# How to Use Steel Forcing

Patrick Lutz

The purpose of these notes is to explain how to use the technique of Steel forcing, a.k.a. “forcing with tagged trees.” This method was first introduced by Steel in [7] and has been used in reverse math, set theory and computability theory [5, 2, 3, 6].

Steel’s original paper remains a good introduction to the technique. However, Steel’s paper may be difficult for some readers to understand due to its use of a ramified forcing language. In particular, Steel introduces a forcing language with names for every element of $L_\alpha[G]$ where $G$ is the generic. In some applications this is necessary, but in many cases it is possible to use a simpler language. We take an approach inspired by [4] which makes use of a restricted infinitary logic which is relatively simple but still expressive enough to describe all Borel sets.

In section 2 we introduce the Steel forcing partial order. In section 3 we introduce the language that we will use, define the forcing relation for this language and prove the usual theorems guaranteeing that the forcing relation works as intended. In section 4 we prove the key lemma about Steel forcing (the “retagging lemma”) and in section 5 we provide an example application of Steel forcing.

In this exposition we will assume that readers have at least some familiarity with forcing. However, we will not assume any familiarity with forcing in computability theory (where “generic” means “meets every one of some specified countable family of dense sets” rather than all dense sets in the ground model). We review this in section 1. Immediately below we list some of our notational choices in case they cause any confusion.

**Notation.** If $\sigma$ is a finite string, we will use $|\sigma|$ to denote the length of $\sigma$. If $\sigma$ and $\tau$ are finite strings, we will use $\sigma \leq \tau$ to mean that $\sigma$ is a prefix of $\tau$ and $\sigma < \tau$ to mean that $\sigma$ is a proper prefix of $\tau$. By a *tree on* $\omega$ we mean a subset $T$ of $\omega^{<\omega}$ such that any prefix of any element of $T$ is also an element of $T$. The *root* of $T$ is simply the empty sequence.

## 1 Forcing in Computability Theory and Descriptive Set Theory

For readers who are only familiar with the use of forcing to build models of set theory, we will now briefly describe how forcing is used in computability theory and descriptive set theory. Instead of being used to construct “potential objects” as in set theory, forcing in computability theory and descriptive set theory is used to construct actual existing objects which satisfy some complicated set of requirements. The key insight is that if we alter the definition of “generic filter” so that “generic” just means “meets every one of a specified countable sequence of dense sets” rather than “meets every dense set” then generic filters always exist. This idea is expressed by the following definition and theorem.

**Definition 1.1.** Suppose $(P, \leq)$ is a partial order and $A$ is a set of filters on $P$. The phrase *every sufficiently generic filter is in* $A$ means that there is some countable sequence $\langle D_n \rangle_{n\in\omega}$ of dense subsets of $P$ such that any filter which meets every $D_n$ is in $A$.

**Theorem 1.2.** Suppose that $(P, \leq)$ is a nonempty partial order and $A$ is a set of filters on $P$ such that every sufficiently generic filter is in $A$. Then $A$ is nonempty.

Here’s how this gets used in practice. Suppose we want to construct a combinatorial object (e.g. a tree, a fast growing sequence, etc) with certain special properties. If we can think of this object as a filter on an appropriate partial order and express the desired properties in terms of meeting a countable collection of dense sets in the partial order then we can say that any sufficiently generic object has the desired properties and the theorem above guarantees that such an object exists.

With an appropriate choice of topological space (namely a completion of the partial order with the topology generated by downwards closed sets), this sort of argument is simply equivalent to using the Baire category theorem to prove the existence of the desired combinatorial object. So what’s the point of framing this argument using the language of forcing? In my view, the basic point is that forcing gives us a systematic way to express many properties to which this sort of Baire category argument can be applied as well as a method for reasoning about such properties (the forcing relation). And even though such arguments can typically be recast in purely topological terms using Baire category arguments, they are often clearer and more intuitive when expressed in the language of forcing.

# 2 Steel Forcing

A condition in the *Steel forcing partial order* is a nonempty finite tree $T$ on $\omega$ together with a function labelling each node in $T$ with a countable ordinal or with $\infty$ such that the ordinal assigned to a node is greater than any of the ordinals assigned to its children. More formally, a condition is a pair $\langle T, f\rangle$ such that

(1) $T$ is a finite subset of $\omega^{<\omega}$ such that if $\sigma \in T$ and $\tau \leq \sigma$ then $\tau \in T$.

(2) $f\colon T \to \omega_1 \cup \{\infty\}$ is a function such that if $\sigma < \tau$ are nodes in $T$ then $f(\sigma) > f(\tau)$. Here we assume for convenience that $\infty < \infty$ and $\alpha < \infty$ for all $\alpha \in \omega_1$.

If $\langle T, f\rangle$ and $\langle S, g\rangle$ are two conditions then $\langle S, g\rangle \leq \langle T, f\rangle$ if and only if $T \subseteq S$ and $g \restriction T = f$.

We now wish to describe what a sufficiently generic filter on the Steel forcing partial order looks like. Suppose $G$ is such a filter. Define

$$T_G := \bigcup\{T \mid \langle T, f\rangle \in G \text{ for some function } f\}$$
$$f_G := \bigcup\{f \mid \langle T, f\rangle \in G \text{ for some tree } T\}.$$

It is not hard to see that $T_G$ is a tree on $\omega$ and $f_G$ is a function $T_G \to \omega_1 \cup \{\infty\}$ such that for all nodes $\sigma < \tau$ in $T_G$, $f_G(\sigma) < f_G(\tau)$.

Informally, if $\langle T, f\rangle$ is a condition in the Steel forcing partial order then you should think of $T$ as a finite fragment of the infinite tree $T_G$ and $f$ as the rank function of $T_G$ restricted to the tree $T$. If a node $\sigma \in T$ is assigned the label $\infty$ by $f$ then $\sigma$ is in the ill-founded part of $T_G$. The following propositions make this informal picture precise. We omit their proofs, which are easy.

