, $f \restriction A_n$ is continuous*[5].

Definition 2.23. *If X and Y are topological spaces then a continuous, injective function $\varphi \colon X \to Y$ is a **topological embedding** of X into Y if it is a homeomorphism between X and its range.*

Definition 2.24. *If X_0, X_1, Y_0, Y_1 are topological spaces and $f \colon X_0 \to Y_0$ and $g \colon X_1 \to Y_1$ are any functions, we say that there is a **topological embedding** of f into g if there are topological embeddings $\varphi \colon X_0 \to X_1$ and $\psi : Y_0 \to Y_1$ such that $\psi \circ f = g \circ \varphi$. In other words, such that the following diagram commutes.*

$$\begin{array}{ccc} Y_0 & \xrightarrow{\psi} & Y_1 \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} \\ X_0 & \xrightarrow{\varphi} & X_1 \end{array}$$

The Solecki dichotomy says that there is a single non-σ-continuous[6] function on a Polish space which embeds into every other non-σ continuous function. This function is sometimes known as the Pawlikowski function, but as we shall see, it is essentially just the Turing jump in disguise.

[5]Continuous in terms of the induced topology on A_n, which is not necessarily the same as the restriction to A_n of a continuous function on X.

[6]One wonders how such a function should be called. σ-discontinuous? Dis-σ-continuous?

The Pawlikowski Function

The Pawlikowski function is usually defined as a function $(\omega+1)^\omega \to \omega^\omega$ where we make $(\omega+1)^\omega$ into a topological space by endowing $(\omega+1)$ with the order topology and then endowing $(\omega+1)^\omega$ with the product topology.

Definition 2.25. *The Pawlikowski function is the function* $P\colon (\omega+1)^\omega \to \omega^\omega$ *defined by*

$$(P(x))(n) = \begin{cases} x(n)+1 & \textit{if } x(n) < \omega \\ 0 & \textit{if } x(n) = \omega. \end{cases}$$

It may not be initially apparent why this function is a version of the Turing jump, or what's even going on with it at all, so let's explain. It is best to start by explaining how to think of the topology on $(\omega+1)^\omega$ and how it compares to the topology on ω^ω.

Recall that in ω^ω, a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ converges to some x only if the x_n's match x on a longer and longer initial segments as n gets larger. In $(\omega+1)^\omega$, things are a bit different. If the k^{th} entry of x is ω (rather than a natural number) then a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ can converge to x even if none of the x_n's match x at position k, as long as the k^{th} entries of the x_n's are getting arbitrarily large as n goes to infinity. This discrepancy between how convergence works in $(\omega+1)^\omega$ and ω^ω is also the reason that the Pawlikowski function is not continuous.

Example 2.26. The following sequence of x_n's converges in $(\omega+1)^\omega$ to the x shown below.

$$\begin{aligned} x_0 &= (0,0,0,0,0,0,0,\ldots) \\ x_1 &= (1,0,1,1,1,1,1,\ldots) \\ x_2 &= (2,0,1,2,2,2,2,\ldots) \\ x_3 &= (3,0,1,2,3,3,3,\ldots) \\ x_4 &= (4,0,1,2,3,4,4,\ldots) \\ x_5 &= (5,0,1,2,3,4,5,\ldots) \\ &\vdots \\ x &= (\omega,0,1,2,3,4,5,\ldots) \end{aligned}$$

Note that even though all then entries of the x_n's past the first entry eventually "stabilize" and become equal to the corresponding entry of x, the first entries of the x_n's are never equal to the first entry of x, but just form a sequence of larger and larger natural numbers.

Also note that this sequences witnesses that the Pawlikowski function is not continuous. Letting P denote the Pawlikowski function, $P(x_n) = x_n$ for each n but the x_n's do not converge in the topology on ω^ω.

We can think of the Pawlikowski function as a version of the jump as follows. The basic idea is that, given an element of $(\omega+1)^\omega$, we should think of its entries as telling us how many steps it takes different programs to converge, with an entry of ω meaning that the program does not converge. The Pawlikowski function then tells us which of the programs fail to converge by replacing the ω with a 0. The next proposition makes this idea precise. Before proving it, however, we need to specify exactly what we mean by the "Turing jump."

Definition 2.27. *Let $\Phi_0, \Phi_1, \Phi_2, \ldots$ denote some standard computable enumeration of Turing functionals (i.e. Turing machines with an oracle tape or something like that). Let $J\colon 2^\omega \to 2^\omega$ denote the usual Turing jump which only records whether programs converge or not and let $J_\omega\colon 2^\omega \to \omega^\omega$ denote a modified version which also records how long programs take to converge. In other words,*

$$(J(x))(n) = \begin{cases} 0 & \textit{if } \Phi_n^x(n) \textit{ diverges} \\ 1 & \textit{if } \Phi_n^x(n) \textit{ converges} \end{cases}$$

$$(J_\omega(x))(n) = \begin{cases} 0 & \textit{if } \Phi_n^x(n) \textit{ diverges} \\ k & \textit{if } \Phi_n^x(n) \textit{ converges in exactly } k \textit{ steps.} \end{cases}$$

Here we have adopted the convention that any program takes at least 1 *step to converge.*

Hopefully it is clear that $J(x) \equiv_T J_\omega(x)$. Thus from the perspective of Martin's conjecture, it doesn't matter which one we use. However it does make some difference in terms of topology, so in the next proposition we will use J_ω.

Proposition 2.28. *Let P denote the Pawlikowski function and let J_ω denote the Turing jump as defined above. Then there is a topological embedding of J_ω into P.*

Proof. Define functions $\psi\colon 2^\omega \to (\omega+1)^\omega$ and $\varphi\colon \omega^\omega \to \omega^\omega$ as follows.

$$(\psi(x))(n) = \begin{cases} \omega & \text{if } \Phi_n^x(n) \text{ diverges} \\ k & \text{if } \Phi_n^x(n) \text{ converges in exactly } k \text{ steps} \end{cases}$$

$$(\varphi(x))(n) = \begin{cases} 0 & \text{if } x(n) = 0 \\ x(n)+1 & \text{otherwise.} \end{cases}$$

It is easy to see that for all $x \in 2^\omega$, $\varphi(J_\omega(x)) = P(\psi(x))$. It is also clear that φ is a topological embedding. So what remains is to show that ψ is a topological embedding. The main point is that ψ is continuous because of the special role of ω in the order topology on $\omega+1$.

First, let's show that ψ is continuous. Let $\langle x_i\rangle_{i\in\mathbb{N}}$ be any sequence of elements of 2^ω which converge to some $x \in 2^\omega$ and let n be any natural number. Note that if $\Phi_n^x(n)$ converges in k steps then there is some initial segment σ of x which is enough to witness this fact. Since x_m will have σ as an initial segment for all large enough m, the n^{th} entry of $\psi(x_m)$ will eventually always be k. On the other hand, if $\Phi_n^x(n)$ diverges then for each k there is an initial segment of x witnessing that $\Phi_n^x(n)$ takes more than k steps to converge and so for all large enough m, $\Phi_n^{x_m}(n)$ takes more than k steps to converge and so $(\psi(x_m))(n)$ must be larger than k (either ω or a natural number larger than k). Thus, in the topology on $(\omega+1)$, the sequence $\psi(x_0)(n), \psi(x_1)(n), \psi(x_2)(n), \ldots$ converges to $\psi(x)(n)$. Since this holds for each n, $\psi(x_0), \psi(x_1), \psi(x_2), \ldots$ converges to $\psi(x)$ and thus ψ is continuous.

Now we need to show that for each basic open subset of 2^ω is mapped by ψ to an open set in the relative topology on range(ψ). Since it is easy to see that ψ is injective, this is enough to show that ψ is a topological embedding. A basic open set in 2^ω just consists of all reals which have some fixed string, σ, as an initial segment. Note that for each bit of a real x, there is a number n such that $\Phi_n^x(n)$ converges if and only if that bit of x is 0. Using this, we can find an open subset of $(\omega+1)^\omega$ whose intersection with range(ψ) is exactly the image of those reals which have σ as an initial segment. □

The Dichotomy

Having defined the Pawlikowski function, we can now state the Solecki dichotomy.

Theorem 2.29 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$; [Zap04] Corollary 2.3.48)**.** *If X and Y are Polish spaces and $f\colon X \to Y$ is any function then either f is σ-continuous or there is a topological embedding of the Pawlikowski function into f.*

This theorem was first proved for Baire class one functions by Solecki [Sol98], extended to all Borel functions (and to all functions under AD) in Zapletal's book *Descriptive Set Theory and Definable Forcing* [Zap04] and extended by Sabok and Pawlikowski to all functions with analytic graphs [PS12].

For our application of the Solecki dichotomy to Martin's conjecture, we wish to use a slightly different form of the Solecki dichotomy which uses a different notion of reduction than "topological embedding" and which is stated in terms of the Turing jump rather than the Pawlikowski function. We will now state this version of the Solecki dichotomy and prove that it is implied by the version we stated above.

Definition 2.30. *If X_0, X_1, Y_0, Y_1 are Polish spaces and $f\colon X_0 \to Y_0$ and $g\colon X_1 \to Y_1$ are any functions then we say f is **continuous strong Weihrauch reducible** to g if there are partial continuous functions[7] $\varphi\colon X_0 \to X_1$ and $\psi\colon Y_1 \to X_1$ such that for all $x \in X_0$, x is in the domain of φ, $g(\varphi(x))$ is in the domain of ψ and $\psi(g(\varphi(x))) = f(x)$. In other words, the following diagram commutes.*

$$\begin{array}{ccc} Y_0 & \xleftarrow{\psi} & Y_1 \\ \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} \\ X_0 & \xrightarrow{\varphi} & X_1 \end{array}$$

[7] I.e. partial functions which are continuous when restricted to their domains.

Theorem 2.31 (ZF + AD; Solecki dichotomy, modified version)**.** *If $f\colon 2^\omega \to 2^\omega$ is any function then either f is σ-continuous or J is continuous strong Weihrauch reducible to f.*

Proof. Let P denote the Pawlikowski function and let J and J_ω denote the two versions of the Turing jump as defined above.

By the regular Solecki dichotomy, either f is σ-continuous or there are topological embeddings φ_1 and ψ_1 such that $\psi_1 \circ P = f \circ \varphi_1$. In the former case, we are done, so let's assume we're in the latter case. By proposition 2.28, there are also topological embeddings φ_2 and ψ_2 such that $\psi_2 \circ J_\omega = P \circ \varphi_2$. Composing these two topological embeddings, we get $\psi_1 \circ \psi_2 \circ J = f \circ \varphi_1 \circ \varphi_2$, as shown in the diagram below.

$$\begin{array}{ccccc} \omega^\omega & \xrightarrow{\psi_2} & \omega^\omega & \xrightarrow{\psi_1} & 2^\omega \\ \uparrow{\scriptstyle J_\omega} & & \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle f} \\ 2^\omega & \xrightarrow{\varphi_2} & (\omega+1)^\omega & \xrightarrow{\varphi_1} & 2^\omega \end{array}$$

Since $\psi_1 \circ \psi_1$ is a topological embedding, its inverse is a partial continuous function. Thus J_ω is continuous strong Weihrauch reducible to f. Since it is easy to show that J is continuous strong Weihrauch reducible to J_ω, this shows that J is continuous strong Weihrauch reducible to f. □

2.5 Σ^1_1-bounding and Pseudo-Wellorders

To work with hyperarithmetic reducibility, we will need to make use of a few facts about computable linear orders and computable well-orders. Proofs can be found in [Sac90].

One of the most important facts about computable linear orders is the Σ^1_1-bounding theorem. Essentially it says that every Σ^1_1-definable collection of well-orders is bounded below a computable ordinal. The theorem comes in multiple flavors, depending on whether we are talking about sets of programs which compute presentations of linear orders, or real numbers which *are* presentations of linear orders and depending on whether the Σ^1_1 definition is boldface, lightface, or lightface relative to some fixed real. Below, we just state the two versions that we will need in this thesis.

Theorem 2.32. *Suppose that x is a real and A is a $\Sigma^1_1(x)$ definable set of codes for programs such that for every e in A, $\Phi_e(x)$ is a presentation of a well-order. Then there is some $\alpha < \omega_1^x$ which is greater than every ordinal with a presentation coded by an element of A.*

Theorem 2.33. *If A is a $\mathbf{\Sigma^1_1}$ definable set of presentations of well-orders then there is some $\alpha < \omega_1$ which is greater than every ordinal with a presentation in A.*

We will also need some ideas originally introduced by Harrison in [Har68].

Definition 2.34. *If x is a real and r is a real computable from x that codes a presentation of a linear order, then r is a **pseudo-well-order** relative to x if it is ill-founded but contains no infinite descending sequence which is hyperarithmetic in x.*

Lemma 2.35. *If r is a presentation of a linear order that is computable from a real x then the assertion "r has no infinite descending sequence which is hyperarithmetic in x" is equivalent to a $\Sigma^1_1(x)$ formula.*

Lemma 2.36. *If r is a pseudo-well-order relative to x and H is a jump hierarchy on r that starts with x then H computes every real which is hyperarithmetic in x.*

2.6 Minimal Upper Bounds

In this section we will review a few classical theorems of computability theory on the existence of upper bounds for sets of Turing degrees.

Minimal Upper Bounds for Countable Sets

First we will mention a theorem which states that every countable set of Turing degrees has a minimal upper bound—and in some cases has continuum many minimal upper bounds. This was first proved by Sacks [Sac63] using forcing with pointed perfect trees.

Definition 2.37. *Suppose $A \subset 2^\omega$ is countable. A* ***minimal upper bound*** *for A is a real x which computes every element of A and such that if $y \leq_T x$ also computes every element of A then $y \equiv_T x$.*

Theorem 2.38 (Sacks; [Sac63] Theorem 8.1)**.** *Every countable set of reals has a minimal upper bound in the Turing degrees.*

Upper Bounds of Subsets of Turing Independent Sets

We will now state another kind of upper bound theorem (which we will also prove since we are not aware of a proof in the literature which has the exact form we would like). This says that if we have a set of Turing degrees which are sufficiently independent of each other and represented by reals forming a nice enough set (specifically a perfect tree) then for each countable subset of the original set, we can find an upper bound in the Turing degrees which is not above any other element of the original set. We will use this in chapter 7 to prove a theorem about embedding partial orders into the Turing degrees. First we need to make a definition to help us state the theorem.

Definition 2.39. *A set $X \subset 2^\omega$ of reals is called a* ***Turing independent set*** *if no finite subset of X computes any other element of X—i.e. if $a_0, \ldots, a_n \in X$ and b is any element of X not equal to any a_i then $a_0 \oplus \ldots \oplus a_n$ does not compute b.*

The idea of the proof is originally due to Spector, who used it to show that every increasing sequence of Turing degrees has an exact pair of upper bounds (see Theorem 6.5.3 in Soare [Soa16]).

Theorem 2.40. *Suppose T is a perfect tree such that $[T]$ is Turing independent. Then every countable subset of $[T]$ has an upper bound in the Turing degrees which does not compute any other element of $[T]$.*

Proof. Suppose A is a countable subset of $[T]$ and $x_0, x_1, x_2, \ldots$ is an enumeration of the elements of A. To keep things simple, we'll say that repetitions are allowed in this enumeration, which allows us to treat the cases where A is finite or infinite in the same way.

Here's the idea of the proof. We will construct an element y of $2^{\omega\times\omega}$ such that column n of y consists of some finite string followed by x_n and this y will be the upper bound we are after. It is easy to see that any such y computes each element of A and so the bulk of the proof consists of showing that if we choose the finite strings in a sufficiently generic way then y does not compute any other element of $[T]$. The proof crucially depends on the fact that $[T]$ is Turing independent.

Formally, we will construct y in a series of stages. At the end of stage n we will have constructed a finite list of finite strings, $\sigma_1, \ldots, \sigma_k$ (where k may not be equal to n) and on stage $n+1$ we will add some more strings onto the end of this list. At the end, we will define column i of y to be $\sigma_i{}^\frown x_i$. The idea is that on stage $n+1$ we will ensure that if we run the n^{th} program with oracle y it either does not compute any element $[T]$ or it computes one of $x_1, \ldots, x_k$.

We will now explain how to complete one step of this construction. Suppose we have just completed stage n and our list of finite strings is $\sigma_0, \sigma_1, \ldots, \sigma_k$. Let us say that a finite string τ in $2^{<\omega\times<\omega}$ (i.e. a finite initial segment of an element of $2^{\omega\times\omega}$) *agrees with y so far* if for each $i \leq k$, column i of τ agrees with σ_i followed by x_i. In other words, τ is a possible initial segment of y given what we have built by the current stage. Let Φ denote the n^{th} Turing functional. We will use the notation Φ^a (where a is a real) to mean the partial function computed by Φ when given oracle a. If τ is a finite string, we will use Φ^τ to mean the partial function computed by Φ when given oracle τ (where the program halts without output when it tries to make a query outside the domain of τ and is allowed to run for at most $|\tau|$ steps on any input). We have four cases to consider.

Case 1: There is some finite string $\tau \in 2^{<\omega\times<\omega}$ which agrees with y so far such that Φ^τ cannot be extended to a path through T. In this case, extend the list $\sigma_0, \sigma_1, \ldots, \sigma_k$ to ensure that y is an extension of τ. This guarantees that Φ^y is not in $[T]$.

Case 2: There is some finite string τ which agrees with y so far and some $m \in \mathbb{N}$ such that for every extension τ' of τ which agrees with y so far, $\Phi^{\tau'}$ does not converge on input m. In this case, extend the list $\sigma_0, \sigma_1, \ldots, \sigma_k$ to ensure that y is an extension of τ. This guarantees that Φ^y is not total.

Case 3: For every finite string τ which agrees with y so far, Φ^τ is compatible with one of $x_0, x_1, \ldots, x_k$. In this case, do nothing; we are already guaranteed that Φ^y is either not total or is equal to one of $\sigma_0, \sigma_1, \ldots, \sigma_k$.

Case 4: None of the first three cases holds. We claim this case actually cannot happen. Let's explain why.

Because Case 3 does not hold, we can find some finite string τ which agrees with y so far and such that Φ^τ is not compatible with any of $x_0, x_1, \ldots, x_k$. Because Case 2 does not hold, for each m we can find some extension τ' of τ which is agrees with y so far and such that $\Phi(\tau')$ converges on input m; what's more, for any given finite string extending τ which agrees with y so far, we can require τ' to actually extend this string. And because Case 1 does not hold, $\Phi^{\tau'}$ is compatible with some element of $[T]$. We will use this to show that we can use $x_0, x_1, \ldots, x_k$ to compute some element of $[T]$ which is not equal to any of them. This contradicts the fact that $[T]$ is Turing independent.

Let's now describe how to use $x_0, x_1, \ldots, x_k$ to compute another element of $[T]$, thus finishing the proof. Form a sequence $\tau = \tau_0 \prec \tau_1 \prec \tau_2 \prec \ldots$ of finite strings which agree with y so far as follows. Given τ_m, look for some extension τ_{m+1} of τ_m which agrees with y so far and such that $\Phi^{\tau_{m+1}}$ converges on input m. We can always find such a string because Case 2 does not hold.

Let z be the real whose m^{th} bit is equal to $\Phi^{\tau_{m+1}}$ on input m. Note that the first m bits of z are equal to the first m bits of $\Phi^{\tau_{m+1}}$. Because Case 1 does not hold, this means that $z \upharpoonright m$ agrees with some element of $[T]$. Since $[T]$ is closed, this means that z is in $[T]$. And z is not equal to any of $x_0, x_1, \ldots, x_k$ because of our choice of τ.

The final point is that to carry out this whole process, we only need to be able to check when a finite string agrees with y so far. And if we know $x_0, x_1, \ldots, x_k$ then we can compute this, so the whole process is computable in $x_0 \oplus x_1 \oplus \ldots \oplus x_k$. $\square$

Chapter 3

Uniformly Invariant Functions

Note: The results in this chapter are joint with Vittorio Bard.

So far in this thesis we have discussed the statement of Martin's conjecture and laid out some of the tools we will use to approach it. All of that was a prelude to the main concern of the thesis: actually proving instances of Martin's conjecture. In this chapter we will properly begin our journey by proving some results on uniformly Turing invariant functions. Since some of the earliest results on Martin's conjecture were on this special case, it seems like a natural place to start. In fact, the first result we will prove in this chapter is an update on the very first published proof of any special case of Martin's conjecture—Lachlan's proof of part 2 of Martin's conjecture for uniformly Turing invariant r.e. operators.

Recall that a Turing invariant function $f\colon 2^\omega \to 2^\omega$ is called uniformly Turing invariant if there is a function $u\colon \mathbb{N}^2 \to \mathbb{N}^2$ such that for all $x, y \in 2^\omega$ and $i, j \in \mathbb{N}$, if $x \equiv_T y$ via (i, j) (meaning that $\Phi_i(x) = y$ and $\Phi_j(y) = x$) then $f(x) \equiv_T f(y)$ via $u(i, j)$.

As we mentioned above, the first result on uniformly invariant functions was Lachlan's proof that part 2 of Martin's conjecture holds for uniformly Turing invariant r.e. operators.

Theorem 3.1 (Lachlan [Lac75])**.** *Suppose W is an r.e. operator such that $W^x \geq_T x$ for all x and $x \mapsto W^x$ is a uniformly Turing invariant function. Then either $W^x \equiv_T x$ on a cone or $W^x \equiv_T x'$ on a cone.*

This theorem also gave a partial answer to a question first asked by Sacks. Sacks' question was this: is there a Turing invariant r.e. operator W such that for every real x, the Turing degree of W^x is strictly in-between the Turing degrees of x and x'? Note that the Friedberg-Muchnik construction of an intermediate r.e. degree (as well as every other known construction) does give an r.e. operator W such that for every x, W^x is strictly inbetween x and x', but this W is *not* Turing invariant. Thus Sacks' question asks whether it is possible to improve the Friedberg-Muchnik construction so that the degree of W^x only depends on the degree of x. Lachlan's theorem says that this is impossible if W is required to be uniformly invariant.

Subsequently, Steel extended Lachlan's result to all uniformly Turing invariant functions [Ste82] and Slaman and Steel finished the job by proving part 1 of Martin's conjecture for all uniformly Turing invariant functions [SS88].

Theorem 3.2 (ZF + AD; Steel [Ste82])**.** *Part 2 of Martin's conjecture holds when restricted to the uniformly Turing invariant functions—i.e. uniformly Turing invariant functions which are above the identity on a cone are prewellordered by the Martin order and the successor in this prewellorder is the jump.*

Theorem 3.3 (ZF + AD; Slaman-Steel [SS88])**.** *Part 1 of Martin's conjecture holds when restricted to the uniformly Turing invariant functions—i.e. if $f\colon 2^\omega \to 2^\omega$ is a uniformly Turing invariant function then either f is constant on a cone or f is above the identity on a cone.*

Later, Becker gave a thorough analysis of all uniformly Turing invariant functions, showing that they all arise as "pointclass jumps" of fairly well-behaved pointclasses [Bec88]. This analysis implies, among other things, that every uniformly Turing invariant function is order preserving (which is not known to follow directly from Martin's conjecture in general).

For many years, it seemed as though these results had completely finished the story of Martin's conjecture for uniformly invariant functions. However, recent work by the Vittorio Bard has given a new way to understand some of these old results [Bar20]. In particular, it turns out that Slaman and Steel's result—that part 1 of Martin's conjecture holds for all uniformly Turing invariant functions—can be broken into two parts. First, a "local" version of their result: there is a version of part 1 of Martin's conjecture that holds for uniformly Turing invariant functions defined only on the reals in a single Turing degree rather than on all reals. And second, a proof that the "local" result implies the "global" one. The first of these two parts is provable in ZF (or even much weaker systems) and does not require any determinacy. And the second is very easy to prove using determinacy. One interesting consequence of this is that it shows that the full axiom of determinacy is not needed to prove Slaman and Steel's result; all that is needed is Turing determinacy.

All of this raises an obvious question. If there is a local version of part 1 of Martin's conjecture for uniformly invariant functions, is there also a local version of part 2? The main focus of this chapter is addressing that question.

So far, we have not been able to actually answer the question, but we will present some partial progress towards doing so. First, we will prove a local version of Lachlan's result and show that the local version easily implies the original theorem. Second, we discuss what a local version of part 2 of Martin's conjecture for uniformly invariant functions could look like and prove some negative results showing that two particular attempts to choose a precise formulation of it do not work. We will also pose a test question that seems to us to be a reasonable first step towards proving such a theorem.

3.1 Local Martin's Conjecture

In this section, we will explain in more detail what we mean by "local" versions of Martin's conjecture for uniformly Turing invariant functions, and in particular we will give the statement of the local version of part 1 of Martin's conjecture for uniformly Turing invariant functions which was proved by Bard in [Bar20].

Uniformly Turing Invariant Functions on a Single Degree

One of Bard's key insights in [Bar20] was that the definition of "uniformly Turing invariant function" still makes sense for functions whose domain is only a small subset of 2^ω and, in particular, for functions whose domain consists only of the reals in a single Turing degree. Let's now spell this out explicitly.

Notation 3.4. If x is a real, let $\deg_T(x)$ denote the set $\{y \in 2^\omega \mid x \equiv_T y\}$, i.e. the set of reals in the same Turing degree as x.

Definition 3.5. *Suppose x and y are reals. A function $f\colon deg_T(x) \to deg_T(y)$ is **uniformly Turing invariant** if there is a function $u\colon \mathbb{N}^2 \to \mathbb{N}^2$ such that for all $x_0, x_1 \in \deg(x)$ and all $i, j \in \mathbb{N}$, if $x_0 \equiv_T x_1$ via (i, j) then $f(x_0) \equiv_T f(x_1)$ via $u(i, j)$. We will call the function u a **uniformity function** for f.*

We can think of a Turing degree, together with the action of all the Turing functionals on it, as a kind of two-sorted algebraic structure, with one sort consisting of the reals in the degree and the other sort consisting of the Turing functionals[1]. A uniformly invariant function defined on a single degree, together with a uniformity function for it, can be seen as a homomorphism of such structures.

Bard's Theorem

As we have said, Bard proved a local version of Slaman and Steel's theorem on part 1 of Martin's conjecture for uniformly Turing invariant functions. This theorem does not require any use of determinacy.

Theorem 3.6 (Bard [Bar20])**.** *Suppose x and y are reals and $f\colon deg_T(x) \to deg_T(y)$ is a uniformly Turing invariant function. Then either $x \leq_T y$ or f is constant[2].*

Thinking again of a uniformly Turing invariant function as a homomorphism of algebraic structures, this theorem says that there can only be a nontrivial homomorphism $\deg_T(x) \to$

[1]Making this precise is a bit messy because, of course, $\Phi(x)$ is not always total, and even when it is it may not be of the same Turing degree as x. It is possible to deal with these issues but we will not need to do so here and thinking of a Turing degree as a two-sorted algebraic structure will be used merely as a source of intuition.

[2]As in literally constant, not constant on a cone (which doesn't make sense in this context).

$\deg_T(y)$ if $x \leq_T y$. We now note that Theorem 3.6 plus Turing determinacy implies a slightly strengthened version of Theorem 3.3.

Theorem 3.7 ($\mathsf{ZF} + \mathsf{TD}$; Bard [Bar20])**.** *If $f\colon 2^\omega \to 2^\omega$ is uniformly Turing invariant then either $f(x)$ is literally constant on a cone (not just of constant Turing degree) or $f(x) \geq_T x$ on a cone.*

A key lemma used in the proof of Theorem 3.6, which we will use again in this paper, is that if $f\colon \deg_T(x) \to \deg_T(y)$ is uniformly Turing invariant then f has not just a uniformity function (which is just the definition of "uniformly Turing invariant"), but actually a *computable* uniformity function.

Lemma 3.8 (Bard [Bar20])**.** *Suppose x and y are reals. If $f\colon \deg_T(x) \to \deg_T(y)$ is uniformly Turing invariant then there is a total computable function $u\colon \mathbb{N}^2 \to \mathbb{N}^2$ which is a uniformity function for f.*

Why It's Hard to Find a Local Version of Part 2

Now that we have seen a precise statement of a local version of part 1 of Martin's conjecture for uniformly Turing invariant functions, let's briefly discuss why it's not immediately clear what a local version of part 2 should look like. Naively, one might expect that we could more-or-less directly translate the statement of part 2 of Martin's conjecture to the local setting. That is, we might state it as something like the following.

> For any real x, the set of Turing invariant functions on $\deg_T(x)$ is prewellordered by Turing reducibility (that is, we consider f to be below g if $f(x) \leq_T g(x)$) and the successor in this prewellorder is given by the jump.

The problem with this statement is that it is false for trivial reasons. For if x is any real and y is any real which computes x then it is easy to check that the function

$$z \mapsto z \oplus y$$

is a uniformly Turing invariant function on $\deg_T(x)$. Thus the uniformly Turing invariant functions on $\deg_T(x)$ are, up to Turing equivalence, in bijection with the Turing degrees above $\deg_T(x)$.

One response to this example might be to give up any hope of proving a local version of part 2. But there are some reasons not to give up just yet. Note that the function in the example above was a continuous function. Recall that a continuous function is computable relative to some oracle. So in the context of global Martin's conjecture, showing that a function f is continuous is good enough to show that it is regressive on a cone (and thus constant on a cone by Slaman and Steel's theorem on regressive functions).

Given these considerations, a reasonable first step to trying to prove a local version of part 2 of Martin's conjecture for uniformly Turing invariant functions might be to show that if $f\colon \deg_T(x) \to \deg_T(y)$ is uniformly Turing invariant then either f is continuous or $y \geq_T x'$. This would be enough to show that every uniformly Turing invariant function is either below the identity on a cone or above the jump on a cone.

Proposition 3.9 (ZF + TD)**.** *Suppose that for all reals x and y and all uniformly Turing invariant functions $f\colon \deg_T(x) \to \deg_T(y)$ either f is continuous or $y \geq_T x'$. Then every uniformly Turing invariant function on 2^ω is either below the identity on a cone or above the jump on a cone.*

Proof. Let $f\colon 2^\omega \to 2^\omega$ be a uniformly Turing invariant function. By assumption, for every real x either f is continuous when restricted to $\deg_T(x)$ or $f(x) \geq_T x'$. By Turing determinacy, one of these two options must hold for all reals in some cone.

First, suppose that for all x in some cone, f is continuous when restricted to $\deg_T(x)$. Thus for every x, there is some oracle a and some Turing functional Φ such that $f(y) = \Phi(x \oplus a)$ for all $y \in \deg_T(x)$. Note that there is actually a canonical choice for this oracle a (i.e. the oracle which just lists, for every finite string σ, which strings define basic open sets in the preimage of the basic open set defined by σ). In other words, for each Turing degree in the cone we are working on, we can choose in a canonical way an oracle and a Turing machine witnessing that f is continuous on that degree. By Turing determinacy, these canonical choices must become fixed on a cone. Thus there is an oracle a and a Turing functional Φ such that for all x in some cone, $f(x) = \Phi(x \oplus a)$. By increasing the base of the cone if necessary, we can assume x computes a and thus x also computes $f(x)$. And by definition, that means f is below the identity on a cone.

Now suppose instead that for all x in some cone, $f(x) \geq_T x'$. Then by definition, f is above the jump on a cone. □

In light of the discussion above, it would be interesting to know whether every uniformly Turing invariant function defined on a single degree is either continuous or above the jump. In the next section, we will prove that this holds for all r.e. operators, which can be considered the local version of Lachlan's theorem on uniformly Turing invariant r.e. operators. At present, we do not know how to extend this result beyond r.e. operators.

3.2 A Local Solution to Sacks' Problem for Uniformly Invariant R.E. Operators

In this section, we will show that a local version of Lachlan's theorem (Theorem 3.1) holds for all Turing degrees above $0'$.

Here's how the proof works. Suppose W is an r.e. operator which is uniformly Turing invariant on a Turing degree $\deg_T(x)$. We split into two cases depending on whether or not W is continuous on $\deg_T(x)$. If so, then we will show that $0' \oplus x$ can compute W^x. Hence if $x \geq_T 0'$ then $x \geq_T W^x$, and we can apply the local version of part 1 of Martin's conjecture (Theorem 3.6). On the other hand, if W is not continuous on $\deg_T(x)$ then we will show how to use this discontinuity together with a computable uniformity function for f to compute x' from W^x. It is key in this part of the proof that we are able to assume we have a *computable* uniformity function (thanks to Lemma 3.8).

We will start by just proving that either W is continuous on $\deg_T(x)$ or W^x computes x' and then derive our local version of Lachlan's theorem as a corollary of this.

Theorem 3.10. *Suppose W is an r.e. operator and x is a real such that W is uniformly Turing invariant on $\deg_T(x)$. Then either W is continuous on $\deg_T(x)$ or $W^x \geq_T x'$.*

Proof. Suppose that W is not continuous on $\deg_T(x)$. Without loss of generality, we can assume that x itself is a point of discontinuity. Because W is an r.e. operator, it is not hard to see that this implies that there is some $n \in \mathbb{N}$ such that $n \notin W^x$ but for all finite initial segments σ of x, there is some finite string τ extending σ such that $n \in W^\tau$. We will now show how to use this to compute x' from W^x.

First, for each e, let y_e be the real defined as follows.

$$y_e = \begin{cases} x & \text{if } \Phi_e^x(e) \text{ diverges} \\ (x \upharpoonright k)^\frown \tau^\frown x & \text{if } \Phi_e^x(e) \text{ converges after } k \text{ steps, where } \tau \text{ is the least string} \\ & \text{such that } n \in W^{(x \upharpoonright k)^\frown \tau}. \end{cases}$$

Note that y_e is uniformly computable from x—that is there is a computable function $r\colon \mathbb{N} \to \mathbb{N}$ such that for each e, $r(e)$ is the index of a program computing y_e from x. To see why, note that given a number e we can let $r(e)$ be the following program: given an oracle x, just keep outputting bits of x until you see $\Phi_e^x(e)$ converge, at which point search for an appropriate string τ and then continue outputting the bits of x.

Also, x is uniformly computable from y_e—i.e. there is a computable function $s\colon \mathbb{N} \to \mathbb{N}$ such that for each e, $s(e)$ is the index of a program computing x from y_e. To see why, note that given a number e we can let $s(e)$ be the following program: given an oracle y, start outputting the bits of y until you see $\Phi_e^y(e)$ converge, at which point search for the first string τ such that $n \in W^{(y \upharpoonright k)^\frown \tau}$ (where k is the stage on which $\Phi_e^y(e)$ converged) and then skip ahead in y to position $k + |\tau| + k$ and keep outputting the bits starting at that position (since if y is really y_e then there will be a copy of x starting at position $k + |\tau|$).

In other words, for each e, x is Turing equivalent to y_e via $(r(e), s(e))$. Now let u be a computable uniformity function for W (guaranteed to exist by Lemma 3.8). So for each e, $u(r(e), s(e))$ gives an index of a program that can be used to compute W^{y_e} from W^x. Thus, it is computable in W^x to check whether n is in W^{y_e} or not. Now note that by construction, n is in W^{y_e} if and only if $\Phi_e^x(e)$ converges. So we can compute x' from W^x by checking whether n is in the real computed from W^x using the program given by $u(r(e), s(e))$ and thus $x' \leq_T W^x$. □

Corollary 3.11. *Suppose W is an r.e. operator, $x \geq_T 0'$, and W is uniformly Turing invariant on $\deg_T(x)$. Then one of the following must hold.*

- *W is constant on $\deg_T(x)$.*

- $W^x \equiv_T x$.
- $W^x \equiv_T x'$.

Proof. By Theorem 3.10, either W is continuous on $\deg_T(x)$ or $W^x \geq_T x'$. Since W^x is r.e. in x, the latter case implies that W^x is actually just Turing equivalent to x'. So we will assume we are in the former case and show that $W^x \leq_T x \oplus 0'$. Since we are assuming x computes $0'$, this implies that $W^x \leq_T x$ and hence we can apply Bard's theorem (Theorem 3.6) to see that either W is constant on $\deg_T(x)$ or $W^x \geq_T x$ and hence $W^x \equiv_T x$.

So to finish the proof we just need to describe how to use x together with $0'$ to compute W^x, assuming that W is continuous on $\deg_T(x)$. (Note that in this part of the proof, we will not make any use of the assumption that x computes $0'$.) So suppose we have some number n and we are trying to determine if $n \in W^x$. If n is in W^x then since W is r.e. there is some finite initial segment σ of x which witnesses this fact—i.e. some $\sigma \prec x$ such that $n \in W^\sigma$. On the other hand, if n is not in W^x then there must be some finite initial segment σ of x such that for all strings τ extending σ, $n \notin W^\tau$. For if not, then for each m, we can find some element y_m of $\deg_T(x)$ which matches x on its first m bits but for which n is not in W^{y_m}. This yields a sequence $\{y_m\}_{m\in\mathbb{N}}$ which converges to x but for which $\{W^{y_m}\}_{m\in\mathbb{N}}$ does not converge to W^x, contradicting the assumption that W is continuous on $\deg_T(x)$.

Thus the following algorithm computes W^x:

> For each n, search for a finite initial segment σ of x such that either $n \in W^\sigma$ or for all τ extending σ, $n \notin W^\tau$. In the former case, n is in W^x and in the latter case n is not in W^x.

Because of the observations we made above, this search is always guaranteed to terminate with the correct answer. And clearly the search can be carried out using just x and $0'$, so we are done. □

Note that it is not hard to show that combining the above corollary with Turing determinacy implies Lachlan's theorem.

An obvious question raised by this theorem is whether it's possible to remove the requirement that $x \geq_T 0'$. In one sense, it is clear that we *cannot* remove this requirement. To see why, consider the example of an incomplete r.e. degree x whose jump is not equal to $0'$ and the r.e. operator W which sends every real y to $y \oplus 0'$. This W is uniformly Turing invariant but it is not constant on $\deg(x)$, and $W^x \equiv_T 0'$, which is not Turing equivalent to x or x'.

This counterexample is not very satisfying, though—it depends on an overly literal reading of what it means to remove the requirement that $x \geq_T 0'$. A more sensible interpretation of the question, which we have not been able to answer, is the following.

Question 3.12. *Suppose W is an r.e. operator and x is any real. If W is uniformly Turing invariant on* $\deg(x)$*, must one of the following always hold?*

- *W is constant on x.*

- *There is some real y of r.e. degree such that $W^x \equiv_T x \oplus y$.*
- $W^x \equiv_T x'$.

3.3 A Local Version of Part 2?

In section 3.1 of this chapter, we explained why it is hard to formulate a local version of part 2 of Martin's conjecture for uniformly invariant functions and we suggested a test question: is every uniformly invariant function defined on a single degree either continuous or above the jump? In the previous section we proved that this question has a positive answer for r.e. operators but left open the question in general. In this section, we will discuss how to generalize this question to get a more complete local version of part 2 and prove a few (easy) theorems indicating why this is hard.

One way to try to extend the question would be to make use of the analogy that continuous functions are to computable functions as Baire class α functions are to functions computable from α jumps. More precisely, a function f is Baire class α if and only if there is an oracle a and a Turing functional Φ such that $f(x) = \Phi((x \oplus a)^{(\alpha)})$ for all x. Thus it sounds like a natural extension of the question of whether every uniformly invariant function on a single degree is either continuous or above the jump would be the following.

> If $f\colon \deg_T(x) \to \deg_T(y)$ is a uniformly Turing invariant function then either f is of Baire class α or $f(x) \geq_T x^{(\alpha+1)}$.

Unfortunately, when $\alpha \geq 1$ this is true for trivial reasons and (unlike the version for continuous functions) does not imply the corresponding global statement about uniformly invariant functions. The problem, briefly, is that while for continuous functions there is a canonical way to choose an oracle, there is no such canonical choice for Baire class α functions when $\alpha \geq 1$. The following proposition shows what goes wrong.

Proposition 3.13. *Every function with countable domain is Baire class one.*

Proof. Let $A \subset 2^\omega$ be a countable set and let $f\colon A \to 2^\omega$ be any function. We want to show f is of Baire class one. Recall that the meaning of Baire class one is just "the pointwise limit of a countable sequence of continuous functions." Let $x_0, x_1, x_2, \ldots$ be an enumeration of A. For each n, let f_n be a continuous function which is equal to f at $x_0, x_1, \ldots, x_n$ (for example, f_n could be a piecewise linear function). Then clearly f is the pointwise limit of the f_n's and so f is of Baire class one. □

Here is one natural (though naive) way to try to fix this problem. Recall that a function f is Baire class α if and only if there is an oracle a and a Turing functional Φ such that $f(x) = \Phi((x \oplus a)^{(\alpha)})$ for all x. We said above that the problem is that the real a cannot be chosen in a canonical way. But if we consider the following stronger condition then this problem disappears: there is an oracle a and a Turing functional Φ such that $f(x) = \Phi(x^{(\alpha)} \oplus a)$ for all x (note that this condition is no longer equivalent to being Baire class α, and is in fact much stronger).

Here's what we mean when we say "the problem disappears." Suppose that you could prove the following:

> If $f\colon \deg_T(x) \to \deg_T(y)$ is uniformly Turing invariant then either there is some a such that $f(z) = \Phi(z^{(\alpha)} \oplus a)$ for all $z \in \deg_T(x)$ or $y \geq_T x^{(\alpha+1)}$.

Then using Turing determinacy you could show that every uniformly Turing invariant function $f\colon 2^\omega \to 2^\omega$ is either below the α-jump or above the $(\alpha+1)$-jump.

Unfortunately, the statement above is false for all $\alpha \geq 1$, and for fairly easy reasons. We will prove it just in the case where $\alpha = 1$, but the proof can be easily modified to work for all larger α.

Proposition 3.14. *There is a real x and a uniformly Turing invariant function $f\colon \deg_T(x) \to \deg_T(x')$ such that there is no oracle a and Turing functional Φ for which $f(y) = \Phi(y' \oplus a)$ for all $y \in \deg_T(x)$.*

Proof. Let x be a 1-generic real. We will show that $y \mapsto (x \oplus y)'$ is a counterexample. First note that as a function from $\deg_T(x)$ to $\deg_T(x')$, $y \mapsto (x \oplus y)'$ is clearly uniformly Turing invariant.

Now suppose for contradiction that there is some oracle a and Turing functional Φ such that $(x \oplus y)' = \Phi(y' \oplus a)$ for all $y \in \deg_T(x)$. First note that if y is 1-generic then $y' \oplus a$ is uniformly computable from $y \oplus 0' \oplus a$, so if we replace a with $0' \oplus a$ and restrict our attention to y's in $\deg_T(x)$ which are 1-generic then we may assume that $(x \oplus y)' = \Phi(y \oplus a)$. In other words, $y \mapsto (x \oplus y)'$ is continuous on the 1-generics in $\deg_T(x)$. The rest of the proof just consists of showing that this is impossible.

Next, note that there is some n such that the n^{th} bit of $(x \oplus y)'$ indicates whether $x = y$ or not. Thus $(x \oplus x)'(n) = 1$ and since x is 1-generic that implies that $\Phi^{x \oplus a}(n) = 1$. Let σ be a finite initial segment which witnesses this computation. Let y be a 1-generic in the same degree as x which also extends σ but is not equal to x. Since y is 1-generic, $(x \oplus y)' = \Phi(y \oplus a)$. And since y extends σ, $\Phi^{y \oplus a}(n) = \Phi^{x \oplus a}(n) = 1$. But since y is not equal to x, $(x + y)'(n) = 0$ and hence $(x + y)'$ is not equal to $\Phi(y \oplus a)$. ∎

We will now discuss one final incorrect way to try to state a local version of part 2 of Martin's conjecture for uniformly invariant functions. This idea was originally proposed by Bard in [Bar20]. It is inspired by the results of Becker that we mentioned in the introduction of this chapter. Recall that Becker showed that under AD, every uniformly Turing invariant function on the reals is equal to a pointclass jump on a cone. Based on this, Bard asked the following question.

> Let x be any real and consider the smallest class $C(x)$ of functions on $\deg_T(x)$ that contains the identity function, all constant functions, all pointclass jumps and which is closed under finite joins, infinite joins, and certain types of computable

transformations. Does $C(x)$ contain all uniformly Turing invariant functions on $\deg_T(x)$?

The problem with this question is that the class $C(x)$ is much too large (it actually contains all uniformly Turing invariant functions on $\deg_T(x)$, and many other functions besides). The essential reason for this is that taking infinite joins is a very brutal operation. Even if each of $f_0, f_1, f_2, \dots$ is a nicely behaved function on its own, the infinite join $x \mapsto \bigoplus_i f_i(x)$ may be very wild. For example, the first bits of each of the $f_i(x)$ could be coding some complicated function of x. However, if infinite joins are removed from the definition above then it is not clear that the class $C(x)$ contains all uniformly Turing invariant functions on $\deg_T(x)$ (and it seems likely that it does not).

In light of the discussion in this section, it would be interesting to find even a candidate for a local version of part 2 of Martin's conjecture for uniformly invariant functions. For a statement to be considered a candidate, it should combine with Turing determinacy to easily imply some fraction of Steel's theorem (Theorem 3.2) and (obviously) should not be known to be false. At present, the only statement along these lines that we are aware of is the question about continuous functions that we posed in section 3.1.

Chapter 4

Regressive Functions

In this chapter we will discuss Martin's conjecture for regressive functions. Recall that a Turing invariant function $f\colon 2^\omega \to 2^\omega$ is called regressive if $f(x) \leq_T x$ for all reals x. In [SS88], Slaman and Steel showed that Martin's conjecture holds for all regressive functions.

Theorem 4.1 ($\mathsf{ZF}+\mathsf{AD}$; Slaman and Steel)**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing-invariant function such that $f(x) \leq_T x$ for all x then either f is constant on a cone or $f(x) \equiv_T x$ on a cone.*

In this chapter we will prove that their theorem can be extended to the hyperarithmetic degrees. In other words, we will prove the following theorem.

Theorem 4.2 ($\mathsf{ZF}+\mathsf{AD}$)**.** *Let $f\colon 2^\omega \to 2^\omega$ be a hyp-invariant function such that $f(x) \leq_H x$ for all x. Then either f is constant on a cone of hyperarithmetic degrees or $f(x) \equiv_H x$ on a cone of hyperarithmetic degrees.*

This theorem answers a question asked by Slaman and Steel in [SS88]. Before giving the proof, I want to explain what motivated them to ask this question.

Slaman and Steel's proof of Martin's conjecture for regressive functions on the Turing degrees starts from the observation that regressive functions are basically just continuous functions. This statement is not literally true, but it is true in spirit and is embodied in a number of precise mathematical statements. For Slaman and Steel's purposes, the relevant fact was that if f is a regressive function on the Turing degrees then f is continuous on some pointed perfect tree. The rest of their proof involves an ingenious coding argument and this coding argument relies heavily on the fact that by working on a pointed perfect tree, you can assume f is continuous.

They therefore wished to know whether it is possible to carry out the kind of coding argument they used without being able to assume that the function is continuous. Martin's conjecture for regressive functions on the hyperarithmetic degrees seemed like a natural test case for this question. Just as regressive functions on the Turing degrees are "basically" continuous functions, regressive functions on the hyperarithmetic degrees are "basically" Borel functions. A regressive function on the hyperarithmetic degrees may not be continuous on any pointed perfect tree, but it *is* guaranteed to be Borel on a pointed perfect tree. Thus it seemed as though handling the case of regressive functions on the hyperarithmetic degrees would require modifying Slaman and Steel's techniques to work for non-continuous functions while still providing some structure—namely the structure of a Borel function.

However, even though we solved Slaman and Steel's test question, we did not do it in the way they expected. Instead of adapting their methods to work with non-continuous functions, we show that a hyp-regressive function f—despite potentially being quite far from continuous—can be replaced by a hyp-equivalent function which *is* continuous. This does not quite finish the proof—we still have to replace Slaman and Steel's coding argument with a new coding argument that works with hyperarithmetic reducibility rather than Turing reducibility—but the fact that we can replace f with a continuous function is a key step.

This might seem like a rather disappointing outcome: Slaman and Steel's problem was solved but not in a way that addressed their underlying question of how to extend their techniques to non-continuous functions. However, the idea of replacing the function f with a related function that is continuous (rather than just restricting to a domain where f itself is continuous) has proved very useful in other cases of Martin's conjecture. In fact, the proof in this section directly inspired the results on order preserving functions and measure preserving functions in chapters 5 and 6.

Another interesting feature of our proof is that it casts a small amount of doubt on the idea that any use of determinacy in proving Martin's conjecture will be "local" (that is, the idea that only Borel determinacy is needed when dealing with Borel functions, and so on). If our proof is directly modified to deal only with Borel functions, it seems to use more than just Borel determinacy—specifically, it seems to require Π^1_1 determinacy. In section 4.2, we show that Borel determinacy *is* sufficient, but this requires a more careful analysis that was not needed for the AD proof. In chapters 5 and 6 we will see more instances of this phenomenon. And in one case—the case of measure preserving functions on the Turing degrees—we give a proof for all functions under AD but we do not know if the version restricted to Borel functions is provable in ZF (the best proof we know uses $\mathbf{\Pi}^1_1$ determinacy).

We also wish to note that our reduction to the case of a continuous function is quite flexible and seems to work in many different degree structures, including the arithmetic degrees. Somewhat surprisingly, it seems much harder to adapt the coding arguments used by Slaman and Steel, even once we are allowed to assume we are dealing with a continuous function. In the hyperarithmetic case we had to use a new coding argument which relies on the Σ^1_1-bounding theorem. And in the arithmetic case we have so far not been able to find a coding argument which works and, in our opinion, the regressive case of Martin's conjecture on the arithmetic degrees is an interesting open question.

4.1 Regressive Functions on the Hyperarithmetic Degrees: AD Case

In this section, we will prove theorem 4.2. Before we launch into the details of the proof, we will give an outline of the general strategy. And before we do that, we will recall the general strategy followed by Slaman and Steel in their proof of theorem 4.1. The steps of their proof are essentially as follows.

- First, use determinacy to show that there is a pointed perfect tree on which f is computable. Then use lemma 2.7 to show that if f is not constant on a cone then we can also assume f is injective.
- Next, show that if x is a path through the pointed perfect tree, then every function computable from x is dominated by a function computable from $f(x)$. The idea is that if x computes a function which is not dominated by any function computable from $f(x)$ then x can diagonalize against $f(x)$ by using this function to guess convergence times for $f(x)$ programs. The diagonalization produces a real y in the same Turing degree as x such that $f(x)$ cannot compute $f(y)$, contradicting the Turing invariance of f.
- Once you can assume that f is computable and injective on a pointed perfect tree and that if x is a path through this tree then every function computed by x is dominated by a function computed by $f(x)$, use a coding argument to show that $f(x) \geq_T x$. The coding argument works by coding bits of x into the relative growth rates of two fast growing functions computed by $f(x)$.

Our proof makes three main modifications to this outline. First, instead of showing that f is computable on some pointed perfect tree, we show that f is hyp-equivalent to some computable function on a pointed perfect tree. Thus we may work with that function instead of f. Second, instead of showing that every fast growing function computed by x is dominated by a function computed by $f(x)$, we show that every well-order computed by x embeds into a well-order computed by $f(x)$—in other words that $\omega_1^x = \omega_1^{f(x)}$. Third, instead of coding bits of x into the relative growth rates of fast growing functions computed by $f(x)$, we code the bits of x into the Kolmogorov complexities of initial segments of reals computed by $f(x)$ (though it is not necessary to know anything about Kolmogorov complexity to follow our argument). Also, to be able to carry out the coding argument, we will first have to use a trick involving Σ_1^1-bounding. To sum up, here's an outline of our proof.

- First, use determinacy to replace f with a hyp-equivalent function which is computable on a pointed perfect tree. By using lemma 2.7, we can also assume that f is injective.
- Next show that $\omega_1^{f(x)} = \omega_1^x$ for all paths x through the pointed perfect tree. The idea is that if $\omega_1^{f(x)}$ is less than ω_1^x then x can diagonalize against $f(x)$ by using $\omega_1^{f(x)}$ jumps.

This diagonalization produces a real y which is hyp-equivalent to x but for which $f(y)$ is not hyp-equivalent to $f(x)$ (because it takes too many jumps to compute), contradicting the hyp-invariance of f.

- Once you can assume that f is computable and injective and that $\omega_1^{f(x)} = \omega_1^x$, use a coding argument to show that $f(x) \geq_H x$. In our coding argument, it will be important to know that there is a single ordinal $\alpha < \omega_1^x$ such that for every real y in the same Turing degree as x, $f(x)^{(\alpha)}$ computes $f(y)$. We will prove this fact using Σ^1_1-bounding.

And now it's time to present the actual proof.

Replacing f with an injective, computable function

First we will show that f can be replaced by a computable function. This is the only part of the proof that uses determinacy.

Lemma 4.3 (ZF + AD)**.** *Suppose $f\colon 2^\omega \to 2^\omega$ is hyp-invariant and hyp-regressive. Then there is a Turing functional Φ and a pointed perfect tree T such that for all $x \in [T]$, $\Phi(x)$ is total and $\Phi(x) \equiv_H f(x)$.*

Proof. Consider the following binary relation, R.

$$R(x, z) \iff x \geq_T z \text{ and } \exists y\, (y \equiv_H x \wedge f(y) = z).$$

So basically R just says that x computes z and z is in the image of f on the hyperdegree of x. I claim that the domain of R is cofinal in the Turing degrees and thus lemma 2.11 applies. To prove this, we need to start with an arbitrary x and show that some real which computes x is in the domain of R. So let $x \in 2^\omega$. Since $f(x) \leq_H x$, there is some $\alpha < \omega_1^x$ such that $f(x) \leq_T x^{(\alpha)}$. And this $x^{(\alpha)}$ is the real we are after: it clearly computes x and since $R(x^{(\alpha)}, f(x))$ holds, $x^{(\alpha)}$ is indeed in the domain of R.

So by lemma 2.11, there is a pointed perfect tree T and a Turing functional Φ such that for all $x \in [T]$, $\Phi(x)$ is total and is in the image of f on the hyperdegree of x. Since f is hyp-invariant, this implies that $f(x) \equiv_H \Phi(x)$ for every $x \in [T]$. □

For the rest of the proof we will simply assume that f is computable on a pointed perfect tree. It will also be convenient to assume that f is injective on a pointed perfect tree, which we show next.

Lemma 4.4 (ZF)**.** *Suppose T is a pointed perfect tree and $f\colon 2^\omega \to 2^\omega$ is a hyp-invariant function which is computable on $[T]$. Then either f is constant on a cone of hyperdegrees or f is injective on a pointed perfect subtree of T.*

Proof. By lemma 2.7, either f is constant on a pointed perfect subtree of T or f is injective on a pointed perfect subtree of T. In the former case, f is constant on a cone of hyperdegrees and in the latter case, we are done. □

For the rest of the proof, we will deal with the case of a hyp-invariant function, f, which is computable and injective on a pointed perfect tree, T. We will show that for any x in $[T]$, $f(x) \geq_H x$. There are two cases: when $\omega_1^{f(x)} < \omega_1^x$ and when $\omega_1^{f(x)} = \omega_1^x$. We will show that the first case is impossible and that if we are in the second case then we can complete the proof. A key fact, which we will use several times, is that since f is computable and injective on $[T]$, if $x \in [T]$ then $f(x) \oplus T \geq_T x$ (by Lemma 2.3).

Proving that f preserves ω_1^x

We will now show that for any $x \in [T]$, f preserves ω_1^x. We will do this by deriving a contradiction from the assumption that $\omega_1^{f(x)} < \omega_1^x$. The basic idea is that in this case we can diagonalize against $f(x)$. Namely, we can use $\omega_1^{f(x)}$ jumps of x to compute a real y so that $f(x)$ cannot compute $f(y)$ with fewer than $\omega_1^{f(x)}$ jumps (and hence $f(x)$ cannot be hyp-equivalent to $f(y)$). Since $\omega_1^x > \omega_1^{f(x)}$, this y can be made hyp-equivalent to x, which violates the hyp-invariance of f. We now give the formal proof.

Lemma 4.5 (ZF)**.** *Suppose T is a pointed perfect tree and f is a hyp-invariant function which is computable and injective on $[T]$. Then for every $x \in [T]$, $\omega_1^{f(x)} = \omega_1^x$.*

Proof. Suppose for contradiction that for some $x \in [T]$, $\omega_1^{f(x)} < \omega_1^x$. Let $\alpha = \omega_1^{f(x)}$. The key point is that for every $y \in [T]$ which is hyp-equivalent to x, $x^{(\alpha)}$ computes y.

Why is that? Well, if y is in the same hyperdegree as x then $f(y)$ is in the same hyperdegree as $f(x)$. So by definition of α, there is some $\beta < \alpha$ such that $f(x)^{(\beta)} \geq_T f(y)$. We then have the following calculation.

$$\begin{aligned} x^{(\alpha)} &\geq_T x^{(\beta)} && \text{because } \beta < \alpha \\ &\geq_T x^{(\beta)} \oplus T && \text{because } T \text{ is pointed} \\ &\geq_T f(x)^{(\beta)} \oplus T && \text{because } f(x) \leq_T x \\ &\geq_T f(y) \oplus T && \text{by definition of } \beta \\ &\geq_T y && \text{by lemma 2.3.} \end{aligned}$$

We can now finish the proof easily. Since T is pointed, we can pick some $y \in [T]$ which is Turing equivalent to $x^{(\alpha+1)}$. Since $\alpha < \omega_1^x$, this y is hyp-equivalent to x. But it obviously is not computable from $x^{(\alpha)}$, so we have reached a contradiction. □

Coding argument

In this part of the proof, we will explain how to code x into an real in the same hyperarithmetic degree as $f(x)$. The argument has some similarity to the proof of a basis theorem for perfect sets given by Groszek and Slaman in [GS98] (which itself has some similarity to the coding argument given in [SS88]). Before giving the coding argument, however, we will first show that for every $x \in [T]$ there is a uniform bound on the number of jumps that $f(x)$ takes to compute $f(y)$ for any $y \in [T]$ which is Turing equivalent to x.

Lemma 4.6 (ZF)**.** *Suppose T is a pointed perfect tree, f is a hyp-invariant function which is computable on $[T]$, and $x \in [T]$. Then there is some $\alpha < \omega_1^x$ such that if $y \in [T]$ is Turing equivalent to x then $f(y) \leq_T f(x)^{(\alpha)}$.*

Proof. The main idea is just to use Σ^1_1-bounding. Let A be the set of codes for linear orders r which are computable from $f(x)$ and such that

- r has no infinite descending sequence which is hyperarithmetic in $f(x)$
- and there is some $y \equiv_T x$ in $[T]$ and some jump hierarchy H on r such that $H_0 = f(x)$ and H does not compute $f(y)$.

By lemma 2.35 (and since $f(x)$ is computable in x), A is $\Sigma^1_1(x)$. I claim that A only contains well-orders.

Suppose instead that A contains an ill-founded order, r. Thus r is a pseudo-well-order relative to $f(x)$. Since r is in A, there must be some $y \equiv_T x$ in $[T]$ and some jump hierarchy on r starting with x which does not compute $f(y)$. And since f is hyp-invariant, we must have $f(x) \equiv_H f(y)$. But by lemma 2.36, any jump hierarchy on r which starts with $f(x)$ computes everything in the hyperdegree of $f(x)$, in particular $f(y)$. This is a contradiction, so A must contain only well-orders.

Since A is $\Sigma^1_1(x)$ and contains only well-orders, Σ^1_1-bounding implies that there is some $\alpha < \omega_1^x$ which bounds every well-order in A. This implies that for every $y \equiv_T x$ in $[T]$, $f(y)$ is computable from $f(x)^{(\alpha+1)}$. □

We now come to the coding argument. It replaces a different coding argument in the proof of Slaman and Steel's theorem on regressive functions on the Turing degrees. That argument coded information into the relative growth rates of two fast-growing functions. This coding argument codes information into the relative Kolmogorov complexities of initial segments of three reals (though the reader does not need to be familiar with Kolmogorov complexity to understand the proof below). The proof is an easier version of the proof of Theorem 2.19 and in fact, was the direct inspiration for that proof.

As with many arguments of this sort, the construction below is easier to explain one-on-one in front of a blackboard than to communicate in writing; we hope the reader will forgive us.

Lemma 4.7 (ZF)**.** *Suppose T is a pointed perfect tree and f is a hyp-invariant function which is computable and injective on $[T]$. Then $f(x) \geq_H x$ for all $x \in [T]$.*

Proof. Let $x \in [T]$. Our goal is to show that $f(x) \geq_H x$. By 4.5, we know that $\omega_1^x = \omega_1^{f(x)}$ and we will use this fact in the proof. By thinning T, we may assume that x is the base of T (i.e. T is a pointed perfect tree such that $x \equiv_T T$), and hence that any element of T can compute x. We will use this fact below without further comment.

By lemma 4.6, there is some $\alpha < \omega_1^x$ such that for all $y \in [T]$ in the same Turing degree as x, we have $f(x)^{(\alpha)} \geq_T f(y)$. For the remainder of the proof, we will explain how to find reals $a, b, c \in [T]$ which are hyp-equivalent to x such that $x \leq_T f(x)^{(\alpha+2)} \oplus f(a) \oplus f(b) \oplus f(c)$.

To see why this is sufficient to complete the proof, first note that since f is hyp-invariant, $f(a), f(b)$, and $f(c)$ are all hyp-equivalent to $f(x)$. Next, note that since $\omega_1^{f(x)} = \omega_1^x$, α is less than $\omega_1^{f(x)}$ and thus $f(x)^{(\alpha+2)}$ is also hyp-equivalent to $f(x)$. Therefore $f(x)^{(\alpha+2)} \oplus f(a) \oplus f(b) \oplus f(c)$ is hyp-equivalent to $f(x)$ and so if x is Turing below the former then it is hyp below the latter.

We will build a, b, and c in stages. At each stage we will keep track of the following data (supposing that the current stage is n):

- Initial segments A_n, B_n, and C_n of a, b, and c.
- Reals a_n, b_n, and c_n in $[T]$ and Turing equivalent to x, which A_n, B_n, and C_n, respectively, are initial segments of. Think of a_n, b_n, c_n as the current "targets" for a, b, c.
- Initial segments $\widetilde{A}_n, \widetilde{B}_n$, and $\widetilde{C}_n$ of $f(a), f(b)$, and $f(c)$. These are the longest initial segments of $f(a), f(b)$, and $f(c)$ that can be determined from knowing the initial segments A_n, B_n, and C_n of a, b, and c (recall that f is continuous on T).
- Indices for programs $e_{a,n}, e_{b,n}$, and $e_{c,n}$. Think of these as "guesses" as to which programs compute $f(a_n), f(b_n)$, and $f(c_n)$ from $f(x)^{(\alpha)}$.

At the same time, $f(x)$ will be using $f(a), f(b)$, and $f(c)$ to try to follow along with this construction by keeping track of the initial segments $\widetilde{A}_n, \widetilde{B}_n$, and $\widetilde{C}_n$ and the "guesses" $e_{a,n}, e_{b,n}$, and $e_{c,n}$. On each step of the construction we will update the data to code the next bit of x.

On each step, two of a, b, and c will be used to code the next bit of x and the third will play a "helper" role of coding some information to help $f(x)$ follow along with the construction. Which of a, b, and c is playing this "helper" role will simply rotate between them on each step. So, for instance, a will play the helper role every third step.

We will make sure that at the beginning of step n, the "guess" corresponding to whichever real is playing the helper role on step n is correct. E.g. if a is in the helper role on step n then we will need that $e_{a,n}$ is really the index of a program computing $f(a_n)$ from $f(x)^{(\alpha)}$. We will see that the construction ensures this.

We will code the next bit of x into the relative sizes of the guesses for the two reals which are not playing the helper role. E.g. if a is playing the helper role on step n then we will code the next bit of x into which of $e_{b,n+1}$ and $e_{c,n+1}$ is larger—if $x(n) = 0$ then we will make sure $e_{b,n+1} > e_{c,n+1}$ and if $x(n) = 1$ then we will make sure $e_{b,n+1} < e_{c,n+1}$.