**Proposition 2.1.** For any sufficiently generic filter $G$, $T_G$ is infinite.

**Proposition 2.2.** For any sufficiently generic filter $G$, if $\sigma$ is a node in $T_G$ such that $f_G(\sigma) = \infty$ then $\sigma$ is in the ill-founded part of $T_G$—i.e. there is an infinite path through $T_G$ extending $\sigma$.

**Proposition 2.3.** If $\alpha$ is a countable ordinal then for any sufficiently generic filter $G$, if $\sigma$ is a node in $T_G$ such that $f_G(\sigma) = \alpha$ then the rank of $\sigma$ in $T_G$ is $\alpha$.

# 3 The Forcing Language

We will now describe a certain language in infinitary logic which we will use to formulate statements about $T_G$ for $G$ a sufficiently generic filter in the Steel forcing partial order. We will also formally define the forcing relation for this language. In order to simplify later proofs, the language we define will be especially pared down (though still quite expressive, as we will see). If you are familiar with infinitary logic, you will recognize that our language is a minimalist version of $\mathcal{L}_{\omega_1,\omega}$.

**The language.** The language $\mathcal{L}_{\text{tree}}$ can be defined inductively as follows:

(1) For each $\sigma \in \omega^{<\omega}$, $\mathcal{L}_{\text{tree}}$ contains a constant symbol $\underline{\sigma}$.

(2) $\mathcal{L}_{\text{tree}}$ contains one unary relation symbol, $X$.

(3) For each constant symbol $\underline{\sigma}$, $\mathcal{L}_{\text{tree}}$ contains a formula $\underline{\sigma} \in X$. These formulas are called *atomic formulas*.

(4) For each formula $\varphi$, $\mathcal{L}_{\text{tree}}$ contains a formula $\neg\varphi$, called the *negation* of $\varphi$.

(5) For each countable sequence of formulas $\langle \varphi_n\rangle_{n\in\omega}$, $\mathcal{L}_{\text{tree}}$ contains a formula $\bigwedge_{n\in\omega} \varphi_n$, called the *infinite conjunction* of the sequence $\langle \varphi_n\rangle_{n\in\omega}$.

Note that $\mathcal{L}_{\text{tree}}$ has no variable symbols whatsoever. The intended meaning of a formula in this language is that $X$ is interpreted as a subset of $\omega^{<\omega}$—usually a tree on $\omega$—and atomic formulas are statements about which elements of $\omega^{<\omega}$ are elements of this set. The next definition formalizes this.

**Definition 3.1.** If $T$ is any subset of $\omega^{<\omega}$ and $\varphi$ is a formula in $\mathcal{L}_{\text{tree}}$, we define $T \vDash \varphi$ inductively as follows.

(1) If $\varphi$ is the atomic formula $\underline{\sigma} \in X$ then $T \vDash \varphi$ if and only if $\sigma \in T$.

(2) If $\varphi$ is of the form $\neg\psi$ then $T \vDash \varphi$ if and only if $T \nvDash \psi$.

(3) If $\varphi$ is of the form $\bigwedge_{n\in\omega} \psi_n$ then $T \vDash \varphi$ if and only if for all $n \in \omega$, $T \vDash \psi_n$.

**The expressiveness of the language.** It may appear at first that $\mathcal{L}_{\text{tree}}$ is somewhat deficient as a language, and perhaps even too impoverished to really be of much use. For one thing, it lacks some of the usual logical symbols like $\wedge$ and $\vee$. For another, it doesn't have any quantifiers.

However, both of these deficiencies are merely cosmetic: we can use $\bigwedge$ to simulate both $\wedge$ and the first order quantifier $\forall$ (and since we have negation, this means we can also simulate $\vee$ and $\exists$).

For example, suppose we want to express that $T$ contains all elements of $\omega^{<\omega}$. In first order logic we might use the formula $\forall y\,(y \in X)$. Here, we can instead do the following. Pick some enumeration $\sigma_0, \sigma_1, \sigma_2, \ldots$ of the elements of $\omega^{<\omega}$. Then it is easy to check that

$$T = \omega^{<\omega} \iff T \vDash \bigwedge_{n\in\omega} \underline{\sigma_n} \in X.$$

In fact, the language $\mathcal{L}_{\text{tree}}$ is extremely expressive: the subsets of $\mathcal{P}(\omega^{<\omega})$ which are definable by formulas in $\mathcal{L}_{\text{tree}}$ are exactly the Borel subsets of $\mathcal{P}(\omega^{<\omega})$. We will now prove one direction of this equivalence.

**Proposition 3.2.** Every Borel subset of $\mathcal{P}(\omega^{<\omega})$ is definable by an $\mathcal{L}_{\text{tree}}$ formula.

**Proof.** It is enough to show that the $\mathcal{L}_{\text{tree}}$ definable sets form a $\sigma$-algebra which contains all the basic open sets of $\mathcal{P}(\omega^{<\omega})$. Recall that a basic open set of $\mathcal{P}(\omega^{<\omega})$ is always of the form $\{T \subseteq \omega^{<\omega} \mid A \subseteq T \text{ and } B \subseteq \omega^{<\omega} \setminus T\}$ where $A$ and $B$ are finite, disjoint subsets of $\omega^{<\omega}$. We can define such a set with the following formula

$$\bigwedge_{\sigma\in A} \underline{\sigma} \in X \wedge \bigwedge_{\sigma\in B} \neg(\underline{\sigma} \in X).$$

To show that the $\mathcal{L}_{\text{tree}}$ definable sets form a $\sigma$-algebra, it is enough to show that they are closed under complement and countable intersections. This is easy to verify: complement corresponds to $\neg$ and countable intersection to $\bigwedge$.

**Rank and subformulas.** We will now give some definitions relating to the syntax of $\mathcal{L}_{\text{tree}}$. These definitions are not particularly exciting, but will be useful later when formulating and proving theorems.