To make things more concrete, let's suppose that we are on step n, a is in the helper role, and the next bit of x is a 0 (so we need to make sure $e_{b,n+1} > e_{c,n+1}$). We can assume that $e_{a,n}$ is correct—i.e. that $\Phi_{e_{a,n}}(f(x)^{(\alpha)}) = a_n$—and we need to make sure that this holds of $e_{b,n+1}$ at the end of this step. Here's what we do.

- The target for c will stay the same—i.e. set $c_{n+1} = c_n$.

- Let $e_{c,n+1}$ be the true guess for $c_n = c_{n+1}$—i.e. the least e such that $\Phi_e(f(x)^{(\alpha)} = f(c_n)$ (we know that such an e must exist because we are assuming c_n is in $[T]$ and Turing equivalent to x and thus $f(c_n)$ is computable from $f(x)^{(\alpha)}$).
- Choose some new target b_{n+1} in $[T]$ and of the same Turing degree as x so that b_{n+1} extends B_n and so that for the least e for which $\Phi_e(f(x)^{(\alpha)}) = f(b_{n+1})$, we have $e > e_{c,n+1}$. We can do this because f is injective on T and there are infinitely many reals in $[T]$ extending B_n which are Turing equivalent to x.
- Let m be a number large enough that $e_{c,n+1}$ is the least e such that $\Phi_e(f(x)^{(\alpha)})$ is total and agrees with the first m bits of $f(c_{n+1})$ and likewise for $e_{b,n+1}$.
- Choose some new target a_{n+1} in $[T]$ of the same Turing degree as x which also agrees with the old initial segment A_n of a but which disagrees with a_n and for which the first place such that $f(a_{n+1})$ disagrees with $f(a_n)$ is greater than m, say m'.
- Let $\widetilde{A}_{n+1} = f(a_{n+1}) \restriction m'$.
- Let A_{n+1} be a long enough initial segment of a_{n+1} to ensure that $f(a) \restriction m' = f(a_{n+1}) \restriction m'$ and thus that the first place at which $f(a_n)$ and $f(a_{n+1})$ disagree is m'.
- Set $\widetilde{B}_{n+1}$ and $\widetilde{C}_{n+1}$ to be $f(b_{n+1}) \restriction m'$ and $f(c_{n+1}) \restriction m'$.
- Let B_{n+1} and C_{n+1} be long enough initial segments of b_{n+1} and c_{n+1} to ensure that $f(b)$ and $f(c)$ agree with the first m' bits of $f(b_{n+1})$ and $f(c_{n+1})$ (recall that f is continuous on $[T]$).

Note that by construction, the guesses $e_{b,n+1}$ and $e_{c,n+1}$ are correct. Now let's describe what's happening from $f(x)$'s perspective.

- First we look at $f(a)$. Since it's a's turn to be the helper, we (as $f(x)$) know we should look for the first place where $f(a)$ disagrees with $\Phi_{e_{a,n}}(f(x)^{(\alpha)})$ (which, recall, agrees with $f(a_n)$). So this allows us to retrieve m'.
- Now look at $f(b) \restriction m'$ and $f(c) \restriction m'$. These are the new $\widetilde{B}_n$ and $\widetilde{C}_n$. Calculate the least e such that $\Phi_e(f(x)^{(\alpha)})$ is total and agrees with $f(b)$ up to m'. This is $e_{b,n+1}$. Do the same thing for c.
- Now we check which of $e_{b,n+1}$ and $e_{c,n+1}$ is bigger. That tells us the next bit of x.
- At this point we have the correct guesses for b and c. We may not have a correct guess for a (or even a guess at all) but that doesn't really matter. The only one for which it is vital we have a correct guess at the beginning of the next step is the one that is going to be in helper mode and this will not be a twice in a row (since helper mode always rotates).

To carry out the entire construction to build a, b, and c, we just need to know x and $f(x)^{(\alpha+2)}$ (to figure out which programs are total). Since x computes $f(x)$ and $\alpha < \omega_1^x$, this means that a, b, and c are hyperarithmetic in x. And since $x \equiv_T T$ and T is pointed, x is computable by a, b, and c and thus they are all in the same hyperdegree. At the same time, all that is required to do the parts "from $f(x)$'s perspective" is $f(x)^{(\alpha+2)}$ (to check which programs are total) along with $f(a), f(b)$, and $f(c)$. Hence $x \leq_T f(x)^{(\alpha+2)} \oplus f(a) \oplus f(b) \oplus f(c)$. □

4.2 Regressive Functions on the Hyperarithmetic Degrees: Borel Case

It is popular to suppose that any proof of Martin's conjecture will only use determinacy in a "local" way—that is, the proof will still work in ZF when restricted to Borel functions, just by replacing the original uses of AD with analogous uses of Borel determinacy. In this section, we will see that this only sort of holds for Martin's conjecture for regressive functions on the hyperarithmetic degrees. It is true that the result holds in ZF when restricted to Borel functions, but proving this requires using a trick not present in the AD proof presented above. The trouble is that even if we only consider Borel functions, the proof of Lemma 4.3 (which allowed us to replace f with a computable function) appears to require analytic determinacy rather than Borel determinacy. However, this can be avoided by a more careful analysis and an appeal to Σ^1_1-bounding.

Here's the key idea. If f is hyp-regressive then we know that for each x there is some $\alpha < \omega_1^x$ such that $x^{(\alpha)}$ computes $f(x)$. We will use Σ^1_1-bounding to find a single α which works for all x. After this, it will be straightforward to modify the proof of lemma 4.3 to only use Borel determinacy.

In the next lemma we will prove this key point that there is a bound on the number of jumps that x needs to compute $f(x)$. Note that since we are restricting ourselves to Borel functions, we can drop the "hyp-regressive" requirement—every Borel function f is automatically regressive on a cone of hyperarithmetic degrees.

Lemma 4.8 (ZF)**.** *Let $f\colon 2^\omega \to 2^\omega$ be a Borel function. Then there is some $\alpha < \omega_1$ such that for all x on a cone of hyperdegrees, $\alpha < \omega_1^x$ and $x^{(\alpha)} \geq_T f(x)$.*

Proof. As noted above, since f is Borel, $f(x) \leq_H x$ on a cone of hyperdegrees. For the rest of the proof, we will implicitly work on this cone and thus we may assume $f(x) \leq_H x$ for all x.

We start by simply writing down the definition of hyperarithmetic reducibility: for each x, we know that $f(x) \leq_H x$ and hence that there is some $\alpha < \omega_1^x$ such that $x^{(\alpha)}$ computes $f(x)$. Our goal is to show that there is some $\alpha < \omega_1$ which is large enough to work for all x. We will do so by using Σ^1_1-bounding.

Let A be the set of presentations of linear orders r such that for some x,

- x computes r
- r has no infinite descending sequences which are hyperarithmetic in x
- and there is a jump hierarchy H on r such that $H_0 = x$ and H does not compute $f(x)$.

By lemma 2.35 plus the fact that f is Borel, the set A is $\boldsymbol{\Sigma^1_1}$ definable (note that this is boldface rather than lightface because f is Borel but not necessarily lightface Δ^1_1).

Next, I claim that A only contains well-orders. Suppose not and that A contains an ill-founded order, r. Let x witness that r is in A. Then r is a pseudo-well-order relative to x. But by lemma 2.36, this means that any jump hierarchy on r starting with x computes everything hyperarithmetic in x, and in particular, computes $f(x)$. This contradicts the definition of A.

Since A is $\boldsymbol{\Sigma^1_1}$ and contains only well-orders, $\boldsymbol{\Sigma^1_1}$-bounding implies that there is some $\alpha < \omega_1$ which bounds everything in A. By the definition of A this means that for every x either $\omega_1^x \leq \alpha$ or $x^{(\alpha+1)} \geq_T f(x)$. And if we go to a cone on which everything computes a presentation of α then we obtain the conclusion of the lemma. □

We can now prove the Borel version of Theorem 4.2.

Theorem 4.9 (ZF)**.** *Let $f\colon 2^\omega \to 2^\omega$ be a hyp-invariant Borel function. Then either f is constant on a cone of hyperdegrees or $f(x) \geq_H x$ on a cone of hyperdegrees.*

Proof. By the previous lemma, we can assume there is some $\alpha < \omega_1$ such that for all x on a cone of hyperdegrees, $\alpha < \omega_1^x$ and $x^{(\alpha)} \geq_T f(x)$. Let a be the base of such a cone and let r be a presentation of α computable from a. For the rest of the proof, we will work on the cone above a and we will interpret $x^{(\alpha)}$ to mean the unique jump hierarchy on r that starts with x.

The main idea of the proof is to go through the proof of Theorem 4.2 and make sure that every time that proof used determinacy, we can actually get by with just Borel determinacy. The only part of that proof in which we used determinacy was in the proof of Lemma 4.3. In particular, we used determinacy by applying lemma 2.11 to the binary relation R defined by

$$R(x, z) \iff x \geq_T z \text{ and } \exists y\, (y \equiv_H x \wedge f(y) = z).$$

The problem is that even if f is Borel, this relation is not $\boldsymbol{\Delta^1_1}$, but only $\boldsymbol{\Pi^1_1}$. We will remedy this problem by showing that the relation R can be replaced by the relation S defined by

$$S(x, z) \iff x \geq_T z \text{ and } \exists y \leq_T x\, (y \geq_T a \wedge x \leq_T y^{(\alpha)} \wedge f(y) = z).$$

In particular, we will show that the domain of S is cofinal in the Turing degrees. The requirement that y must compute a is necessary to ensure that $y^{(\alpha)}$ is well-defined (and note that it implies that x must also compute a).

Why is this sufficient? Let's first assume that we can show that the domain of S is cofinal and see why that is enough to complete the proof. Since the definition of S is $\boldsymbol{\Delta^1_1}$, and satisfies the conditions of lemma 2.11, there is a pointed perfect tree T and a Turing functional Φ such that for all $x \in [T]$, $S(x, \Phi(x))$ holds. Furthermore, the definition of S and the hyp-invariance of f guarantee that $\Phi(x) \equiv_H f(x)$ for all $x \in [T]$. Thus we have recovered the conclusion of lemma 4.3 and the rest of the proof works unchanged.

Why is this true? Now we will show that S has cofinal domain. The proof is very similar to the proof of Lemma 4.3. Let x be any real. By joining with a if necessary, we may assume that x is in the cone above a. Since x is in the cone above a, we know that $f(x) \leq_T x^{(\alpha)}$ and so $S(x^{(\alpha)}, f(x))$ holds. Since $x \leq_T x^{(\alpha)}$, we have succeeded in finding something in the domain of S which is above x. □

Chapter 5

Measure Preserving Functions

Note: The results of this chapter are joint work with Benjamin Siskind.

In this chapter, we will discuss Martin's conjecture for measure preserving functions. Our main result is that part 1 of Martin's conjecture holds for all measure preserving functions on the Turing degrees and and our principal thesis is that measure preserving functions are worth studying. Here's a summary of the evidence we present for this thesis.

Measure preserving functions are natural. The class of measure preserving functions has a few different appealing characterizations.

- First, it has a relatively simple combinatorial definition formalizing the informal idea of functions on the Turing degrees which are "going to infinity in the limit."
- Second, it has a characterization in terms of the Martin order: measure preserving functions are exactly those functions which are above every constant function.
- Third, it has a characterization in terms of the Martin measure: measure preserving functions are exactly those functions which preserve Martin measure in the sense of ergodic theory. It is because of this last characterization that we chose the name "measure preserving."

Measure preserving functions give insight into Martin's conjecture. The main result of this chapter—that part 1 of Martin's conjecture holds for measure preserving functions—has several consequences for Martin's conjecture.

- In section 6.1, we will show that every order preserving function is either constant on a cone or measure preserving. This implies part 1 of Martin's conjecture for order preserving functions (we will discuss this more in the next chapter). This result also implies that the Turing degrees are not a universal locally countable Borel partial order.
- In section 5.8 we show that our result about part 1 of Martin's conjecture for measure preserving functions implies a special case of part 2 of Martin's conjecture.

- In section 5.9 we use the characterization of measure preserving functions in terms of the Martin measure to show that our result implies that part 1 of Martin's conjecture is equivalent to a statement about the Rudin-Keisler order on ultrafilters on the Turing degrees.

- In sections 5.5 and 5.6 we show that our proof of part 1 of Martin's conjecture for measure preserving functions is quite general: it works in other degree structures (for example, for the arithmetic degrees and the hyperarithmetic degrees) and even for non-Turing invariant functions on the reals.

We will actually give two proofs of part 1 of Martin's conjecture for measure preserving functions. The first proof is very simple—it does not use much more than the general framework that we set up in section 2.1—but it does not work in $\mathsf{ZF} + \mathsf{AD}$ and instead requires that we either work in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ or $\mathsf{ZF} + \mathsf{AD}^+$ (two strengthenings of AD). The second proof works in $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, but is a bit more complicated. In particular, it uses the notion of ordinal invariants of Turing degrees (see section 2.2).

Before we discuss either proof in more detail, let's first recall our general framework for proving instances of part 1 of Martin's conjecture. We are given a Turing invariant function f which we want to show is above the identity in the Martin order. The basic idea is that we should try to find a computable function g which is below f and injective on a pointed perfect tree, T. The point of finding such a g is that its existence implies that $f(x) \oplus T$ computes x for all x on a cone. The main difficulties in using this strategy lie in figuring out how to find g so that it can be made injective and in figuring out how to use the fact that $f(x) \oplus T$ computes x to show that $f(x)$ by itself computes x.

So how do we get around these difficulties in the case of measure preserving functions? First note that the definition of "measure preserving" is tailor-made to make the second difficulty vanish—if f is measure preserving then $f(x)$ computes any constant on a cone, so in particular there is a cone on which $f(x)$ computes T. So the entire difficulty in proving part 1 of Martin's conjecture for measure preserving functions consists of figuring out how to get around the first problem—how to find a computable function below f which can be made injective on a pointed perfect tree.

As a warm-up to our main result, we will first present a special case of part 1 of Martin's conjecture for measure preserving functions which was first proved by Martin not long after formulating his conjecture. In this special case, we assume that the function f is also regressive[1]. This extra assumption implies that f is already basically a computable function and so by the tree thinning lemma (Lemma 2.7), it is either constant on a cone or injective on a pointed perfect tree. Thus in this special case, we can just let the function g that we are trying to find be f itself.

When we consider an arbitrary measure preserving function, things are not so simple, of course. Instead of letting g be f itself we will have to be a bit more clever to find a suitable g. In our first proof, we will find g as follows. For any measure preserving function f, we can

[1]So this case follows from Slaman and Steel's result on regressive functions.

find an associated function called a "modulus" for f, which is essentially a Skolem function witnessing that f is measure preserving. We will pick a modulus for f and then use Lemma 2.11 (and more specifically, Corollary 2.12 of that lemma) to find a right inverse for the modulus. This inverse is the function g that we are trying to find. It is injective because it is a right inverse of another function, and we will see that it is computable and below f due to the definition of "modulus."

There is one catch to this proof, however: it is not clear that $\mathsf{ZF} + \mathsf{AD}$ implies that every measure preserving function has a modulus. If we are willing to work in a stronger theory like $\mathsf{AD}_{\mathbb{R}}$ or AD^+ then it is provable and so, as we have mentioned, our first proof is really a proof from either $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ or $\mathsf{ZF} + \mathsf{AD}^+$.

We will remedy this flaw in our second proof, but at the cost of making the proof a bit more complicated. Here is the idea of the second proof. Instead of finding an actual modulus for f, we will instead define an ordinal invariant that allows us to approximate the effect of having a modulus. Very roughly, the ordinal invariant will be a function $\alpha\colon 2^\omega \to 2^\omega$ such that $\alpha(x)$ is the minimal value of ω_1^y over all y that could have been the image of x under a modulus for f. We will then use the computable uniformization lemma (Lemma 2.11) to find a computable function g which is below f and which preserves α.

Later in this chapter, we will explain much more carefully what we mean by all of this. But first we will discuss the definition of a measure preserving function on the Turing degrees.

5.1 Equivalent Definitions of Measure Preserving

In the introduction to this chapter, we mentioned that the class of measure preserving functions has several different equivalent definitions. In this section we will formally present these definitions and prove that they are equivalent. Each definition emphasizes a different aspect of measure preserving functions, and we will also comment on this.

Combinatorial Definition

Let's start with our official definition of measure preserving functions.

Definition 5.1. *A Turing invariant function $f\colon 2^\omega \to 2^\omega$ is* ***measure preserving*** *if for every z there is some y such that*

$$x \geq_T y \implies f(x) \geq_T z.$$

In other words, for every z, f gets above z on a cone.

We chose this as our official definition of "measure preserving" because in practice it is often the most useful to work with. Intuitively, it can be thought of as saying that the function f is "going to infinity in the limit" or "eventually getting above every constant."

Martin Order Definition

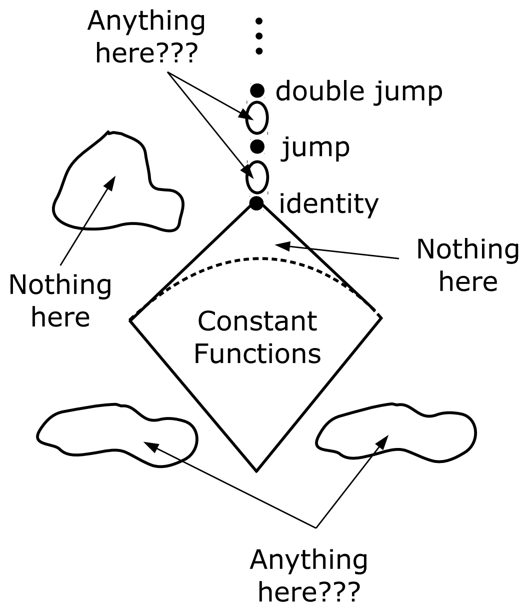
Our next definition of measure preserving functions emphasizes their role in the Martin order: they are just those functions which are Martin above every constant functions. The proof basically just consists of unrolling all the definitions involved and checking that these two characterizations are equivalent.

Proposition 5.2. *A Turing invariant function $f\colon 2^\omega \to 2^\omega$ is measure preserving if and only if f is Martin above every constant function.*

Proof. Let a be any Turing degree. A Turing invariant function f is above the constantly-a function in the Martin order if and only if there is a cone on which f is above a. Therefore a Turing invariant function f is above every constant function in the Martin order if and only if for every Turing degree a, there is a cone on which f is above a. And this condition is identical to the definition of "measure preserving." □

This characterization of measure preserving functions allows us to fit our proof of part 1 of Martin's conjecture for measure preserving functions into a developing picture of what the Martin order looks like under determinacy.

It is relatively easy to use determinacy to show that if a Turing invariant function is Martin below a constant function then it must be constant on a cone. Thus the constant functions form an initial segment of the Martin order which is identical to the partial order of the Turing degrees. This initial segment has a natural upper bound—the identity function. The results of the previous section show that this is actually a *minimal* upper bound: any function which is below the identity but above every constant function must be equivalent to the identity. Part 1 of Martin's conjecture for measure preserving functions shows that it is, in fact, a *least* upper bound: any function which is an upper bound for all the constant functions must be above the identity. And Slaman and Steel's result on regressive functions shows that it is not an upper bound for any non-constant function. The remaining case of part 1 of Martin's conjecture is to rule out functions which are "off to the side" of the constant functions in the Martin order—for example, functions which are incomparable to all nonzero constant functions. And part 2 of Martin's conjecture asks to show anything strictly above the identity is also above the jump (so there are no functions in-between the identity and the jump or above the identity but incomparable to the jump), and similarly for the jump and the double jump, and so on. The following picture summarizes all of this.



Measure Theoretic Definition

Our third definition of measure preserving functions emphasizes their relationship to the Martin measure: they are exactly those functions which preserve Martin measure (which is the reason that we chose to call such functions "measure preserving"). To make sense of this, we first need to review some concepts from measure theory.

First, if we have a measure μ on a set X and a function $f\colon X \to Y$, then there is a canonical way of getting a measure on Y, called the pushforward of μ by f.

Definition 5.3. *If μ is a measure on a space X and $f\colon X \to Y$ is a function then the* ***pushforward*** *of μ by f, denoted $f_*\mu$, is the measure on Y given by*

$$f_*\mu(A) = \mu(f^{-1}(A)).$$

Proposition 5.4. *The pushforward of a measure is itself a measure.*

Proposition 5.5. *Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are functions and μ is a measure on X. Then $g_*(f_*(\mu)) = (g \circ f)_*(\mu)$.*

We can now define "measure preserving."

Definition 5.6. *If μ is a measure on a space X and $f\colon X \to X$ is a function then f is* ***measure preserving*** *for μ, or sometimes is said to* ***preserve*** *μ, if $f_*\mu = \mu$.*

In this thesis we are mostly only concerned with measures that are ultrafilters on the Turing degrees. Recall that an ultrafilter on a set X can be considered a $\{0, 1\}$-valued measure on X (more precisely, on the σ-algebra $\mathcal{P}(X)$). The next proposition tells us that we can talk about the pushforwards of ultrafilters without worrying about measures which are not ultrafilters.

Proposition 5.7. *The pushforward of an ultrafilter is itself an ultrafilter.*

Suppose that U is an ultrafilter on a set X, V is an ultrafilter on a set Y and $f\colon X \to Y$ is any function. If we want to know whether $f_*U = V$ then if we simply use the definition of pushforward, we must check that for each $A \in V$, $f^{-1}(A) \in U$ and for each $A \notin V$, $f^{-1}(A) \notin U$. The following lemma tells us that because U and V are ultrafilters, this condition can be simplified somewhat.

Lemma 5.8. *If U is an ultrafilter on X, V is an ultrafilter on Y and $f\colon X \to Y$ then $f_*U = V$ if and only if for all $A \in U$, $f(A) \in V$.*

Proof. ($\Longrightarrow$) First, suppose that $f_*U = V$ and let A be any set in U. By the definition of pushforward, $f(A)$ is in V if and only if $f^{-1}(f(A))$ is in U. Since $f^{-1}(f(A))$ clearly contains A and A is in U, we can conclude that $f(A)$ is in V.

($\Longleftarrow$) Now suppose that for all $A \in U$, $f(A) \in V$. We need to show that for each $B \subset Y$, $B \in V$ if and only if $f^{-1}(B) \in U$.

First assume $f^{-1}(B)$ is in U. Then by our earlier assumption, $f(f^{-1}(B))$ is in V and therefore so is B, since it is a superset of $f(f^{-1}(B))$.

Now assume that $f^{-1}(B)$ is not in U. Since U is an ultrafilter, this means $(f^{-1}(B))^C = f^{-1}(B^C)$ *is* in U. By the reasoning in the preceding paragraph, this means that B^C is in V and hence that B is not. □

We can now present the third equivalent definition of a measure preserving function on the Turing degrees.

Proposition 5.9. *A Turing invariant function $f\colon 2^\omega \to 2^\omega$ is measure preserving if and only if the function F that it induces on the Turing degrees is measure preserving for Martin measure.*

Proof. ($\Longrightarrow$) Suppose f is measure preserving (in the sense of Definition 5.1). By proposition 5.8, we just need to show that if a subset A of the Turing degrees contains a cone then $F(A)$ contains a cone. By determinacy, it is enough to show that $F(A)$ is cofinal in the Turing degrees. So let $\mathbf{x}$ be any Turing degree and we need to show that there is some degree above $\mathbf{x}$ which is in $F(A)$. Since f is measure preserving, there is some degree $\mathbf{y}$ such that on the cone above $\mathbf{y}$, F is above $\mathbf{x}$. To finish the proof, just take some such $\mathbf{z}$ which is in the intersection of A with the cone above $\mathbf{y}$.

($\Longleftarrow$) Suppose F is measure preserving for Martin measure. Let x be any real. We need to find some real so that on the cone above that real, f is always above x. Since F preserves Martin measure, $F^{-1}(\mathrm{Cone}(x))$ must contain a cone. Let y be a base of this cone. Then for any $z \geq_T y$, $F(z) \in \mathrm{Cone}(x)$ and hence $z \geq_T x$. □

This third characterization allows us to connect part 1 of Martin's conjecture for measure preserving functions to a partial order on ultrafilters, called the Rudin-Keisler order. We will pick up this thread in a later section and comment on possible strategies that it suggests for proving the full part 1 of Martin's conjecture.

5.2 Warm-up: Regressive Measure Preserving Functions

As we mentioned in the introduction to this chapter, this section can be seen as a warm-up to our main result that part 1 of Martin's conjecture holds for all measure preserving functions. It also forms perhaps the simplest case study of how to use the framework we established in section 2.1.

In this section, we will prove that part 1 of Martin's conjecture holds for all measure preserving functions which are also regressive. The assumption that the function is regressive removes the main obstacle from the proof of the theorem for all measure preserving functions and gives a clearer picture of how the general framework works and why it is useful to assume that the function is measure preserving.

It turns out that this result actually has a long history. It was first proved by Martin in the 1970s (though he did not use the term "measure preserving" in stating it). Martin never published this result, but a proof was included in a later paper by Steel [Ste82]. Later, Slaman and Steel proved that a modified version of this theorem is still true even in ZFC [SS88]. This is somewhat surprising because other special cases of Martin's conjecture tend to have counterexamples in ZFC (even when the statement is modified in various ways to avoid trivial counterexamples). We will explain Slaman and Steel's result later on in this section.

Theorem 5.10 (ZF + AD; Martin)**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is regressive and measure preserving then $f(x) \geq_T x$ on a cone.*

Proof. If we follow the framework from section 2.1, the first step is to find a computable function below f which is injective on a pointed perfect tree. The idea is that since f is regressive, we can just use f itself. Let's explain this more carefully.

By determinacy, we can find a pointed perfect tree, T, and a Turing functional, Φ, such that for all $x \in [T]$, $f(x) = \Phi(x)$. By the tree thinning lemma (Lemma 2.7), either Φ is constant on a pointed perfect subtree of T or Φ is injective on a pointed perfect subtree of T. In the former case, Φ is constant on a cone, which contradicts the assumption that f is measure preserving. So let's assume we are in the latter case and replace T with a pointed perfect subtree on which Φ is injective.

Now that we have a computable, injective function below f, we are in business. By the lemma on inverting computable injective functions (Lemma 2.3), for all $x \in [T]$, $\Phi(x) \oplus T$ computes x. Since f is equal to Φ on $[T]$, this means that $f(x) \oplus T$ computes x on a cone.

And since f is measure preserving, $f(x)$ computes T on a cone. Taking the intersection of these two cones, we get a cone where $f(x)$ computes x and thus we are done. □

It is easy to construct a counterexample to Theorem 5.10 in ZFC. But, as we mentioned above, Slaman and Steel showed that it is possible to modify the statement of the theorem to get a statement that *is* provable in ZFC. The modification is simply to replace "on a cone" with "cofinally in the Turing degrees." As we mentioned above, the fact that this result remains true in ZFC once we make this modification seems to be a rather rare phenomenon among special cases of Martin's conjecture.

Theorem 5.11 (ZFC; Slaman-Steel [SS88]). *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is regressive and measure preserving then for cofinally many x, $f(x) \geq_T x$.*

Proof. Let x be any Turing degree. We need to show that for some $y \geq_T x$, $f(y) \geq_T y$. We will choose y to be a minimal upper bound for a certain sequence of degrees.

We choose this sequence of degrees as follows. Since f is measure preserving, there is some a_0 large enough that on the cone above it, f is above x. By increasing a_0 if necessary, we may also assume that $a_0 \geq_T x$. Likewise, there is some $a_1 \geq_T a_0$ such that on the cone above a_1, f is above a_0. We can continue in this way to pick a sequence $a_0 \leq_T a_1 \leq_T a_2 \leq_T \ldots$ such that for each n, f is above a_n everywhere in the cone above a_{n+1}.

Now, as we said, let y be a minimal upper bound for the sequence $a_0, a_1, a_2, \ldots$ (such minimal upper bounds were first constructed by Sacks). By our choice of the sequence, $y \geq_T x$. And for each a_n, since y is in the cone above a_{n+1}, our choice of a_{n+1} implies that $f(y) \geq_T a_n$. Therefore $f(y)$ is also an upper bound for the sequence and since f is regressive, $f(y) \leq_T y$. So minimality of y implies that $f(y) \equiv_T y$, as desired. □

One final comment: at first it may appear that the two proofs in this section are quite different, but there is actually a hidden similarity. First recall that Sacks' proof that every countable set of Turing degrees has a minimal upper bound uses forcing with pointed perfect trees. Now observe that pointed perfect trees also play a key role in most uses of determinacy in computability theory, including the proof of Theorem 5.10. It is easy to dismiss this methodological consonance as mere coincidence, but I believe something deeper is going on. We will return to this issue later in this chapter when we discuss ultrafilters on the Turing degrees.

5.3 Part 1 of Martin's Conjecture for Measure Preserving Functions

In this section we will give a proof of part 1 of Martin's conjecture for measure preserving functions. As we mentioned in the introduction to this chapter, the proof we give in this section is relatively straightforward but seems to require that we either assume $\mathsf{AD}_\mathbb{R}$ or AD^+ rather than just AD. In the next section, we will give a proof that works in $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$.

The proof from the previous section, the proof from this section and the proof from the next section can all be seen as following the framework from section 2.1 but getting around the main obstacle from that framework in three different ways. Recall that in that framework we are given a Turing invariant function f and we want to find a computable injective function below f. In the previous section, we assumed that f is regressive and so we could basically let this function be f itself. In this section we will find this function by inverting a certain function associated to f, called a modulus for f. In the next section, we will use the computable uniformization lemma (Lemma 2.11) to find a computable function below f which preserves a certain ordinal invariant and then use the tree thinning lemma (Lemma 2.7) to find a pointed perfect tree on which this function is injective. The ordinal invariant is used to make sure we can carry out this last step without getting stuck (i.e. to make sure the function we've found is not constant on any pointed perfect tree).

Modulus of a Measure Preserving Function

We will now explain what we mean be a "modulus for a measure preserving function"—essentially this is just a Skolem function witnessing that the original function is measure preserving.

Definition 5.12. *Suppose $f\colon 2^\omega \to 2^\omega$ is a measure-preserving function. A* ***modulus*** *for f is a function $g\colon 2^\omega \to 2^\omega$ such that for all x and all $y \geq_T g(x)$ we have $f(y) \geq_T x$. In other words, for each x, $g(x)$ is the base of a cone that f takes into the cone above x.*

For convenience, we will only use moduluses which are above the identity. We will call such a modulus an increasing modulus.

Definition 5.13. *If $f\colon 2^\omega \to 2^\omega$ is a measure preserving function then a modulus g for f is an* ***increasing modulus*** *for f if for all x, $g(x) \geq_T x$.*

We will first show that every measure preserving function has an increasing modulus. This is the only part of the proof where something stronger than AD is needed.

Lemma 5.14 ($\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$)**.** *If f is a measure preserving function then f has an increasing modulus.*

Proof. This lemma would be obvious if we could use the Axiom of Choice; the tricky part is to get it without choice. To do so, we need to use a uniformization principle which is not provable from AD, but is provable from stronger hypotheses such as $\mathsf{AD}_\mathbb{R}$. Let me explain. Let $R(x, y)$ be the binary relation defined by

$$R(x, y) \iff x \leq_T y \text{ and } \forall z \geq_T y\, (f(z) \geq_T x).$$

Finding an increasing modulus for f just means finding a function g such that for each x, $R(x, g(x))$ holds. Since f is measure preserving, we know that for each x, the set $\{y \mid R(x, y)\}$ is nonempty. The trouble is finding some way to pick on element from each of these sets.

In other words, how can we uniformize R? If we are working under AD then it is not clear what to do—$\mathsf{ZF}+\mathsf{AD}$ cannot prove that all binary relations on the reals can be uniformized. This is why we must invoke $\mathsf{AD}_{\mathbb{R}}$, which *does* prove that every binary relation on the reals can be uniformized. $\square$

The Proof

We are now ready to prove the theorem. Here's how the proof works. We start with a measure preserving function f. Using $\mathsf{AD}_{\mathbb{R}}$, we can pick an increasing modulus, g, for f. We can then use lemmas from section 2.1 to invert g on a pointed perfect tree and show that this inverse is below f in the Martin order, but also (up to the addition of a constant) above the identity. Since f is eventually above every constant, this implies that f is above the identity.