First, we will define the notion of the *rank* of an $\mathcal{L}_{\text{tree}}$ formula $\varphi$, denoted $\text{rank}(\varphi)$, which roughly corresponds to the usual notion of quantifier complexity (i.e. the rank of a formula is the number of alternations between universal and existential quantifiers after the formula is put into prenex normal form). However, our definition will not quite match this notion—in order to keep the definition (and later proofs) as simple as possible, we will assign some formulas a rank which is higher than their quantifier complexity.

**Definition 3.3.** Formally, the rank of a formula is a countable ordinal. We define it inductively as follows.

$$\text{rank}(\varphi) := \begin{cases} 0 & \text{if } \varphi \text{ is atomic} \\ \text{rank}(\psi) + 1 & \text{if } \varphi = \neg\psi \\ \sup\{\text{rank}(\psi_n) + 1 \mid n \in \omega\} & \text{if } \varphi = \bigwedge_{n\in\omega} \psi_n. \end{cases}$$

Next we define the notion of the set of *subformulas* of an $\mathcal{L}_{\text{tree}}$ formula $\varphi$. This notion means exactly what you think it does—with the possible exception of the detail that we consider a formula $\varphi$ to be a subformula of itself. For the sake of completeness we give a formal definition below.

**Definition 3.4.** The set of subformulas of a formula $\varphi$ is defined inductively as follows.

(1) If $\varphi$ is an atomic formula then the only subformula of $\varphi$ is $\varphi$ itself.

(2) If $\varphi$ has the form $\neg\psi$ then the subformulas of $\varphi$ are $\varphi$ itself along with all subformulas of $\psi$.

(3) If $\varphi$ has the form $\bigwedge_{n\in\omega} \psi_n$ then the subformulas of $\varphi$ are $\varphi$ itself together with any formula that is a subformula of some $\psi_n$.

**The forcing relation.** We will now define the forcing relation for Steel forcing and the language $\mathcal{L}_{\text{tree}}$. First, a preliminary definition: if $\sigma$ is a nonempty finite sequence on $\omega$ then we will use $\sigma^-$ to denote the sequence obtained from $\sigma$ by removing the last element—e.g. $\langle 4,3,4,4,5\rangle^- = \langle 4,3,4,4\rangle$. Now suppose $\langle T, f\rangle$ is a condition in the Steel forcing partial order and $\varphi$ is a formula. We inductively define the forcing relation, $\langle T, f\rangle \Vdash \varphi$, as follows.

(1) If $\varphi$ is an atomic formula of the form $\underline{\sigma} \in X$ then $\langle T, f\rangle \Vdash \varphi$ if and only if either $\sigma \in T$ or $\sigma$ is a nonempty sequence, $\sigma^- \in T$ and $f(\sigma^-) > 0$.

(2) If $\varphi$ is of the form $\neg\psi$ then $\langle T, f\rangle \Vdash \varphi$ if and only if for all $\langle S, g\rangle \leq \langle T, f\rangle$, $\langle S, g\rangle \nVdash \psi$.

(3) If $\varphi$ is of the form $\bigwedge_{n\in\omega} \psi_n$ then $\langle T, f\rangle \Vdash \varphi$ if and only if for all $n \in \omega$, $\langle T, f\rangle \Vdash \psi_n$.

Note that the definition of the forcing relation for atomic formulas looks a little complicated: it seems more natural to just say that $\langle T, f\rangle \Vdash \underline{\sigma} \in X$ if and only if $\sigma \in T$. The problem with this definition is that if $\sigma \notin T$ but $\sigma^- \in T$ and $f(\sigma^-) > 0$ then $\sigma$ will be in any sufficiently generic filter extending $\langle T, f\rangle$.

We now want to prove that the forcing relation behaves as it should. In particular, we want to prove the following theorem.

**Theorem 3.5.** Suppose $\varphi$ is an $\mathcal{L}_{\text{tree}}$ formula. For every sufficiently generic filter $G$, $T_G \vDash \varphi$ if and only if there is some condition $\langle T, f\rangle \in G$ such that $\langle T, f\rangle \Vdash \varphi$.

The remainder of this section is devoted to proving this theorem. Any reader who is willing to take this theorem on faith can safely skip the rest of this section. The proof of the theorem is entirely ordinary; it is only included because the details of the proofs involving atomic formulas and infinite conjunctions may differ somewhat from other presentations of forcing that the reader has seen.

As usual, we first need to prove several technical lemmas about the forcing relation. We omit the proofs of the first two, which are both easy.

**Lemma 3.6** (Monotonicity of the forcing relation)**.** For any $\mathcal{L}_{\text{tree}}$ formula $\varphi$ and condition $\langle T, f\rangle$, if $\langle T, f\rangle \Vdash \varphi$ then for any condition $\langle S, g\rangle$ extending $\langle T, f\rangle$, $\langle S, g\rangle \Vdash \varphi$.

**Lemma 3.7** (Consistency of the forcing relation)**.** For any $\mathcal{L}_{\text{tree}}$ formula $\varphi$ and condition $\langle T, f\rangle$, if $\langle T, f\rangle \Vdash \varphi$ then $\langle T, f\rangle \nVdash \neg\varphi$.

**Lemma 3.8** (Invariance under double negation)**.** For any $\mathcal{L}_{\text{tree}}$ formula $\varphi$ and condition $\langle T, f\rangle$, we have the following equivalence

$$\langle T, f\rangle \Vdash \varphi \iff \text{for all } \langle S, g\rangle \leq \langle T, f\rangle \text{ there is some } \langle R, h\rangle \leq \langle S, g\rangle \text{ such that } \langle R, h\rangle \Vdash \varphi.$$

**Proof.** The forward direction follows from monotonicity of the forcing relation (lemma 3.6). To prove the backwards direction, we will use induction on $\varphi$. Also, we will prove the contrapositive: if $\langle T, f\rangle \nVdash \varphi$ then there is some $\langle S, g\rangle \leq \langle T, f\rangle$ such that no extension of $\langle S, g\rangle$ forces $\varphi$.

**Atomic formulas.** Suppose that $\varphi$ is the atomic formula $\underline{\sigma} \in X$. Let $\langle T, f\rangle$ be a condition such that $\langle T, f\rangle \nVdash \varphi$. We want to find an extension $\langle S, g\rangle$ of $\langle T, f\rangle$ such that no extension of $\langle S, g\rangle$ forces $\varphi$. There are a few cases to consider.