Theorem 5.15 (ZF + $\mathsf{AD}_{\mathbb{R}}$)**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant, measure preserving function then f is above the identity on a cone.*

Proof. Suppose f is a measure preserving function. By lemma 5.14 we can find an increasing modulus g for f. By Corollary 2.12 we can invert g on a pointed perfect tree—that is, there is a pointed perfect tree T and a Turing functional Φ such that for each $x \in [T]$, $\Phi(x)$ is total and $g(\Phi(x)) = x$. Now let's review what we know about this Φ.

- Φ is injective on $[T]$: if $\Phi(x) = \Phi(y)$ for $x, y \in [T]$ then $x = g(\Phi(x)) = g(\Phi(y)) = y$. Hence by lemma 2.3, for all $x \in [T]$, $\Phi(x) \oplus T \geq_T x$.
- Φ is below f on $[T]$: if $x \in [T]$ then since g is a modulus for f, $f(g(\Phi(x)))$ computes $\Phi(x)$. And since $g(\Phi(x)) = x$, this just means that $f(x)$ computes $\Phi(x)$.

Furthermore, since f is measure preserving, $f(x)$ is above T on a cone. Putting all this together, for any x in $[T]$ which is in this cone,

$$f(x) \geq_T \Phi(x) \oplus T \geq_T x.$$

Since f is Turing invariant and $[T]$ contains a representative of every Turing degree on a cone, this implies that f is above the identity on a cone. $\square$

Using AD^+ Instead of $\mathsf{AD}_{\mathbb{R}}$

We claimed earlier that the proof in this section works under either $\mathsf{AD}_{\mathbb{R}}$ or AD^+. The proof we gave above used $\mathsf{AD}_{\mathbb{R}}$, but we will now briefly indicate how to modify the proof to make it work in $\mathsf{ZF} + \mathsf{AD}^+$. This is intended only for those already familiar with AD^+; we will make no attempt to explain what AD^+ is or the facts about it that we invoke.

The main AD^+ fact that we will use is the Σ^2_1 reflection theorem. The key point is that the existence of a Turing invariant, measure preserving function $f\colon 2^\omega \to 2^\omega$ such that f is not above the identity on a cone is a Σ^2_1 sentence and so if it holds then it must be witnessed by a function whose graph is Suslin.

The idea of the proof is to fix such a function f (i.e. a Turing invariant, measure preserving function which is not above the identity on a cone and whose graph is Suslin) and follow the argument we gave above to show that f is above the identity on a cone, thus reaching a contradiction. At the point in the proof above when we had to invoke $\mathsf{AD}_{\mathbb{R}}$, we can instead use the fact that f is Suslin.

More precisely, the one place in the proof when we needed to use $\mathsf{AD}_{\mathbb{R}}$ was to show that there is an increasing modulus for f. But if f is Suslin then we no longer need $\mathsf{AD}_{\mathbb{R}}$ to establish this. To see why, consider the same binary relation, R, that we considered in the $\mathsf{AD}_{\mathbb{R}}$ proof:

$$R(x, y) \iff x \leq_T y \text{ and } \forall z \geq_T y\,(f(z) \geq_T x).$$

Since f is Suslin, the closure properties of Suslin sets imply that R is also Suslin. In other words, there is a cardinal κ and a tree S on $\omega \times \omega \times \kappa$ such that

$$R(x, y) \iff (x, y) \in p[S].$$

Taking the first coordinate of the left most branch of S_x gives the desired increasing modulus for f.

5.4 Part 1 of Martin's Conjecture for Measure Preserving Functions: AD Proof

In the previous section, we saw how to prove part 1 of Martin's conjecture for measure preserving functions under either $\mathsf{AD}_{\mathbb{R}}$ or AD^+. In this section we will see how to prove it from just $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ (though we really do need $\mathsf{DC}_{\mathbb{R}}$ rather than just countable choice, in contrast to most proofs of statements related to Martin's conjecture).

Recall that the reason we needed $\mathsf{AD}_{\mathbb{R}}$ or AD^+ in the previous section was to show that every measure preserving function has a modulus. In this section we will get around that difficulty by using an ordinal invariant (a concept which was introduced in section 2.2) to approximate the role of the modulus in the previous proof.

We will start by defining something called a "modulus sequence," which can be thought of as a countable fragment of a modulus function for f.

Definition 5.16. *Suppose $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is measure preserving and $x \in 2^\omega$. A* ***modulus sequence*** *for x is a sequence of reals $x = x_0 \leq_T x_1 \leq_T x_2 \leq_T \ldots$ which is increasing in the Turing degrees and such that for all $n \in \mathbb{N}$ and all $y \in 2^\omega$,*

$$y \geq_T x_{n+1} \implies f(y) \geq_T x_n.$$

In other words, x_1 is large enough that f is above x on the cone above x_1, x_2 is large enough that f is above x_1 on the cone above x_2, and so on.

The idea is that if g is an increasing modulus for f (see the definitions from the last section for context) then $x, g(x), g(g(x)), \ldots$ is a modulus sequence for x, but that it also makes sense to talk about modulus sequences even when we cannot prove that f has a modulus. And it is easy to see that the amount of choice required to prove that modulus sequences exist is much weaker than the amount of choice that seems to be required to prove that modulus functions exist. This is expressed by the following lemma.

Lemma 5.17 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$)**.** *Suppose $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is measure preserving. Then for all $x \in 2^\omega$, there is a modulus sequence for x.*

Proof. Let x be an arbitrary real. Note that a sequence $x_0, x_1, x_2, \ldots$ is a modulus sequence for x as long as $x_0 = x$ and for each n, x_{n+1} satisfies a certain condition with respect to x_n. It is easy to see that since f is measure preserving, no matter what x_n we have picked, there is some x_{n+1} which satisfies this condition with respect to it. Thus we can use $\mathsf{DC}_{\mathbb{R}}$ to pick a modulus sequence for x. □

We can now prove the theorem. Before we actually give the proof, let's briefly recall how ordinal invariants can be used to carry out the general strategy from section 2.1. Recall that to prove that a measure preserving function f is above the identity, the main thing we need to do is find a computable function g which is below f and can be made injective on a pointed perfect tree. It is easy to use the computable uniformization theorem to find functions g which are computable and below f. But it is hard to ensure that they are not just constant on a cone. This is where ordinal invariants come in. If we can find an ordinal invariant, α, and a function g such that $\alpha(x) = \alpha(g(x))$ for all x then as long as α is not constant on a cone, g cannot be constant on any pointed perfect tree. And as long as g is computable, this means we can use the tree thinning lemma to find a pointed perfect tree on which g is injective.

So the goal of the proof below is to come up with an ordinal invariant α for which we can find a computable function g which is both below f and preserves α. And by the computable uniformization lemma, to find such a g, it is enough to show that for cofinally many x, there is some y which is below both x and $f(x)$ and such that $\alpha(y) = \alpha(x)$. The crux of the proof, then, is to construct some ordinal invariant α that will allow us to prove this.

Theorem 5.18 ($\mathsf{ZF}+\mathsf{AD}$)**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant, measure preserving function then f is above the identity on a cone.*

Proof. First we will define our ordinal invariant. Let $\alpha\colon 2^\omega \to \mathrm{Ord}$ be the function defined by

$$\alpha(x) = \min\{\sup_{n\in\mathbb{N}} \omega_1^{x_n} \mid \langle x_n \rangle_n \text{ is a modulus sequence for } x\}.$$

Observe that $\alpha(x)$ is always a countable ordinal which is at least ω_1^x and therefore $\alpha(x)$ is not constant on any cone (because any cone contains reals whose ω_1's are bigger than any fixed countable ordinal).

Our goal now is to find a computable function defined on a pointed perfect tree which is below f and which preserves α. It is enough to show that the following set is cofinal in the Turing degrees.

$$A = \{x \mid \exists y\,(y \leq_T x \text{ and } y \leq_T f(x) \text{ and } \alpha(y) = \alpha(x))\}.$$

Why is this enough? Suppose for the moment that A is cofinal, and we will explain how to finish the proof. Since A is cofinal, the computable uniformization lemma (Lemma 2.11) implies that there is a Turing functional Φ and a pointed perfect tree T such that for each $x \in [T]$, $\Phi(x)$ is total, computable from $f(x)$ and satisfies $\alpha(\Phi(x)) = \alpha(x)$.

We now want to apply the tree thinning lemma (Lemma 2.7) to find a pointed perfect subtree of T on which Φ is injective. To do this, it is enough to show that Φ is not constant on any pointed perfect tree. And since $\alpha(\Phi(x)) = \alpha(x)$ for all x and α is Turing invariant, it is enough to show that α is not constant on any cone, which we have already observed to be the case.

So we may assume that Φ is injective on $[T]$. Therefore by Lemma 2.3, for every $x \in [T]$, $\Phi(x) \oplus T$ computes x. And since f is measure preserving, $f(x)$ computes T on a cone. Let x be any real in this cone. By increasing the base of the cone if necessary, we may assume $x \geq_T T$. And since $[T]$ contains a representative for every Turing degree in the cone above T (and f is Turing invariant), we may assume x is actually in $[T]$. Thus $f(x)$ computes both $\Phi(x)$ (by our choice of Φ) and T. Hence we have

$$f(x) \geq_T \Phi(x) \oplus T \geq_T x.$$

In other words, $f(x)$ is above the identity on a cone.

Why is the set cofinal? We now want to show A is cofinal in the Turing degrees, which will complete the proof. So let x be an arbitrary real and we need to find an element of A which computes x. To do so, let $x = x_0 \leq_T x_1 \leq_T x_2 \leq_T \ldots$ be a modulus sequence for x which witnesses the value of $\alpha(x)$ (i.e. such that $\alpha(x) = \sup_n \omega_1^{x_n}$).

We now claim that x_1 is in A, as witnessed by x. By the definition of modulus sequence, it is clear that $x \leq_T x_1$ and $x \leq_T f(x_1)$. So we just need to show that $\alpha(x) = \alpha(x_1)$. First observe that $\alpha(x_1)$ cannot be larger than $\alpha(x)$ because $x_1, x_2, x_3, \ldots$ is a modulus sequence for x_1 and so

$$\alpha(x_1) \leq \sup\{\omega_1^{x_1}, \omega_1^{x_2}, \ldots\} = \sup\{\omega_1^{x}, \omega_1^{x_1}, \omega_1^{x_2}, \ldots\} = \alpha(x).$$

And next observe that $\alpha(x_1)$ cannot be smaller than $\alpha(x)$ because if $x_1 = y_0 \leq_T y_1 \leq_T y_2 \leq_T \ldots$ is a modulus sequence for x_1 witnessing the value of $\alpha(x_1)$ then $x, y_0, y_1, y_2, \ldots$ is a modulus sequence for x and so

$$\alpha(x) \leq \sup\{\omega_1^{x}, \omega_1^{y_0}, \omega_1^{y_1}, \ldots\} = \sup\{\omega_1^{y_0}, \omega_1^{y_1}, \omega_1^{y_2}, \ldots\} = \alpha(x_1). \qquad \square$$

What about Borel Functions?

In the previous section, we saw how to prove part 1 of Martin's conjecture for measure preserving functions using either $\mathsf{AD}_{\mathbb{R}}$ or AD^+. A careful examination of that proof shows that when the functions considered are restricted to Borel functions, the proof requires $\mathbf{\Pi}^1_1$ determinacy. And the reason that the proof requires $\mathbf{\Pi}^1_1$ rather than Borel determinacy is similar to the reason that the proof for all functions required something more than just AD.

Recall that to prove a measure preserving function f has a modulus, we had to uniformize the following relation.

$$R(x, y) \iff x \leq_T y \text{ and } \forall z \geq_T y\, (f(z) \geq_T x).$$

When f is Borel, this relation is $\mathbf{\Pi}^1_1$ and so the Kondo-Addison theorem says that it has a $\mathbf{\Pi}^1_1$ uniformization (but not necessarily a Borel uniformization). Thus every Borel measure preserving function has a $\mathbf{\Pi}^1_1$ modulus. Since the rest of the proof needs to apply determinacy to sets defined using the modulus, the fact that the modulus is only guaranteed to be $\mathbf{\Pi}^1_1$ rather than Borel causes the proof to require $\mathbf{\Pi}^1_1$ determinacy rather than Borel determinacy.

In this section, we saw how to prove part 1 of Martin's conjecture for measure preserving functions using just $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ rather than $\mathsf{AD}_{\mathbb{R}}$ or AD^+. In light of the above discussion, it is reasonable to ask whether this yields a proof for Borel functions that only requires Borel determinacy (and thus works in ZF). Somewhat surprisingly, the answer seems to be "no." That is, the proof in this section seems to use more than Borel determinacy even when the functions considered are Borel. This is because the definition of the ordinal invariant that we use in the proof is rather complicated and so the set that we need to apply determinacy to in the proof is not Borel.

This is interesting in part because it seems to somewhat contradict the idea that proofs of Martin's conjecture should only use determinacy in a "local" way (that is, the proof for Borel functions should only require Borel determinacy, and so on). It would be interesting to know whether part 1 of Martin's conjecture for measure preserving Borel functions can be proved using just Borel determinacy.

5.5 Other Degree Structures

One of the running themes of this thesis is to track the differences in how Martin's conjecture works in different degree structures, with a particular focus on the Turing degrees, arithmetic degrees and hyperarithmetic degrees. As we saw in the introduction, there are some big differences: there are known counterexamples to Martin's conjecture on the arithmetic degrees, including in cases which are known to hold for the Turing degrees (in particular, uniformly invariant functions and order preserving functions not above the identity). In this section, we will see a rare instance of concurrence between all three degree structures. Namely, part 1 of Martin's conjecture for measure preserving functions holds in all three, with essentially the same proof (the only other instance of this that we are aware of is part 2 of Martin's conjecture for order preserving functions).

More generally, both of our proofs of part 1 of Martin's conjecture for measure preserving functions seem to be very flexible and can be made to work for most degree structures. Rather than recapitulate our proof in detail for lots of different degree structures, we will just give a sketch of the key points for the case of the arithmetic degrees. We will use the $\mathsf{AD}_{\mathbb{R}}$ proof since it is simpler to explain, but our second proof can also be easily adapted to the arithmetic degrees.

For the sake of completeness, we begin by stating the definition of "measure preserving" for arithmetically invariant functions.

Definition 5.19. *An arithmetically invariant function $f\colon 2^\omega \to 2^\omega$ is* ***measure preserving*** *if for every z there is some y such that*

$$x \geq_A y \implies f(x) \geq_A z.$$

In other words, for every z, f is arithmetically above z on a cone of arithmetic degrees.

Theorem 5.20 ($\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$)**.** *If $f\colon 2^\omega \to 2^\omega$ is an arithmetically invariant, measure preserving function then $f(x) \geq_A x$ on a cone of arithmetic degrees.*

Proof. First, we use uniformization for binary relations on the reals (which is provable in $\mathsf{AD}_{\mathbb{R}}$) to pick an increasing modulus, g, for f. There is one subtlety here: we want g to be increasing not only on the arithmetic degrees, but also on the Turing degrees. In other words, we want g to be a function such that

$$g(x) \geq_T x \text{ and } y \geq_A g(x) \implies f(y) \geq_A x.$$

This may look a little unintuitive: why not just require $g(x) \geq_A x$? The reason is that some of the lemmas we would like to invoke depend on finding a *computable* injective function on a pointed perfect tree and we cannot find a computable inverse for g unless $g(x) \geq_T x$. In the case of the arithmetic degrees, this is sort of a fake problem—it is actually possible to prove versions of all the theorems we need that work with arithmetic functions rather than computable functions. But doing so requires more work than was required for the original lemmas and does not seem to work in all degree structures, whereas the approach of requiring $g(x)$ to compute x *does* seem to work in great generality.

In any case, there is no problem with requiring $g(x)$ to compute x. If we have found a real y such that on the cone above y, f is above x, then we can always take $y \oplus x$ to get the base of another such cone which computes x.

Now that we have found g, we can proceed with the rest of the proof more or less unchanged. By Corollary 2.12 we can find a pointed perfect tree, T, and a Turing functional, Φ, such that on $[T]$, Φ is a right inverse for g (in this case, it does not matter whether T is pointed in the sense of the Turing degrees or in the sense of the arithmetic degrees). For all $x \in [T]$, the definition of modulus implies that $f(x) \geq_A \Phi(x)$ and Lemma 2.3 implies that $\Phi(x) \oplus T \geq_T x$. Putting these together we have that $f(x) \oplus T \geq_A x$ on a cone of arithmetic degrees. Since f is measure preserving, it gets above T on a cone and thus $f(x) \geq_A x$ on a cone, as desired. □

The only things about the degree structure that seem required to make this proof work are that it satisfies something like Martin's pointed perfect tree theorem (Lemma 2.9) and its notion of reduction is reasonable enough to prove something like Corollary 2.12.

5.6 Non-invariant Functions

In the previous section, we saw that our proof of part 1 of Martin's conjecture for measure preserving functions is robust in the sense that it works in many different degree structures (and in particular, in the arithmetic degrees). In this section, we will see that it is robust in another sense: it works even for functions which are not Turing-invariant but which still satisfy the definition of a "measure preserving function on the Turing degrees." As in the previous section, both proofs we have given can easily be modified to work in this new setting, but we will just explain how to modify the $\mathsf{AD}_{\mathbb{R}}$ proof since it's a bit simpler.

Before we actually give the proof, we will mention one reason why this sort of result is interesting. If we remove the requirement of Turing invariance from Martin's conjecture then there are many counterexamples. In fact, practically every construction of classical computability theory gives a counterexample. As a concrete example, consider the Friedberg jump inversion theorem. The proof of this theorem is quite constructive, and in fact produces a (non-Turing-invariant) function $f\colon 2^\omega \to 2^\omega$ such that for each x, $f(x)$ is a real whose jump is x. Thus f is in fact a regressive function which is not constant on any cone nor ever above the identity. An interesting feature of most of these classical constructions is that the produce reals which are, in some sense, "generic." Sometimes this is true in a precise technical sense. For example, if f is the function produced by Friedberg jump inversion then $f(x)$ is actually 1-generic. In other cases, it is more of a vague intuition.

It is reasonable to ask whether there is a constructive proof of jump inversion that doesn't essentially just always produce a generic real. Of course, this is a hard question to make precise, but we can at least note some ways that reals can fail to look generic. One feature of most types of generic reals is that they avoid a cone—for example, if g is a function such that $g(x)$ is always x-generic then g avoids the cone above $0'$. And if $g(x)$ is instead always x-random then g does not necessarily avoid the cone above $0'$ (there are 1-random reals in every degree above $0'$), but as soon as x is above $0'$, it does. Thus any proof of jump inversion which produces a (non-invariant) measure preserving function would seem to be producing reals that were not generic in any reasonable sense. The theorem we prove below on non-Turing-invariant measure preserving functions shows this is not possible. In fact, it shows that if $f(x)$ is always inverting the jump of x then there actually is a cone that is disjoint from the range of f. In other words, that feature of the proof of jump inversion seems to be inescapable.

We will now turn to the proof itself. For completeness, we will begin by stating the definition of "measure preserving" for functions which are not Turing invariant.

Definition 5.21. *A function $f\colon 2^\omega \to 2^\omega$ is called* ***Turing measure preserving*** *if for all* *z there is some y such that*

$$x \geq_T y \implies f(x) \geq_T z.$$

Theorem 5.22 ($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ or $\mathsf{ZF} + \mathsf{AD}^+$)**.** *If $f\colon 2^\omega \to 2^\omega$ is Turing measure preserving (but not necessarily Turing invariant) then $f(x) \geq_T x$ on a cone.*

Proof. The proof is nearly identical to the proof for Turing invariant functions. Let g be an increasing modulus for f, and Φ a Turing functional which inverts g on a pointed perfect tree, T. The key point is that it follows from the definition of modulus that if y is in $[T]$ and $x \geq_T y$ then $f(x) \geq_T \Phi(y)$ (even though f is not Turing invariant).

To see why this is enough, suppose x is large enough to be Turing equivalent to something in T and so that $f(x) \geq_T T$. Let $y \in [T]$ be Turing equivalent to x. Then

$$f(x) \geq_T \Phi(y) \oplus T \geq_T y \equiv_T x.$$

In other words, $f(x) \geq_T x$ for all large enough x. □

The fact that our proof of part 1 of Martin's conjecture for measure preserving functions still works for functions which are not Turing invariant points to an interesting feature of the proof: it relies mostly on manipulating functions which are *not* themselves Turing invariant (even when the function f is). The way that we chose the modulus, g, came with no guarantees that the resulting function is Turing invariant. What's more, it follows from Slaman and Steel's theorem on regressive functions that no inverse for g can be Turing invariant (otherwise it would yield a non-constant, regressive function on the Turing degrees). It will be interesting to see whether non-invariant functions have other roles to play in resolving Martin's conjecture. As one example of what might be possible, Andrew Marks has conjectured that under AD, every function on the reals is either constant on a pointed perfect tree or injective on a pointed perfect tree. As we will explain later, if this is true it would have some implications for Martin's conjecture. Those implications would only use the conjecture applied to Turing invariant functions, but it seems plausible that the conjecture could be true in general and not a special feature of Turing invariant functions specifically.

5.7 ZFC Counterexample

Earlier in this chapter, we saw that it is possible to prove some cases of Martin's conjecture in ZFC rather than $\mathsf{ZF} + \mathsf{AD}$, as long as we replace "on a cone" with "cofinally." In particular, Theorem 5.11 showed that ZFC proves a version of Martin's conjecture for regressive measure preserving functions—i.e. if $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is regressive and measure preserving then $f(x) \geq_T x$ on a cofinal set of Turing degrees. In this section, we will show that the assumption that f is regressive cannot be removed. In other words, the main results of this chapter do not have a natural counterpart provable in ZFC. We prove this by showing that in ZFC there is a measure preserving function f such that $f(x)$ never computes x (except when x is computable).

The main idea is to write the Turing degrees as an increasing union of Turing ideals and define $f(x)$ to be a minimal upper bound for all the reals computable from x which first show up in a strictly earlier ideal than x itself. First we will review the definition of Turing ideals and a few facts about them.

Definition 5.23. *A **Turing ideal** is a set of Turing degrees I such that*

- *I is closed under Turing reducibility: if $\mathbf{x} \in A$ and $\mathbf{y} \leq_T \mathbf{x}$ then $\mathbf{y} \in I$.*
- *I is closed under finite joins: if $\mathbf{x}_1, \ldots, \mathbf{x}_n \in I$ then $\mathbf{x}_1 \oplus \ldots \oplus \mathbf{x}_n \in I$.*

Of course, to verify that a set I is a Turing ideal it is enough to check that I is closed under joining any two elements rather than all finite joins. Next we will state the facts about Turing ideals that we need.

Lemma 5.24 (ZFC)**.** *There is a strictly increasing, wellordered sequence of Turing ideals whose union is all of $\mathcal{D}_T$.*

Proof. Let $\langle \mathbf{r}_\alpha \rangle_{\alpha < 2^{\aleph_0}}$ be a wellordering of the Turing degrees. For each α, define I_α to be the smallest Turing ideal containing all $\mathbf{r}_\beta$'s for $\beta \leq \alpha$. That is, define

$$I_\alpha = \{\mathbf{x} \mid \exists \beta_1, \ldots, \beta_n \leq \alpha\, (\mathbf{x} \leq_T \mathbf{r}_{\beta_1} \oplus \ldots \oplus \mathbf{r}_{\beta_n}\}.$$

It is easy to show by induction that these ideals are increasing. And by removing duplicates, we can make sure they are strictly increasing. □

Lemma 5.25 (Spector; [Soa16] Exercise 6.5.12)**.** *If I is a nonempty, countable Turing ideal then there are Turing degrees $\mathbf{a}$ and $\mathbf{b}$ which form an exact pair for I—i.e. for all $\mathbf{x}$,*

$$\mathbf{x} \in I \iff (\mathbf{x} \leq_T \mathbf{a} \text{ and } \mathbf{x} \leq_T \mathbf{b}).$$

Lemma 5.26. *If I is a countable Turing ideal and $\mathbf{x}$ is an uncomputable Turing degree which is not contained in I then I has an upper bound which does not compute $\mathbf{x}$.*

Proof. If I is empty then this is trivial. Otherwise, let $\mathbf{a}$ and $\mathbf{b}$ be an exact pair for I as in the lemma above. They are both upper bounds for I and at most one of them can compute $\mathbf{x}$ since otherwise it would imply $\mathbf{x}$ is in I. □

Theorem 5.27 (ZFC)**.** *There is a Turing invariant function $f \colon 2^\omega \to 2^\omega$ such that f is measure preserving and for all uncomputable x, $f(x)$ does not compute x.*

Proof. Instead of defining a Turing invariant function on the reals, we will define a function on the Turing degrees (which is equivalent under ZFC). Let $\langle I_\alpha \rangle$ be a strictly increasing sequence of Turing ideals as in the lemma above.

Given a Turing degree $\mathbf{x}$, we will now explain how to define $F(\mathbf{x})$. First, let α be the least ordinal such that $\mathbf{x} \in I_\alpha$. Since the union of all the I_α's is all of $\mathcal{D}_T$, such an α must exist.

Next, let I_x be the following set

$$I_x = \{\mathbf{y} \mid \mathbf{y} <_T \mathbf{x} \text{ and } \exists \beta < \alpha\, (\mathbf{y} \in I_\beta)\}.$$

In other words, I_x consists of degrees $\mathbf{y}$ which are computable from $\mathbf{x}$ and are contained in a strictly earlier ideal than $\mathbf{x}$. It is easy to check that I_x is a countable Turing ideal which does not contain $\mathbf{x}$. The main point is that it is closed under finite joints because if $\mathbf{y}_1 \in I_{\beta_1}$ and $\mathbf{y}_2 \in I_{\beta_2}$ where $\beta_1 \leq \beta_2$ then $\mathbf{y}_1$ and $\mathbf{y}_2$ are both in I_{β_2} (since the sequence of ideals is increasing) and hence $\mathbf{y}_1 \oplus \mathbf{y}_2 \in I_{\beta_2}$ (by the definition of "Turing ideal").

If $\mathbf{x} = \mathbf{0}$ then we will simply set $F(\mathbf{x}) = \mathbf{0}$. Otherwise, by Lemma 5.26, we can find an upper bound for I_x which does not compute $\mathbf{x}$. Set $F(\mathbf{x})$ equal to this upper bound.

It is clear that $F(\mathbf{x})$ does not compute $\mathbf{x}$ for any uncomputable $\mathbf{x}$. So all that remains is to check that F is measure preserving. To this end, let $\mathbf{z}$ be any degree. We need to show that F gets above $\mathbf{z}$ on a cone.

Let α be the least ordinal such that $\mathbf{z} \in I_\alpha$ and let $\mathbf{y}$ be a Turing degree which computes $\mathbf{z}$ but is not contained in I_α (note that this automatically implies $\mathbf{y}$ is uncomputable). We claim that on the cone above $\mathbf{y}$, F is above $\mathbf{z}$. Let $\mathbf{x} \geq_T \mathbf{y}$. Since $\mathbf{y}$ is not in I_α and I_α is closed under Turing reducibility, $\mathbf{x}$ is not in I_α either. And since we chose an *increasing* sequence of ideals, this means that the least α' such that $\mathbf{x} \in I_{\alpha'}$ must be greater than α. Since $\mathbf{x}$ also computes $\mathbf{z}$, this implies $\mathbf{z} \in I_x$. Therefore $F(\mathbf{x})$ was chosen as an upper bound for a set that contains $\mathbf{z}$ and hence computes $\mathbf{z}$. □

5.8 Application to Part 2 of Martin's Conjecture

In this section we will show that our result on part 1 of Martin's conjecture for measure preserving functions has some applications to part 2 of Martin's conjecture. This application is due to Benjamin Siskind.

Recall that part 2 of Martin's conjecture says that the Martin order is a prewellorder on Turing invariant functions which are above the identity and that the successor in this prewellorder is given by the Turing jump. As a first step towards proving this, we could try to show that all functions above the identity are comparable in the Martin order. This would not show that the quotient of the Martin order by Martin equivalence is a wellorder, but it would at least show it's a linear order. We will not even show this, but we will show that if we have two Turing invariant functions f and g which are above the identity and which satisfy an additional assumption then they are comparable in the Martin order.

To understand the idea of the proof, consider the following way that one might try to show any two functions above the identity are comparable. Suppose we are given Turing invariant functions f and g and want to show $f \leq_M g$. Naively, we might try to "subtract" f from g and show that the resulting function is above the identity. That is, we might try to define a function h as follows. Given a real x, we first try to find a y such that $f(y) = x$ and then set $h(x) = g(y)$. This h can be thought of as the "difference" between g and f because for the y used to define $h(x)$, we have $h(f(y)) = g(y)$. If we could show that h is above the identity then it would be some indication that g is above f since their "difference" is "positive."

There are a number of obvious problems with this strategy. First, there is no reason to expect that any h we define this way will be Turing invariant, much less amenable to known techniques for proving functions are above the identity. And second, even if h is a Turing invariant function above the identity, it is not clear that this really implies that g is above f on a cone (for example, the y's we use to define h could all lie outside of some cone). The key insight of the proof below is that if f and g satisfy a certain additional condition then all these problems disappear.

Theorem 5.28 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *Suppose $f, g\colon 2^\omega \to 2^\omega$ are Turing invariant functions which are above the identity and such that for all $x, y \in 2^\omega$,*

$$f(x) \equiv_T f(y) \implies g(x) \equiv_T g(y).$$

Then $f \leq_M g$.

Proof. First note that since f and g are above the identity, their ranges must be cofinal and thus by determinacy must each contain a cone. Also note that since f is above the identity, we can use Corollary 2.12 to the computable uniformization lemma to find a (possibly non-Turing invariant) right inverse to f defined on a pointed perfect tree, T. Let's call this right inverse k.

Now define a function $h\colon 2^\omega \to 2^\omega$ as follows. Given any real x of sufficiently high Turing degree, let y be some real such that $f(y) \equiv_T x$ and define $h(x) = g(y)$. You might be worried that we cannot define h without using the axiom of choice because we have no way to choose a real y, but one way to do it is to use the function k we mentioned above (briefly, for each x of sufficiently high Turing degree, we can pick an element $\widetilde{x}$ of $[T]$ which is of the same degree as x and use $k(\widetilde{x})$ as our y).

Note first that because of the condition that f and g satisfy, the Turing degree of $h(x)$ does not depend on which y we used: if y_1 and y_2 are two degrees such that $f(y_1) \equiv_T x$ and $f(y_2) \equiv_T x$ then $g(y_1) \equiv_T g(y_2)$. Hence h is actually Turing invariant. Note that this also implies that for any real y, $h(f(y)) \equiv_T g(y)$.

Next, we will show that h is measure preserving. Let z be an arbitrary degree. We want to show that h gets above z on a cone. By determinacy, it is enough to show that it gets above z cofinally. So let y be an arbitrary real, and we will show that h is above z on some degree above y. I claim that $f(y \oplus z)$ is one such degree. Since f is above the identity, $f(y \oplus z)$ is above y. By the observation we have already made about h, $h(f(y \oplus z)) \equiv_T g(y \oplus z)$. And since g is above the identity, this implies $h(f(y \oplus z))$ is above z.

Since h is measure preserving, Theorem 5.18 implies that h is above the identity on a cone. We will now use this fact to show that f is below g on a cone. Let x be any degree in a cone on which h is above the identity. Since f is above the identity, $f(x)$ is above x. Since x was in a cone on which h is above the identity, this implies that $h(f(x)) \geq_T f(x)$. Since $h(f(x)) \equiv_T g(x)$, we have shown that $g(x) \geq_T f(x)$, as desired. $\square$

One might hope to use the theorem above to show that the Martin order is linear above the identity by showing that for every pair of Turing invariant functions f and g which are above the identity, the relationship required by the theorem holds for f and g in some order (or at least, holds on a cone).

At first, this does not seem like such an unreasonable hope. It does hold for many pairs of functions on the Turing degrees. For example, if two Turing degrees have the same Turing jump then they also have the same ω-jump. Likewise, if they have the same ω-jump then they also have the same hyperjump. But if we go just a little bit higher than the hyperjump, we can find examples of pairs of Turing invariant functions which do not have this sort of relationship. We give one such example below (which is also due to Siskind).

Example 5.29. Let $f\colon 2^\omega \to 2^\omega$ be the hyperjump, i.e. $f(x) = \mathcal{O}^x$, and let $g\colon 2^\omega \to 2^\omega$ be the function defined by

$$g(x) = (\mathcal{O}^x)^{(\omega_1^x)}.$$

The function g is well-defined because $\omega_1^{\mathcal{O}^x}$ is always strictly greater than ω_1^x and thus the ω_1^x-th jump of $\mathcal{O}^x$ is well-defined.

On the one hand, it is easy to see that there are reals x and y such that $f(x) \neq f(y)$ but $g(x) = g(y)$. To find such x and y we can take reals $\widetilde{x}$ and $\widetilde{y}$ which are above Kleene's $\mathcal{O}$ and not Turing equivalent but such that $\widetilde{x}^{(\omega_1^{\mathrm{CK}})} \equiv_T \widetilde{y}^{(\omega_1^{\mathrm{CK}})}$ and then use hyperjump inversion to find x and y which are low for ω_1^{CK} such that $\mathcal{O}^x = \widetilde{x}$ and $\mathcal{O}^y = \widetilde{y}$.

On the other hand, we can *also* find reals x and y such that $f(x) = f(y)$ but $g(x) \neq g(y)$. To see why, let x be some real such that $\omega_1^x > \omega_1^{\mathrm{CK}}$. We can use hyperjump inversion to find a real y such that $\omega_1^y = \omega_1^{\mathrm{CK}}$ and $\mathcal{O}^y = \mathcal{O}^x$. Since $\omega_1^x \neq \omega_1^y$, it is clear that $(\mathcal{O}^x)^{(\omega_1^x)} \neq (\mathcal{O}^y)^{(\omega_1^y)}$.

Actually, without even bothering to construct these examples, it should have been apparent that f and g cannot have the relationship required by the theorem above. Since $f <_M g$, we know that there must be x and y such that $g(x) = g(y)$ and $f(x) \neq f(y)$ since otherwise the theorem would imply that g is below f on a cone. On the other hand, if we look at the proof of the theorem we can also see it provides reason to believe that there are reals x and y such that $f(x) = f(y)$ and $g(x) \neq g(y)$. If not, then the proof of the theorem would imply that there is a Turing invariant function h such that $h(f(x)) = g(x)$ on a cone. Such an h would have to be above every function of the form $x \mapsto x^{(\alpha)}$ for a fixed countable ordinal α, but also below the hyperjump. But it seems plausible that no such function exists at all and it is known that such a function cannot be uniformly invariant or order preserving so we should not expect to be able to find it so easily.

5.9 Martin's Conjecture and the Rudin-Keisler Order

In the previous section we saw an application of part 1 of Martin's conjecture for measure preserving functions to part 2 of Martin's conjecture. In this section we will see that it also has another interesting consequence: it implies that the full part 1 of Martin's conjecture is equivalent to a statement about ultrafilters on the Turing degrees. This also suggests some possible approaches to making progress on proving part 1 of Martin's conjecture, which we will discuss below.

The Rudin-Keisler Order

To explain the connection between part 1 of Martin's conjecture and ultrafilters on the Turing degrees, we first need to give some background on something called the "Rudin-Keisler ordering" on ultrafilters.

Definition 5.30. *Suppose U is an ultrafilter on a set X and V is an ultrafilter on a set Y. Then U is* ***Rudin-Keisler below*** *V, written $U \leq_{RK} V$, if there is a function $f\colon Y \to X$ such that*

$$f_*(V) = U.$$

Example 5.31. If U is a principal ultrafilter on a set X then U is Rudin-Keisler below every other ultrafilter. To see why, suppose U concentrates on the point $a \in X$ and suppose V is an ultrafilter on a set Y. It is easy to check that if $f\colon Y \to X$ is the constant function with constant value a then $f_*(V) = U$.

Note that in the definition of $\leq_{RK}$, the function f is going in the opposite direction from what one might naively expect. This makes more sense if one considers embeddings of ultrapowers: if $U \leq_{RK} V$ then for every structure M there is an embedding $M^X/U \to M^Y/V$.

Also note that it is possible to have distinct ultrapowers U and V such that $U \leq_{RK} V$ and $V \leq_{RK} U$; in other words, $\leq_{RK}$ is only a quasi order rather than a partial order. In case this happens we will say that U and V are Rudin-Keisler equivalent.

Definition 5.32. *Suppose U is an ultrafilter on a set X and V is an ultrafilter on a set Y. Then U is* ***Rudin-Keisler equivalent*** *to V, written $U \equiv_{RK} V$, if $U \leq_{RK} V$ and $V \leq_{RK} U$.*

The simplest example of Rudin-Keisler equivalent ultrafilters is to simply pick sets X and Y of the same cardinality, let f be a bijection between them and take $V = f_*(U)$. Then it is easy to verify that $U = (f^{-1})_*(V)$ and hence $U \equiv_{RK} V$. In fact, the normal definition of Rudin-Keisler equivalence is actually that ultrafilters U and V on sets X and Y respectively are Rudin-Keisler equivalent if there is a bijection $f\colon X \to Y$ such that $f_*(U) = V$. Under ZFC, this is equivalent to the definition we gave above, but under $\mathsf{ZF} + \mathsf{AD}$ they are not equivalent. We chose to use the definition above even though it disagrees with the normal definition in ZF + AD because it will allow us to state things more conveniently.

If we restrict our attention to a ultrafilters on a single set and ignore the principal ultrafilters, then the class of ultrafilters which are minimal in the Rudin-Keisler order is often an important class which has other natural characterizations.

Example 5.33. The minimal nonprincipal ultrafilters on ω are exactly the Ramsey ultrafilters.

Example 5.34. Every normal ultrafilter on a cardinal κ is Rudin-Keisler minimal among nonprincipal ultrafilters on κ and every minimal nonprincipal ultrafilter on κ is Rudin-Keisler equivalent to either a normal ultrafilter or to a Ramsey ultrafilter on ω.

Part 1 of Martin's Conjecture and the Rudin-Keisler Order

We will now show that part 1 of Martin's conjecture is essentially equivalent to the statement about the position of Martin measure in the Rudin-Keisler order on nonprincipal ultrafilters on the Turing degrees. The statement is essentially a stronger version of asserting that Martin measure is minimal in this order.

Theorem 5.35 ($\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$)**.** *Part 1 of Martin's conjecture is equivalent to the following statement: if V is a nonprincipal ultrafilter on $\mathcal{D}_T$ such that $V \leq_{RK} U_M$ then $V = U_M$.*

Proof. Since we are working under AD_R, part 1 of Martin's conjecture is equivalent to the statement that every function $F\colon \mathcal{D}_T \to \mathcal{D}_T$ is either constant on a cone or above the identity on a cone (the point is that under $\mathsf{AD}_\mathbb{R}$ every function on the Turing degrees is induced by a Turing invariant function on the reals). We have the following equivalences.

- F is constant on a cone if and only if $F_*(U_M)$ is a principal ultrafilter.
- By Theorem 5.18, F is above the identity on a cone if and only if F is measure preserving. By one of the alternative characterizations of measure preserving, this means F is above the identity on a cone if and only if $F_*(U_M) = U_M$.

Thus F is either constant on a cone or above the identity on a cone if and only if $F_*(U_M)$ is either a principal ultrafilter or U_M itself. Since every ultrafilter which is Rudin-Keisler below U_M is by definition the pushforward of U_M along some function on the Turing degrees, this latter statement is equivalent to saying that no nonprincipal ultrafilter on $\mathcal{D}_T$ other than U_M itself is Rudin-Keisler below U_M. □

We can understand this theorem as saying that part 1 of Martin's conjecture is equivalent to the conjunction of the following two statements.

- U_M is $\leq_{RK}$-minimal among nonprincipal ultrafilters on $\mathcal{D}_T$.

- No ultrafilter besides U_M itself is Rudin-Keisler equivalent to U_M.

Possible Approaches to Part 1 of Martin's Conjecture

The equivalence we proved in the previous section suggests a few ways that one could try to make progress on part 1 of Martin's conjecture.

(1) The structure of the Rudin-Keisler order and the $\leq_{RK}$-minimal ultrafilters in particular has been the focus of a lot of research within set theory. Perhaps some of the ideas from that line of research can be applied to Martin's conjecture. This does not seem entirely hopeless. For example, we have already mentioned that normal ultrafilters on cardinals are $\leq_{RK}$-minimal. This is notable for us because Slaman and Steel's result on Martin's conjecture for regressive functions on the Turing degrees can be seen as proving a kind of analogue of normality for Martin measure.

(2) To better understand the position of Martin measure in the Rudin-Keisler order, we could try to look specific ultrafilters on the Turing degrees and prove that they are not Rudin-Keisler below Martin measure. We will pick up this idea again in the next section. One could also try to look at entire classes of ultrafilters with certain structural features instead of just looking at individual ultrafilters. We will discuss this more in section 5.11.

(3) The theorem suggests a novel way to split part 1 of Martin's conjecture into two parts: it consists of showing both that no nonprincipal ultrafilter on the Turing degrees is *strictly below* Martin measure in the Rudin-Keisler order and that no nonprincipal ultrafilter on the Turing degrees is *equivalent* to Martin measure in the Rudin-Keisler order. Perhaps one of these two is easier to show than the full part 1 of Martin's conjecture.

We will now discuss this last idea a bit more. Andrew Marks has made a conjecture which would imply that no nonprincipal ultrafilter on the Turing degrees is strictly below Martin measure in the Rudin-Keisler order.

Conjecture 5.36 (Marks)**.** *It is provable in* $\mathsf{ZF} + \mathsf{AD}$ *that every function* $f \colon 2^\omega \to 2^\omega$ *is either constant on a pointed perfect tree or injective on a pointed perfect tree.*

Proposition 5.37 ($\mathsf{ZF}+\mathsf{AD}_\mathbb{R}$)**.** *If the conjecture above is true then no nonprincipal ultrafilter on the Turing degrees is strictly below Martin measure in the Rudin-Keisler order.*

Proof. Suppose V is an ultrafilter on $\mathcal{D}_T$ such that $V \leq_{RK} U_M$. Let $F \colon \mathcal{D}_T \to \mathcal{D}_T$ be a function such that $F_*(U_M) = V$. Since we are working under $\mathsf{AD}_\mathbb{R}$, we may assume that F is induced by a Turing invariant function $f \colon 2^\omega \to 2^\omega$. Since we are assuming that Marks' conjecture holds, there is a pointed perfect tree T such that either f is constant on $[T]$ or f is injective on $[T]$. In the former case, F is constant on a cone and thus $V = F_*(U_M)$ is a principal ultrafilter. So let's assume we are in the latter case.

The idea is that since we can invert f on $[T]$, we can find a function G such that U_M is the pushforward of V along G. Define $G\colon \mathcal{D}_T \to \mathcal{D}_T$ as follows.

$$G(\deg_T(x)) = \begin{cases} \deg_T(y) & \text{if } y \in [T] \text{ and } f(y) \equiv_T x \\ \mathbf{0} & \text{if } x \notin f([T]). \end{cases}$$

The function G is well-defined because f is Turing invariant and injective on $[T]$. Also, on the cone above T, $G(F(\mathbf{x})) = \mathbf{x}$. It is straightforward to verify that this implies that $G_*(F_*(U_M)) = U_M$. Since $F_*(U_M) = V$, this means that $G_*(V) = U_M$ and hence $U_M \leq_{RK} V$ as desired. □

Note that Marks' conjecture is true if "pointed perfect tree" is replaced by "perfect tree" (the argument is just a version of the tree thinning lemma from section 2.1). Thus it seems at least superficially plausible that it holds. However, it has turned out to be surprisingly difficult to prove and in my opinion a proof or disproof of it would constitute a major advance in our understanding of Martin's conjecture.

5.10 The Lebesgue and Baire Ultrafilters

In this section we will look at two particular ultrafilters on the Turing degrees. In light of the last section, it would be interesting to show that neither of these ultrafilters is below Martin measure in the Rudin-Keisler order. At present, we don't know how to do that but we can show that neither of them is *above* Martin measure in the Rudin-Keisler order. Which might sound like the wrong direction. But it's not as bad as it sounds—this still shows that neither of these ultrafilters is Rudin-Keisler equivalent to Martin measure.

So where do these two ultrafilters come from? In fact, they are just Lebesgue measure and Baire category! It turns out that under AD, they both induce ultrafilters on the Turing degrees.

We will now explain why they induce ultrafilters on the Turing degrees, prove that they are not Rudin-Keisler above Martin measure and speculate on how one might show they are not below Martin measure. Because the two cases parallel each other pretty closely, we will first concentrate on the Lebesgue measure.

The Lebesgue Ultrafilter on the Turing Degrees

So, first—why is the Lebesgue measure an ultrafilter on the Turing degrees? It comes down to two facts: Kolmogorov's zero-one law and the fact that under AD, every set of reals is Lebesgue measurable.

Definition 5.38. *A set $A \subset 2^\omega$ is* ***closed under tail equivalence*** *if for all $x, y \in 2^\omega$ which differ at only finitely many positions*

$$x \in A \iff y \in A.$$

Notation 5.39. Let λ denote the Lebesgue measure on 2^ω.

Theorem 5.40 (Kolmogorov's zero-one law)**.** *If $A \subset 2^\omega$ is Lebesgue measurable and closed under tail equivalence then either $\lambda(A) = 0$ or $\lambda(A) = 1$.*

Theorem 5.41 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Every subset of 2^ω is Lebesgue measurable.*

Definition 5.42. *Let U_L be the collection of subsets of $A \subseteq \mathcal{D}_T$ such that the collection of reals which are in some Turing degree in A is a set of Lebesgue measure 1. In other words,*

$$A \in U_L \iff \lambda(\{x \in 2^\omega \mid deg_T(x) \in A\}) = 1.$$

We will refer to U_L as the ***Lebesgue ultrafilter****.*

Proposition 5.43 ($\mathsf{ZF} + \mathsf{AD}$)**.** *$U_L$ is an ultrafilter.*

Proof. Let A be any set of Turing degrees. We want to show that either A is in U_L or $\mathcal{D}_T \setminus A$ is in U_L. Since Turing degrees are closed under tail equivalence, the set $\{x \mid \deg_T(x) \in A\}$ is also closed under tail equivalence. By determinacy, it is Lebesgue measurable and thus by Kolmogorov's zero-one law it either has measure 1 or measure 0. In the former case, $A \in U_L$. In the latter case, the complement of A is in U_L. □

Lebesgue is Not Above Martin

We will now prove that the Lebesgue ultrafilter is not Rudin-Keisler above Martin measure. Actually, we know of several proofs of this. We will present one in full and sketch the others since they may be of some interest. Our proof uses the following lemma.

Lemma 5.44 ($\mathsf{ZF}{+}\mathsf{AD}$)**.** *Every function $f : 2^\omega \to \omega_1$ is constant on a set of positive Lebesgue measure.*

Proof. Suppose for contradiction that f is not constant on any set of positive measure. Note that for every $\alpha \in \omega_1$, AD implies that $f^{-1}(\alpha)$ is Lebesgue measurable and so our assumption implies that it has measure 0. By countable additivity of the Lebesgue measure, this implies that for any countable set $A \subset \omega_1$, $f^{-1}(A)$ has measure 0.

Now let B be the subset of $2^\omega \times 2^\omega$ defined by

$$B = \{(x, y) \mid f(x) \leq f(y)\}.$$

Again, since we are working under AD, we know B is Lebesgue measurable. We will now use Fubini's theorem to compute the measure of B in two different ways to arrive at a contradiction. We have:

$$\int \lambda(\{y \mid (x, y) \in B\})\, dx = \int \lambda(\{x \mid (x, y) \in B\})\, dy.$$

Now note that for any y, we have

$$\{x \mid (x, y) \in B\} = f^{-1}(\{\alpha \mid \alpha \leq f(y)\}.$$

Since this is the inverse image under f of a countable set, its measure must be 0. Thus

$$\int \lambda(\{x \mid (x, y) \in B\})\, dy = \int 0\, dy = 0.$$

On the other hand, for any x,

$$\{y \mid (x, y) \in B\} = f^{-1}(\{\alpha \mid f(x) \leq \alpha\}).$$

Since this is the complement of the inverse image under f of a countable set, its measure must be 1. Thus

$$\int \lambda(\{y \mid (x, y) \in B\})\, dx = \int 1\, dx = 1.$$

Therefore we have calculated that the measure of B is both 0 and 1, a contradiction. □

Corollary 5.45. *Suppose $F \colon \mathcal{D}_T \to \omega_1$ is any function. Then F is constant on a set in U_L.*

Theorem 5.46 ($\mathsf{ZF} + \mathsf{AD}$)**.** *The Lebesgue filter is not Rudin-Keisler above Martin measure, i.e. $U_M \not\leq_{RK} U_L$.*

Proof. Suppose for contradiction that $U_M \leq_{RK} U_L$, as witnessed by some function F (so $F_*(U_L) = U_M$). Let $G \colon \mathcal{D}_T \to \omega_1$ be the map defined by

$$G(\mathbf{x}) = \omega_1^{\mathbf{x}}.$$

It is straightforward to check that $G_*(U_M)$ is a countably complete ultrafilter on ω_1. But by the lemma above, $G \circ F$ is constant on a set of Lebesgue measure 1 and hence $(G \circ F)_*(U_L)$ is a principal ultrafilter on ω_1. This is a contradiction since we have

$$(G \circ F)_*(U_L) = G_*(F_*(U_L)) = G_*(U_M).$$ □

We will now briefly sketch a few other proofs of this theorem.

- First, instead of using Fubini's theorem to prove Lemma 5.44, we can use forcing, at least if we work in a sufficiently strong theory, such as $\mathsf{AD} + V = L(\mathbb{R})$. The idea is roughly as follows. Let $f \colon 2^\omega \to \omega_1$ be any function. We want to show that it is constant on a set of positive measure. If f is sufficiently definable (and in particular, if it has a definition which is absolute between nice enough inner models) then we can work over some inner model in which the powerset of the reals is countable. Let r be a generic for random real forcing over this model and take any condition which forces $f(r) = \alpha$. If we intersect this condition (which is a set of positive Lebesgue measure) with the set of random reals over our original model then we get a set of positive measure on which f has constant value α.

- We can also do without Lemma 5.44. Let $f: 2^\omega \to 2^\omega$ be a Turing invariant function. To show that the pushforward of the Lebesgue ultrafilter along f is not the Martin measure, it is enough to show that there is a set of positive Lebesgue measure whose image under f is disjoint from a cone. By Luzin's theorem, we can find a continuous function g which is equal to f on a set of positive measure. Recall that a continuous function is just a function which is computable relative to some oracle. Let a be such an oracle—in other words a is a real such that $g(x)$ is computable from $x \oplus a$ for all x. By a theorem of Sacks, the set of reals x such that $x \oplus a$ does not compute a' has measure 1. Intersecting this set with the set on which f is equal to g gives a set of positive measure whose image under f avoids the cone above a'.

- There is a variation on the proof above due to Andrew Marks. If $f: 2^\omega \to 2^\omega$ is any Lebesgue measurable function then the pushforward of Lebesgue measure along f always yields a Borel probability measure. It turns out that the theorem of Sacks we used in the previous paragraph actually applies to any Borel probability measure and shows that for any noncomputable real a, the set of reals which compute a has measure 0. This shows that no Borel probability measure on 2^ω can induce the Martin measure on the Turing degrees and thus that f does not push the Lebesgue ultrafilter forward to Martin measure.

Speculations

Given all of the above, it would be very interesting to show that the Lebesgue ultrafilter is not strictly below Martin measure in the Rudin-Keisler order. Andrew Marks has shown under $\mathsf{AD}_\mathbb{R}$ that $U_L <_{RK} U_M$ holds if and only if there is a Turing invariant function $f: 2^\omega \to 2^\omega$ such that for all x, $f(x)$ is x-random (in the sense of algorithmic randomness).

It would also be interesting to know how much the results above can be generalized. Here's an example. We have just shown that U_L is not above U_M in the Rudin-Keisler order. Could it be that *no* ultrafilter on the Turing degrees is above U_M? I.e. that U_M is maximal in the Rudin-Keisler order on ultrafilters on $\mathcal{D}_T$? In the next section, we will discuss some restricted forms of this question.

The Baire Ultrafilter on the Turing Degrees

All of the above analysis can be repeated almost verbatim for one other ultrafilter on the Turing degrees—the ultrafilter that arises from Baire category. Briefly, define the **Baire ultrafilter** to be the collection U_B of subsets of the Turing degrees defined as follows.

$$A \in U_B \iff \{x \in 2^\omega \mid \deg_T(x) \in A\} \text{ is comeager.}$$

Since Baire category also satisfies a zero-one law and AD implies that every set of reals has the property of Baire, U_B is also an ultrafilter on the Turing degrees. And all the proofs above to show that $U_M \not\leq_{RK} U_L$ work more-or-less unchanged to show that $U_M \not\leq_{RK} U_B$ (for example, Baire category also satisfies a version of Fubini's theorem, we can replace random real forcing with Cohen forcing, and so on). It would be interesting to know if U_B is strictly below U_M in the Rudin-Keisler order and if there are any differences between the cases of the Lebesgue and Baire ultrafilters.

5.11 Generic Ultrapowers and Forcing with Positive Sets

In this section, we will discuss some ideas from set theory that could offer a useful perspective on Martin's conjecture and its connection ultrafilters on the Turing degrees and ultrapowers by Martin measure. The development in this section will be extremely sketchy.

First, we will introduce something called "forcing with positive sets" and then we will introduce the related idea of a "generic ultrapower." Some good references for this are the book "Descriptive Set Theory and Definable Forcing" by Zapletal [Zap04], the Handbook of Set Theory Chapter "Ideals and Generic Elementary Embeddings" by Foreman [For10], and the article "Generic Codes for Uncountable Ordinals, Partition Properties, and Elementary Embeddings," by Kechris and Woodin [KW08].

Forcing with Positive Sets

Suppose I is an ideal on the reals—i.e. I is a collection of sets of reals which is closed under subset and finite unions. A set $A \subset 2^\omega$ is called **I-positive** if it is not in I. By **forcing with I-positive sets** we mean the notion of forcing given by taking the quotient of the algebra of Borel subsets of 2^ω by the ideal I (note that we are only using Borel sets, not all subsets of 2^ω).

If the ideal I is countably complete then a generic for this notion of forcing can be thought of as specifying a single real number. That is, for every generic G, there is a real r such that $V[G] = V[r]$. The real r is defined by setting $r(n) = 0$ if (the I-equivalence class of) the set of reals which have a 0 at position n is in G, and 1 otherwise. Conversely, G can be recovered from r because a Borel set B is in G if and only if r is in B. We will often conflate this real r with the generic G.

Example 5.47. Let I be the ideal of sets of Lebesgue measure 0. In ZFC, the I-positive sets consist of the measurable sets with positive measure, together with the non-measurable sets. In $\mathsf{ZF} + \mathsf{AD}$, all sets of reals are Lebesgue measurable so the I-positive sets are exactly the sets of positive Lebesgue measure. Forcing with I-positive Borel sets is equivalent to random real forcing and the reals corresponding to generics for forcing with I-positive sets are exactly the random reals.

Example 5.48. Let I be the ideal of meager sets. Under $\mathsf{ZF} + \mathsf{AD}$, the I-positive sets are exactly those sets which are comeager in some open set. Forcing with I-positive Borel sets is equivalent to Cohen forcing and the reals corresponding to generics for this forcing are exactly the Cohen generics.

Example 5.49. Let I is the ideal of countable sets. Under $\mathsf{ZF} + \mathsf{AD}$, the I-positive sets are exactly the sets which contain a perfect set. Forcing with I-positive Borel sets is equivalent to forcing with perfect sets—a.k.a. Sacks forcing—and the reals corresponding to generics for this forcing have minimal constructibility degree over the ground model.

In the context of Martin's conjecture, we are interested in ideals on the reals which arise from ultrafilters on the Turing degrees. Suppose U is an ultrafilter on $\mathcal{D}_T$. Let I be the ideal on 2^ω consisting of sets A for which the set of Turing degrees of elements of A is not in U (in other words, the image of A in $\mathcal{D}_T$ under the quotient map $2^\omega \to \mathcal{D}_T$ is not in U).

Recall from the last section that U_L denotes the Lebesgue ultrafilter on the Turing degrees, which is the ultrafilter induced by Lebesgue measure, and that U_B denotes the Baire ultrafilter on the Turing degrees, which is the ultrafilter induced by the filter of comeager sets on the reals. Let I_L denote the ideal on the reals corresponding to U_L and likewise for I_B. Forcing with I_L-positive sets is similar to, but not quite the same as, forcing with positive sets for the ideal of sets of Lebesgue measure 0—the difference is that a set of measure 0 is not in I_L if the smallest Turing invariant set which contains it happens to have positive Lebesgue measure. However, generic reals for forcing with I_L-positive sets are exactly the reals which are Turing equivalent to a random real and thus the two forcings are equivalent. Likewise, generic reals for forcing with I_B-positive sets are exactly the reals which are Turing equivalent to a Cohen generic real.

Of course, the ultrafilter on the Turing degrees that we are most interested in is Martin measure, U_M. So what does it mean to force with positive sets for the ideal on the reals induced by U_M? It turns out that it is equivalent to forcing with pointed perfect trees and the generic real it adds is a minimal upper bound in the Turing degrees for the set of reals in the ground model. This forcing is quite different from the ones we have listed above in several respects. For example, it collapses the reals of the ground model to a countable set in the generic extension and thus also collapses ω_1.

Also note that both pointed perfect trees and minimal upper bounds for ideals in the Turing degrees seem to play important roles in the study of Martin's conjecture. We believe that this is connected to some of the facts we have just mentioned.

Proper Forcing

In the previous two sections, we discussed the fact that part 1 of Martin's conjecture is equivalent to a statement about the structure of the Rudin-Keisler order on ultrafilters on the Turing degrees. We also investigated two particular ultrafilters on the Turing degrees—the Lebesgue ultrafilter and the Baire ultrafilter—and showed that neither one is above Martin measure in the Rudin-Keisler order. In this section we will discuss a generalization of these results which we believe is likely to be provable under some strengthening of AD, such as $\mathsf{AD} + V = L(\mathbb{R})$.

Recall that a notion of forcing is called **proper** if it preserves stationarity of subsets of $[\lambda]^\omega$ for all uncountable cardinals λ. Thus both c.c.c. forcings and ω_1-closed forcings are proper, but so are many other notions of forcing, such as Sacks forcing. There are many equivalent characterizations of forcing. The one we are most interested in here is the following (see [Zap04] for a proof): if I is an ideal on 2^ω then forcing with I-positive sets is proper if and only if for every countable model M which is an elementary submodel of a big enough fragment of V and any I-positive Borel set B, there is an I-positive subset of B consisting of reals which are generic over M.

Suppose U is an ultrafilter on the Turing degrees and I is the associated ideal on the reals. We believe it is possible to show in $\mathsf{AD} + V = L(\mathbb{R})$ that if forcing with I-positive sets is proper then U is not Rudin-Keisler above the Martin measure. We are not currently able to give a complete proof, so we will just sketch the main idea here.

First, we observe that it is enough to show that every function from the 2^ω to ω_1 is constant on an I-positive set. This can be shown by following the proof of Theorem 5.46 from the previous section (essentially the proof is to note that if U is Rudin-Keisler above U_M then U pushes forward to a countably complete ultrafilter on ω_1, but this contradicts every function into ω_1 being constant on an I-positive set).

Now we will sketch a possible way to prove that every function $f\colon 2^\omega \to \omega_1$ is constant on an I-positive set. Suppose we may assume f is Σ^2_1-definable in the following sense: the binary relation R on 2^ω which holds of (x, y) whenever y is a code for the countable ordinal $f(x)$ is Σ^2_1-definable. Suppose we may also assume that I is Σ^2_1-definable when restricted to the Borel sets (since every Borel set is coded by a real, we may then think of I as a set of reals). If we are working in $V = L(\mathbb{R})$ then this seems like a reasonable assumption.

Now consider forcing over HOD with I-positive Borel sets (which we can do because the powerset of the reals in HOD is countable). Let r be a generic real for this forcing and let $\alpha = f(r)$. Since f is Σ^2_1-definable as a relation on the reals and Σ^2_1 formulas are absolute for HOD, there must be some y in $\mathsf{HOD}(r)$ such that y codes $f(r)$ and thus α is countable in $\mathsf{HOD}(r)$. (Note that since forcing with I-positive sets is proper, it doesn't collapse ω_1 and so this implies that α was already countable in HOD). Let B be a condition forcing that $f(r) = \alpha$. Since B is I-positive and forcing with I-positive sets is proper, B contains an I-positive Borel set C consisting only of reals which are generic over HOD. We can then use absoluteness to show that for every x in C, $f(x) = \alpha$ and thus f is constant on C.

The discussion in this section suggests that one way to try to work on Martin's conjecture would be to identify features of forcing so that if the notion of forcing associated to an ultrafilter on the Turing degrees has one of those features then the ultrafilter is not Rudin-Keisler below Martin measure. Properness is one candidate for such a feature. Slaman has suggested that the property of not collapsing $\mathfrak{c}$ might also be such a feature.

The Generic Ultrapower

We will now mention one other idea from set theory that may prove useful in the study of Martin's conjecture: the generic ultrapower. As above, let I be an ideal on the reals and consider forcing with I-positive sets (this time with all I-positive sets rather than just the Borel ones). Let G be a generic filter for this notion of forcing. G can be used to define an ultrafilter U on $\mathcal{P}(\mathbb{R})^V$. This allows us to define, in $V[G]$, the ultrapower of V by U. More precisely, we start with the class of functions $f\colon 2^\omega \to V$ which are in V and identify two functions f and g if they agree on a set in U (note that we are only using functions in V, but that this equivalence relation is itself not definable n V, but only in $V[G]$). The resulting structure is called the **generic ultrapower**.

Woodin has shown—assuming $V = L(\mathcal{P}(\mathbb{R}))$, Θ is regular, and $\mathsf{AD}_\mathbb{R}$—that if the generic ultrapower is well-founded then it satisfies Łos's theorem and that the embedding of V into the generic ultrapower is elementary. Also, if I is the ideal of Lebesgue null sets or meager sets then the embedding is the identity on the ordinals. On the other hand, if the ideal is the one induced by Martin measure then the embedding is *not* the identity on the ordinals and, in fact, ω_1 is the critical point (ω_1 in the generic ultrapower is represented by the function $x \mapsto \omega_1^x$).

Martin's conjecture can be rephrased as asking about the structure of the generic ultrapower by the ideal corresponding to Martin measure. It would be interesting to know whether this perspective can be used to help prove any new results about Martin's conjecture.

Chapter 6

Order Preserving Functions

Note: The results of this chapter are joint work with Benjamin Siskind.

In this chapter, we will discuss Martin's conjecture for order preserving functions. Recall that a Turing invariant function $f\colon 2^\omega \to 2^\omega$ is called **order preserving** if for all $x \geq_T y$, $f(x) \geq_T f(y)$. In [SS88], Slaman and Steel proved part 2 of Martin's conjecture for a large class of order preserving functions.

Theorem 6.1 ($\mathsf{ZF} + \mathsf{AD}$; Slaman and Steel)**.** *If $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is order preserving and above the identity in the Martin order then either there is some $\alpha < \omega_1$ such that f is Martin equivalent to the α-jump or f is above the hyperjump in the Martin order.*

In particular, this theorem implies that part 2 of Martin's conjecture holds for all order preserving Borel functions.

Our main goal in this chapter will be to prove a complementary result to Slaman and Steel's theorem, namely that part 1 of Martin's conjecture holds for all order preserving functions on the Turing degrees. In fact, we will actually present three different proofs of this result. This may seem odd, but we believe each proof sheds light on possible approaches to the rest of Martin's conjecture.