(1) Suppose $\sigma$ is nonempty, $\sigma^-$ is in $T$ and $f(\sigma^-) = 0$. In this case we can let $\langle S, g\rangle = \langle T, f\rangle$: no extension of $\langle T, f\rangle$ can include $\sigma$ in the tree nor assign an ordinal greater than 0 to $\sigma^-$.

(2) Suppose $\sigma$ is nonempty and $\sigma^-$ is not in $T$. Let $\tau$ be the last ancestor of $\sigma$ in $T$ and let $\tau'$ be the unique child of $\tau$ which is an ancestor of $\sigma$. Define $\langle S, g\rangle$ by adding $\tau'$ to $T$ and setting $g(\tau') = 0$. Note that no extension of $\langle S, g\rangle$ can add any descendants of $\tau'$ and in particular, no extension of $\langle S, g\rangle$ can include $\sigma$ or assign a nonzero ordinal to $\sigma^-$. Thus no extension of $\langle S, g\rangle$ forces $\varphi$.

(3) If neither of the two cases above hold then either $\sigma$ is the empty sequence, $\sigma \in T$ or $\sigma^- \in T$ and $f(\sigma^-) > 0$. But all of these cases are impossible by the definition of the forcing relation and the assumption that $\langle T, f\rangle \nVdash \varphi$.

**Negations.** Suppose that $\varphi$ has the form $\neg\psi$. Let $\langle T, f\rangle$ be a condition such that $\langle T, f\rangle \nVdash \varphi$. Then by definition, there is some extension $\langle S, g\rangle$ of $\langle T, f\rangle$ such that $\langle S, g\rangle \Vdash \psi$. By monotonicity (lemma 3.6), every extension of $\langle S, g\rangle$ forces $\psi$ and by consistency (lemma 3.7), no condition which forces $\psi$ can force $\neg\psi$. Thus no extension of $\langle S, g\rangle$ forces $\varphi$ and so we are done.

**Infinite conjunctions.** Suppose that $\varphi$ has the form $\bigwedge\!\!\!\bigwedge_{n\in\omega}\psi_n$. Let $\langle T, f\rangle$ be a condition such that $\langle T, f\rangle \nVdash \varphi$. Thus there is some $n \in \omega$ such that $\langle T, f\rangle \nVdash \psi_n$. So by the inductive hypothesis applied to $\psi_n$, there is some condition $\langle S, g\rangle \leq \langle T, f\rangle$ such that no condition extending $\langle S, g\rangle$ forces $\psi_n$. By the definition of the forcing relation, no condition extending $\langle S, g\rangle$ forces $\varphi$ and so we are done.

We can now almost prove theorem 3.5. However, before we do so we need to determine the countable sequence of dense sets that give a strong enough notion of "sufficiently generic" to make the statement of that theorem true. We will do that now. The idea is that for each formula $\varphi$, we will define a set $D_\varphi$ which is a dense subset of the Steel forcing partial order. We will then show that theorem 3.5 holds for any filter which meets $D_\psi$ for every subformula $\psi$ of $\varphi$.

**Definition 3.9.** For each $\mathcal{L}_{\text{tree}}$ formula $\varphi$, define a subset $D_\varphi$ of the Steel forcing partial order as follows.

(1) If $\varphi$ is an atomic formula of the form $\underline{\sigma} \in X$ then $D_\varphi := \{\langle T, f\rangle \mid \sigma \in T \text{ or } \exists\tau < \sigma\,(\tau \in T \text{ and } f(\tau) = 0)\}$.

(2) If $\varphi$ is of the form $\neg\psi$ then $D_\varphi := \{\langle T, f\rangle \mid \langle T, f\rangle \Vdash \psi \text{ or } \langle T, f\rangle \Vdash \neg\psi\}$.

(3) If $\varphi$ is of the form $\bigwedge\!\!\!\bigwedge_{n\in\omega}\psi_n$ then $D_\varphi := \{\langle T, f\rangle \mid \langle T, f\rangle \Vdash \varphi \text{ or for some } n \in \omega, \langle T, f\rangle \Vdash \neg\psi_n\}$.

**Lemma 3.10.** For each $\mathcal{L}_{\text{tree}}$ formula $\varphi$, the set $D_\varphi$ is dense.

**Proof.** Let $\varphi$ be an $\mathcal{L}_{\text{tree}}$ formula and $\langle T, f\rangle$ be any condition. We need to find an extension of $\langle T, f\rangle$ which is in $D_\varphi$. There are three cases to consider, depending on the form of $\varphi$. We will prove each of the three cases separately. Note that this proof does not use induction; it is just a proof by cases.

**Atomic formulas.** Suppose $\varphi$ is the atomic formula $\underline{\sigma} \in X$. There are a few cases to consider.

(1) Suppose $\sigma \notin T$, $\sigma$ is nonempty, $\sigma^- \in T$ and $f(\sigma^-) > 0$. In this case we can extend $\langle T, f\rangle$ by adding $\sigma$ to $T$ with label 0.

(2) Suppose $\sigma \notin T$, $\sigma$ is nonempty and $\sigma^- \notin T$. Let $\tau$ be the last ancestor of $\sigma$ which is in $T$ and let $\tau'$ be the unique child of $\tau$ which is an ancestor of $\sigma$. In this case, we can extend $\langle T, f\rangle$ by adding $\tau'$ to $T$ with label 0.

(3) In all other cases, $\langle T, f\rangle$ is already in $D_\varphi$.

**Negations.** Suppose $\varphi$ has the form $\neg\psi$. If there is some $\langle S, g\rangle$ extending $\langle T, f\rangle$ such that $\langle S, g\rangle \Vdash \psi$ then we are done by definition of $D_\varphi$. And if not, then by definition of the forcing relation, $\langle T, f\rangle \Vdash \neg\psi$ and so we are done in this case as well.

**Infinite Conjunctions.** Suppose $\varphi$ has the form $\bigwedge\!\!\!\bigwedge_{n\in\omega}\psi_n$. We have two cases to consider.

(1) Suppose there is some $n \in \omega$ and condition $\langle S, g\rangle \leq \langle T, f\rangle$ such that no extension of $\langle S, g\rangle$ forces $\psi_n$. Then by definition of the forcing relation, $\langle S, g\rangle \Vdash \neg\psi_n$. Since $\langle S, g\rangle$ extends $\langle T, f\rangle$ we are done.

(2) Suppose that for every $n \in \omega$ and condition $\langle S, g\rangle \leq \langle T, f\rangle$, there is some condition extending $\langle S, g\rangle$ which forces $\psi_n$. Then by lemma 3.8, for each $n \in \omega$, $\langle T, f\rangle \Vdash \psi_n$. Thus $\langle T, f\rangle \Vdash \varphi$ and so we are done.

**Lemma 3.11.** Suppose $\varphi$ is an $\mathcal{L}_{\text{tree}}$ formula and $G$ is a filter such that $G$ meets $D_\psi$ for every subformula $\psi$ of $\varphi$. Then $T_G \vDash \varphi$ if and only if there is some condition $\langle T, f\rangle \in G$ such that $\langle T, f\rangle \Vdash \varphi$.

**Proof.** We will use induction on $\varphi$.

**Atomic formulas.** Suppose $\varphi$ is the atomic formula $\underline{\sigma} \in X$ and $G$ is a filter which meets $D_\varphi$.

($\Longrightarrow$) First suppose that $T_G \vDash \varphi$. Thus $\sigma \in T_G$ and so there is some $\langle T, f\rangle \in G$ such that $\sigma \in T$. By definition of the forcing relation, this condition forces $\varphi$.

($\Longleftarrow$) Now suppose that there is some $\langle T, f\rangle \in G$ such that $\langle T, f\rangle \Vdash \varphi$. Since $G$ meets $D_\varphi$, there is some $\langle S, g\rangle \in G$ such that either $\sigma \in S$ or some ancestor of $\sigma$ is in $S$ and is assigned the label 0 by $g$. In the former case, we have $\sigma \in T_G$ and thus $T_G \vDash \varphi$ and the latter case is impossible because it is not possible to find a common extension of $\langle T, f\rangle$ and $\langle S, g\rangle$.

**Negations.** Suppose $\varphi$ has the form $\neg\psi$ and $G$ is a filter which meets $D_\theta$ for every subformula $\theta$ of $\varphi$ (and in particular, every subformula of $\psi$).

( $\Longrightarrow$ ) First suppose that $T_G \vDash \varphi$. Since $G$ meets $D_\varphi$, there is some $\langle T, f\rangle \in G$ such that either $\langle T, f\rangle \Vdash \psi$ or $\langle T, f\rangle \Vdash \neg\psi$. The former case is impossible: by the inductive hypothesis, the existence of such a condition in $G$ would imply $T_G \vDash \psi$. In the latter case we are done since we have found a condition in $G$ which forces $\varphi$.

( $\Longleftarrow$ ) Now suppose that for some $\langle T, f\rangle \in G$, $\langle T, f\rangle \Vdash \varphi$. This implies that no condition in $G$ forces $\psi$: if we had such a condition $\langle S, g\rangle$ then we could find a common extension of $\langle T, f\rangle$ and $\langle S, g\rangle$ in $G$. By monotonicity this condition would force both $\psi$ and $\neg\psi$ which violates consistency. By the inductive hypothesis applied to $\psi$, if no condition in $G$ forces $\psi$ then $T_G \nvDash \psi$. Thus $T_G \vDash \neg\psi$ and so we are done.

**Infinite conjunctions.** Suppose $\varphi$ has the form $\bigwedge\!\!\!\bigwedge_{n\in\omega} \psi_n$ and $G$ is a filter which meets $D_\theta$ for every subformula $\theta$ of $\varphi$.

( $\Longrightarrow$ ) First suppose that $T_G \vDash \bigwedge\!\!\!\bigwedge_{n\in\omega} \psi_n$. Since $G$ meets $D_\varphi$, there is some condition $\langle T, f\rangle \in G$ such that either $\langle T, f\rangle \Vdash \varphi$ or $\langle T, f\rangle \Vdash \neg\psi_n$ for some $n \in \omega$. In the former case we are done. The latter case is impossible: by monotonicity and consistency of the forcing relation, it implies that no condition in $G$ forces $\psi_n$. By the induction hypothesis applied to $\psi_n$, this means that $T_G \nvDash \psi_n$, which contradicts the assumption that $T_G \vDash \varphi$.

( $\Longleftarrow$ ) Now suppose that for some $\langle T, f\rangle \in G$, $\langle T, f\rangle \Vdash \varphi$. Thus for each $n \in \omega$, $\langle T, f\rangle \Vdash \psi_n$. So by the induction hypothesis applied to each $\psi_n$, we have that $T_G \vDash \psi_n$ for each $n$. By definition this means that $T_G \vDash \varphi$.

The proof of theorem 3.5 now follows easily from what we have proved so far.

**Proof of Theorem 3.5.** This follows immediately from the preceding lemma, the density of $D_\psi$ for each formula $\psi$ and the fact that each $\mathcal{L}_{\text{tree}}$ formula has only countably many subformulas.

# 4 Retagging Lemma

In this section we will prove a lemma which is crucial to just about every application of Steel forcing. Informally, it says that the forcing relation for formulas of rank $\alpha$ cannot tell the difference between ordinals larger than $\omega \cdot \alpha$, or between such ordinals and $\infty$. To make this precise, we first define a notion of equivalence of labelled trees if we ignore the differences between all ordinals larger than some fixed ordinal $\alpha$.