Each of these proofs depends on the following fact, which we will prove in section 6.1: every order preserving function is either measure preserving or constant on a cone. To prove this, we will use the basis theorem for perfect sets that we proved in section 2.3.

Since we already proved part 1 of Martin's conjecture for all measure preserving functions, the fact that every order preserving function is either measure preserving or constant on a cone immediately implies that part 1 of Martin's conjecture holds for all order preserving functions. This constitutes our first proof. This proof has the virtue of simplicity, but it has two minor downsides. First, our proof of part 1 of Martin's conjecture for measure preserving functions used $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$ rather than just AD and the version restricted to Borel functions used $\boldsymbol{\Pi}^1_1$ determinacy rather than Borel determinacy.

We will give a second proof using the concept of ordinal invariants of Turing degrees. This proof no longer requires $\mathsf{DC}_\mathbb{R}$ for the general case, but still seems to require $\boldsymbol{\Pi}^1_1$ determinacy for the case of Borel functions. We include it mostly because it provides another example of using ordinal invariants (it is somewhat different than the second proof of part 1 of Martin's conjecture for measure preserving functions in the last chapter, which also used ordinal invariants) and because some parts of the argument can be viewed as taking place in ultrapowers by the Martin measure in an interesting way (which we will comment on more in section 6.3).

We will then give a third proof using the Solecki dichotomy introduced in section 2.4. The main idea for this proof is due to Takayuki Kihara (essentially the proof consists of combining a theorem proved by Kihara with the fact that order preserving functions are measure preserving). It works for all functions in AD and for all Borel functions in ZF.

6.1 Order Preserving Functions are Measure Preserving

In this section, we will prove that every order preserving function is either measure preserving or constant on a cone. The proof uses the basis theorem for perfect sets that we proved in section 2.3 in conjunction with the fact that in AD, every set of reals has the perfect set property. As a reminder, here's what that means.

Theorem 6.2 (AD; [Jec03] theorem 33.3)**.** *Every subset of* 2^ω *is either countable or contains a perfect set.*

We can now give our proof of the theorem.

Theorem 6.3 (AD)**.** *If* $f\colon 2^\omega \to 2^\omega$ *is an order-preserving function then either* f *is Martin-equivalent to a constant function or* f *is measure-preserving.*

Proof. Before giving the proof in detail, here's a sketch. Suppose that $f\colon 2^\omega \to 2^\omega$ is an order preserving function. By Theorem 6.2, range(f) is either countable or contains a perfect set (this is the only use of determinacy in the proof). We will show that if range(f) is countable then f is constant on a cone and if range(f) contains a perfect set then f is measure preserving.

The case where range(f) is countable is straightforward. In the case where range(f) contains a perfect set we will use the basis theorem for perfect sets that we proved in section 2.3 (more specifically, we will use corollary 2.21). The key point is that since f is order preserving, its range is countably directed for Turing reducibility.

Case 1: the range of f is countable. Let F denote the function on the Turing degrees induced by f. Since the set of preimages of elements of the range of F is a countable set whose union contains all Turing degrees (and thus is cofinal), there must be some element of the range of F whose preimage is cofinal, and so by determinacy contains a cone. On this cone, F is constant.

Case 2: the range of f contains a perfect set. The main point here is that the range of an order preserving function is countably directed. To see why, suppose that $x_0, x_1, \ldots$ are all in the range of f. So we can pick reals $y_0, y_1, \ldots$ such that $f(y_0) = x_0$, $f(y_1) = x_1$, and so on. Let y be the Turing join of all the y_i's. Since y computes each y_i and since f is order-preserving, $f(y)$ computes each x_i. In other words, $f(y)$ is an upper bound for $\{x_0, x_1, \ldots\}$.

Since range(f) contains a perfect set and is countably directed, corollary 2.21 implies that it is cofinal. Now we want to show f is measure preserving. In other words, we start with an arbitrary x and we want to show that there is some y so that f sends everything in the cone above y into the cone above x. Since the range of f is cofinal, there is some $z \geq_T x$ in the range of f. Since z is in the range of f, there is some y such that $f(y) = z$. And since f is order preserving, it takes the cone above y into the cone above z and hence into the cone above x. □

6.2 Three Proofs of Part 1 of Martin's Conjecture for Order Preserving Functions

In this section, we will give three proofs of part 1 of Martin's conjecture for order preserving functions. All three of the proofs rely on the result from the previous section that every order preserving function is either constant on a cone or measure preserving.

Proof 1: As a Corollary of Theorem 5.18

Our first proof of this theorem just consists of the observation that it is a corollary of two facts which we have already established, namely: every order preserving function is either constant on a cone or measure preserving, and part 1 of Martin's conjecture holds for all measure preserving functions.

Theorem 6.4 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$). *Part 1 of Martin's conjecture holds for all order preserving functions.*

Proof. Suppose $f : 2^\omega \to 2^\omega$ is a Turing invariant function which is order preserving. We want to show that f is either constant on a cone or above the identity on a cone. By Theorem 6.3, f is either constant on a cone or measure preserving. If f is constant on a cone then we are done and if f is measure preserving then by Theorem 5.18 it is above the identity on a cone. □

Proof 2: Ordinal Invariants

Our second proof uses the idea of ordinal invariants of Turing degrees, which we introduced in section 2.2. This is a feature it shares with the proof in section 5.4. However, we will choose a fairly different ordinal invariant from the one in that proof, which only works for order preserving functions and which does not require $\mathsf{DC}_{\mathbb{R}}$ to be well-defined.

Before we give the proof, we will first prove a lemma about how ordinal invariants interact with measure preserving functions. Briefly, it says that if f is measure preserving and α is an ordinal invariant then $\alpha(f(x)) \geq \alpha(x)$ for all x on a cone. We express this by saying that f **preserves** α. As we will see in the next section, this lemma can be proved easily by considering the ultrapower of the ordinals by Martin measure. But for now, we will establish in a more down-to-earth manner. We first need to prove a few basic facts about measure preserving functions.

Lemma 6.5. *If $f\colon 2^\omega \to 2^\omega$ and $g\colon 2^\omega \to 2^\omega$ are both measure preserving then so is $f \circ g$.*

Proof. This is an immediate consequence of the characterization of measure preserving functions in terms of measure theory, but we will give a direct proof anyway. Let x be an arbitrary real. We need to find y large enough that on the cone above y, $f \circ g$ is above x. Since f is measure-preserving we can pick y_0 large enough that on the cone above y_0, f is above x. And since g is measure-preserving we can then pick y large enough that on the cone above y, g is above y_0. So on the cone above y, $f \circ g$ is above x. □

Corollary 6.6. *If f is measure preserving then so is $f^{(n)}$ for each $n \in \mathbb{N}$.*

Proof. By induction on n, using Lemma 6.5 for the induction step. □

Lemma 6.7 ($\mathsf{ZF} + \mathsf{AD}$). *If $f\colon 2^\omega \to 2^\omega$ is a measure preserving function and α is an ordinal invariant then on some cone, $\alpha(f(x)) \geq \alpha(x)$.*

Proof. By determinacy, either $\alpha(f(x)) \geq \alpha(x)$ on a cone or $\alpha(f(x)) < \alpha(x)$ on a cone. Suppose for contradiction that the latter holds, say on the cone above some z.

Since each $f^{(n)}$ is measure-preserving, we can find reals $y_0, y_1, y_2, \ldots$ so that for each n, if $x \geq_T y_n$ then $f^{(n)}(x) \geq_T z$. Let y be an upper bound for all these y_n's.

Because $f^{(n)}(y) \geq_T z$ for each n, we have that

$$\alpha(f(f^{(n)}(y))) < \alpha(f^{(n)}(y))$$

and so

$$\alpha(y) > \alpha(f(y)) > \alpha(f(f(y))) > \ldots$$

is a descending sequence of ordinals. □

We are now almost ready to give our second proof of part 1 of Martin's conjecture for order preserving functions. The basic idea is that we will find some ordinal invariant α such that we can use the computable uniformization lemma to find a computable function g which is below f and which preserves α.

Theorem 6.8 ($\mathsf{ZF} + \mathsf{AD}$). *Part 1 of Martin's conjecture holds for all order preserving functions.*

Proof. Let $f\colon 2^\omega \to 2^\omega$ be an order preserving function. Since we proved that order-preserving functions are either constant on a cone or measure preserving (theorem 6.3), we may assume that f is a measure preserving function and therefore that f preserves ordinal invariants. We will now construct an ordinal invariant, α, for use in the proof.

First, for each $x \in 2^\omega$, define $C(x)$ to be the smallest set of reals such that

- $C(x)$ contains x
- $C(x)$ is closed under Turing reducibility: if $y \in C(x)$ and $z \leq_T y$ then $z \in C(x)$
- $C(x)$ is closed under finite joins: if y and z are both in $C(x)$ then so is $y \oplus z$
- $C(x)$ is closed under f: if $y \in C(x)$ then so is $f(y)$.

Now define $\alpha(x) = \sup\{\omega_1^y \mid y \in C(x)\}$. Note that $\alpha(x) \geq \omega_1^x$ and that $C(x)$ is countable so $\alpha(x) < \omega_1$.

We now want to apply the computable uniformization lemma to get a pointed perfect tree, T, and a Turing functional, Φ, such that for every $x \in [T]$, $\Phi(x)$ is total, below $f(x)$ and $\alpha(x) = \alpha(\Phi(x))$. To do so, we need to show that the following set is cofinal in the Turing degrees.

$$A = \{x \mid \exists y\,(y \leq_T x \wedge y \leq_T f(x) \wedge \alpha(y) = \alpha(x))\}.$$

So why is A cofinal? Well, let z be an arbitrary real. We need to find $x \geq_T z$ which is in the set. By increasing z if necessary, we can assume we are on a cone where f preserves α (i.e. on which $\alpha(f(x)) \geq \alpha(x)$). Let $x = z \oplus f(z)$ and let $y = f(z)$. Now observe:

- By the definition of x, we have $z \leq_T x$ and $y = f(z) \leq_T x$.
- Since f is order-preserving we have $y = f(z) \leq_T f(x)$
- Since $C(x) = C(z) \supseteq C(y)$, we have $\alpha(x) = \alpha(z) \geq \alpha(y)$ and since z is on a cone where f preserves α, we have $\alpha(y) \geq \alpha(z)$ and therefore $\alpha(y) = \alpha(z) = \alpha(x)$.

So y witnesses that $x \in A$ and hence A is cofinal.

Thus we can apply the computable uniformization theorem to get T and Φ as described above.

We now want to apply the tree thinning lemma to find a pointed perfect subtree on which Φ is injective. To do this, it is enough to show that Φ is not constant on any pointed perfect tree. And since $\alpha(x) = \alpha(\Phi(x))$ and α is Turing invariant[1], it is enough to show that α is not constant on any cone. To see why this holds, note that $\alpha(x)$ is always a countable ordinal greater than or equal to ω_1^x. So by increasing x enough, we can always make $\alpha(x)$ increase (specifically, for any x we can choose $y \geq_T x$ such that ω_1^y, and hence also $\alpha(y)$, is strictly greater than $\alpha(x)$).

[1]Note that Φ itself may not be Turing invariant.

So we may assume that Φ is injective on $[T]$. Therefore by Lemma 2.3, for every $x \in [T]$, $\Phi(x) \oplus T$ computes x. Since f is measure preserving, it gets above T on a cone. So for any x on this cone which is in $[T]$, $f(x)$ computes $\Phi(x) \oplus T$ and hence $f(x)$ computes x. Thus f is above the identity on a cone. □

Here's one interesting feature of this proof. Even though the proof works in AD, it does *not* appear to work in ZF when restricted to Borel functions. The problem is that the ordinal invariant in the proof is defined using ω_1^x and relations like $\omega_1^x = \omega_1^y$ are not Borel definable. Thus even when f is Borel, the proof involves applying determinacy to sets which are not Borel. Just as with the results of chapters 4 and 5, this casts a bit of doubt on the idea that any proof of Martin's conjecture is guaranteed to use determinacy in a strictly "local" way.

In our third and final proof, we will at last see a proof that does work in ZF when restricted to Borel functions.

Proof 3: Solecki Dichotomy

Our third proof uses a theorem of descriptive set theory known as the "Solecki dichotomy," which we originally introduced in section 2.4. The proof is due to Kihara who used it to show that if f is an order preserving function then either f is constant on a cone or there is some real a such that $f(x) \oplus a \geq_T x$ on a cone [Kih21]. Combined with our proof that order preserving functions are either constant on a cone or measure preserving, this yields a proof of part 1 of Martin's conjecture for order preserving functions. Since Kihara's proof does not use determinacy in any way outside of the Solecki dichotomy, and since the Solecki dichotomy has a proof using determinacy in a "local" way, this at last gives us a proof for Borel functions that works in ZF (as well as an alternate proof for all functions from AD).

Recall that the Solecki dichotomy more or less says that for any function $f\colon 2^\omega \to 2^\omega$, either f is a countable union of partial continuous functions or the Turing jump can be embedded into f. The idea of the proof is that in the first case, one of the pieces must contain a pointed perfect tree and hence f is regressive on a cone (so we can apply Slaman and Steel's Theorem about regressive functions on the Turing degrees), and in the second case, f is above the jump on a cone. The assumption that f is order preserving is needed to make the second case work.

Theorem 6.9 (ZF + AD)**.** *Suppose $f\colon 2^\omega \to 2^\omega$ is an order preserving function. Then either f is constant on a cone or f is equal to the identity on a cone or f is above the jump on a cone.*

Proof. First, since f is order preserving, it is either constant on a cone or measure preserving. If it is constant on a cone then we are done, so we may assume it is measure preserving. Next, by the Solecki dichotomy, either there is a countable collection of subsets of 2^ω such that f is continuous on each one or there are partial continuous functions φ and ψ such that for all x, $\varphi(f(\psi(x))) = x'$. We will show that in the former case, f is regressive and in the latter case, f is above the jump.

Case 1: f is σ-continuous: First assume that we can break 2^ω into countably many pieces such that f is continuous on each one. Since there are only countably many, one of these pieces must contain the set of paths through some pointed perfect tree, T. Since being continuous is equivalent to being computable relative to some oracle, there is some Turing functional Φ and some oracle, a, such that for all $x \in [T]$, $f(x) = \Phi(x \oplus a)$. Thus for all x in $[T]$ which compute a, $f(x) \leq_T x$. Therefore f is regressive on a cone so we may apply Slaman and Steel's theorem to conclude that either f is constant on a cone or f is equal to the identity on a cone.

Case 2: the jump embeds into f: Now assume that there are partial continuous functions φ and ψ such that for all $x \in 2^\omega$, $\varphi(f(\psi(x))) = x'$. We can again use the fact that (partial) continuous functions are computable relative to an oracle[2] to show that there are Turing functionals Φ and Ψ and oracles a and b such that for all x, $\Phi(f(\Psi(x \oplus a)) \oplus b) = x'$.

Let's try to understand a little better what this is saying. For convenience, we will work on a high enough cone such that x always computes the oracle a and thus x and $x \oplus a$ are Turing equivalent. We start with a real x in this cone. This x then computes another real, let's call it y, which we feed to f. We take the result, $f(y)$, and use it together with b to compute x'.

This seems pretty close to saying that $f(x)$ computes x', but there are two problems. First, we are using $f(y)$ to compute x', not $f(x)$ itself. Second, we also have to use b when computing x'. Sure, we can work on a cone on which x computes b, but this gives us no guarantees that $f(x)$ will also compute b. What if the range of f completely misses the cone above b?

The solution to the first problem is to note that f is order preserving and y is computable from x. Thus $f(y)$ is computable from $f(x)$. The solution to the second problem is to use the fact that f is measure preserving, so on a high enough cone, $f(x)$ computes b.

Let's put all of this more formally. Let x be large enough that x computes a and $f(x)$ computes b. Thus $\Psi(x \oplus a)$ is computable from x and since f is order preserving, this means that $f(x)$ computes $f(\Psi(x \oplus a))$. Since $f(x)$ also computes b, $f(x)$ computes $\Phi(f(\Psi(x \oplus a)) \oplus b)$, which is equal to x'. So $f(x)$ computes x' on a cone. □

6.3 Ultrapowers by the Martin Measure

As part of our second proof in the previous section, we proved a lemma stating that measure preserving functions preserve ordinal invariants of Turing degrees (Lemma 6.7). In this section we will provide an alternate proof of that Lemma in which we reason about the ultrapower of the ordinals by Martin measure. It may seem a little odd to include an alternate proof for a lemma which was not very complicated in the first place and which has only one application in this thesis. But we wish to include it because it gives a short and

[2]Note that since the continuous functions are partial, the Turing functional may not be total relative to all oracles.

appealing demonstration of another connection between Martin's conjecture and ultrafilters and ultrapowers on the Turing degrees, complementary to the connections already mentioned in chapter 5.

There are three key ideas to the alternate proof. The first is that an ordinal invariant can be thought of as (a representative of) an element of the ultrapower of the ordinals by Martin measure, i.e. $\mathrm{Ord}^{\mathcal{D}_T}/U_M$. The second idea is that because Martin measure is countably complete, $\mathrm{Ord}^{\mathcal{D}_T}/U_M$ is well-founded. And the third idea is that a measure preserving function induces an embedding of $\mathrm{Ord}^{\mathcal{D}_T}/U_M$ into itself. To explain this last idea, we need to introduce a few much more general concepts related to ultrapowers.

Definition 6.10. *Suppose U is an ultrafilter on X, $f\colon X \to Y$ is a function and M is some structure. Then f induces a map*

$$\begin{aligned} f^*\colon M^Y/f_*(U) &\to M^X/U \\ [h]_{f_*(U)} &\mapsto [h \circ f]_U \end{aligned}$$

*called the **pullback** of $M^Y/f_*(U)$ along f.*

In the definition above, h denotes a function $Y \to M$ and $[h]_{f_*(U)}$ denotes its $f_*(U)$-equivalence class (recall that all elements of $M^Y/f_*(U)$ have this form). Technically it is necessary to check that the map f^* in the definition above is well-defined—i.e. that if h_1 and h_2 are two functions $Y \to M$ such that $[h_1]_{f_*(U)} = [h_2]_{f_*(U)}$ then $[h_1 \circ f]_U = [h_2 \circ f]_U$—but fortunately doing so is straightforward. In fact, we can prove a bit more: f^* is actually an embedding from $M^Y/f_*(U)$ to M^X/U. In ZFC, it is even an elementary embedding but since this depends on Łos's theorem, it does not always hold in ZF (or $\mathsf{ZF} + \mathsf{AD}$, which is our context).

Lemma 6.11 (ZF)**.** *Suppose U is an ultrafilter on a set X, $f\colon X \to Y$ is a function and M is a structure. Then $f^*\colon M^Y/f_*(U) \to M^X/U$ is an embedding.*

Proof. We want to show that f^* preserves all quantifier free formulas in the language of M. Let $\varphi(x)$ be such a quantifier free formula. For convenience we assume it has one variable. Let $h\colon Y \to M$ be any function. Then we have

$$\begin{aligned} M^Y/f_*(U) \vDash \varphi([h]) &\iff \{y \in Y \mid M \vDash \varphi(h(y))\} \in f_*(U) \\ &\iff f^{-1}(\{y \in Y \mid M \vDash \varphi(h(y))\}) \in U \\ &\iff \{x \in X \mid M \vDash \varphi(h(f(x)))\} \in U \\ &\iff M^X/U \vDash \varphi([h \circ f]) \end{aligned}$$

where the first and last lines follow from Łos's theorem restricted to quantifier free formulas (which does hold in ZF, unlike the full Łos's theorem). □

So how does this all apply to Martin's conjecture and specifically to measure preserving functions? First, Martin's conjecture can be interpreted as a statement about the structure of the ultrapower of the Turing degrees by the Martin measure. Elements of this ultrapower are exactly Martin-equivalence classes of functions from the Turing degrees to the Turing degrees. Martin's conjecture says, more or less, that under ZF+AD, this ultrapower looks like an initial segment that is a copy of the Turing degrees, followed by a well-ordered sequence whose elements are equivalence classes of the identity and transfinite iterates of the Turing jump.

Thus if f is a Turing invariant function then it can be thought of as an element of the ultrapower $\mathcal{D}_T^{\mathcal{D}_T}/U_M$. But if f is measure preserving then it can be thought of in another way as well. The idea is that if U is an ultrafilter on a set X and $g\colon X \to X$ is a function which is measure preserving for U and M is any structure then g^* is an embedding from M^X/U to itself. Thus measure preserving functions on the Turing degrees lead a kind of double life. On the one hand, they represent elements of a specific ultrapower by the Martin measure. On the other hand, they give rise to endomorphisms of ultrapowers by the Martin measure. In our case, we are particularly interested in the fact that they give rise to embeddings of $\mathrm{Ord}^{\mathcal{D}_T}/U_M$ to itself.

We can now provide the alternate proof we promised at the beginning of this section.

Theorem 6.12 (ZF + AD)**.** *Suppose $f\colon 2^\omega \to 2^\omega$ is a Turing invariant function which is measure preserving and $\alpha\colon 2^\omega \to$ Ord is an ordinal invariant. Then on some cone, $\alpha(f(x)) \geq \alpha(x)$.*

Proof. We can think of α as representing an element of $\mathrm{Ord}^{\mathcal{D}_T}/U_M$, the ultrapower of the ordinals by Martin measure. Seen in this light, the theorem states that $[\alpha]_{U_M} \leq f^*([\alpha]_{U_M})$.

We can prove this as follows. Since U_M is countably complete, $\mathrm{Ord}^{\mathcal{D}_T}/U_M$ is wellfounded. And since f is measure preserving, its pullback, $f^*\colon \mathrm{Ord}^{\mathcal{D}_T}/U_M \to \mathrm{Ord}^{\mathcal{D}_T}/U_M$, is an embedding. Since f^* is an embedding of a wellorder into itself, it must never be decreasing. Thus $[\alpha] \leq f^*([\alpha])$, as desired. □

Note that all the elements of the original proof are still present. For example, in the original proof we showed that if $\alpha(f(x)) < \alpha(x)$ on a cone then $\alpha(f(f(x))) < \alpha(f(x))$ on a cone as well. In the proof we just gave, this is subsumed into the fact that f^* is an embedding. Likewise, the argument we gave before to arrive at a descending chain of ordinals is replaced by the fact that since U_M is countably complete, $\mathrm{Ord}^{\mathcal{D}_T}/U_M$ is wellfounded. It would be interesting to see if any other arguments related to Martin's conjecture can be reinterpreted as arguments about ultrapowers by Martin measure in this way.

6.4 ZFC Counterexample

In the last chapter we saw that if it is possible to prove at least one case of Martin's conjecture in ZFC as long as we replace "on a cone" with "cofinally." We also saw (in section 5.7) that this cannot be done for measure preserving functions: it is provable in ZFC+CH that there is a measure preserving function which is not above the identity on any noncomputable degree.

In this section we will see that it also cannot be done for order preserving functions: it is provable in ZFC that there is an order-preserving function on the Turing degrees which is not constant on any cofinal set nor above the identity on any noncomputable degree (note that unlike for the case of measure preserving functions, we do not need to assume CH to construct this counterexample). What's more, the function we construct is also not measure preserving so the results of section 6.1 also cannot be made to work in ZFC. The proof is due to Siskind and will also appear in a forthcoming joint paper by the two of us.

Theorem 6.13 (ZFC; Siskind)**.** *There is a Turing invariant function $f\colon 2^\omega \to 2^\omega$ such that f is order preserving, not constant on any cofinal set and for all noncomputable x, $f(x)$ does not compute x. Also, the range of f is disjoint from a cone and hence f is not measure preserving.*

Proof. By a well-known theorem of Sacks, every locally countable partial order of size at most ω_1 embeds into the Turing degrees. Let $(P, \leq_P)$ be the partial order consisting of ω_1 plus one point, q, which is incomparable to everything in ω_1 (this extra point is there to ensure that the range of f is disjoint from a cone). Let ψ be an embedding of P into the Turing degrees.

For any x, let α_x be the least ordinal below ω_1 such that x does not compute $\psi(\alpha_x)$. Note that such an α_x must exist since x can only compute countably many things, but the range of ψ is uncountable. Now define $f(x) = \psi(\alpha_x)$ (note that technically $\psi(\alpha_x)$ is a Turing degree, not a real, but we can use choice to pick a real in the Turing degree).

We now have a few things to check. First, it should be clear that f is Turing invariant. Next, if $x \leq_T y$ and x computes $\psi(\alpha)$ for some α then so does y. This shows that $\alpha_x \leq \alpha_y$. Since ψ is an order embedding, this implies that $f(x) \leq_T f(y)$ and hence f is order preserving.

Also, f is not constant on any cofinal set. Suppose it were, with constant value a. Let α be an ordinal such that a does not compute $\psi(\alpha)$. But there is a cone of x which compute $\psi(\alpha)$ (and hence $\psi(\beta)$ for all $\beta < \alpha$). On this cone, $\alpha_x > \alpha$ and hence $f(x) = \psi(\alpha_x)$ computes $\psi(\alpha)$ (since ψ is an embedding). Since a does not compute $\psi(\alpha)$, this shows that $f(x)$ is not equal to a, contradicting the fact that there is a cofinal set of reals which are sent to a.

Finally, the range of f is disjoint from a cone, namely the cone above $\psi(q)$. This is because the range of f is contained in the image of ω_1 under ψ and since ψ is an order embedding, nothing in this image computes $\psi(q)$.

You might notice that we have not quite satisfied one of the constraints of the theorem statement. As we have defined f, it is entirely possible (actually, certain) that $f(x)$ computes x for some noncomputable x (i.e. anything in the image of ω_1 under f). We can fix this problem by modifying f as follows. On the cone above $\psi(q)$, use the original definition of f. But if x is not in this cone then set $f(x) = 0$. It is easy to check that with this modification, f is still Turing invariant, order preserving, not constant on any cofinal set (every cofinal set is still cofinal when you take its intersection with the cone above $\psi(q)$) and its range is disjoint from a cone (the modification we made does not change the range of f except possibly to add 0 to it). □

Chapter 7

Application 1: Sacks' Question on Embedding Partial Orders Into the Turing Degrees

Note: The results of this chapter are joint work with Kojiro Higuchi.

In this chapter, we will describe an application of some of the techniques introduced in this thesis to Sacks' question about embedding partial orders into the Turing degrees.

Sacks' Question

A perennial question in computability theory is to determine which structures can be embedded into the Turing degrees. When the structures under consideration are partial orders, there are two obvious restrictions: the partial order of Turing degrees has size continuum and every Turing degree has at most countably many predecessors. Thus any partial order which embeds into the Turing degrees must have size at most continuum and all of its elements must have at most countably many predecessors. A long-open question of Sacks asks whether these are the only restrictions.

More precisely, if every element of a partial order has at most countably many predecessors then we say that the partial order is **locally countable**. In 1963, Sacks asked whether every locally countable partial order of size continuum can be embedded into the Turing degrees [Sac63]. Sacks himself proved that this holds in $\mathsf{ZFC} + \mathsf{CH}$ (because it holds in ZFC for all locally countable partial orders of size ω_1) and it is known to be independent of ZF. But whether it is independent of ZFC is an open question.

For an introduction to past results about embedding various partial orders and related structures into the Turing degrees, see the survey by Montalbán [Mon09].

What We Will Prove

We will not resolve Sacks' question in this thesis. But in this chapter we will present an interesting phenomenon related to it. First, we will show (in ZFC) that every size continuum, locally countable partial order *of height two* can be embedded into the Turing degrees. Superficially it appears plausible that this proof could be generalized substantially, but we will provide evidence that it is hard to extend even to partial orders of height three. Here's what we mean. We will show that the result for partial orders of height two is fairly robust: it works not only in ZFC but also in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$. However, we will show that the corresponding statement for partial orders of height three fails—it is provable in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ that there is a height three, locally countable partial order of size continuum that does not embed into the Turing degrees. Our proof of this non-embeddability result will use techniques introduced earlier in this thesis, in particular the basis theorem of section 2.3.

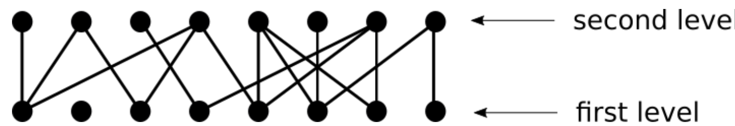
It turns out that this non-embeddability result also has consequences for embedding partial orders into the Turing degrees in ZFC. We will see that our proof actually shows that the technique we used for the height two case cannot be extended to work for height three partial orders, even in ZFC. More precisely, the embedding we construct for a height two partial order embeds the first level of the partial order as (the Turing degrees of) a perfect set of reals. But we will see that our non-embeddability proof for height three partial orders shows that there is a certain height three partial order such that no embedding of this partial order into the Turing degrees can embed the first level as a set of Turing degrees which contains a perfect set of reals. This complements another obstacle to embedding partial orders into the Turing degrees in ZFC discovered by Groszek and Slaman [GS83] and extended by Kumar in an unpublished note [Kum19]. We will discuss this more in section 7.4.

In the next chapter, we will also briefly revisit our results on height two and height three partial orders in the context of Borel relations on the reals. We will see that our results have parallels in the Borel world, which are part of a more general story about the theory of countable Borel equivalence relations and locally countable Borel quasi orders.

7.1 Embedding Height Two Partial Orders: ZFC Case

In this section we will explain how to embed any height two, locally countable partial order of size continuum into the Turing degrees. For now, we will just prove this theorem in ZFC. In the next section we will see how to refine the proof to work in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$.

Before we dive into the details of the proof, let's discuss the strategy. Recall that a partial order of height two is a partial order with no chains of length greater than two. We can think of such a partial order as consisting of two "levels": a first level which contains all elements with no predecessors and a second level which contains those elements which have one or more predecessors. Note that the elements of a single level are all incomparable with each other. Here's a picture.



The basic idea of the proof is that we will map the elements of the first level to a bunch of mutually generic reals (or, rather, the Turing degrees of these reals) and map each element of the second level to a sufficiently generic upper bound of the images of its predecessors.

There is one wrinkle in the proof: we need to ensure that even when two elements of the second level have exactly the same predecessors, they get mapped to incomparable Turing degrees[1]. One way to handle this is to insert a unique point below each element of the second level, which is not below any other elements of the partial order. This ensures that the elements of the second level all have distinct sets of predecessors and an embedding of this new partial order yields an embedding of the original partial order by forgetting about the new elements that we added. When we give the proof in detail, we will not quite explain things in this way, but it is essentially what will happen.

We will break the proof into two lemmas, the first of which tells us that we can find mutually generic reals to map the elements of the first level to and the second of which tells us that we can always find sufficiently generic upper bounds to map the elements of the second level to. Both lemmas are essentially folklore.

In a little more detail, the idea is that we will embed the first level as the set of paths through some carefully chosen perfect set of reals. Essentially we want to pick a perfect tree whose paths are all mutually generic, but for simplicity we will only require that the paths form a Turing independent set (see Definition 2.39), which will be sufficient for our purposes here. Since the set of paths through a perfect tree has size continuum, this will provide a set of size continuum to map the first level of the partial order to.

Lemma 7.1 (Sacks [Sac61] Theorem 3)**.** *There is a perfect tree T such that $[T]$ is a Turing independent set.*

Proof. First consider how one might construct a perfect tree none of whose branches is computable. To do this, we need to ensure that every branch of the tree disagrees with each total computable function in at least one place. We can accomplish this by "growing" the tree from the root node up in a series of stages. At each stage we have built a finite tree and we continue growing it by extending the leaf nodes (i.e. by adding children to the leaf nodes, children to those children, and so on). On alternate stages we can add incomparable children below every leaf node (to make sure the tree is perfect) and extend each leaf node to make sure any branch which extends it disagrees with the next total computable function (note that there is no need to make the tree itself computable).

To make sure that no finite set of branches computes any other branch, we can do something similar but now instead of extending leaf nodes one at a time to make them

[1]A similar problem arises if an element of the second level has exactly one predecessor: we need to make sure it gets sent to a different Turing degree than its predecessor.

disagree with the next computable function, we need to extend finite sets of leaf nodes at the same time to make sure the next computable function which uses those branches as an oracle disagrees with the branches of the tree which extend the other leaf nodes.

We will now describe this a bit more formally. We will form a sequence of finite subtrees of $2^{<\omega}$, $T_0 \subset T_1 \subset T_2 \subset \ldots$ such that T_{n+1} is an end extension of T_n (every node in $T_{n+1} \setminus T_n$ extends a leaf node of T_n). The final tree will be obtained as $T = \bigcup_{n \in \mathbb{N}} T_n$. We can start with T_0 as the tree just consisting of a single root node and nothing else (i.e. just the empty sequence).

Now we will explain how to extend T_n to T_{n+1}. The idea, again, is to first split every leaf node in T_n and then extend all of them without splitting in order to make sure that no finite subset of them is correctly computing any of the others using the the n^{th} Turing functional. To this end, first let T_n^0 be the tree formed by adding incomparable children below each leaf node of T_n. In other words,

$$T_n^0 = T_n \cup \{\sigma^\frown 0 \mid \sigma \text{ is a leaf node of } T_n\} \cup \{\sigma^\frown 1 \mid \sigma \text{ is a leaf node of } T_n\}.$$

Next, let Φ be the n^{th} Turing functional in some standard enumeration and let $\text{Leaves}(T_n^0)$ be the set of leaves of T_n^0. We will form a series of end extensions $T_n^0 \subset T_n^1 \subset \ldots \subset T_n^k$ (where k is the number of nonempty subset of $\text{Leaves}(T_n^0)$) and take $T_{n+1} = T_n^k$. In each of these extensions, we will not split any nodes. In other words, each leaf of T_n^0 will have at most one descendant at each level of T_n^i.

Suppose we have already formed T_n^i and let S be the i^{th} nonempty finite subset of $\text{Leaves}(T_n^0)$. We will now explain how to form T_n^{i+1}. Our goal is to ensure that no set of branches extending the nodes in S can compute any branch extending any other leaf node of T_n^0.

Since we never split any nodes in any of the previous extensions of T_n^0, each element of S corresponds to a unique leaf of T_n^i. Let $\sigma_1, \ldots, \sigma_l$ denote these leaves of T_n^i. Let N be a number larger than the height of T_n^i. Now either we can find extensions $\tau_1, \ldots, \tau_l$ of $\sigma_1, \ldots, \sigma_l$ such that $\Phi^{\tau_1 \oplus \ldots \oplus \tau_l}(N)$ converges or we can't find such extensions. In the former case, define T_n^{i+1} from T_n^i by extending each σ_j to τ_j and extending all other leaf nodes of T_n^i to strings whose N^{th} bit disagrees with $\Phi^{\tau_1 \oplus \ldots \oplus \tau_l}(N)$. In the latter case, set $T_n^{i+1} = T_n^i$.

Now let's check that $[T]$ is really a Turing independent set. Let $x_1, \ldots, x_l$ and y be distinct elements of $[T]$ and let Φ be any Turing functional. Let n be some number large enough that all of $x_1, \ldots, x_l$ and y correspond to distinct leaf nodes in T_n and chosen so that the n^{th} Turing functional is equivalent to Φ (we are assuming that every computable function shows up infinitely often in whatever enumeration we are using). Suppose that $x_1, \ldots, x_l$ correspond to the i^{th} set of leaves of T_n^0. Then our definition of T_n^{i+1} ensures that either $\Phi^{x_1 \oplus \ldots \oplus x_l}$ disagrees with y in at least one place (this corresponds to the first case in our construction above) or $\Phi^{x_1 \oplus \ldots \oplus x_l}$ is not a total function (this corresponds to the second case). □

Our second lemma guarantees we can find sufficiently generic upper bounds to map the elements of the second level of our partial order to. We stated and proved this lemma in section 2.6 as Theorem 2.40, but we will restate it here for convenience.

Lemma 7.2. *Suppose T is a perfect tree such that $[T]$ is Turing independent. Then every countable subset of $[T]$ has an upper bound in the Turing degrees which does not compute any other element of $[T]$.*

We will now explain how to put together the two lemmas to prove the main theorem of this section.

Theorem 7.3. *Every height two, locally countable partial order of size continuum embeds into the Turing degrees.*

Proof. Let $(P, \leq_P)$ be a height two, locally countable partial order of size continuum. We will define a map $\psi\colon P \to 2^\omega$ such that $x \leq_P y$ if and only if $\psi(x) \leq_T \psi(y)$. So even though ψ is a map of P into the reals, it induces an embedding of P into the Turing degrees.

First, let T be a perfect tree such that $[T]$ is Turing independent, as in Lemma 7.1. Since P and $[T]$ both have size continuum, we can find an injective map $f\colon P \to [T]$.

Now, let P_0 be the first level of P and let P_1 be the second level. We will define $\psi(x)$ by cases depending on whether x is in P_0 or P_1. If x is in P_0 then we will simply set $\psi(x) = f(x)$. If x is in P_1 then define $\psi(x)$ as follows. Let $P_{\leq x} = \{y \in P \mid y \leq_P x\}$ be the set of predecessors of x in P; note that $P_{\leq x}$ includes x itself. By Lemma 7.2, we can find a real which computes every element of the image of $P_{\leq x}$ under f but which computes no other elements of $[T]$. Set $\psi(x)$ equal to this real.

Now we need to check that ψ is an embedding. Let x and y be any two distinct elements of P. We need to show that $x \leq_P y$ if and only if $\psi(x) \leq_T \psi(y)$.

First suppose $x \leq_P y$. Notice that x must be in the first level of P and y must be in the second level. Therefore $\psi(x) = f(x)$ and $\psi(y)$ is an upper bound in the Turing degrees for a set which includes $f(x)$, so $\psi(y)$ computes $\psi(x)$.

Now suppose that $x \not\leq_P y$. We know that no matter which level x is in, $\psi(x)$ computes $f(x)$. So to show that $\psi(y)$ doesn't compute $\psi(x)$, it is enough to show that $\psi(y)$ doesn't compute $f(x)$. If y is in the first level of P then this is guaranteed by the fact that $f(x)$ and $f(y)$ are distinct elements of the Turing independent set $[T]$. And if y is in the second level of P then since x is not a predecessor of y, our choice of $\psi(y)$ ensures that it cannot compute $f(x)$. □

The theorem we have just proved is very similar to Theorem 2.2 of the paper "Separating Families and Order Dimension of Turing Degrees" by Kumar and Raghavan [KR21]. That theorem states that a specific height two, locally countable partial order of size continuum—which the authors refer to as $\mathbb{H}_\mathfrak{c}$—embeds into the Turing degrees. Obviously this theorem is implied by our Theorem 7.3. But, in fact, it is not too hard to show that every height two, locally countable partial order of size continuum embeds into $\mathbb{H}_\mathfrak{c}$ and so the two theorems are actually equivalent.

An especially attentive reader of both papers may note that the proofs of our Theorem 7.3 and Kumar and Raghavan's Theorem 2.2 are somewhat different. Essentially, our proof first embeds the first level of the partial order in "one shot" and then embeds the elements of the second level independently of each other. In contrast, Kumar and Raghavan use transfinite recursion to choose the image of each element of the partial order one at a time in such a way that the image of each element may depend on all the choices already made. Moreover, the elements of the two levels are interleaved in this recursion and so the choices made for elements of the first level may depend on the choices made for elements of the second level.

There are advantages and disadvantages to each approach. Our approach can be extended to show that the result still holds in $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$ and in the Borel setting, which is not apparent for Kumar and Raghavan's approach. On the other hand, because our approach embeds the first level of the partial order as a perfect set, it falls prey to the nonembedding result we describe in section 7.3 and so cannot be extended to handle height three partial orders whereas it is not clear whether this is a problem for the Kumar and Raghavan approach. But on the *other* other hand, the Kumar and Raghavan approach falls prey to a different obstacle to embedding partial orders in the Turing degrees discovered Groszek-Slaman and extended by Kumar [GS83; KR21]. We will have more to say about this in section 7.4.

7.2 Embedding Height Two Partial Orders: $\mathsf{AD}_{\mathbb{R}}$ Case

In this section we will show that the proof from the previous section also works in $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$. This is interesting because, as we will see in the next section, it is provable in $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$ that not every height three, locally countable partial order of size continuum embeds into the Turing degrees.

Actually, the theory $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$ is much stronger than necessary to prove the results of this section and the next section. All that is needed beyond ZF to show that height two partial orders can be embedded into the Turing degrees is $\mathsf{Uniformization}_{\mathbb{R}}$, the statement that every binary relation on the reals can be uniformized. And all that is needed beyond ZF to show that not every height three partial order can be embedded into the Turing degrees is the statement that every set of reals has the perfect set property. On their own, each of these statements is much weaker than $\mathsf{AD}_{\mathbb{R}}$. We chose to state our results in $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$ rather than these weaker theories because it is a well-known theory which suffices to prove both results and it demonstrates that the two results are consistent with each other.

So how do we modify the proof of Theorem 7.3 to work in $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$? There is actually only one part of the proof that cannot be carried out in ZF. In Theorem 2.40, we saw how to find an upper bound for any countable subset of a Turing independent perfect set. The proof required choosing a specific enumeration of the elements of the countable subset. In the context of the proof of Theorem 7.3, this means that we had to be able to choose an enumeration of the predecessors of each element of the second level of the partial order that we are trying to embed. The proof still works in $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$ because we have access to a sufficiently strong form of choice (namely $\mathsf{Uniformization}_{\mathbb{R}}$) to pick these enumerations.

We will now give the details of the proof. We start by stating the consequence of $\mathsf{Uniformization}_{\mathbb{R}}$ that we will need.

Corollary 7.4 (ZF+$\mathsf{AD}_{\mathbb{R}}$)**.** *If R is a binary relation on 2^ω with nonempty, countable sections (i.e. for each $x \in 2^\omega$, the set $\{y \mid R(x,y)\}$ is nonempty and countable) then there is a function $f\colon 2^\omega \to (2^\omega)^\omega$ such that for each $x \in \mathrm{dom}(R)$, $f(x)$ is an enumeration of the section of R at x (where repeats are allowed in the enumeration).*

Proof. Let S be the relation on $2^\omega \times (2^\omega)^\omega$ defined by

$$S(x,y) \iff y \text{ is an enumeration of the set } \{z \mid R(x,z)\}.$$

Because there is a bijection between 2^ω and $(2^\omega)^\omega$, we can think of S as a binary relation on 2^ω. Since $\mathsf{Uniformization}_{\mathbb{R}}$ is provable in $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$, S can be uniformized. Since a uniformization of S is a function that picks an enumeration of the sections of R, we are done. □

The theorem now follows from the proof of Theorem 7.3, replacing the one use of choice in that proof with the corollary above.

Theorem 7.5 (ZF+$\mathsf{AD}_{\mathbb{R}}$)**.** *Every height two, locally countable partial order of size continuum embeds into the Turing degrees.*

7.3 Nonembedding for Height Three Partial Orders: $\mathsf{AD}_{\mathbb{R}}$ Case

In this section, we will show that it is provable in $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$ that there is some height three, locally countable partial order of size continuum which does not embed into the Turing degrees. The only special fact about $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$ that we need is the perfect set theorem[2]. We will combine this with our basis theorem for perfect sets (Theorem 2.19) much as we did in the proof that order preserving functions are either constant on a cone or measure preserving (Theorem 6.3).

Our proof proceeds in two steps. First we will show that no height three partial order with certain properties embeds into the Turing degrees and then we will show that there is a height three, locally countable partial order of size continuum which has those properties.

Lemma 7.6 ($\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$)**.** *Suppose $(P, \leq_P)$ is a height three partial order with the following properties.*

1. *The first level of P (i.e. the set of elements in P which have no predecessors) has size continuum.*
2. *Every countable subset of the first level of P has an upper bound in the second level.*
3. *For every finite subset S of the first two levels of P and every x in the second level which is not in S, there is some y in the third level which is above every element of S but not above x.*

[2] Which, as we have noted, is actually provable in much weaker theories than ZF+$\mathsf{AD}_{\mathbb{R}}$, including ZF+AD.

Then P cannot be embedded into the Turing degrees.

Proof. Suppose for contradiction that f is an embedding of P into the Turing degrees. Roughly the argument will go as follows: the image of the first level of P under f is uncountable and so it must contain a perfect set. We can then use the basis theorem for perfect sets stated above to conclude something about the image of P in the Turing degrees which contradicts the assumptions we are making about the structure of P.

There is something odd about this sketch that we have to clear up, though: the image of the first level of P under f is a set of *Turing degrees* but the perfect set theorem and the basis theorem for perfect sets are both statements about sets of *reals*. The solution to this discrepancy is to consider not the image of f, but rather the set of reals which are in some Turing degree in the image of f. In the rest of the proof, we will implicitly identify a set of Turing degrees with the set of reals which are contained in those Turing degrees.

As we have already said, the image under f of the first level of P is an uncountable set of reals and hence contains a perfect set (again, what we really mean here is not quite the image of the first level of P under f, but rather the set of reals contained in one of the Turing degrees in this set). Pick a countable dense subset of this perfect set and let A be the elements of the first level of P which map to this countable dense subset. Let x be an element of the second level of P which is an upper bound for A (and thus $f(x)$ computes every element of the countable dense subset).

Now let y be an element of the second level of P which is not equal x[3]. By the basis theorem for perfect sets that we proved in chapter 2 (Theorem 2.19), there are four elements, a_0, a_1, a_2, a_3, of the perfect set which, together with $f(x)$, compute $f(y)$. Since the perfect set is contained in the image of the first level of P under f, this means there are four elements, w_0, w_1, w_2, w_3, of the first level of P which map to (the Turing degrees of) a_0, a_1, a_2, a_3.

Pick an element z of the third level of P which is above these w_0, w_1, w_2, w_3 and above x, but not above y. Since z is above w_0, w_1, w_2, w_3 and x, $f(z)$ must compute a_0, a_1, a_2, a_3 and $f(x)$ and hence also computes $f(y)$. But since we chose z to not be above y, this contradicts the fact that f is an embedding. □

Lemma 7.7 (ZF). *There is a height three, locally countable partial order of size continuum with the properties listed in the statement of Lemma 7.6.*

Proof. We can essentially take the free height three, locally countable partial order on continuum-many generators. Let's explain what that means.

Let P_0 be a set of size continuum (2^ω, say), let P_1 be the set of countable sequences of elements of P_0 (with repetitions allowed—i.e. $P_1 = P_0^\omega$) and let P_2 be the set of countable sequences of elements of $P_0 \cup P_1$ which have at least one element in P_1. Let $(P, \leq_P)$ be the partial order whose domain is $P_0 \cup P_1 \cup P_2$ and where $x \leq_P y$ if one of the three following conditions holds

- $x = y$

[3] Notice that the properties of P imply the existence of two distinct elements in the second level of P.

- y is in $P_1 \cup P_2$ and for some $n \in \mathbb{N}$, $y_n = x$ (recall that the elements of P_1 and P_2 are countable sequences)
- y is in $P_1 \cup P_2$ and for some $n, m \in \mathbb{N}$, y_n is in P_1 and $(y_n)_m = x$.

It is easy to check that $(P, \leq_P)$ is a height three, locally countable partial order of size continuum such that the conditions required by Lemma 7.6 hold. □

Together, Lemmas 7.6 and 7.7 imply the main theorem of this section.

Theorem 7.8 ($\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$)**.** *There is a height three, locally countable partial order of size continuum which does not embed into the Turing degrees.*

7.4 Obstacles to Embedding Partial Orders in the Turing Degrees

Our proof of Theorem 7.8 actually shows something stronger than that not every height three partial order embeds into the Turing degrees in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$. It shows that if $(P, \leq_P)$ is the partial order from Lemma 7.7, then any strategy for embedding P into the Turing degrees which ends up with a perfect set contained in the image of the first level of P is guaranteed to fail, even if we are working in ZFC. But our proof that height two partial orders embed into the Turing degrees began by embedding the first level of the partial order as a perfect set! Thus there can be no direct extension of our proof of Theorem 7.3 to work for partial orders of height three. In this section, we will discuss how this observation fits into other known obstacles to embedding partial orders into the Turing degrees.

We will begin our discussion with fairly general considerations. Suppose you want to embed an arbitrary locally countable partial order into the Turing degrees. How might you go about it? A reasonable approach is to pick a well-ordering of the elements of the partial order and proceed by transfinite recursion. In other words, pick up elements of the partial order one at a time and show that as long as you have embedded fewer than continuum many elements so far, there is always a place to send the next element. This is essentially the approach taken by Sacks to embed locally countable partial orders of size ω_1 in [Sac63] and also the approach taken by Kumar and Raghavan to embed height two partial orders [KR21].

A fundamental obstacle to using this approach to embed an arbitrary locally countable partial order was discovered by Groszek and Slaman [GS83]. They showed that it is consistent with ZFC that there is a maximal Turing independent set of size less than continuum. Thus if you want to embed the elements of a partial order into the Turing degrees by a transfinite induction of length continuum, you have to be careful not to end up with this particular Turing independent set in the range of your embedding after fewer than continuum many steps. For suppose you did end up with this set in the range of your embedding. Then if you later encounter another element of the partial order which is sufficiently independent of all the others you have seen so far, there will be nowhere to send it. A potential solution to this is to make the transfinite recursion satisfy some stronger inductive assumption that prevents this situation from occurring, but no suitable condition has been identified so far. In [Kum19], Kumar used Groszek and Slaman's technique to show that even if you are embedding a height three partial order whose first level has size ω_1, a similar obstacle may occur.

From these results of Groszek, Slaman and Kumar, we know that it is tricky to embed locally countable partial orders into the Turing degrees by using transfinite recursion. This suggests that a more structural approach may work better. To make things simple, suppose that we are trying to embed a locally countable partial order of finite height. A reasonable-sounding approach might be to find an especially "nice" subset of the Turing degrees to map the first level to, and then use the features of this "nice" subset to find a "nice" subset of the Turing degrees to map the second level to, and so on. Note that this is exactly the approach we have taken in section 7.1.

But as we have noted, the results of the previous section show that there are also obstacles to extending this approach. In particular, if by "nice" subset we mean "perfect set" then it is guaranteed to fail for sufficiently complicated partial orders of height at least three. This is made precise in the following theorem, whose proof is just a subset of our proof of Theorem 7.8.

Theorem 7.9. *There is a height three, locally countable partial order of size continuum, $(P, \leq_P)$, such that for any function f from P into the Turing degrees, if the image of f on the first level of P contains a perfect set of reals then f cannot be an embedding.*

It is common to phrase obstacles to embedding partial orders into the Turing degrees in terms of obstacles to extending or modifying Turing independent sets. We can also do that here. The following theorem easily implies Theorem 7.9 and its proof is more or less the same.

Theorem 7.10. *Suppose A is a perfect subset of 2^ω which is Turing independent, B is a countable dense subset of A and x is a real which computes every element of B. Then $(A \setminus B) \cup \{x\}$ is not Turing independent.*

It would be interesting to know if this obstacle could be combined with the obstacles identified by Groszek, Slaman, and Kumar to resolve Sacks' question, but at present we have no idea how to do so. One rather incredible possibility is indicated by the following question: Is it consistent with ZFC that there is a height three, locally countable partial order of size continuum which cannot be embedded into the Turing degrees? It sounds rather unbelievable that this question could have a positive answer, but it is surprisingly difficult to rule it out.

Chapter 8

Application 2: The Theory of Locally Countable Borel Quasi Orders

In the last chapter we saw an application of the techniques of this thesis to questions about embedding partial orders into the Turing degrees. In this chapter, we will discuss similar results in the Borel context. The idea is to investigate which *Borel* partial orders have *Borel* embeddings into the Turing degrees.

At first it may not even be clear what this question means. The Turing degrees do not form a topology in any obvious way (the quotient topology they inherit from the reals is trivial) and thus it is not clear what we mean by a "Borel embedding." The usual solution is to say that a Borel partial order $\leq_P$ on the reals has a Borel reduction to the Turing degrees[1] if there is a Borel map on the reals which induces an embedding of $\leq_P$ into the Turing degrees—i.e. into the quotient of the reals by Turing equivalence (and if φ is injective then it is called a Borel embedding). Thus what we are really asking for is a Borel map, φ, such that $x \leq_P y$ if and only if $\varphi(x) \leq_T \varphi(y)$. This can be thought of as an instance of a more general notion of Borel reductions between Borel relations on the reals (which of course could itself be generalized much further).

Definition 8.1. *If R and S are n-ary relations on 2^ω which are Borel (when considered as subsets of $(2^\omega)^n$) then a* ***Borel reduction*** *of R into S is a Borel map $\varphi\colon 2^\omega \to 2^\omega$ such that for all $x_1, \ldots, x_n \in 2^\omega$,*

$$R(x_1, \ldots, x_n) \iff S(\varphi(x_1), \ldots, \varphi(x_n)).$$

If there is a Borel reduction of R into S then we say that R is ***reducible*** *to S, written $R \leq_B S$. Note that φ here is not required to be injective on 2^ω. If it is, then it is called a* ***Borel embedding*** *and we say that R* ***embeds*** *into S, written $R \sqsubseteq_B S$.*

A natural question, which can be thought of as a Borel version of Sacks' question, is whether every locally countable Borel partial order on the reals is Borel reducible to Turing

[1]Note the slight change in terminology here; it might be reasonable to call this a "Borel embedding" instead, but that term is reserved for a stricter notion.

reducibility[2]. However, this is arguably not quite the right question to ask. When we ask about reducing a Borel partial order, $\leq_P$, to Turing reducibility, there is an obvious discrepancy. We are assuming that $\leq_P$ is a partial order on the reals, but $\leq_T$ itself is not a partial order on the reals, only a quasi order (essentially a partial order where some non-equal elements are allowed to be equivalent). Thus a fairer question might be whether every locally countable Borel quasi order on the reals has a Borel reduction to Turing reducibility (note that every partial order is also a quasi order, so this question is more general). This second question is typically phrased as asking whether Turing reducibility is a universal locally countable Borel quasi order (which just means a locally countable Borel quasi order which every other locally countable Borel quasi order reduces to).

Later in this chapter, we will see that both the answer to both questions is "no." In sections 8.1 and 8.2 we will see that there are locally countable Borel quasi orders which do not Borel reduce to Turing reducibility, and we can even find some which are actually partial orders (in fact, partial orders of height three, a la the results of the previous chapter).

Besides the connection to Sacks' question, there is another motivation to care about the topics of this chapter. The question of whether Turing reducibility is a universal locally countable Borel quasi order has a parallel in the world of countable Borel equivalence relations (i.e. Borel equivalence relations whose equivalence classes are all countable). The theory of countable Borel equivalence relations has received a lot of attention in recent years and a well-developed structure theory has emerged (see for instance the survey by Kechris [Kec21]). A major open question in the area, known as Kechris' conjecture, asks whether Turing equivalence is a universal countable Borel equivalence relation.

Conjecture 8.2 (Kechris; [DK00])**.** *Turing equivalence is a universal countable Borel equivalence relation—that is, every countable Borel equivalence relation on the reals is Borel reducible to Turing equivalence.*

This conjecture is interesting for a few reasons. First, it would refute the Borel version of Martin's conjecture: if Turing equivalence is universal then it is easy to construct Turing invariant Borel functions which violate part 1 of Martin's conjecture (for example by using a Borel reduction of the disjoint union of two copies of Turing equivalence to Turing equivalence—see the paper by Dougherty and Kechris [DK00]). Second, it may have consequences for the general theory of countable Borel equivalence relations. Slaman and Steel's counterexample to Martin's conjecture on the arithmetic degrees can be extended to show that arithmetic equivalence is a universal countable Borel equivalence relation and this fact was used by Marks to show that a countable Borel equivalence relation is universal for Borel reductions if and only if it is universal for Borel embeddings (a priori a stronger notion). Marks has also shown that polynomial time Turing equivalence is universal and used this to derive other facts about countable Borel equivalence relations.

[2] By the way, we are only restricting to Borel relations on the reals for convenience. There is no meaningful change to the question if we ask instead about locally countable Borel partial orders on any Polish space, for instance.

Thus far, when a countable Borel equivalence relation arises from a locally countable Borel quasi order, it has generally been expected that universality as an equivalence relation and as a quasi order go together. For example, both the "hereditarily a column of" quasi order and arithmetic reducibility induce universal countable Borel equivalence relations and are themselves universal locally countable Borel quasi orders. Therefore the fact that Turing reducibility is *not* a universal locally countable Borel quasi order can be seen as evidence that Turing equivalence is not a universal countable Borel equivalence relation. Later in this chapter (in section 8.2) we will discuss another statement related to locally countable Borel quasi orders which, if true, could be seen as even stronger evidence in this direction.

8.1 Turing Reducibility is not a Universal Locally Countable Borel Quasi Order

In this section we will show that Turing reducibility is not a universal locally countable Borel quasi order. We will begin with one particular example of a locally countable Borel quasi order that does not reduce to Turing reducibility and then prove the same result for a class of quasi orders generalizing this example. The main idea of these proofs is essentially to combine the perfect set theorem for analytic sets with the basis theorem for perfect sets proved in section 2.3. This basic technique can actually be used to prove a number of other Borel quasi orders do not reduce to Turing reducibility—we will see another instance of this in the next section. Rather than trying to state some overly technical theorem capturing all cases that this technique can handle, we will just provide some examples to demonstrate how it works. But first, for the sake of completeness, we will give a formal definition of "locally countable quasi order."

Locally Countable Quasi Orders

First we will define "quasi order."

Definition 8.3. *A* ***quasi order*** *is a binary relation which is transitive and reflexive.*

This definition, while not complicated, somehow does not immediately convey the flavor of a quasi order. A quasi order should be thought of as a partial order where some elements are allowed to be equivalent. Once we quotient by this equivalence, we get an honest partial order. Here's what we mean. If $(P, \leq_P)$ is a quasi order then the relation $x \sim_P y$ which holds whenever both $x \leq_P y$ and $y \leq_P x$ hold is an equivalence relation on P. And if we mod out by $\sim_P$ then $\leq_P$ induces a partial order on the resulting quotient. This is exactly the process of passing from the structure of Turing reducibility on the reals (a quasi order) to Turing reducibility on the Turing degrees (the induced partial order on the quotient).

Definition 8.4. *Suppose $(P, \leq_P)$ is a quasi order. Then it is* ***locally countable*** *if for every $x \in P$, the set $\{y \in P \mid y \leq_P x\}$ is countable. Note that this includes elements of P which are not strictly below x.*

If a quasi order $(P, \leq_P)$ is locally countable then all equivalence classes of the associated relation, $\sim_P$, must be countable and the induced partial order on the quotient by $\sim_P$ must be locally countable.

Note that this means that every locally countable quasi order has associated to it a countable equivalence relation (namely $\sim_P$). And if the original quasi order happens to be Borel then so is the associated equivalence relation. Thus locally countable Borel quasi orders can be thought of as countable Borel equivalence relations with some extra structure added and Borel reductions between Borel quasi orders are just Borel reductions of the associated equivalence relations which are required to satisfy more stringent conditions. We will return to this point in the next section.

Example 1: The Turing Degrees Plus a Point

We will now show that Turing reducibility is not universal. More specifically, we will show that if we add one extra point onto the Turing degrees which is not comparable with anything else then the resulting Borel quasi order does not have a Borel reduction to Turing reducibility.

Here's the main idea of the proof. If we have a Borel reduction from Turing reducibility plus a point to regular Turing reducibility then by ignoring the extra point, we get an injective, order preserving Borel function on the Turing degrees. By the Borel version of the results of section 6.1, this function must be measure preserving. But this means that this function eventually gets above the image of the extra point, which contradicts the fact that it is a reduction (since the extra point is supposed to be incomparable to everything else).

We will now fill in the details in this sketch. First, let's recall the main result of section 6.1: every Turing invariant function which is an order preserving function on the Turing degrees is either constant on a cone or measure preserving. This theorem (Theorem 6.3) was proved for all functions using $\mathsf{ZF} + \mathsf{AD}$, but it is not hard to see that the proof also holds for all Borel functions in ZF. The key point is that the perfect set theorem that we used in that proof (Theorem 6.2) has a Borel version provable in ZF:

Theorem 8.5 (Perfect set theorem for analytic sets; [Kec95] Exercise 14.13)**.** *Every $\mathbf{\Sigma^1_1}$ definable subset of 2^ω is either countable or contains a perfect subset.*

Next we will define precisely what we mean by "Turing reducibility plus a point." Let 0 denote the element of 2^ω whose bits are all 0s and let $\leq_T^*$ be the binary relation on 2^ω defined as follows.

$$x \leq_T^* y \iff \begin{cases} x \leq_T y \text{ and } x, y \neq 0 \\ x = y = 0. \end{cases}$$

In other words, $\leq_T^*$ is exactly like Turing reducibility except that there is a special point, 0, which is not comparable to anything else. It is easy to see that this is a locally countable Borel quasi order. We will now show that it does not Borel reduce to Turing reducibility.

Theorem 8.6. *The quasi order $\leq_T^*$ is not Borel reducible to $\leq_T$.*

Proof. Suppose for contradiction that $f\colon 2^\omega \to 2^\omega$ is a Borel reduction from $\leq_T^*$ to $\leq_T$. Let f^* denote the function f restricted to all the reals not equal to 0. By the definition of "Borel reduction," f^* is a Borel order preserving function which is injective on the Turing degrees (though not necessarily on the reals). By Theorem 6.3, either f^* is constant on a cone or measure preserving. But since it is injective on the Turing degrees it cannot be constant on a cone and thus must be measure preserving.

Since f^* is measure preserving, its range is cofinal in the Turing degrees. Thus there is some $x \neq 0$ such that $f(0) \leq_T f^*(x)$. But $f^*(x)$ is just $f(x)$ and hence we have $f(0) \leq_T f(x)$. Since f is a Borel reduction, this implies that $0 \leq_T^* x$, which contradicts the definition of $\leq_T^*$ (since 0 is supposed to be incomparable to all other elements). □

Corollary 8.7. *Turing reducibility is not a universal locally countable Borel quasi order.*

Generalization: Any Countably Directed Order Plus a Point

We will now prove a generalization of Theorem 8.6. It may seem like that theorem is hard to generalize much since it depended on special properties of order preserving functions on the Turing degrees and these may not carry over to embeddings of more general quasi orders into the Turing degrees. However, if you "unroll" the proof of that theorem you will find that the basis theorem for perfect sets is doing most of the work and that it doesn't use that much about the structure of $\leq_T^*$. In fact, the main feature of $\leq_T^*$ that is important is that it has an uncountable Borel subset which is countably directed. This observation is the basis for the next theorem, which essentially says that if a Borel quasi order contains a large countably directed subset plus a point not below that subset then it does not have a Borel reduction to Turing reducibility.

Definition 8.8. *A quasi order $(P, \leq_P)$ is countably directed if every countable subset has an upper bound.*

Theorem 8.9. *Suppose $\leq_P$ is a Borel quasi order on 2^ω, A is a Borel subset of 2^ω which is uncountable and countably directed under $\leq_P$ and x is an element of 2^ω which is not $\leq_P$-below any element of A. Then $\leq_P$ does not Borel reduce to Turing reducibility.*

Proof. Suppose for contradiction that $f\colon 2^\omega \to 2^\omega$ is a Borel reduction from $\leq_P$ to $\leq_T$. The image of A under f is an uncountable $\mathbf{\Sigma^1_1}$ set and thus by Theorem 8.5 contains a perfect set. And by the definition of "reduction," it is also countably directed for Turing reducibility. Thus by Theorem 2.21, $f(A)$ is cofinal in the Turing degrees. This implies that $f(A)$ contains an element which computes $f(x)$. In other words, there is some $y \in A$ such that $f(x) \leq_T f(y)$. Since f is a reduction, this means that $x \leq_P y$, which contradicts one of our assumptions. $\square$

By the way, note that implicit in this proof is the fact that every uncountable, countably directed $\mathbf{\Sigma^1_1}$ subset of the Turing degrees is cofinal.