**Definition 4.1.** Suppose $\alpha$ is a countable ordinal and $T$ is a finite tree on $\omega$. Steel forcing conditions $\langle T, f\rangle$ and $\langle T, g\rangle$ *agree below* $\alpha$ if for all $\sigma \in T$,

$$f(\sigma) \neq g(\sigma) \implies f(\sigma) > \alpha \text{ and } g(\sigma) > \alpha.$$

As usual, in this definition we regard $\infty$ as larger than any ordinal. We sometimes refer to $\langle T, g\rangle$ as an *$\alpha$-relabelling* or *$\alpha$-retagging* of $\langle T, f\rangle$.

**Lemma 4.2** (Retagging Lemma)**.** Suppose $\varphi$ is an $\mathcal{L}_{\text{tree}}$ formula of rank $\alpha$ and that $\langle T, f\rangle$ and $\langle T, g\rangle$ are conditions which agree below $\omega \cdot \alpha$. Then $\langle T, f\rangle \Vdash \varphi$ if and only if $\langle T, g\rangle \Vdash \varphi$.

**Proof.** We will prove this by induction on the rank of $\varphi$. There are three cases, depending on the form of $\varphi$, but only the third case (negation) requires any real work.

**Atomic formulas.** Suppose that $\varphi$ is the atomic formula $\underline{\sigma} \in X$. Let $\langle T, f\rangle$ and $\langle T, g\rangle$ be conditions which agree below $\omega \cdot \text{rank}(\varphi) = 0$. Note that this means $f$ and $g$ agree on any node which is labelled 0 by either $f$

or $g$. Thus by definition of the forcing relation for atomic formulas we have

$$\begin{aligned}\langle T, f\rangle \Vdash \varphi &\iff \sigma \in T \text{ or } (|\sigma| > 0 \text{ and } \sigma^- \in T \text{ and } f(\sigma^-) = 0)\\ &\iff \sigma \in T \text{ or } (|\sigma| > 0 \text{ and } \sigma^- \in T \text{ and } g(\sigma^-) = 0)\\ &\iff \langle T, g\rangle \Vdash \varphi.\end{aligned}$$

**Infinite conjunctions.** Suppose that $\varphi$ is a formula of the form $\bigwedge\!\!\!\bigwedge_{n\in\omega} \psi_n$. Let $\langle T, f\rangle$ and $\langle T, g\rangle$ be conditions which agree below $\omega \cdot \mathrm{rank}(\varphi)$. Note that each $\psi_n$ has strictly lower rank than $\varphi$ and thus we may apply the induction hypothesis to all of them. By the definition of the forcing relation for infinite conjunctions and the inductive hypothesis applied to the $\psi_n$'s, we have

$$\begin{aligned}\langle T, f\rangle \Vdash \varphi &\iff \forall n \in \omega,\ \langle T, f\rangle \Vdash \psi_n\\ &\iff \forall n \in \omega,\ \langle T, g\rangle \Vdash \psi_n\\ &\iff \langle T, g\rangle \Vdash \varphi.\end{aligned}$$

**Negations.** Suppose $\varphi$ is a formula of the form $\neg\psi$. Let $\alpha$ be the rank of $\psi$ and thus $\alpha + 1$ is the rank of $\varphi$. Let $\langle T, f\rangle$ and $\langle T, g\rangle$ be conditions which agree below $\omega \cdot (\alpha + 1) = \omega \cdot \alpha + \omega$. We want to show that $\langle T, f\rangle \Vdash \varphi$ if and only if $\langle T, g\rangle \Vdash \varphi$. By symmetry, it suffices to only prove one direction.

So suppose that $\langle T, f\rangle \nVdash \varphi$ and we will show that $\langle T, g\rangle \nVdash \varphi$. By definition, we have that there is some $\langle T', f'\rangle \leq \langle T, f\rangle$ such that $\langle T', f'\rangle \Vdash \psi$. Our goal is to find some condition extending $\langle T, g\rangle$ which also forces $\psi$. We will do so by modifying $f'$ to get some $g'$ which is compatible with $g$ and only disagrees with $f'$ above $\omega \cdot \alpha$.

To see why this is sufficient to finish the proof, note that since $\langle T', f'\rangle$ and $\langle T', g'\rangle$ only disagree above $\omega \cdot \alpha$, we can apply the inductive hypothesis to conclude that $\langle T', g'\rangle \Vdash \psi$. Since $\langle T', g'\rangle$ extends $\langle T, g\rangle$ we are done.

We will now explain how to construct $g'$. Let $\sigma$ be a node in $T$. For the label of $\sigma$ itself, we have no choice but to set $g'(\sigma) = g(\sigma)$. Since $\langle T, f\rangle$ and $\langle T, g\rangle$ agree below $\omega \cdot \alpha + \omega$, this is safe to do. Now we will explain how to label all descendants of $\sigma$ which are in $T'$ and whose last ancestor in $T$ is $\sigma$. There are two cases to consider.

(1) First, suppose $g(\sigma) = f(\sigma)$. Then we are free to assign $g'(\tau) = f'(\tau)$ for all $\tau$'s in $T'$ whose last ancestor in $T$ is $\sigma$.

(2) Second, suppose $g(\sigma) \neq f(\sigma)$. Since $f$ and $g$ agree below $\omega \cdot \alpha + \omega$, this means both $g(\sigma)$ and $f(\sigma)$ must be strictly greater than $\omega \cdot \alpha$. Let $n$ be the maximum height among all nodes in $T'$ whose last ancestor in $T$ is $\sigma$. For all such nodes $\tau$, define $g'(\tau) = \omega \cdot \alpha + n + 1 - |\tau|$.

It is straightforward to verify that $g'$ is a valid labelling of $T'$ which extends $g$ and which only disagrees with $f'$ above $\omega \cdot \alpha$.

# 5 Application: $\mathsf{WF}$ and $\mathsf{UB}$ are Borel Inseparable

We will now show how Steel forcing can be used to prove a theorem in descriptive set theory. Recall that $\mathsf{WF}$ is the set of trees on $\omega$ with no infinite path and $\mathsf{UB}$ is the set of trees on $\omega$ which have exactly one infinite path[1]. It is a standard result that both of these sets are complete $\mathbf{\Pi}^1_1$ sets in $\mathcal{P}(\omega^{<\omega})$. We will show that they cannot be separated by a Borel set, which was first proved by Becker in [1].

**Theorem 5.1.** There is no Borel set $A$ such that $\mathsf{UB} \subseteq A$ and $\mathsf{WF} \subseteq \omega^\omega \setminus A$.

We will first describe our strategy for proving this theorem. Suppose there was a Borel set $A$ separating $\mathsf{WF}$ and $\mathsf{UB}$. Since $A$ is Borel, it is defined by an $\mathcal{L}_{\text{tree}}$ formula, $\varphi$. Let $\alpha$ be the rank of $\varphi$.

Now imagine for a moment that we can find a condition $\langle T, f\rangle$ such that for all sufficiently generic filters $G$ containing $\langle T, f\rangle$, $T_G$ is a tree with exactly one infinite path. Then for any such $T_G$ we have $T_G \vDash \varphi$ and so there must be some condition $\langle S, g\rangle$ such that $\langle S, g\rangle \Vdash \varphi$. However, we can relabel the nodes of $S$ so that no node has label $\infty$ but the new labelling agrees with the old labelling below $\omega \cdot \alpha$. This yields a new condition $\langle S, \tilde{g}\rangle$. By the retagging lemma, $\langle S, \tilde{g}\rangle \Vdash \varphi$. However, it is easy to check that for any sufficiently generic filter $F$ extending $\langle S, \tilde{g}\rangle$, $T_F$ is well-founded

[1] $\mathsf{WF}$ stands for "well founded" and $\mathsf{UB}$ for "unique branch."

and thus we must have $T_F \vDash \neg\varphi$. If $F$ is sufficiently generic then this contradicts the fact that $F$ contains a condition forcing $\varphi$, so we are done.

The trouble with this argument is that there is no condition $\langle T, f\rangle$ with the property that we wanted—in general if $\langle T, f\rangle$ has no nodes labelled $\infty$ then any filter containing it will yield a well-founded tree and if $\langle T, f\rangle$ has a node labelled $\infty$ then any sufficiently generic filter containing it will have infinitely many paths extending that node.

The solution is to define a variation on Steel forcing, which we will call UB-forcing, such that generic filters for UB-forcing do yield trees with exactly one infinite path. We will then show that the forcing relation for UB-forcing matches the forcing relation for Steel forcing closely enough for us to carry out something like the argument above.

First, let's define the UB-forcing partial order. A condition in the UB-forcing partial order consists of a condition $\langle T, f\rangle$ in the Steel forcing partial order which satisfies both of the following properties.

(1) The root of $T$ (i.e. the empty sequence) has label $\infty$.

(2) On each level in $T$, there is at most one node labelled $\infty$. In other words, for all $\sigma, \tau \in T$, if $|\sigma| = |\tau|$ and $f(\sigma) = f(\tau) = \infty$ then $\sigma = \tau$.

The ordering on the UB-forcing partial order is the same as for Steel forcing.

We can now retrace our development of basic definitions and facts about Steel forcing for UB forcing instead. Instead of doing this in detail, we will simply give a summary and let the reader fill in the details should they wish to do so.

First, for any filter $G$ on the UB-forcing partial order, we can define a tree $T_G$ in exactly the same way that we did for Steel forcing:

$$T_G := \bigcup\{T \mid \langle T, f\rangle \in G \text{ for some function } f\}.$$

If $G$ is sufficiently generic then this tree is infinite and has exactly one infinite branch.

**Lemma 5.2.** For any sufficiently UB-generic filter $G$, $T_G$ has exactly one infinite branch.

Next, we can define a forcing relation $\Vdash_{\mathsf{UB}}$ for UB-forcing. Its definition follows exactly the same scheme as the definition of the forcing relation for Steel forcing:

(1) If $\varphi$ is an atomic formula of the form $\underline{\sigma} \in X$ then $\langle T, f\rangle \Vdash_{\mathsf{UB}} \varphi$ if and only if $\underline{\sigma} \in T$ or $\sigma$ is nonempty, $\sigma^- \in T$ and $f(\sigma^-) > 0$.

(2) If $\varphi$ is of the form $\neg\psi$ then $\langle T, f\rangle \Vdash_{\mathsf{UB}} \varphi$ if and only if for all UB-forcing conditions $\langle S, g\rangle$ such that $\langle S, g\rangle \leq \langle T, f\rangle$, $\langle S, g\rangle \nVdash_{\mathsf{UB}} \psi$.

(3) If $\varphi$ is of the form $\bigwedge\!\!\!\bigwedge_{n\in\omega} \psi_n$ then $\langle T, f\rangle \Vdash_{\mathsf{UB}} \varphi$ if and only if for all $n \in \omega$, $\langle T, f\rangle \Vdash_{\mathsf{UB}} \psi_n$.

It is now possible to prove the fundamental fact about the forcing relation for the UB-forcing partial order. The proof is identical to the proof we gave for Steel forcing.

**Lemma 5.3.** Suppose $\varphi$ is an $\mathcal{L}_{\text{tree}}$ formula. For any sufficiently UB-generic filter $G$, $T_G \vDash \varphi$ if and only if there is some $\langle T, f\rangle \in G$ such that $\langle T, f\rangle \Vdash_{\mathsf{UB}} \varphi$.

We now come to the key lemma. First note that for an $\mathcal{L}_{\text{tree}}$ formula $\varphi$ and a condition $\langle T, f\rangle$ in the UB-forcing partial order, we have two possible forcing relations between $\langle T, f\rangle$ and $\varphi$, namely $\langle T, f\rangle \Vdash \varphi$ and $\langle T, f\rangle \Vdash_{\mathsf{UB}} \varphi$. The definitions of these two relations look very similar, but with one key difference: in the definition of $\langle T, f\rangle \Vdash_{\mathsf{UB}} \neg\varphi$, instead of quantifying over all extensions of $\langle T, f\rangle$ in the Steel forcing partial order, we only quantify over all extensions of $\langle T, f\rangle$ which are in the UB-forcing partial order.