8.2 Borel Quasi Orders of Finite Height

In this section we will investigate which locally countable Borel quasi orders of finite height have Borel reductions to Turing reducibility.

Definition 8.10. *A quasi order $(P, \leq_P)$ is said to have* ***height*** *n if the longest strictly descending chain in P has length n. It is said to have* ***finite height*** *if it has height n for some $n \in \mathbb{N}$.*

As we mentioned earlier, one of our main motivations for studying Borel quasi orders of finite height comes from Kechris' conjecture on countable Borel equivalence relations. Before we state our main results, let us explain this connection a bit more. Recall that Kechris' conjecture states that every countable Borel equivalence relation has a Borel reduction to Turing equivalence. Now note that any countable equivalence relation can be thought of as a locally countable quasi order of height one by setting all the equivalence classes to be incomparable to each other. There is one subtlety to note here—namely, a Borel reduction of an equivalence relation to Turing equivalence is not quite the same as a Borel reduction of that same equivalence relation, considered as a height one quasi order, to Turing reducibility. In the former case, distinct equivalence classes only need to be sent to distinct Turing degrees, while in the latter case they must be sent to *incomparable* degrees.

Thus the question of whether every locally countable Borel quasi order of height one has a Borel reduction to Turing reducibility can be seen as a mild strengthening of Kechris' conjecture. And the question of whether every locally countable Borel quasi order of finite height has a Borel reduction to Turing reducibility can be seen as a further strengthening, intermediate between Kechris' conjecture and the conjecture (which we have just seen is false) that every locally countable Borel quasi order (of any height, not necessarily finite) has a Borel reduction to Turing reducibility.

In this section, we will see that this question about quasi orders of finite height also has a negative answer. In fact, there is a locally countable Borel quasi order of height three that has no Borel reduction to Turing reducibility. The proof of this fact closely parallels the proof of the analogous fact which we proved in the previous chapter in the $\mathsf{AD}_{\mathbb{R}}$ setting.

As in the previous chapter, we will also prove a funny difference between height two and height three partial orders. As we will see, there is actually a height three, locally countable Borel *partial order* (not just quasi order) which does not have a Borel reduction to Turing reducibility, but, on the other hand, every *height two*, locally countable Borel partial order does have a Borel reduction to Turing reducibility. Note that this second fact is significantly weaker than the analogous fact for height two quasi orders (and does not imply Kechris' conjecture).

The proofs are overall quite similar to the proofs in the previous chapter, but we will occasionally need to invoke some new tools to get around obstacles in the Borel setting that are not present in the $\mathsf{AD}_{\mathbb{R}}$ setting.

Borel Embedding Height Two Partial Orders

We will start with the case of height two partial orders. Suppose that $\leq_P$ is a height two, locally countable Borel partial order on 2^ω. We will show that there is a Borel reduction[3] of $\leq_P$ to Turing reducibility, $\leq_T$.

There are two obstacles to adapting the proof of Theorem 7.3 to the Borel case. The first is an obstacle that we also encountered when adapting the proof to work in $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$—namely that for each element of the second level of the partial order, we need to choose an enumeration of its predecessors. However, we do not have access to the same kind of uniformization result that we used in the $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ case; it is simply not true that every binary Borel relation has a Borel uniformization. However, it turns out that corollary 7.4, which is all that we really needed in the proof, does have a Borel version, called the Lusin-Novikov theorem (which also has applications in the theory of countable Borel equivalence relations).

Theorem 8.11 (Lusin-Novikov uniformization theorem; [Kec95] Theorem 18.10)**.** *Suppose R is a Borel subset of $2^\omega \times 2^\omega$ with countable sections (i.e. for each $x \in 2^\omega$, the set $\{y \mid (x, y) \in R\}$ is countable). Then both of the following hold:*

1. *The domain of R (i.e. the set $\{x \mid \exists y\, (x, y) \in R\}$) is Borel.*
2. *R can be written as a countable union of Borel subsets of $2^\omega \times 2^\omega$ which are graphs of partial functions.*

The second obstacle is that in both the ZFC and $\mathsf{ZF}+\mathsf{AD}_{\mathbb{R}}$ cases, we defined the embedding $\psi(x)$ by cases depending on whether x was in the first or second level of the partial order. So to ensure that we actually get a Borel reduction, we need to show that the first and second levels of the partial order are both Borel sets. Because the partial order is locally countable, this follows from the first part of the Lusin-Novikov theorem.

We will now proceed with a formal proof. For clarity, we will begin by stating a refined version of Theorem 2.40 (which guaranteed the existence of sufficiently generic upper bounds). The proof just consists of noting that other than choosing an enumeration of the countable set, every part of the construction of the upper bound in Theorem 2.40 was arithmetically definable and thus yields a Borel map from enumerations to upper bounds.

[3] Which is actually also automatically a Borel embedding, but we won't pay much attention to that here.

Lemma 8.12. *Suppose T is a perfect tree such that $[T]$ is Turing independent. Then there is a Borel function $g\colon (2^\omega)^\omega \to 2^\omega$ such that for any sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ of elements of $[T]$, $g(\langle x_n\rangle_{n\in\mathbb{N}})$ computes all of the x_n's but does not compute any other element of $[T]$.*

Theorem 8.13. *Every height two, locally countable Borel partial order on 2^ω has a Borel embedding into Turing reducibility.*

Proof. Let $\leq_P$ be a height two, locally countable Borel partial order on 2^ω. Recall that in the proof of Theorem 7.3, we constructed a map $\psi\colon P \to 2^\omega$ such that $x \leq_P y$ if and only if $\psi(x) \leq_T \psi(y)$. In this proof, we will simply give a definition of the function ψ which makes it clear why it is Borel; the proof that it is an embedding is unchanged (and we no longer need to worry that it maps into the reals rather than the Turing degrees, because a Borel reduction to Turing reducibility *is* a map into the reals).

Let T be a perfect tree such that $[T]$ is Turing independent, as in Lemma 7.1. Let $f\colon 2^\omega \to [T]$ be a homeomorphism. By the Lusin-Novikov theorem, there is a Borel function $F\colon 2^\omega \to (2^\omega)^\omega$ such that for each $x \in 2^\omega$, $\langle F(x,n)\rangle_{n\in\mathbb{N}}$ enumerates the set of $\leq_P$-predecessors of x (possibly with duplicates)[4]. By Lemma 8.12, there is a Borel function $g\colon (2^\omega)^\omega \to 2^\omega$, which, when given any sequence of elements of $[T]$, outputs a real which computes all of them but does not compute any other element of $[T]$. Now define ψ as follows

$$\psi(x) = \begin{cases} f(x) & \text{if } \forall y \in 2^\omega\,(y \not<_P x)\\ g(n \mapsto f(F(x,n))) & \text{if } \exists y \in 2^\omega\,(y <_P x). \end{cases}$$

By the first part of the Lusin-Novikov theorem, the set $\{x \mid \exists y\,(y <_P x)\}$ is Borel and thus the function ψ defined above is a Borel function. $\square$

Borel Nonembedding for Height Three Partial Orders

We will now explain how to reproduce the results of section 7.3 in the Borel setting. Namely, we will show that there is a height three, locally countable Borel partial order on 2^ω which does not have a Borel reduction to Turing reducibility[5]. The proof of Theorem 7.8 actually requires little modification to adapt to the Borel setting. Recall that the proof was broken into two parts: the first showed that in $\mathsf{ZF} + \mathsf{AD}_\mathbb{R}$, no partial order with certain properties can be embedded into the Turing degrees and the second showed that in ZF there is a height three, locally countable partial order of size continuum with those properties.

A close examination of the proof of the second part (the construction of the height three partial order) reveals that it actually yields a Borel partial order on 2^ω. For the first part, the only fact used in the proof that is not provable in ZF is that every uncountable subset

[4]It may seem that there is a slight problem here since the Lusin-Novikov theorem only promises a *partial* function. But by the first part of the Lusin-Novikov theorem, the domain of each $F(-,n)$ is Borel so we can extend F to a total Borel function by setting $F(x,n) = x$ anytime it is undefined.

[5]And note that every partial order is also a quasi order, so this gives the result mentioned earlier about height three quasi orders.

of 2^ω contains a perfect set. But we can simply replace this with Theorem 8.5, as we have already seen in the previous section.

We will now give the proofs of the Borel versions of Lemmas 7.6 and 7.7 in greater detail.

Lemma 8.14. *Suppose $\leq_P$ is a height three Borel partial order on 2^ω with the following properties.*

1. *The first level of $\leq_P$ (i.e. the elements of 2^ω which have no $\leq_P$-predecessors) has size continuum.*
2. *Every countable subset of the first level of $\leq_P$ has an upper bound in the second level.*
3. *For every finite subset $S \subset 2^\omega$ which is contained in the first two levels of $\leq_P$ and every $x \in 2^\omega$ in the second level which is not contained in S, there is some $y \in 2^\omega$ in the third level which is above every element of S but not above x.*

Then there is no Borel reduction of $\leq_P$ to Turing reducibility.

Proof. Suppose for contradiction that f is a Borel reduction of $\leq_P$ to $\leq_T$. Just as in the proof of Lemma 7.6, we want to begin by showing that the image of the first level of $\leq_P$ under f contains a perfect set. To do so, we just use the Lusin-Novikov theorem to show that the first level of $\leq_P$ is Borel and then use the perfect set theorem for $\mathbf{\Sigma^1_1}$ sets to show that the image of the first level under f must contain a perfect set. Let's now explain slightly more carefully how this works.

Let P_0, P_1 and P_2 denote the first, second and third levels of $\leq_P$, respectively. By the first part of the Lusin-Novikov theorem, the set

$$P_1 \cup P_2 = \{x \mid \exists y\, (y <_P x)\}$$

is Borel (this uses the fact that $\leq_P$ is locally countable). Therefore $P_0 = 2^\omega \setminus (P_1 \cup P_2)$ is also Borel. And since f is Borel, $f(P_0)$ is $\mathbf{\Sigma^1_1}$ definable. Since P_0 is uncountable and f is injective, the perfect set theorem for $\mathbf{\Sigma^1_1}$ sets implies that $f(P_0)$ contains a perfect set.

The rest of the proof is identical to the proof of Lemma 7.6. □

Lemma 8.15. *There is a height three, locally countable Borel partial order on 2^ω with the properties listed in the statement of Lemma 8.14.*

Proof. The partial order described in the proof of Lemma 7.7 is actually Borel, but it is not a partial order on 2^ω. In some sense this is an artificial difficulty because everything that we are doing in the section still works for Borel partial orders on arbitrary Polish spaces; we only restricted to the case of Borel partial orders on 2^ω to keep things simple. In any case, it is easy to adapt the proof of Lemma 7.7 to give a Borel partial order on 2^ω, as we will now explain.

By picking a bijection $f\colon \omega \to \omega \times \omega$, we can think of elements of 2^ω as elements of $(2^\omega)^\omega$. If x is an element of 2^ω then we will use the phrase "the n^{th} column of x" to refer to the element of 2^ω defined by $\{m \in \omega \mid f(n,m) \in x\}$, which we will also write as x_n. Also, if $x \in 2^\omega$, let $1^\frown x$ denote the element of 2^ω obtained by appending 1 at the front of x (i.e. $1^\frown x = \{0\} \cup \{n > 0 \mid n-1 \in x\}$) and similarly for $11^\frown x$. We will assume that the first two bits of the first column of each real are the first two bits of the real (i.e. that $f(0,0) = 0$ and $f(0,1) = 1$).

Now define three sets: P_0, P_1, and P_2. P_0 is the set of elements of 2^ω whose first column starts with 11, P_1 is the set of elements of 2^ω whose first column starts with 10 and P_2 is the set of elements of 2^ω whose first column starts with 0. Now define a relation $\leq_P$ on 2^ω by setting $x \leq_P y$ if and only if one of the following holds.

- $x = y$
- y is in P_1 and x is in P_0 and for some $n > 0$, $11^\frown y_n = x$
- y is in P_2 and x is in P_0 or P_1 and for some n, $1^\frown y_n = x$.
- y is in P_2 and x is in P_0 and for some z in P_1, $x \leq_P z \leq_P y$ by one of the two previous rules.

The explicit definition given above makes it clear that $\leq_P$ is Borel and it is easy to check that $\leq_P$ has the required properties (one key point is that the specific definition of P_0, P_1 and P_2 given above guarantees that every element of P_2 is above at least one element of P_1—recall that the first column of an element of P_2 begins with 0 and thus when we append a 1 to the beginning of that column we get an element of P_1). □

The Borel version of Theorem 7.8 now follows from the previous two lemmas.

Theorem 8.16. *There is a height three, locally countable Borel partial order on 2^ω which has no Borel embedding into Turing reducibility.*

Chapter 9

Open Questions

In this chapter we will collect questions related to Martin's conjecture that we asked earlier in the thesis and add a couple more that are near to my heart but don't fit clearly into any chapter of the thesis. It is my hope that these questions will help stimulate work on Martin's conjecture and that several of them will be answered in the coming years. I would be interested to see a solution to any one of them.

By the way, there are questions here that I have thought long and hard about and others that I have thought about for only a little while[1]. Caveat lector, or something like that.

Uniformly Invariant Functions

Question 9.1. *Suppose x and y are reals and $f\colon deg_T(x) \to deg_T(y)$ is a uniformly Turing invariant function. Must it be the case that either f is continuous or $y \geq_T x'$?*

Question 9.2. *Is there a reasonable version of part 2 of Martin's conjecture for uniformly Turing invariant functions defined on a single degree?*

For the second question, even finding a statement which sounds reasonable, easily implies part 2 of Martin's conjecture for uniformly Turing invariant functions and does not have an easy counterexample would be interesting. One way to answer this question in the negative would be to prove that part 2 of Martin's conjecture for uniformly Turing invariant functions is not provable in $\mathsf{ZF}+\mathsf{TD}$. This is because it seems reasonable to expect that a "local" version of part 2 for uniformly invariant functions should be provable in ZF and should imply the global version using only Turing determinacy (and not the full Axiom of Determinacy). Note that this might be difficult since it is not even known that TD does not imply AD.

The next question is intended to probe how far we can push the proof of Martin's conjecture for uniformly Turing invariant functions. It is similar to a question asked by Marks in his thesis [Mar12].

[1]And keep in mind that it usually takes me about a year to answer even the simplest question.

Question 9.3. *Suppose $f\colon 2^\omega \to 2^\omega$ is Turing invariant. Say that f is* ***2-uniformly invariant*** *if there are functions $u_1, u_2\colon \mathbb{N}^2 \to \mathbb{N}^2$ such that for all $x, y \in 2^\omega$ and $i, j \in \mathbb{N}$, if $x \equiv_T y$ via (i, j) then either $f(x) \equiv_T f(y)$ via $u_1(i, j)$ or $f(x) \equiv_T f(y)$ via $u_2(i, j)$. Does Martin's conjecture hold for all 2-uniformly Turing invariant functions?*

We can't make a list of open questions on uniformly Turing invariant functions without mentioning Steel's conjecture.

Conjecture 9.4 (Steel; [Ste82])**.** *Every Turing invariant function on the reals is Martin equivalent to a uniformly Turing invariant function.*

Regressive Functions

Question 9.5. *Does Martin's conjecture hold for regressive functions on the arithmetic degrees?*

Order Preserving Functions

Question 9.6. *Does part 1 of Martin's conjecture hold for all order preserving functions on the hyperarithmetic degrees?*

Question 9.7. *Can part 2 of Martin's conjecture be proved for* all *Turing invariant functions rather than just those that are not above the hyperjump? How about arithmetically invariant or hyperarithmetically invariant functions?*

Measure Preserving Functions

Question 9.8. *Is part 1 of Martin's conjecture for measure preserving functions provable in* ZF *when restricted to Borel functions?*

The Martin Measure and the Rudin Keisler Order

In section 5.9 we saw that part 1 of Martin's conjecture is equivalent to the statement that no non-principal ultrafilter on the Turing degrees other than Martin measure itself is below Martin measure in the Rudin-Keisler order. Recall that this means that no non-principal ultrafilter on the Turing degrees is strictly below Martin measure *and* that no ultrafilter on the Turing degrees besides Martin measure itself is equivalent to Martin measure. This suggests that one way to work on Martin's conjecture would be to try to prove these two assertions separately.

Question 9.9. *Is there any non-principal ultrafilter on the Turing degrees which is strictly below Martin measure in the Rudin-Keisler order?*

Andrew Marks has made a conjecture that would imply a negative answer to this question.

Conjecture 9.10 (Marks; [Mar20])**.** *Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*. If* $f\colon 2^\omega \to 2^\omega$ *is any function then there is either a pointed perfect tree on which* f *is constant or a pointed perfect tree on which* f *is injective.*

We can also ask about whether any ultrafilter is equivalent to Martin measure, of course.

Question 9.11. *Is there any ultrafilter on the Turing degrees besides Martin measure itself which is Rudin-Keisler equivalent to Martin measure?*

One way to make this question easier is to restrict the class of ultrafilters considered.

Question 9.12. *Suppose* U *is an ultrafilter on the Turing degrees such that the associated notion of forcing is proper (see section 5.11). Can* U *be Rudin-Keisler below Martin measure? What if the associated notion of forcing does not collapse* $\mathfrak{c}$*?*

In light of our results on the Lebesgue and Baire ultrafilters on the Turing degrees, there are a couple of other questions we can ask about the place of Martin measure in the Rudin-Keisler order. The first of these is a special case of the question about whether any non-principal ultrafilter is strictly below Martin measure. And the second, while not directly relevant to Martin's conjecture, seems like it could be interesting to know.

Question 9.13. *Are the Lebesgue and Baire ultrafilters on the Turing degrees below Martin measure in the Rudin-Keisler order?*

Question 9.14. *Is Martin measure maximal in the Rudin-Keisler order on ultrafilters on the Turing degrees?*

Extending the Solecki Dichotomy

It seems possible that the Solecki dichotomy could be extended to help prove part 2 of Martin's conjecture for all order preserving functions. The following question suggests one route to achieving this.

Question 9.15. *Define an ordering on functions on* 2^ω *by setting* $f \prec g$ *if there is a countable partition of* 2^ω *such that* f *restricted to each part of the partition has a topological embedding into* g *(where* g *is still thought of as a function on all of* 2^ω*). Is this ordering linear? Is it wellfounded?*

Sacks' Question and Locally Countable Borel Quasi Orders

Question 9.16. *Is it provable in* ZFC *that every size continuum, locally countable partial order of height three can be embedded into the Turing degrees?*

Question 9.17. *Does every locally countable Borel quasi order of height one have a Borel reduction to Turing reducibility?*

Does Martin's Conjecture Say What We Think It Does?

The questions below are intended to get at the following basic concern: even if Martin's conjecture is true, how much does this really tell us about what Turing invariant functions look like? Note that Steel's conjecture (every Turing invariant function is equivalent to a uniformly invariant one) implies that the first two questions below have positive answers (by work of Steel and Becker [Ste82; Bec88]).

Question 9.18. *Does Martin's conjecture imply that every Turing invariant function is order preserving on a cone?*

Question 9.19. *Does Martin's conjecture imply that there are no "pseudo ω-jumps"—i.e. that there is no Turing invariant function $f\colon 2^\omega \to 2^\omega$ which is Martin above every finite jump but strictly below the ω jump? What about pseudo α-jumps for general countable limit ordinals α? What about "pseudo hyperjumps"?*

The next question is motivated by the following idea. Martin's conjecture says that the only natural functions on the Turing degrees are iterates of the jump. However, if we ask instead for functions that take Turing degrees and give back *countable sets of Turing degrees* then there are many examples that are quite different from the jump. For example, the function that maps a real x to all the reals that are low relative to x, or K-trivial relative to x, or whatever (pick your favorite computability-theoretic property relativized to x)[2]. It seems natural to wonder where the line is between "the jump is the only natural function" and "there are lots of natural functions." If Martin's conjecture is true then we know that functions which take a Turing degree and give back a Turing degree are on one side of this line and functions which take a Turing degree and give back a countable set of Turing degrees are on the other. But what about functions that take a Turing degree and give back an unordered set of two Turing degrees? Or some other finite number? Are these functions also composed of iterates of the jump in a simple way?

Question 9.20. *Suppose f is a function on 2^ω such that for each x, $f(x)$ is a size two subset of 2^ω (it is very important that $f(x)$ is an* unordered *set of size two—otherwise this question becomes trivial). Suppose furthermore that f is Turing invariant in the following sense: if $x \equiv_T y$, then the Turing degrees of the reals in $f(x)$ are the same as the Turing degrees of the reals in $f(y)$. Does Martin's conjecture imply that f can be decomposed into two Turing invariant functions on the reals—i.e. that there are two Turing invariant functions from reals to reals, f_0 and f_1, such that $f(x) = \{f_0(x), f_1(x)\}$ for all x?*

The Landscape of Martin's Conjecture

These next questions are about how the behavior of Martin's conjecture changes on different degree structures. There has been a fair amount of work on Martin's conjecture on a few

[2] Steve Simpson has sloganized this idea by saying that Turing degrees besides the jump "don't have names," but Muchnik degrees do.

specific degree structures—in this thesis we focused on the Turing degrees, arithmetic degrees and hyperarithmetic degrees, but the many-one degrees and the polynomial time degrees have also been studied—but little work on how Martin's conjecture behaves across all reasonable degree structures. It would be very interesting, for example, to find any degree structure at all where it is possible to prove Martin's conjecture, even if such a degree structure looks very different from the Turing degrees.

Question 9.21. *Is there a computability theoretic degree structure on which Martin's conjecture fails but the uniformly invariant case (defined in some appropriate way for the degree structure) is true? What about where Martin's conjecture fails but the order preserving case holds?*

Question 9.22. *Is there some computability theoretic degree structure where a version of Martin's conjecture is provable?*

Bibliography

[Bar20] Vittorio Bard. “Uniform Martin’s conjecture, locally”. In: *Proc. Amer. Math. Soc.* 148.12 (2020), pp. 5369–5380. ISSN: 0002-9939. DOI: 10.1090/proc/15159.

[Bec88] Howard Becker. “A Characterization of Jump Operators”. In: *J. Symbolic Logic* 53.3 (Sept. 1988), pp. 708–728.

[CWY10] C. T. Chong, Wei Wang, and Liang Yu. “The strength of the projective Martin conjecture”. In: *Fund. Math.* 207.1 (2010), pp. 21–27. ISSN: 0016-2736. DOI: 10.4064/fm207-1-2.

[DK00] Randall Dougherty and Alexander S. Kechris. “How many Turing degrees are there?” In: *Computability theory and its applications (Boulder, CO, 1999)*. Vol. 257. Contemp. Math. Amer. Math. Soc., Providence, RI, 2000, pp. 83–94. DOI: 10.1090/conm/257/04029.

[Dow+05] Rod Downey et al. “Relativizing Chaitin’s halting probability”. In: *J. Math. Log.* 5.2 (2005), pp. 167–192. ISSN: 0219-0613. DOI: 10.1142/S0219061305000468.

[EP70] H. B. Enderton and Hilary Putnam. “A Note on the Hyperarithmetical Hierarchy”. In: *J. Symbolic Logic* 35.3 (Sept. 1970), pp. 429–430.

[Esc07] Martin Escardo. “Infinite Sets That Admit Fast Exhaustive Search”. In: *Proceedings of the 22nd Annual IEEE Symposium on Logic in Computer Science*. LICS ’07. USA: IEEE Computer Society, 2007, pp. 443–452. ISBN: 0769529089. DOI: 10.1109/LICS.2007.25.

[For10] Matthew Foreman. “Ideals and Generic Elementary Embeddings”. In: *Handbook of Set Theory*. Ed. by Matthew Foreman and Akihiro Kanamori. Dordrecht: Springer Netherlands, 2010, pp. 885–1147. ISBN: 978-1-4020-5764-9. DOI: 10.1007/978-1-4020-5764-9_14.

[GS83] Marcia J. Groszek and Theodore A. Slaman. “Independence results on the global structure of the Turing degrees”. In: *Trans. Amer. Math. Soc.* 277.2 (1983), pp. 579–588. ISSN: 0002-9947. DOI: 10.2307/1999225.

[GS98] Marcia J. Groszek and Theodore A. Slaman. “A Basis Theorem for Perfect Sets”. In: *The Bulletin of Symbolic Logic* 4.2 (1998), pp. 204–209. ISSN: 10798986.

[Har68] Joseph Harrison. “Recursive Pseudo-Well-Orderings”. In: *Transactions of the American Mathematical Society* 131.2 (1968), pp. 526–543. ISSN: 00029947.

[Jec03] Thomas Jech. *Set theory*. Springer Monographs in Mathematics. The third millennium edition, revised and expanded. Springer-Verlag, Berlin, 2003, pp. xiv+769. ISBN: 3-540-44085-2.

[Kec21] Alexander S. Kechris. “The Theory of Countable Borel Equivalence Relations”. Preprint available online at author’s website. 2021.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*. Vol. 156. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. xviii+402. ISBN: 0-387-94374-9. DOI: 10.1007/978-1-4612-4190-4.

[Kih21] Takayuki Kihara. Personal Communication. 2021.

[KM18] Takayuki Kihara and Antonio Montalbán. “The uniform Martin’s conjecture for many-one degrees”. In: *Trans. Amer. Math. Soc.* 370.12 (2018), pp. 9025–9044. ISSN: 0002-9947. DOI: 10.1090/tran/7519.

[KR21] Ashutosh Kumar and Dilip Raghavan. “Separating families and order dimension of Turing degrees”. In: *Ann. Pure Appl. Logic* 172.5 (2021), p. 102911. ISSN: 0168-0072. DOI: 10.1016/j.apal.2020.102911.

[Kum19] Ashutosh Kumar. “Suborders of Turing Degrees”. Unpublished note available at the author’s website. 2019.

[KW08] Alexander S. Kechris and W. Hugh Woodin. “Generic codes for uncountable ordinals, partition properties, and elementary embeddings”. In: *Games, Scales and Suslin Cardinals: The Cabal Seminar, Volume I*. Ed. by Alexander S. Kechris, Benedikt Löwe, and John R.Editors Steel. Lecture Notes in Logic. Cambridge University Press, 2008, pp. 379–397. DOI: 10.1017/CBO9780511546488.019.

[Lac75] A. H. Lachlan. “Uniform enumeration operations”. In: *J. Symbolic Logic* 40.3 (1975), pp. 401–409. ISSN: 0022-4812. DOI: 10.2307/2272164.

[Mar12] Andrew Scott Marks. *Recursion theory and countable Borel equivalence relations*. Thesis (Ph.D.)–University of California, Berkeley. ProQuest LLC, Ann Arbor, MI, 2012, p. 74. ISBN: 978-1267-60511-5.

[Mar18] Andrew Marks. “The universality of polynomial time Turing equivalence”. In: *Mathematical Structures in Computer Science* 28.3 (2018), pp. 448–456. DOI: 10.1017/S0960129516000232.

[Mar20] Andrew Marks. Personal Communication. 2020.

[Mar68] Donald A. Martin. “The axiom of determinateness and reduction principles in the analytical hierarchy”. In: *Bull. Amer. Math. Soc.* 74.4 (July 1968), pp. 687–689.

[Mar85] Donald A. Martin. “A purely inductive proof of Borel determinacy”. In: *Recursion theory (Ithaca, N.Y., 1982)*. Vol. 42. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 1985, pp. 303–308. DOI: 10.1090/pspum/042/791065.

[Mon09] Antonio Montalbán. "Embeddings into the Turing degrees". In: *Logic Colloquium 2006*. Vol. 32. Lect. Notes Log. Assoc. Symbol. Logic, Chicago, IL, 2009, pp. 229–246. DOI: 10.1017/CBO9780511605321.012.

[Mon17] Antonio Montalbán. "Effectively existentially-atomic structures". In: *Computability and complexity*. Vol. 10010. Lecture Notes in Comput. Sci. Springer, Cham, 2017, pp. 221–237. DOI: 10.1007/978-3-319-50062-1.

[Mos09] Yiannis N. Moschovakis. *Descriptive set theory*. Second. Vol. 155. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2009, pp. xiv+502. ISBN: 978-0-8218-4813-5. DOI: 10.1090/surv/155.

[MSS16] Andrew Marks, Theodore A. Slaman, and John R. Steel. "Martin's conjecture, arithmetic equivalence, and countable Borel equivalence relations". In: *Ordinal Definability and Recursion Theory: The Cabal Seminar, Volume III*. Vol. 3. Lecture Notes in Logic. Cambridge University Press, 2016, pp. 493–520. DOI: 10.1017/CBO9781139519694.017.

[Odi89] Piergiorgio Odifreddi. *Classical recursion theory*. Vol. 125. Studies in Logic and the Foundations of Mathematics. The theory of functions and sets of natural numbers, With a foreword by G. E. Sacks. North-Holland Publishing Co., Amsterdam, 1989, pp. xviii+668. ISBN: 0-444-87295-7.

[Pos44] Emil L. Post. "Recursively enumerable sets of positive integers and their decision problems". In: *Bull. Amer. Math. Soc.* 50 (1944), pp. 284–316. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1944-08111-1.

[PS12] Janusz Pawlikowski and Marcin Sabok. "Decomposing Borel functions and structure at finite levels of the Baire hierarchy". In: *Ann. Pure Appl. Logic* 163.12 (2012), pp. 1748–1764. ISSN: 0168-0072. DOI: 10.1016/j.apal.2012.03.004.

[PY20] Yinhe Peng and Liang Yu. *TD implies CCR*. 2020. arXiv: 2012.10003 [math.LO].

[Sac61] Gerald E. Sacks. "On suborderings of degrees of recursive unsolvability". In: *Z. Math. Logik Grundlagen Math.* 7 (1961), pp. 46–56. ISSN: 0044-3050. DOI: 10.1002/malq.19610070109.

[Sac63] Gerald E. Sacks. *Degrees of unsolvability*. Princeton University Press, Princeton, N.J., 1963, pp. ix+174.

[Sac90] Gerald E. Sacks. *Higher recursion theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990, pp. xvi+344. ISBN: 3-540-19305-7. DOI: 10.1007/BFb0086109.

[Soa16] Robert I. Soare. *Turing computability*. Theory and Applications of Computability. Theory and applications. Springer-Verlag, Berlin, 2016, pp. xxxvi+263. ISBN: 978-3-642-31932-7; 978-3-642-31933-4. DOI: 10.1007/978-3-642-31933-4.

[Sol78] Robert M. Solovay. “The independence of DC from AD”. In: *Cabal Seminar 76–77*. Ed. by Alexander S. Kechris and Yiannis N. Moschovakis. Berlin, Heidelberg: Springer Berlin Heidelberg, 1978, pp. 171–183. ISBN: 978-3-540-35626-4.

[Sol98] Sawomir Solecki. “Decomposing Borel sets and functions and the structure of Baire class 1 functions”. In: *J. Amer. Math. Soc.* 11.3 (1998), pp. 521–550. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-98-00269-0.

[SS88] Theodore A. Slaman and John R. Steel. “Definable functions on degrees”. In: *Cabal Seminar 81–85*. Ed. by Alexander S. Kechris, Donald A. Martin, and John R. Steel. Berlin, Heidelberg: Springer Berlin Heidelberg, 1988, pp. 37–55. ISBN: 978-3-540-45896-8.

[SS99] Richard A. Shore and Theodore A. Slaman. “Defining the Turing jump”. In: *Math. Res. Lett.* 6.5-6 (1999), pp. 711–722. ISSN: 1073-2780. DOI: 10.4310/MRL.1999.v6.n6.a10.

[Ste82] John R Steel. “A classification of jump operators”. In: *The Journal of Symbolic Logic* 47.2 (1982), pp. 347–358.

[Woo88] W. Hugh Woodin. “Supercompact cardinals, sets of reals, and weakly homogeneous trees”. In: *Proc. Nat. Acad. Sci. U.S.A.* 85.18 (1988), pp. 6587–6591. ISSN: 0027-8424. DOI: 10.1073/pnas.85.18.6587.

[Zap04] Jindich Zapletal. “Descriptive set theory and definable forcing”. In: *Mem. Amer. Math. Soc.* 167.793 (2004), pp. viii+141. ISSN: 0065-9266. DOI: 10.1090/memo/0793.