We will now show that despite this apparent difference, the two relations are actually identical. That is, $\Vdash_{\mathsf{UB}}$ is simply the restriction of $\Vdash$ to the UB-forcing partial order. This allows us to move back and forth between UB forcing and Steel forcing and thus to carry out the proof strategy sketched at the beginning of this section.

**Lemma 5.4.** For any $\mathcal{L}_{\text{tree}}$ formula $\varphi$, if $\langle T, f\rangle$ is a condition in the UB-forcing partial order then $\langle T, f\rangle \Vdash_{\mathsf{UB}} \varphi$ if and only if $\langle T, f\rangle \Vdash \varphi$.

**Proof.** We will prove this by induction on $\varphi$. The only case where any work is required is when $\varphi$ has the form $\neg\psi$. So suppose for induction that the lemma's conclusion holds for $\psi$ and let $\langle T, f\rangle$ be a condition in the UB-forcing partial order. We want to show that $\langle T, f\rangle \Vdash \neg\psi$ if and only if $\langle T, f\rangle \Vdash_{\mathsf{UB}} \neg\psi$.

The forwards direction is easy. We will prove the contrapositive. So suppose that $\langle T, f\rangle \nVdash_{\mathsf{UB}} \neg\psi$. Thus for some $\mathsf{UB}$-forcing condition $\langle S, g\rangle$ extending $\langle T, f\rangle$, $\langle S, g\rangle \Vdash_{\mathsf{UB}} \psi$. By the inductive assumption, $\langle S, g\rangle \Vdash \psi$. Since the ordering on the $\mathsf{UB}$-forcing partial order matches the ordering on the Steel forcing partial order, $\langle S, g\rangle$ is a condition extending $\langle T, f\rangle$ in the Steel forcing partial order such that $\langle S, g\rangle \Vdash \psi$ and thus $\langle T, f\rangle \nVdash \neg\psi$.

Now for the harder direction. Suppose that $\langle T, f\rangle \nVdash \neg\psi$. Thus there is some $\langle S, g\rangle \leq \langle T, f\rangle$ such that $\langle S, g\rangle \Vdash \psi$. The problem is that $\langle S, g\rangle$ may not be an element of the $\mathsf{UB}$-forcing partial order (if it was then we would be done by applying the inductive assumption and the definition of $\Vdash_{\mathsf{UB}}$ for negations, as above). The key idea of this proof is that we may retag it so that it is an element of the $\mathsf{UB}$ forcing partial order without changing whether it extends $\langle T, f\rangle$ and whether it forces $\psi$.

Let's consider how $\langle S, g\rangle$ could fail to be an element of the $\mathsf{UB}$-forcing partial order. There are only two requirements that a condition in the Steel forcing partial order needs to meet to be in the $\mathsf{UB}$-forcing partial order: first, the root needs to have label $\infty$ and second, every level of the tree needs to have at most one node labelled $\infty$. Since $\langle S, g\rangle$ extends $\langle T, f\rangle$, its root must have label $\infty$. Thus the only way $\langle S, g\rangle$ can fail to be a $\mathsf{UB}$-forcing condition is if there are one or more levels on which multiple nodes are labelled $\infty$.

By the *number of collisions* in $\langle S, g\rangle$, we mean the total number of nodes in $\langle S, g\rangle$ which are labelled $\infty$ and which have at least one other node on the same level also labelled $\infty$. We will show by induction on the number of collisions that we can find a labelling $\tilde{g}$ of $S$ such that $\langle S, \tilde{g}\rangle$ is a condition in the $\mathsf{UB}$-forcing partial order which extends $\langle T, f\rangle$ and $\langle S, \tilde{g}\rangle \Vdash \psi$.

If $\langle S, g\rangle$ has no collisions then we are done by setting $\tilde{g} = g$. If not, then we can find some node in $S$ which is labelled $\infty$ and which has some other node on its level also labelled $\infty$. Furthermore, we can find at least one such node which is not in $T$. Let $\sigma$ be such a node of maximum height. Note that none of $\sigma$'s children can be labelled $\infty$. Now we can simply modify $g$ by replacing $\sigma$'s label with some countable ordinal larger than the maximum of $\omega \cdot \operatorname{rank}(\psi)$ and the labels of all of $\sigma$'s children. This gives us a new tree which still extends $\langle T, f\rangle$ in the Steel forcing partial order and which, by the retagging lemma, still forces $\psi$. Since this new tree has fewer collisions than $\langle S, g\rangle$ we can now apply the inductive hypothesis to finish.

We can now prove the theorem with which we began this section.

**Proof of theorem 5.1.** Suppose for contradiction that $A$ is a Borel set such that $\mathsf{UB} \subseteq A$ and $\mathsf{WF} \cap A = \varnothing$. Let $\varphi$ be an $\mathcal{L}_{\text{tree}}$ formula which defines $A$. Let $G$ be a filter in the $\mathsf{UB}$-forcing partial order which is sufficiently generic that $T_G$ has exactly one branch and $T_G \vDash \varphi$ if and only if some condition in $G$ forces $\varphi$. Since $T_G \in \mathsf{UB}$, $T_G \in A$ and thus $T_G \vDash \varphi$. Therefore there is some condition $\langle T, f\rangle \in G$ such that $\langle T, f\rangle \Vdash \varphi$.

By an argument similar to the one in the proof of lemma 5.4, we can find a relabelling $g$ of $T$ such that no node is labelled with $\infty$, but $g$ and $f$ agree below $\omega \cdot \operatorname{rank}(\varphi)$. By the retagging lemma, we have $\langle T, g\rangle \vDash \varphi$. Now let $F$ be a filter containing $\langle T, g\rangle$. If $F$ is sufficiently generic, we have that $T_F \vDash \varphi$. But if $F$ is sufficiently generic, we also have that the root node of $T_F$ has the same rank as the label assigned by $g$ to the root node of $T$, and in particular, $T_F$ is well-founded. But if $T_F$ is well-founded then we must have $T_F \notin A$ and hence $T_F \nvDash \varphi$, which is a contradiction.

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